## Pacific

## Journal of

## Mathematics

# HIGHER HOMOTOPY COMMUTATIVITY OF $H$-SPACES AND HOMOTOPY LOCALIZATIONS 

Yusuke Kawamoto

# HIGHER HOMOTOPY COMMUTATIVITY OF $\boldsymbol{H}$-SPACES AND HOMOTOPY LOCALIZATIONS 

Yusuke Kawamoto<br>Dedicated to Professor Takao Matumoto on his sixtieth birthday

In this paper, we prove that the homotopy localization of an $A C_{n}$-space is an $A C_{n}$-space so that the universal map is an $A C_{n}$-map. This result is used to study the higher homotopy commutativity of $\boldsymbol{H}$-spaces with finitely generated cohomology over the Steenrod algebra $\mathscr{A}_{p}^{*}$. Our result shows that for any prime $p$, if $X$ is a connected $A C_{p}$-space whose $\bmod p$ cohomology $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathscr{A}_{p}^{*}$, then $X$ has the mod $\boldsymbol{p}$ homotopy type of a Postnikov $\boldsymbol{H}$-space.

## 1. Introduction

The theory of $H$-spaces has been studied in algebraic topology to understand homotopy properties of Lie groups. Given a prime $p$, a $\mathbb{Z} / p$-finite $H$-space is an $H$-space whose mod $p$ cohomology is finite dimensional. In recent decades, many theorems have been proved about $\mathbb{Z} / p$-finite $H$-spaces [Kane 1988; Lin 1995], which suggest that they have many similar properties to those of Lie groups.

In this paper, we study an $H$-space which need not be $\mathbb{Z} / p$-finite but whose $\bmod p$ cohomology is finitely generated as an algebra over the Steenrod algebra $\mathscr{A}_{p}^{*}$. For example, the $n$-connected covering $X\langle n\rangle$ of a $\mathbb{Z} / p$-finite $H$-space $X$ is not $\mathbb{Z} / p$-finite for $n \geq 3$ but the $\bmod p$ cohomology is finitely generated as an algebra over $\mathscr{A}_{p}^{*}$, by [Castellana et al. 2006, Corollary 4.3]. Eilenberg-Mac Lane spaces $K(\mathbb{Z}, n)$ and $K\left(\mathbb{Z} / p^{i}, n\right)$ are other examples for $n, i \geq 1$.

Using the homotopy localizations of Bousfield [1994] and Dror Farjoun [1996], Castellana, Crespo and Scherer have studied $H$-spaces with finitely generated cohomology over $\mathscr{A}_{p}^{*}$ [Castellana et al. 2007]. In their Theorem 7.3, these authors proved that if $X$ is such an $H$-space, the $B \mathbb{Z} / p$-localization $L_{B \mathbb{Z} / p}(X)$ is a $\mathbb{Z} / p$ finite $H$-space and the homotopy fiber $F\left(\phi_{X}\right)$ of the universal map $\phi_{X}: X \rightarrow$ $L_{B \mathbb{Z} / p}(X)$ is $\bmod p$ homotopy equivalent to a Postnikov $H$-space (see Theorem

[^0]5.1). Here an $H$-space is called Postnikov if the homotopy groups are finitely generated over the $p$-adic integers $\mathbb{Z}_{p}^{\wedge}$ which vanish above some dimension, and $\bmod p$ homotopy equivalence means homotopy equivalence up to $p$-completion in the sense of [Bousfield and Kan 1972].

Moreover, by combining their main result with the mod 2 torus theorem by Hubbuck [1969] and Lin [1985], Castellana et al. generalized results of Slack [1991] and Lin and Williams [1991], as follows:

Theorem 1.1 [Castellana et al. 2007, Corollary 7.4]. If $X$ is a connected homotopy commutative $H$-space whose $\bmod 2$ cohomology $H^{*}(X ; \mathbb{Z} / 2)$ is finitely generated as an algebra over $\mathscr{A}_{2}^{*}$, then $X$ is $\bmod 2$ homotopy equivalent to a Postnikov $H$ space.

On the other hand, the odd prime version of Theorem 1.1 does not hold. In fact, Iriye and Kono [1985] showed that for an odd prime $p$, any connected $H$-space is mod $p$ homotopy equivalent to a homotopy commutative $H$-space. Moreover, $S p(2)_{3}^{\wedge}$ for $p=3$ and $\left(S^{3}\right)_{p}^{\wedge}$ for $p \geq 5$ are examples of homotopy commutative loop spaces which are not Postnikov $H$-spaces by McGibbon [1984], where $Y_{p}^{\wedge}$ denotes the $p$-completion of a space $Y$.

To describe an odd prime version of Theorem 1.1, we use the higher homotopy commutativity of the multiplication. Such notions are first considered by Sugawara [1960] and Williams [1969] in the case of topological monoids. (The higher homotopy commutativity of the third order in the sense of Williams is illustrated by the left hexagon in Figure 1.)

Williams' definition was generalized to the case of $A_{n}$-spaces in [Hemmi and Kawamoto 2004] (see also [Hemmi 1991]). An $A_{n}$-space with a multiplication admitting the higher homotopy commutativity of the $n$-th order is called an $A C_{n^{-}}$ space. By [Hemmi and Kawamoto 2004, Example 3.2(1)], an $A C_{2}$-space is the same as a homotopy commutative $H$-space. Let $X$ be an $A_{3}$-space admitting an $A C_{2}$-structure. Then by using the associating homotopy $M_{3}: I \times X^{3} \rightarrow X$ and the


Figure 1. The higher homotopy commutativity of the third order.
commuting homotopy $Q_{2}: I \times X^{2} \rightarrow X$, we can define a map $\widetilde{Q}_{3}: S^{1} \times X^{3} \rightarrow X$ illustrated by the right dodecagon in Figure 1. For example, the uppermost edge represents the commuting homotopy between $x y$ and $y x$ given by $Q_{2}(t, x, y) z$, and the next right edge is the associating homotopy between $(x y) z$ and $x(y z)$ given by $M_{3}(t, x, y, z)$. Then $X$ is an $A C_{3}$-space if and only if $\widetilde{Q}_{3}$ is extended to a map $Q_{3}: D^{2} \times X^{3} \rightarrow X$. In general, $X$ is an $A C_{n}$-space if and only if there is a family of maps

$$
\left\{Q_{i}: D^{i-1} \times X^{i} \rightarrow X\right\}_{1 \leq i \leq n}
$$

with the relations in [Hemmi and Kawamoto 2004, Proposition 2.1].
To generalize Theorem 1.1 to the case of any prime $p$, we first show:
Theorem A. Let $A$ be a topological space and $n \geq 1$. If $X$ is an $A C_{n}$-space, then the A-localization $L_{A}(X)$ is an $A C_{n}$-space so that the universal map $\phi_{X}: X \rightarrow L_{A}(X)$ is an $A C_{n}$-map.

From Theorem A and [Castellana et al. 2007, Theorem 7.3], we can generalize the $\bmod p$ torus theorem stated in [Hemmi and Kawamoto 2004, Corollary 1.1] to the case of $A C_{p}$-spaces with finitely generated cohomology over $\mathscr{A}_{p}^{*}$.

Theorem B. Let $p$ be a prime. If $X$ is a connected $A C_{p}$-space whose $\bmod p$ cohomology $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathscr{A}_{p}^{*}$, then $X$ is $\bmod p$ homotopy equivalent to a Postnikov $H$-space.

Theorem B is a generalization of Theorem 1.1 to the case of any prime $p$ since an $A C_{2}$-space is the same as a homotopy commutative $H$-space. In the above theorem, the assumption of $A C_{p}$-space cannot be relaxed to $A C_{p-1}$-space. In fact, by [Hemmi and Kawamoto 2004, Proposition 3.8], the $(2 m-1)$-dimensional sphere $\left(S^{2 m-1}\right)_{p}^{\wedge}$ is an $A C_{p-1}$-space but not a Postnikov $H$-space for $m \geq 2$.

Moreover, since the loop space of an $H$-space admits an $A C_{\infty}$-structure by [Hemmi and Kawamoto 2004, Example 3.2(3)], Theorem B implies:

Corollary 1.2 [Castellana et al. 2007, p. 17]. Let p be a prime. Assume that $X$ is a connected loop space whose mod $p$ cohomology $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathscr{A}_{p}^{*}$. If the classifying space $B X$ is an $H$-space, then $X$ is $\bmod$ $p$ homotopy equivalent to a Postnikov $H$-space.

There is an example of a Postnikov loop space $Y$ admitting an $A C_{p}$-structure such that the classifying space $B Y$ is not an $H$-space by [McGibbon 1989, Example 5]. Corollary 1.2 is a generalization of results from [Aguadé and Smith 1986; Kawamoto 1999; Lin 1994].

Bousfield [2001] studied the $K(n)_{*}$-localizations of Postnikov $H$-spaces, where $K(n)_{*}$ denotes the Morava $K$-homology theory for $n \geq 1$. By Theorem B and [Bousfield 2001, Theorem 7.2], we have:

Corollary 1.3. Let $p$ be a prime and $n \geq 1$. If $X$ is a connected $A C_{p}$-space whose $\bmod p$ cohomology $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathscr{A}_{p}^{*}$, then the $K(n)_{*}$-localization $L_{K(n)_{*}}\left(X_{p}^{\wedge}\right)$ is $\bmod p$ homotopy equivalent to the $\Sigma^{n} B \mathbb{Z} / p$-localization $L_{\Sigma^{n} B \mathbb{Z} / p}\left(X_{p}^{\wedge}\right)$. In particular, $X_{p}^{\wedge}$ is $K(n)_{*}$-local if and only if the $n$-fold loop space $\Omega^{n} X_{p}^{\wedge}$ is $B \mathbb{Z} / p$-local.

We also generalize [Hemmi and Kawamoto 2007, Theorem B] to the case of $A_{p}$-spaces with finitely generated cohomology over $\mathscr{A}_{p}^{*}$.

Theorem C. Let $p$ be an odd prime. Assume that $X$ is a connected $A_{p}$-space admitting an $A C_{n}$-structure with $n>(p-1) / 2$ and the $\bmod p$ cohomology $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathscr{A}_{p}^{*}$. If the Steenrod operations $\mathscr{P}^{j}$ act on the indecomposable module $Q H^{*}(X ; \mathbb{Z} / p)$ trivially for $j \geq 1$, then $X$ is $\bmod p$ homotopy equivalent to a finite product of $\left(S^{1}\right)_{p}^{\wedge} s,\left(\mathbb{C} P^{\infty}\right)_{p}^{\wedge} s$ and $B \mathbb{Z} / p^{i} s$ for $i \geq 1$.

In Theorem C, the assumption $n>(p-1) / 2$ is necessary. In fact, by [Hemmi 1991, Theorem 2.4], $\left(S^{3}\right)_{p}^{\wedge}$ is an $A_{p}$-space admitting an $A C_{(p-1) / 2}$-structure for any odd prime $p$.

Outline of article. In Section 2, we recall the associahedra, the multiplihedra and the permuto-associahedra. Then we show that the permuto-associahedra are decomposed by using the multiplihedra in a combinatorial way (see Proposition 2.1). In Section 3, we give the definition of an $A C_{n}$-map between $A C_{n}$-spaces by using Proposition 2.1 (see Definition 3.1). Section 4 is devoted to the proof of Theorem A. We show that if $\phi: X \rightarrow Y$ is an $A_{n}$-map between $A_{n}$-spaces and $Y$ is $\phi$-local, then $\phi$ transmits an $A C_{n}$-structure from $X$ to $Y$ (see Proposition 4.1). By applying Proposition 4.1 to the universal map $\phi_{X}: X \rightarrow L_{A}(X)$ for the $A$-localization of $X$, we prove Theorem A. In Section 5, we first recall the result of [Castellana et al. 2007] on $H$-spaces with finitely generated cohomology over $\mathscr{A}_{p}^{*}$ (Theorem 5.1). From Theorem A, Theorem 5.1 and the results in [Hemmi 1991; Hemmi and Kawamoto 2004], we prove Theorem B. Next Corollary 1.3 is proved by Theorem B and the result from [Bousfield 2001] on the $K(n)_{*}$-localizations of Postnikov H spaces. We finally give the proof of Theorem C by using Theorem A and [Hemmi and Kawamoto 2007, Theorem B].

The content of the paper was first presented in a conference on algebraic topology at Shinshu University in July 2005. The author is grateful to the organizers for their kind invitation and hospitality.

## 2. Decompositions of the permuto-associahedra

We first recall the associahedra $\left\{K_{n}\right\}_{n \geq 2}$ of Stasheff and the multiplihedra $\left\{J_{n}\right\}_{n \geq 1}$ of Iwase and Mimura.


Figure 2. The associahedra $K_{3}$ and $K_{4}$.
Stasheff [1963, p. 283] constructed the associahedra $\left\{K_{n}\right\}_{n \geq 2}$ to introduce the concept of $A_{n}$-space (see Section 3). From the construction, the associahedron $K_{n}$ is an $(n-2)$-dimensional polytope whose boundary $\partial K_{n}$ is given by

$$
\partial K_{n}=\bigcup_{r, s, k} K_{k}(r, s)
$$

for $n \geq 2$, where $r, s \geq 2$ with $r+s=n+1$ and $1 \leq k \leq r$. Here the facet (codimension-one face) $K_{k}(r, s)$ is homeomorphic to the product $K_{r} \times K_{s}$ by a face operator $\partial_{k}(r, s): K_{r} \times K_{s} \rightarrow K_{k}(r, s)$ with the relations in [Stasheff 1963, p. 278, 3(a),(b)]. There is a family of degeneracy operators $\left\{\theta_{j}: K_{n} \rightarrow K_{n-1}\right\}_{1 \leq j \leq n}$ satisfying the relations in [Stasheff 1963, p. 278, Proposition 3].

The associahedra $\left\{K_{n}\right\}_{n \geq 2}$ are also used in [Stasheff 1970, Definition 11.9] to define an $A_{n}$-map from an $A_{n}$-space to a topological monoid.

Iwase and Mimura [1989, §2] introduced the multiplihedra $\left\{J_{n}\right\}_{n \geq 1}$ for the purpose of defining an $A_{n}$-map between $A_{n}$-spaces (see Section 3). From the properties in [Iwase and Mimura 1989, p. 200, (2-a) and (2-b)], the multiplihedron $J_{n}$ is an $(n-1)$-dimensional polytope whose boundary $\partial J_{n}$ is given by

$$
\partial J_{n}=\bigcup_{k, r, s} J_{k}(r, s) \cup \bigcup_{q, r_{1}, \ldots, r_{q}} J\left(q, r_{1}, \ldots, r_{q}\right)
$$

for $n \geq 1$, where $r \geq 1, s \geq 2$ with $r+s=n+1$ and $1 \leq k \leq r$, and $2 \leq q \leq n$, $r_{1}, \ldots, r_{q} \geq 1$ with $r_{1}+\cdots+r_{q}=n$. As in the case of the associahedra, we have face


Figure 3. The multiplihedra $J_{2}$ and $J_{3}$.


Figure 4. The permuto-associahedra $\Gamma_{2}$ and $\Gamma_{3}$.
operators $\delta_{k}(r, s): J_{r} \times K_{s} \rightarrow J_{k}(r, s)$ and $\delta\left(q, r_{1}, \ldots, r_{q}\right): K_{q} \times J_{r_{1}} \times \cdots \times J_{r_{q}} \rightarrow$ $J\left(q, r_{1}, \ldots, r_{q}\right)$ with the relations in (2-c) of the same work. The degeneracy operators $\left\{\xi_{j}: J_{n} \rightarrow J_{n-1}\right\}_{1 \leq j \leq n}$ satisfy the relations in (2-d).

We next recall the permuto-associahedra $\left\{\Gamma_{n}\right\}_{n \geq 1}$ constructed by Kapranov and by Reiner and Ziegler. By [Kapranov 1993, Theorem 2.5] and [Reiner and Ziegler 1994, Theorem 2], the permuto-associahedron $\Gamma_{n}$ is an ( $n-1$ )-dimensional polytope whose faces are described in a combinatorial way for $n \geq 1$ (see also [Ziegler 1995, Definition 9.13, Example 9.14]). In particular, a facet of $\Gamma_{n}$ is represented by a partition of the sequence $\mathbf{n}=(1, \ldots, n)$ into at least two parts. Here a partition of $\mathbf{n}$ of type $\left(t_{1}, \ldots, t_{l}\right)$ is an ordered sequence $\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ consisting of disjoint subsequences $\alpha_{i}$ of $\mathbf{n}$ of length $t_{i}$ with $\alpha_{1} \cup \cdots \cup \alpha_{l}=\mathbf{n}$. See [Hemmi and Kawamoto 2004; Ziegler 1995] for the full details of the partitions.

Let $\Gamma\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ denote the facet of $\Gamma_{n}$ corresponding to a partition $\left(\alpha_{1}, \ldots, \alpha_{l}\right)$. The boundary of $\Gamma_{n}$ is given by

$$
\begin{equation*}
\partial \Gamma_{n}=\bigcup_{\left(\alpha_{1}, \ldots, \alpha_{l}\right)} \Gamma\left(\alpha_{1}, \ldots, \alpha_{l}\right), \tag{2-1}
\end{equation*}
$$

where the union covers all partitions $\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ of $\mathbf{n}$ with $l \geq 2$. If ( $\alpha_{1}, \ldots, \alpha_{l}$ ) is of type $\left(t_{1}, \ldots, t_{l}\right)$, then the facet $\Gamma\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ is homeomorphic to the product $K_{l} \times \Gamma_{t_{1}} \times \cdots \times \Gamma_{t_{l}}$ by a face operator $\epsilon^{\left(\alpha_{1}, \ldots, \alpha_{l}\right)}: K_{l} \times \Gamma_{t_{1}} \times \cdots \times \Gamma_{t_{l}} \rightarrow \Gamma\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ with the relations in Proposition 2.1 of [Hemmi and Kawamoto 2004]. Moreover, there are degeneracy operators $\left\{\omega_{j}: \Gamma_{n} \rightarrow \Gamma_{n-1}\right\}_{1 \leq j \leq n}$ satisfying the relations in Proposition 2.3 of the same reference.

In Definition 3.1, we need the following result:
Proposition 2.1. Let $n \geq 1$.
(1) The permuto-associahedron $\Gamma_{n}$ is decomposed by

$$
\Gamma_{n}=\bigcup_{\left(\beta_{1}, \ldots, \beta_{m}\right)} B\left(\beta_{1}, \ldots, \beta_{m}\right)
$$

where the union covers all partitions $\left(\beta_{1}, \ldots, \beta_{m}\right)$ of $\mathbf{n}$ with $m \geq 1$.
(2) If $\left(\beta_{1}, \ldots, \beta_{m}\right)$ is a partition of $\mathbf{n}$ of type $\left(u_{1}, \ldots, u_{m}\right)$, then $B\left(\beta_{1}, \ldots, \beta_{m}\right)$ is homeomorphic to the product $J_{m} \times \Gamma_{u_{1}} \times \cdots \times \Gamma_{u_{m}}$ by an operator

$$
\iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}: J_{m} \times \Gamma_{u_{1}} \times \cdots \times \Gamma_{u_{m}} \rightarrow B\left(\beta_{1}, \ldots, \beta_{m}\right)
$$

By an inductive argument, we can show:
Lemma 2.2 [Stasheff 1963, p. 288, Proposition 25]. There is a family of homeomorphisms $\left\{\zeta_{m}: I \times K_{m} \rightarrow J_{m}\right\}_{m \geq 2}$ with the relations

$$
\begin{aligned}
\zeta_{m}(0, \sigma) & =\delta_{1}(1, m)(*, \sigma) \\
\zeta_{m}\left(t, \partial_{k}(r, s)(\rho, \sigma)\right) & =\delta_{k}(r, s)\left(\zeta_{r}(t, \rho), \sigma\right)
\end{aligned}
$$

for $r, s \geq 2$ with $r+s=m+1$ and $1 \leq k \leq r$.
Proof of Proposition 2.1. We work by induction on $n$. Since $\Gamma_{1}=J_{1}=*$, the result is clear for $n=1$. We put

$$
\begin{equation*}
U_{n}=\Gamma_{n} \cup_{\{0\} \times \partial \Gamma_{n}} I \times \partial \Gamma_{n}, \tag{2-2}
\end{equation*}
$$

where $I$ is the unit interval and $\{0\} \times \partial \Gamma_{n}$ is identified with $\partial \Gamma_{n} \subset \Gamma_{n}$. It is clear that $U_{n}$ is homeomorphic to the $(n-1)$-dimensional ball.

Let $B(\mathbf{n})=\Gamma_{n} \subset U_{n}$. Then an operator $\iota^{(\mathbf{n})}: J_{1} \times \Gamma_{n} \rightarrow B(\mathbf{n})$ is defined by ${ }_{\iota}{ }^{(\mathbf{n})}(*, \tau)=\tau$ for $\tau \in \Gamma_{n}$. If $m \geq 2$, then by Lemma 2.2, we can identify $J_{m}$ with


Figure 5. The decompositions of $\Gamma_{2}$ and $\Gamma_{3}$.
$I \times K_{m}$ by $\zeta_{m}: I \times K_{m} \rightarrow J_{m}$. Assume that $\left(\beta_{1}, \ldots, \beta_{m}\right)$ is a partition of $\mathbf{n}$ of type $\left(u_{1}, \ldots, u_{m}\right)$ with $m \geq 2$. Put

$$
B\left(\beta_{1}, \ldots, \beta_{m}\right)=I \times \Gamma\left(\beta_{1}, \ldots, \beta_{m}\right) \subset u_{n},
$$

and define an operator $\iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}: J_{m} \times \Gamma_{u_{1}} \times \cdots \times \Gamma_{u_{m}} \rightarrow B\left(\beta_{1}, \ldots, \beta_{m}\right)$ by

$$
\iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}\left(\zeta_{m}(t, \sigma), \tau_{1}, \ldots, \tau_{m}\right)=\left(t, \epsilon^{\left(\beta_{1}, \ldots, \beta_{m}\right)}\left(\sigma, \tau_{1}, \ldots, \tau_{m}\right)\right)
$$

Then by (2-1) and (2-2), we have that

$$
u_{n}=\bigcup_{\left(\beta_{1}, \ldots, \beta_{m}\right)} B\left(\beta_{1}, \ldots, \beta_{m}\right)
$$

where the union covers all partitions $\left(\beta_{1}, \ldots, \beta_{m}\right)$ of $\mathbf{n}$ with $m \geq 1$.
By Lemma 2.2, we see that

$$
\zeta_{m}\left(\{1\} \times K_{m}\right)=\bigcup_{q, r_{1}, \ldots, r_{q}} J\left(q, r_{1}, \ldots, r_{q}\right)
$$

for $2 \leq q \leq m$ and $r_{1}, \ldots, r_{q} \geq 1$ with $r_{1}+\cdots+r_{q}=m$. This implies that
$(2-3) \partial u_{n}=\bigcup_{\left(\beta_{1}, \ldots, \beta_{m}\right)} \iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}\left(\left(\bigcup_{q, r_{1}, \ldots, r_{q}} J\left(q, r_{1}, \ldots, r_{q}\right)\right) \times \Gamma_{u_{1}} \times \cdots \times \Gamma_{u_{m}}\right)$
for $2 \leq q \leq m$ and $r_{1}, \ldots, r_{q} \geq 1$ with $r_{1}+\cdots+r_{q}=m$, where $\left(\beta_{1}, \ldots, \beta_{m}\right)$ are partitions of $\mathbf{n}$ of type $\left(u_{1}, \ldots, u_{m}\right)$ with $m \geq 2$. If we define face operators on $\partial U_{n}$ satisfying the relations in [Hemmi and Kawamoto 2004, Proposition 2.1], then $\partial U_{n}$ is homeomorphic to $\partial \Gamma_{n}$, and it follows that $U_{n}$ is homeomorphic to $\Gamma_{n}$, which implies the required conclusion.

Recall that $\partial \Gamma_{n}$ is given by

$$
\partial \Gamma_{n}=\bigcup_{\left(\alpha_{1}, \ldots, \alpha_{l}\right)} \Gamma\left(\alpha_{1}, \ldots, \alpha_{l}\right),
$$

where the union covers all partitions $\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ of $\mathbf{n}$ with $l \geq 2$.
Assume that $\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ is a partition of $\mathbf{n}$ of type $\left(t_{1}, \ldots, t_{l}\right)$ with $l \geq 2$. Then by inductive hypothesis, we can assume that

$$
\Gamma_{t_{j}}=\bigcup_{\left(\gamma_{j, 1}, \ldots, \gamma_{j, h_{j}}\right)} B\left(\gamma_{j, 1}, \ldots, \gamma_{j, h_{j}}\right)
$$

for $1 \leq j \leq l$, where the union covers all partitions $\left(\gamma_{j, 1}, \ldots, \gamma_{j, h_{j}}\right)$ of $\left(1, \ldots, t_{j}\right)$ with $h_{j} \geq 1$. If $\left(\gamma_{j, 1}, \ldots, \gamma_{j, h_{j}}\right)$ is a partition of $\left(1, \ldots, t_{j}\right)$ of type $\left(v_{j, 1}, \ldots, v_{j, h_{j}}\right)$, then by inductive hypothesis, we have the operator $\iota^{\left(\gamma_{j, 1}, \ldots, \gamma_{j, h_{j}}\right)}: J_{h_{j}} \times \Gamma_{v_{j, 1}} \times \cdots \times$ $\Gamma_{v_{j, h_{j}}} \rightarrow B\left(\gamma_{j, 1}, \ldots, \gamma_{j, h_{j}}\right)$ which is a homeomorphism. Put $m=h_{1}+\cdots+h_{l}$.

We give a partition $\left(\beta_{1}, \ldots, \beta_{m}\right)$ of $\mathbf{n}$ of type $\left(v_{1,1}, \ldots, v_{1, h_{1}}, \ldots, v_{l, 1}, \ldots, v_{l, h_{l}}\right)$ by

$$
\beta_{i}(t)=\alpha_{j} \gamma_{j, i-\left(h_{1}+\cdots+h_{j-1}\right)}(t)
$$

for $h_{1}+\cdots+h_{j-1}+1 \leq i \leq h_{1}+\cdots+h_{j}$ and $1 \leq t \leq i-\left(h_{1}+\cdots+h_{j-1}\right)$. Define a face operator $\epsilon^{\left(\alpha_{1}, \ldots, \alpha_{l}\right)}: K_{l} \times \Gamma_{t_{1}} \times \cdots \times \Gamma_{t_{l}} \rightarrow \partial U_{n}$ by

$$
\begin{aligned}
& \epsilon^{\left(\alpha_{1}, \ldots, \alpha_{l}\right)}\left(\sigma, l^{\left(\gamma_{1,1}, \ldots, \gamma_{1, h_{1}}\right)}\left(\rho_{1}, \tau_{1,1}, \ldots, \tau_{1, h_{1}}\right), \ldots, \iota^{\left(\gamma_{l, 1}, \ldots, \gamma_{l, h_{l}}\right)}\left(\rho_{l}, \tau_{l, 1}, \ldots, \tau_{l, h_{l}}\right)\right) \\
& \quad=\iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}\left(\delta\left(l, h_{1}, \ldots, h_{l}\right)\left(\sigma, \rho_{1}, \ldots, \rho_{l}\right), \tau_{1,1}, \ldots, \tau_{1, h_{1}}, \ldots, \tau_{l, 1}, \ldots, \tau_{l, h_{l}}\right) .
\end{aligned}
$$

By (2-3) and the relation [Iwase and Mimura 1989, p. 201, (c-4)], the face operator satisfies the relations in [Hemmi and Kawamoto 2004, Proposition 2.1]. This implies that $U_{n}$ is homeomorphic to $\Gamma_{n}$, and so we have the required conclusion. This completes the proof.

Remark 2.3. The decomposition of $\Gamma_{n}$ in Proposition 2.1 is compatible with the degeneracy operators $\left\{\omega_{j}: \Gamma_{n} \rightarrow \Gamma_{n-1}\right\}_{1 \leq j \leq n}$. Assume that $\left(\beta_{1}, \ldots, \beta_{m}\right)$ is a partition of $\mathbf{n}$ of type $\left(u_{1}, \ldots, u_{m}\right)$ for $u_{1}, \ldots, u_{m} \geq 1$ with $u_{1}+\cdots+u_{m}=n$. Let $1 \leq j \leq n$. Then $\beta_{k}(t)=j$ for some $1 \leq k \leq m$ and $1 \leq t \leq u_{k}$.
(i) If $u_{k} \geq 2$, then

$$
\omega_{j} \iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}\left(\sigma, \tau_{1}, \ldots, \tau_{m}\right)=\iota^{\left(\tilde{\beta}_{1}, \ldots, \tilde{\beta}_{m}\right)}\left(\sigma, \tau_{1}, \ldots, \tau_{k-1}, \omega_{t}\left(\tau_{k}\right), \tau_{k+1}, \ldots, \tau_{m}\right)
$$

where $\left(\tilde{\beta}_{1}, \ldots, \tilde{\beta}_{m}\right)$ is the partition of $(1, \ldots, n-1)$ of type $\left(u_{1}, \ldots, u_{k-1}, u_{k}-\right.$ $1, u_{k+1}, \ldots, u_{m}$ ) given by

$$
\tilde{\beta}_{k}(s)= \begin{cases}\beta_{k}(s) & \text { if } \beta_{k}(s)<j \\ \beta_{k}(s+1)-1 & \text { if } \beta_{k}(s) \geq j\end{cases}
$$

and for $1 \leq i \leq n$ with $i \neq k$,

$$
\tilde{\beta}_{i}(s)= \begin{cases}\beta_{i}(s) & \text { if } \beta_{i}(s)<j  \tag{2-4}\\ \beta_{i}(s)-1 & \text { if } \beta_{i}(s)>j\end{cases}
$$

(ii) If $u_{k}=1$, then
$\omega_{j} \iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}\left(\sigma, \tau_{1}, \ldots, \tau_{m}\right)=\iota^{\left(\tilde{\beta}_{1}, \ldots, \tilde{\beta}_{k-1}, \tilde{\beta}_{k+1}, \ldots, \tilde{\beta}_{m}\right)}\left(\xi_{k}(\sigma), \tau_{1}, \ldots, \tau_{k-1}, \tau_{k+1}, \ldots, \tau_{m}\right)$,
where $\left(\tilde{\beta}_{1}, \ldots, \tilde{\beta}_{k-1}, \tilde{\beta}_{k+1}, \ldots, \tilde{\beta}_{m}\right)$ is the partition of $(1, \ldots, n-1)$ of type

$$
\left(u_{1}, \ldots, u_{k-1}, u_{k+1}, \ldots, u_{m}\right)
$$

given by (2-4) and $\xi_{k}$ denotes the degeneracy operator of $J_{m}$.

## 3. Higher homotopy commutativity

We first recall the higher homotopy associativity of $H$-spaces and $H$-maps.
Sugawara [1957] gave a criterion for a topological space to have the homotopy type of a loop space. Later Stasheff [1963] expanded his definition, and introduced the concept of $A_{n}$-space by using the associahedra $\left\{K_{i}\right\}_{i \geq 2}$. Let $X$ be an $H$-space whose multiplication is given by $\mu: X \times X \rightarrow X$ with $\mu(x, *)=\mu(*, x)=x$ for $x \in X$. Then an $A_{n}$-form on $X$ is a family of maps $\left\{M_{i}: K_{i} \times X^{i} \rightarrow X\right\}_{2 \leq i \leq n}$ with the relations

$$
M_{2}(*, x, y)=\mu(x, y),
$$

$M_{i}\left(\partial_{k}(r, s)(\rho, \sigma), x_{1}, \ldots, x_{i}\right)=M_{r}\left(\rho, x_{1}, \ldots, x_{k-1}, M_{s}\left(\sigma, x_{k}, \ldots, x_{k+s-1}\right), x_{k+s}, \ldots, x_{i}\right)$
for $r, s \geq 2$ with $r+s=i+1$ and $1 \leq k \leq r$, and

$$
M_{i}\left(\sigma, x_{1}, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_{i}\right)=M_{i-1}\left(\theta_{j}(\sigma), x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i}\right)
$$

for $1 \leq j \leq i$.
An $A_{1}$-space is just a topological space, and an $H$-space which admits an $A_{n^{-}}$ form is called an $A_{n}$-space for $n \geq 2$. From the definition, an $A_{2}$-space and an $A_{3}$-space are an $H$-space and a homotopy associative $H$-space, respectively. Moreover, an $A_{\infty}$-space $X$ has the homotopy type of a loop space admitting the classifying space $B X$ with $\Omega(B X) \simeq X$ (see [Kane 1988, §6-2]).

It is natural to consider the concept of $A_{n}$-map between $A_{n}$-spaces. Sugawara [1960] first considered such a concept for a map between topological monoids. Stasheff [1970] next studied an $A_{n}$-map from an $A_{n}$-space to a topological monoid by using the associahedra $\left\{K_{i}\right\}_{i \geq 2}$ used for the definition of an $A_{n}$-space.

The full generality was described by Iwase and Mimura [1989] by using the multiplihedra $\left\{J_{i}\right\}_{i \geq 1}$. Let $X$ and $Y$ be $A_{n}$-spaces with the $A_{n}$-forms $\left\{M_{i}\right\}_{2 \leq i \leq n}$ and $\left\{N_{i}\right\}_{2 \leq i \leq n}$, respectively. Assume that $\phi: X \rightarrow Y$ is a map. Then an $A_{n}$-form on $\phi$ is a family of maps $\left\{F_{i}: J_{i} \times X^{i} \rightarrow Y\right\}_{1 \leq i \leq n}$ with the relations

$$
\begin{aligned}
& \qquad F_{1}(*, x)=\phi(x) \\
& F_{i}\left(\delta_{k}(r, s)(\rho, \sigma), x_{1}, \ldots, x_{i}\right)=F_{r}\left(\rho, x_{1}, \ldots, x_{k-1}, M_{s}\left(\sigma, x_{k}, \ldots, x_{k+s-1}\right), x_{k+s}, \ldots, x_{i}\right) \\
& \text { for } r \geq 1, s \geq 2 \text { with } r+s=i+1 \text { and } 1 \leq k \leq r \\
& F_{i}\left(\delta\left(q, r_{1}, \ldots, r_{q}\right)\left(\rho, \sigma_{1}, \ldots, \sigma_{q}\right), x_{1}, \ldots, x_{i}\right) \\
& \qquad=N_{q}\left(\rho, F_{r_{1}}\left(\sigma_{1}, x_{1}, \ldots, x_{r_{1}}\right), \ldots, F_{r_{q}}\left(\sigma_{q}, x_{r_{1}+\cdots+r_{q-1}+1}, \ldots, x_{i}\right)\right)
\end{aligned}
$$

for $q \geq 2$ and $r_{1}, \ldots, r_{q} \geq 1$ with $r_{1}+\cdots+r_{q}=n$, and

$$
F_{i}\left(\rho, x_{1}, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_{i}\right)=F_{i-1}\left(\xi_{j}(\rho), x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i}\right)
$$

for $1 \leq j \leq i$.

A map which admits an $A_{n}$-form is called an $A_{n}$-map for $n \geq 1$. An $A_{1}$-map is just a map, and by [Iwase and Mimura 1989, p. 195, P8], an $A_{2}$-map and an $A_{3}$-map are an $H$-map and an $H$-map preserving the homotopy associativity, respectively. Moreover, an $A_{\infty}-\operatorname{map} \phi$ is homotopic to a loop map which induces a map between the classifying spaces $B \phi: B X \rightarrow B Y$ with $\Omega(B \phi) \simeq \phi$ (see [Kane 1988, §6-4]).

We next recall the higher homotopy commutativity of $H$-spaces.
Sugawara [1960] gave a criterion for the classifying space of a topological monoid to have the homotopy type of an $H$-space. His criterion is a higher homotopy commutativity of the multiplication. Later Williams [1969] considered another type of higher homotopy commutativity which is weaker than the one of Sugawara, and defined $C_{n}$-spaces.

In [Hemmi and Kawamoto 2004], we generalized the definition of Williams to the case of $A_{n}$-spaces, and defined $A C_{n}$-spaces by using the permuto-associahedra $\left\{\Gamma_{i}\right\}_{i \geq 1}$. Let $X$ be an $A_{n}$-space with the $A_{n}$-form $\left\{M_{i}\right\}_{2 \leq i \leq n}$. Then an $A C_{n}$-form on $X$ is a family of maps $\left\{Q_{i}: \Gamma_{i} \times X^{i} \rightarrow X\right\}_{1 \leq i \leq n}$ with the relations

$$
\begin{gather*}
Q_{1}(*, x)=x  \tag{3-1}\\
Q_{i}\left(\epsilon^{\left(\alpha_{1}, \ldots, \alpha_{l}\right)}\left(\sigma, \tau_{1}, \ldots, \tau_{l}\right), x_{1}, \ldots, x_{i}\right) \\
=M_{l}\left(\sigma, Q_{t_{1}}\left(\tau_{1}, x_{\alpha_{1}(1)}, \ldots, x_{\alpha_{1}\left(t_{1}\right)}\right), \ldots, Q_{t_{l}}\left(\tau_{l}, x_{\alpha_{l}(1)}, \ldots, x_{\alpha_{l}\left(t_{l}\right)}\right)\right)
\end{gather*}
$$

for a partition $\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ of $\mathbf{i}$ of type $\left(t_{1}, \ldots, t_{l}\right)$ with $l \geq 2$, and

$$
\begin{equation*}
Q_{i}\left(\tau, x_{1}, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_{i}\right)=Q_{i-1}\left(\omega_{j}(\tau), x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i}\right) \tag{3-3}
\end{equation*}
$$

for $1 \leq j \leq i$.
An $A_{n}$-space admitting an $A C_{n}$-form is called an $A C_{n}$-space for $n \geq 1$. By Example 3.2(1) in [Hemmi and Kawamoto 2004], $X$ is an $A C_{2}$-space if and only if $X$ is a homotopy commutative $H$-space. Moreover, if $X$ is a topological monoid, then by Corollary 3.6 of the same work, $X$ is an $A C_{n}$-space if and only if $X$ is a $C_{n}$-space of Williams [1969].

Williams [1969, Definition 20], also considered the concept of $C_{n}$-map between $C_{n}$-spaces. We generalize his definition to the case of maps between $A C_{n}$-spaces.

Definition 3.1. Let $X$ and $Y$ be $A C_{n}$-spaces with the $A C_{n}$-forms $\left\{Q_{i}\right\}_{1 \leq i \leq n}$ and $\left\{R_{i}\right\}_{1 \leq i \leq n}$, respectively. Assume that $\phi: X \rightarrow Y$ is an $A_{n}$-map with the $A_{n}$-form $\left\{F_{i}\right\}_{1 \leq i \leq n}$. Then an $A C_{n}$-form on $\phi$ is a family of maps

$$
\left\{D_{i}: I \times \Gamma_{i} \times X^{i} \rightarrow Y\right\}_{1 \leq i \leq n}
$$

with the relations

$$
\begin{equation*}
D_{1}(t, *, x)=\phi(x) \tag{3-4}
\end{equation*}
$$



Figure 6. The $A C_{n}$-forms on $\phi$ for $n=2,3$.

$$
\begin{align*}
& D_{i}\left(t, \epsilon^{\left(\alpha_{1}, \ldots, \alpha_{l}\right)}\left(\sigma, \tau_{1}, \ldots, \tau_{l}\right), x_{1}, \ldots, x_{i}\right)  \tag{3-5}\\
& \quad=N_{l}\left(\sigma, D_{t_{1}}\left(t, \tau_{1}, x_{\alpha_{1}(1)}, \ldots, x_{\alpha_{1}\left(t_{1}\right)}\right), \ldots, D_{t_{l}}\left(t, \tau_{l}, x_{\alpha_{l}(1)}, \ldots, x_{\alpha_{l}\left(t_{l}\right)}\right)\right)
\end{align*}
$$

for a partition $\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ of $\mathbf{i}$ of type $\left(t_{1}, \ldots, t_{l}\right)$ with $l \geq 2$,

$$
\begin{align*}
& D_{i}\left(0, \iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}\left(\sigma, \tau_{1}, \ldots, \tau_{m}\right), x_{1}, \ldots, x_{i}\right)  \tag{3-6}\\
& \quad=F_{m}\left(\sigma, Q_{u_{1}}\left(\tau_{1}, x_{\beta_{1}(1)}, \ldots, x_{\beta_{1}\left(u_{1}\right)}\right), \ldots, Q_{u_{m}}\left(\tau_{m}, x_{\beta_{m}(1)}, \ldots, x_{\beta_{m}\left(u_{m}\right)}\right)\right)
\end{align*}
$$

for a partition $\left(\beta_{1}, \ldots, \beta_{m}\right)$ of $\mathbf{i}$ of type $\left(u_{1}, \ldots, u_{m}\right)$ with $m \geq 1$,

$$
\begin{equation*}
D_{i}\left(1, \tau, x_{1}, \ldots, x_{i}\right)=R_{i}\left(\tau, \phi\left(x_{1}\right), \ldots, \phi\left(x_{i}\right)\right) \tag{3-7}
\end{equation*}
$$

and
(3-8) $\quad D_{i}\left(t, \tau, x_{1}, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_{i}\right)$

$$
=D_{i-1}\left(t, \omega_{j}(\tau), x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i}\right)
$$

for $1 \leq j \leq i$.
An $A_{n}$-map admitting an $A C_{n}$-form is called an $A C_{n}$-map for $n \geq 1$. If there is a family of maps $\left\{D_{i}\right\}_{i \geq 1}$ such that $\left\{D_{i}\right\}_{1 \leq i \leq n}$ is an $A C_{n}$-form on $\phi$ for any $n \geq 1$, then $\phi$ is called an $A C_{\infty}$-map.

Example 3.2. (1) An $A C_{2}$-space is the same as a homotopy commutative $H$ space by [Hemmi and Kawamoto 2004, Example 3.2(1)]. Then an $A C_{2}$-map
is a map between $A C_{2}$-spaces preserving the homotopy commutativity, and so it is the same as an HC-map of Zabrodsky [1976, p. 62].
(2) Let $\phi: X \rightarrow Y$ be a homomorphism for topological monoids $X, Y$. Then $\phi$ is an $A C_{n}$-map if and only if $\phi$ is a $C_{n}$-map of [Williams 1969, Definition 20].
(3) If $\phi: X \rightarrow Y$ is an $H$-map, then the loop map $\Omega \phi: \Omega X \rightarrow \Omega Y$ is an $A C_{\infty^{-}}$ map.

## 4. Proof of Theorem A

Let $\mathscr{S}_{*}$ denote the category of pointed and connected topological spaces having the homotopy type of $C W$-complexes. Assume that $f: A \rightarrow B$ is a pointed map for $A, B \in \mathscr{S}_{*}$. According to Dror Farjoun [1996, p. 2, A.1], $Z \in \mathscr{S}_{*}$ is called $f$-local if the induced map

$$
f^{\#}: \operatorname{Map}_{*}(B, Z) \longrightarrow \operatorname{Map}_{*}(A, Z)
$$

is a homotopy equivalence. In the case that $B=*$ and $f: A \rightarrow *$ is the constant map, $Z$ is called $A$-local, that is, the pointed mapping $\operatorname{space}^{\operatorname{Map}_{*}}(A, Z)$ is contractible.

Bousfield [1994, §2] and Dror Farjoun [1996, §1] constructed the $A$-localization $L_{A}(X)$ with the universal map $\phi_{X}: X \rightarrow L_{A}(X)$ for $X \in \mathscr{S}_{*}$ (see also [Chachólski 1996, §14]). By [Farjoun 1996, p. 4, A.4], $L_{A}(X)$ is $A$-local, and by [Bousfield 1994, Theorem 2.10(ii)], $\phi_{X}$ induces a homotopy equivalence

$$
\begin{equation*}
\left(\phi_{X}\right)^{\#}: \operatorname{Map}_{*}\left(L_{A}(X), Z\right) \rightarrow \operatorname{Map}_{*}(X, Z) \tag{4-1}
\end{equation*}
$$

for any $A$-local space $Z$ (see also [Chachólski 1996, Theorem 14.1]).
To prove Theorem A, we first show:
Proposition 4.1. Let $\phi: X \rightarrow Y$ be an $A_{n}$-map for $A_{n}$-spaces $X, Y$. If $X$ is an $A C_{n}$-space and $Y$ is $\phi$-local, then $Y$ is an $A C_{n}$-space so that $\phi$ is an $A C_{n}$-map.

Lemma 4.2. Let $\phi: X \rightarrow Y$ be a map. If $Y$ is $\phi$-local, then we have the homotopy equivalences

$$
\begin{gather*}
\left(\phi^{n}\right)^{\#}: \operatorname{Map}_{*}\left(Y^{n}, Y\right) \longrightarrow \operatorname{Map}_{*}\left(X^{n}, Y\right)  \tag{4-2}\\
\left(\phi^{(n)}\right)^{\#}: \operatorname{Map}_{*}\left(Y^{(n)}, Y\right) \longrightarrow \operatorname{Map}_{*}\left(X^{(n)}, Y\right), \tag{4-3}
\end{gather*}
$$

where $Z^{(n)}$ denotes the $n$-fold smash product of $Z \in \mathscr{T}_{*}$ for $n \geq 1$.

Proof. We first show (4-2). From the homotopy commutative diagram of fibrations

it is sufficient to show that the middle horizontal map is a homotopy equivalence for $n \geq 1$, where $e$ and $e^{\prime}$ are the evaluation maps at the base points.

We work by induction on $n$. Since $Y$ is $\phi$-local, the result is clear for $n=1$. Assume that $\left(\phi^{n-1}\right)^{\#}: \operatorname{Map}\left(Y^{n-1}, Y\right) \rightarrow \operatorname{Map}\left(X^{n-1}, Y\right)$ is a homotopy equivalence. By [Farjoun 1996, p. 5, A.8, e.2], $\operatorname{Map}\left(Y^{n-1}, Y\right)$ is $\phi$-local. From the homotopy commutative diagram

we have that $\left(\phi^{n}\right)^{\#}: \operatorname{Map}\left(Y^{n}, Y\right) \rightarrow \operatorname{Map}\left(X^{n}, Y\right)$ is a homotopy equivalence.
In the case of (4-3), by similar arguments to the case of (4-2) and a homotopy equivalence

$$
\operatorname{Map}_{*}(Z \wedge W, U) \simeq \operatorname{Map}_{*}\left(Z, \operatorname{Map}_{*}(W, U)\right)
$$

for $Z, W, U \in \mathscr{S}_{*}$, we have the required conclusion. This completes the proof.
Lemma 4.3. Let $\phi: Z \rightarrow W$ be a homotopy equivalence for $Z, W \in \mathscr{S}_{*}$, and let $(K, L)$ be a relative $C W$-complex.
(1) If there are maps $f: K \rightarrow W$ and $g: L \rightarrow Z$ with $\phi g=\left.f\right|_{L}$, then we have a map $h: K \rightarrow Z$ with $\left.h\right|_{L}=g$ and $\phi h \simeq f$ rel $L$.
(2) If $h, k: K \rightarrow Z$ are maps with $\left.h\right|_{L}=\left.k\right|_{L}$ and $\phi h \simeq \phi k$ rel $L$, then $h \simeq k$ rel $L$. Proof of Proposition 4.1. Let $\left\{M_{i}\right\}_{2 \leq i \leq n}$ and $\left\{N_{i}\right\}_{2 \leq i \leq n}$ be the $A_{n}$-form on $X$ and $Y$, respectively. Since $\phi: X \rightarrow Y$ is an $A_{n}$-map, there is an $A_{n}$-form $\left\{F_{i}\right\}_{1 \leq i \leq n}$ on $\phi$. Moreover, we denote the $A C_{n}$-form on $X$ by $\left\{Q_{i}\right\}_{1 \leq i \leq n}$.

We work by induction on $n$. By (3-1) and (3-4), the result is clear for $n=1$. Assume that there are $A C_{n-1}$-forms $\left\{R_{i}\right\}_{1 \leq i \leq n-1}$ and $\left\{D_{i}\right\}_{1 \leq i \leq n-1}$ on $Y$ and $\phi$, respectively.

Put $\mathscr{V}_{n}(Z)=\left(I \times \partial \Gamma_{n} \cup\{0\} \times \Gamma_{n}\right) \times Z^{n}$ and $\mathscr{W}_{n}(Z)=I \times \Gamma_{n} \times Z^{[n]}$, where $Z^{[n]}$ denotes the $n$-fold fat wedge of $Z \in \mathscr{S}_{*}$ given by

$$
Z^{[n]}=\left\{\left(z_{1}, \ldots, z_{n}\right) \in Z^{n} \mid z_{j}=* \text { for some } 1 \leq j \leq n\right\} .
$$

Let $E_{n}: \mathscr{V}_{n}(X) \cup \mathscr{W}_{n}(X) \rightarrow Y$ be the map defined by

$$
\begin{aligned}
E_{n}\left(t, \epsilon^{\left(\alpha_{1}, \ldots, \alpha_{l}\right)}\right. & \left.\left(\sigma, \tau_{1}, \ldots, \tau_{l}\right), x_{1}, \ldots, x_{n}\right) \\
& =N_{l}\left(\sigma, D_{t_{1}}\left(t, \tau_{1}, x_{\alpha_{1}(1)}, \ldots, x_{\alpha_{1}\left(t_{1}\right)}\right), \ldots, D_{t_{l}}\left(t, \tau_{l}, x_{\alpha_{l}(1)}, \ldots, x_{\alpha_{l}\left(t_{l}\right)}\right)\right)
\end{aligned}
$$

for a partition $\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ of $\mathbf{n}$ of type $\left(t_{1}, \ldots, t_{l}\right)$ with $l \geq 2$,

$$
\begin{aligned}
& E_{n}\left(0, \iota^{\left(\beta_{1}, \ldots, \beta_{m}\right)}\left(\sigma, \tau_{1}, \ldots, \tau_{m}\right), x_{1}, \ldots, x_{n}\right) \\
& \quad=F_{m}\left(\sigma, Q_{u_{1}}\left(\tau_{1}, x_{\beta_{1}(1)}, \ldots, x_{\beta_{1}\left(u_{1}\right)}\right), \ldots, Q_{u_{m}}\left(\tau_{m}, x_{\beta_{m}(1)}, \ldots, x_{\beta_{m}\left(u_{m}\right)}\right)\right)
\end{aligned}
$$

for a partition $\left(\beta_{1}, \ldots, \beta_{m}\right)$ of $\mathbf{n}$ of type $\left(u_{1}, \ldots, u_{m}\right)$ with $m \geq 1$ and

$$
E_{n}\left(t, \tau, x_{1}, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_{n}\right)=D_{n-1}\left(t, \omega_{j}(\tau), x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n}\right)
$$

for $1 \leq j \leq n$.
Since there is a map $\widetilde{E}_{n}: I \times \Gamma_{n} \times X^{n} \rightarrow Y$ with $\widetilde{E}_{n} \mid \mathscr{V}_{n}(X) \cup \mathscr{W}_{n}(X)=E_{n}$ by the homotopy extension property, we define a map $S_{n}: \Gamma_{n} \times X^{n} \rightarrow Y$ by

$$
S_{n}\left(\tau, x_{1}, \ldots, x_{n}\right)=\widetilde{E}_{n}\left(1, \tau, x_{1}, \ldots, x_{n}\right)
$$

Let $\gamma_{n}: \Gamma_{n} \rightarrow \operatorname{Map}_{*}\left(X^{n}, Y\right)_{\left(\phi^{n}\right)^{\#}\left(\mu_{n}\right)}$ be the adjoint of $S_{n}$, where $\mu_{n}: Y^{n} \rightarrow Y$ is the map given by $\mu_{n}\left(y_{1}, \ldots, y_{n}\right)=\left(\cdots\left(y_{1} y_{2}\right) \cdots\right) y_{n}$. If a map $\kappa_{n}: \partial \Gamma_{n} \rightarrow$ $\operatorname{Map}_{*}\left(Y^{n}, Y\right)_{\mu_{n}}$ is defined by

$$
\begin{aligned}
\kappa_{n}\left(\epsilon^{\left(\alpha_{1}, \ldots, \alpha_{l}\right)}(\sigma,\right. & \left.\left.\tau_{1}, \ldots, \tau_{l}\right), y_{1}, \ldots, y_{n}\right) \\
& =N_{l}\left(\sigma, R_{t_{1}}\left(\tau_{1}, y_{\alpha_{1}(1)}, \ldots, y_{\alpha_{1}\left(t_{1}\right)}\right), \ldots, R_{t_{l}}\left(\tau_{l}, y_{\alpha_{l}(1)}, \ldots, y_{\alpha_{l}\left(t_{l}\right)}\right)\right)
\end{aligned}
$$

then $\left(\phi^{n}\right)^{\#}\left(\kappa_{n}\right)=\left.\gamma_{n}\right|_{\partial \Gamma_{n}}$, and so by (4-2) and Lemma 4.3 (1), we have a map $\lambda_{n}: \Gamma_{n} \rightarrow \operatorname{Map}_{*}\left(Y^{n}, Y\right)_{\mu_{n}}$ with $\left.\lambda_{n}\right|_{\partial \Gamma_{n}}=\kappa_{n}$ and $\left(\phi^{n}\right)^{\#}\left(\lambda_{n}\right) \simeq \gamma_{n}$ rel $\partial \Gamma_{n}$.

To construct a map $R_{n}: \Gamma_{n} \times Y^{n} \rightarrow Y$ with the relations (3-1)-(3-3), we need to show that the induced map

$$
\begin{equation*}
\left(\phi^{[n]}\right)^{\#}: \operatorname{Map}_{*}\left(Y^{[n]}, Y\right)_{v_{n}} \longrightarrow \operatorname{Map}_{*}\left(X^{[n]}, Y\right)_{\left(\phi^{[n]}\right)^{\#}\left(v_{n}\right)} \tag{4-4}
\end{equation*}
$$

is a homotopy equivalence, where $v_{n}: Y^{[n]} \rightarrow Y$ denotes the composite of $\mu_{n}$ with the inclusion $\iota_{Y}: Y^{[n]} \rightarrow Y^{n}$. Since $Y$ is an $H$-space, it is sufficient to show the
same homotopy equivalence on the components of the constant maps. Consider the following homotopy commutative diagram of fibrations:

where $C_{1}=\left\{h: Y^{(n)} \rightarrow Y \mid\left(\pi_{Y}\right)^{\#}(h) \simeq c\right\}$ and $C_{2}=\left\{k: X^{(n)} \rightarrow Y \mid\left(\pi_{X}\right)^{\#}(k) \simeq c\right\}$. Since the vertical arrows are fibrations, the bottom horizontal arrow is a homotopy equivalence, which implies (4-4).

Define a map $\rho_{n}: \Gamma_{n} \rightarrow \operatorname{Map}_{*}\left(Y^{[n]}, Y\right)_{v_{n}}$ by

$$
\rho_{n}(\tau)\left(y_{1}, \ldots, y_{j-1}, *, y_{j+1}, \ldots, y_{n}\right)=R_{n-1}\left(\omega_{j}(\tau), y_{1}, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{n}\right)
$$

for $1 \leq j \leq n$. Then $\left(\phi^{[n]}\right)^{\#}\left(\iota_{Y}\right)^{\#}\left(\lambda_{n}\right) \simeq\left(\phi^{[n]}\right)^{\#}\left(\rho_{n}\right)$ rel $\partial \Gamma_{n}$, and so by (4-4) and Lemma 4.3 (2), we have $\left(\iota_{Y}\right)^{\#}\left(\lambda_{n}\right) \simeq \rho_{n}$ rel $\partial \Gamma_{n}$, which implies that there is a map $\psi_{n}: I \times \Gamma_{n} \rightarrow \operatorname{Map}_{*}\left(Y^{[n]}, Y\right)_{v_{n}}$ with

$$
\psi_{n}(t, \tau)= \begin{cases}\left(\iota_{Y}\right)^{\#}\left(\lambda_{n}\right)(\tau) & \text { if }(t, \tau) \in\{0\} \times \Gamma_{n} \cup I \times \partial \Gamma_{n} \\ \rho_{n}(\tau) & \text { if }(t, \tau) \in\{1\} \times \Gamma_{n}\end{cases}
$$

If a map $G_{n}: \mathscr{V}_{n}(Y) \cup \mathscr{W}_{n}(Y) \rightarrow Y$ is given by

$$
G_{n}\left(t, \tau, y_{1}, \ldots, y_{n}\right)= \begin{cases}\lambda_{n}(\tau)\left(y_{1}, \ldots, y_{n}\right) & \text { if }\left(t, \tau, y_{1}, \ldots, y_{n}\right) \in \mathscr{V}_{n}(Y) \\ \psi_{n}(t, \tau)\left(y_{1}, \ldots, y_{n}\right) & \text { if }\left(t, \tau, y_{1}, \ldots, y_{n}\right) \in \mathscr{W}_{n}(Y)\end{cases}
$$

there is an extension $\widetilde{G}_{n}: I \times \Gamma_{n} \times Y^{n} \rightarrow Y$ with $\widetilde{G}_{n} \mid \mathscr{V}_{n}(Y) \cup W_{n}(Y)=G_{n}$. Let $R_{n}: \Gamma_{n} \times Y^{n} \rightarrow Y$ be the map defined by $R_{n}\left(\tau, y_{1}, \ldots, y_{n}\right)=\widetilde{G}_{n}\left(1, \tau, y_{1}, \ldots, y_{n}\right)$. Then $R_{n}$ satisfies the relations (3-1)-(3-3).

Since $R_{n}\left(1_{\Gamma_{n}} \times \phi^{n}\right) \simeq S_{n}$ rel $\partial \Gamma_{n} \times X^{n}$, we have a map $H_{n}: I \times \Gamma_{n} \times X^{n} \rightarrow Y$ with $H_{n}\left|V_{n}(X)=E_{n}\right| V_{n}(X)$ and $H_{n}\left(1, \tau, x_{1}, \ldots, x_{n}\right)=R_{n}\left(\tau, \phi\left(x_{1}\right), \ldots, \phi\left(x_{n}\right)\right)$. Moreover, $\left.H_{n}\right|_{\partial\left(I \times \Gamma_{n}\right) \times X^{[n]}}=\left.E_{n}\right|_{\partial\left(I \times \Gamma_{n}\right) \times X^{[n]}}$, and so by [Williams 1969, Remark 10], we can choose a map $D_{n}: I \times \Gamma_{n} \times X^{n} \rightarrow Y$ with $\left.D_{n}\right|_{\partial\left(I \times \Gamma_{n}\right) \times X^{n}}=\left.H_{n}\right|_{\partial\left(I \times \Gamma_{n}\right) \times X^{n}}$ and $D_{n}\left|W_{n}(X)=E_{n}\right| W_{n}(X)$. Then $D_{n}$ satisfies the relations (3-4)-(3-8), and we have the required conclusion. This completes the proof.

Let $\phi: X \rightarrow Y$ be a homotopy equivalence. Then $Y$ is $\phi$-local, and so by Proposition 4.1, we have:

Proposition 4.4. Let $X, Y$ be $A_{n}$-spaces. Assume that $\phi: X \rightarrow Y$ is an $A_{n}$-map which is a homotopy equivalence. If $X$ is an $A C_{n}$-space, then $Y$ is an $A C_{n}$-space so that $\phi$ is an $A C_{n}$-map.

Remark 4.5. By Proposition 4.4, the property of being an $A C_{n}$-space is an invariant of $A_{n}$-homotopy type. This is a generalization of [Williams 1969, Proposition 8, Theorem 9] for $C_{n}$-spaces in the category of topological monoids.

Proof of Theorem $A$. If $X$ is an $A C_{n}$-space, then the $A$-localization $L_{A}(X)$ is an $A_{n}$-space so that the universal map $\phi_{X}: X \rightarrow L_{A}(X)$ is an $A_{n}$-map, by [Kawamoto 2002, Theorem 2.1(1)]. Since $L_{A}(X)$ is $\phi_{X}$-local by (4-1), we have the required conclusion by Proposition 4.1. This completes the proof of Theorem A.

By [Farjoun 1996, p. 26, E.1], the $S^{m+1}$-localization $L_{S^{m+1}}(X)$ of $X$ is the same as the $m$-th stage $P^{m}(X)$ of the Postnikov system of $X$ for $m \geq 1$, where $S^{t}$ denotes the $t$-dimensional sphere for $t \geq 1$. Then by Theorem A, we have:

Corollary 4.6. Let $n \geq 1$. If $X$ is an $A C_{n}$-space, then the $m$-th stage $P^{m}(X)$ of the Postnikov system of $X$ is an $A C_{n}$-space so that the projection $\rho_{m}: X \rightarrow P^{m}(X)$ is an $A C_{n}$-map for $m \geq 1$.

Let $A \in \mathscr{S}_{*}$. Dror Farjoun [1996, §2] constructed the $A$-colocalization $C W_{A}(X)$ with the universal map $\psi_{X}: C W_{A}(X) \rightarrow X$ for $X \in \mathscr{F}_{*}$ (see also [Chachólski 1996, §7]).

Theorem 4.7. Let $A \in \mathscr{S}_{*}$. If $X$ is an $A C_{n}$-space, then the $A$-colocalization $C W_{A}(X)$ is an $A C_{n}$-space so that the universal map $\psi_{X}: C W_{A}(X) \rightarrow X$ is an $A C_{n}$-map.

Let $f: Z \rightarrow W$ be a pointed map for $Z, W \in \mathscr{S}_{*}$. According to [Farjoun 1996, p. 39, A.1], $f$ is called an $A$-equivalence if the induced map

$$
f_{\#}: \operatorname{Map}_{*}(A, Z) \longrightarrow \operatorname{Map}_{*}(A, W)
$$

is a homotopy equivalence. From the proof of [Farjoun 1996, p. 53, E.1], the universal map $\psi_{X}: C W_{A}(X) \rightarrow X$ is a $C W_{A}(X)$-equivalence (see also [Chachólski 1996, p. 614]), and so we can prove Theorem 4.7 from the following result:

Proposition 4.8. Let $\phi: X \rightarrow Y$ be an $A_{n}$-map for $A_{n}$-spaces $X, Y$. If $Y$ is an $A C_{n}$-space and $\phi$ is an $X$-equivalence, then $X$ is an $A C_{n}$-space so that $\phi$ is an $A C_{n}$-map.

Proposition 4.8 is proved by similar arguments to the proof of Proposition 4.1, and so we omit the proof. In the proof of Proposition 4.8, we need the next lemma instead of Lemma 4.2:

Lemma 4.9. Let $\phi: X \rightarrow Y$ be a map. If $\phi$ is an $X$-equivalence, then we have the homotopy equivalences

$$
\begin{aligned}
\phi_{\#}: \operatorname{Map}_{*}\left(X^{n}, X\right) & \longrightarrow \operatorname{Map}_{*}\left(X^{n}, Y\right), \\
\phi_{\#}: \operatorname{Map}_{*}\left(X^{(n)}, X\right) & \longrightarrow \operatorname{Map}_{*}\left(X^{(n)}, Y\right)
\end{aligned}
$$

for $n \geq 1$.
Proof. By [Farjoun 1996, p. 46, D.2.2],

$$
\mathscr{E}(\phi)=\left\{A \in \mathscr{S}_{*} \mid \phi \text { is an } A \text {-equivalence }\right\}
$$

is a closed class.
Since $\phi$ is an $X$-equivalence, $X \in \mathscr{E}(\phi)$. If $A, B \in \mathscr{E}(\phi)$, then by [Farjoun 1996, p. 52, D.16], the product $A \times B \in \mathscr{E}(\phi)$. Since the wedge sum $A \vee B$ is represented by a homotopy colimit, we have $A \vee B \in \mathscr{E}(\phi)$ by [Farjoun 1996, p. 45, D.1] (see also [Chachólski 1996, Proposition 4.2]), and so $A \wedge B \in \mathscr{E}(\phi)$ by [Farjoun 1996, p. 45, D.1, 3,4]. From these properties, we have $X^{n}, X^{(n)} \in \mathscr{E}(\phi)$ for $n \geq 1$, which implies the required conclusion. This completes the proof.

Dror Farjoun [1996, p. 39, A.3] proved that the $S^{m+1}$-colocalization $C W_{S^{m+1}}(X)$ of $X \in \mathscr{S}_{*}$ is the same as the $m$-connected covering $X\langle m\rangle$ of $X$ for $m \geq 1$, and so by Theorem 4.7, we have:

Corollary 4.10. Let $n \geq 1$. If $X$ is an $A C_{n}$-space, then the $m$-connected covering $X\langle m\rangle$ of $X$ is an $A C_{n}$-space so that the inclusion $\iota_{m}: X\langle m\rangle \rightarrow X$ is an $A C_{n}$-map for $m \geq 1$.
Remark 4.11. In [Hemmi and Kawamoto 2004], we have shown that the universal covering inherits the property of being an $A C_{n}$-space. Corollary 4.10 is a generalization of Lemma 3.9 of that work to the case of any $m \geq 1$.

## 5. Proofs of Theorems B and C

Theorem 5.1 [Castellana et al. 2007, Theorem 7.3]. Let $p$ be a prime. If $X$ is a connected $H$-space whose mod $p$ cohomology $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathscr{A}_{p}^{*}$, then there is an H-fibration

$$
\begin{equation*}
F\left(\phi_{X}\right) \longrightarrow X \xrightarrow{\phi_{X}} L_{B \mathbb{Z} / p}(X), \tag{5-1}
\end{equation*}
$$

where $L_{B \mathbb{Z} / p}(X)$ is a connected $\mathbb{Z} / p$-finite $H$-space and $F\left(\phi_{X}\right)$ is $\bmod p$ homotopy equivalent to a Postnikov $H$-space.
Remark 5.2. In Theorem 5.1, if $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathbb{Z} / p$, then $F\left(\phi_{X}\right)$ is $\bmod p$ homotopy equivalent to a finite product of $\left(\mathbb{C} P^{\infty}\right)_{p}^{\wedge} \mathrm{s}$ and $B \mathbb{Z} / p^{i} \mathrm{~s}$ for $i \geq 1$ by [Broto et al. 2001, Theorem 1.2, Theorem 1.3].

Proof of Theorem B. By Theorem A and Theorem 5.1, $L_{B \mathbb{Z} / p}(X)$ is a connected $\mathbb{Z} / p$-finite $A C_{p}$-space so that the universal map $\phi_{X}: X \rightarrow L_{B \mathbb{Z} / p}(X)$ is an $A C_{p^{-}}$ map. Then $H^{*}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right)$ is an exterior algebra generated by odd dimensional generators by [Kane 1988, §12-3, Corollaries A and B]. Let $Z$ be the universal covering of $L_{B \mathbb{Z} / p}(X)$. Then there is an $H$-fibration

$$
Z \longrightarrow L_{B \mathbb{Z} / p}(X) \longrightarrow K\left(\pi_{1}\left(L_{B \mathbb{Z} / p}(X)\right), 1\right),
$$

where $K\left(\pi_{1}\left(L_{B \mathbb{Z} / p}(X)\right), 1\right)$ has the $\bmod p$ homotopy type of a torus. Since $Z$ is a simply connected $\mathbb{Z} / p$-finite $A C_{p}$-space by [Hemmi and Kawamoto 2004, Lemma 3.9] and [Kane 1988, §3-1, Theorem B], we have $\widetilde{H}^{*}(Z ; \mathbb{Z} / p)=0$ by [Hemmi and Kawamoto 2004, Theorem A (1)] and [Hemmi 1991, Theorem 1.1]. Then $L_{B \mathbb{Z} / p}(X)$ has the $\bmod p$ homotopy type of a torus, and so by Theorem 5.1, $X$ is $\bmod p$ homotopy equivalent to a Postnikov $H$-space. This completes the proof of Theorem B.

Remark 5.3. From Theorem B, we have the $\bmod p$ torus theorem stated in [Hemmi and Kawamoto 2004, Corollary 1.1] since a result of McGibbon and Neisendorfer [1984] on a conjecture of Serre implies that a connected Postnikov $H$-space which is also $\mathbb{Z} / p$-finite has the $\bmod p$ homotopy type of a torus.

The proof of Corollary 1.3 is given as follows:
Proof of Corollary 1.3. By Theorem B, $X$ is mod $p$ homotopy equivalent to a Postnikov $H$-space. Put $Y=X_{p}^{\wedge}$. Then, by [Bousfield 2001, Theorem 7.2], $L_{K(n) *}(Y)$ is homotopy equivalent to the $(n+1)$-st stage $\widetilde{P}^{n+1}(Y)$ of the modified Postnikov system of $Y$ given by

$$
\pi_{j}\left(\widetilde{P}^{n+1}(Y)\right) \cong \begin{cases}\pi_{j}(Y) & \text { for } 1 \leq j \leq n  \tag{5-2}\\ \pi_{n+1}(Y) / T_{n+1}(p) & \text { for } j=n+1 \\ 0 & \text { for } j>n+1\end{cases}
$$

where $T_{n+1}(p)$ denotes the $p$-torsion subgroup of $\pi_{n+1}(Y)$. Since $\Omega^{n} L_{K(n)_{*}}(Y)$ is $B \mathbb{Z} / p$-local by (5-2), there is a map $f: L_{\Sigma^{n} B \mathbb{Z} / p}(Y) \rightarrow L_{K(n)_{*}}(Y)$ with $f \phi_{Y} \simeq \kappa_{Y}$, where $\kappa_{Y}: Y \rightarrow L_{K(n)_{*}}(Y)$ denotes the universal map for the $K(n)_{*}$-localization of $Y$.

Since $\Omega^{n} L_{\Sigma^{n} B \mathbb{Z} / p}(Y)$ is $B \mathbb{Z} / p$-local, there is an $H$-fibration

$$
F\left(\phi_{L_{\Sigma^{n} B \mathbb{Z} / p}(Y)}\right) \longrightarrow L_{\Sigma^{n} B \mathbb{Z} / p}(Y) \xrightarrow{\phi_{L_{\Sigma^{n} B \mathbb{Z} / p}(Y)}} L_{B \mathbb{Z} / p}\left(L_{\Sigma^{n} B \mathbb{Z} / p}(Y)\right),
$$

by [Castellana et al. 2007, Theorem 3.2], where $F\left(\phi_{L_{\Sigma^{n} B Z / p}(Y)}\right)$ is $\bmod p$ homotopy equivalent to a Postnikov $H$-space which satisfies that $\pi_{n+1}\left(F\left(\phi_{L_{\Sigma^{n}}{ }_{B L / p}(Y)}\right)_{p}^{\wedge}\right)$ has no $p$-torsion and $\pi_{j}\left(F\left(\phi_{L_{\Sigma^{n}}}{ }_{B Z / p}(Y)\right)_{p}^{\wedge}\right)=0$ for $j>n+1$.

By [Farjoun 1996, p. 139, B.6], $L_{B \mathbb{Z} / p}\left(L_{\Sigma^{n} B \mathbb{Z} / p}(Y)\right) \simeq L_{B \mathbb{Z} / p}(Y)$, and by the proof of Theorem B, we have that $L_{B \mathbb{Z} / p}(Y)$ has the $\bmod p$ homotopy type of a torus. Then $L_{\Sigma^{n} B \mathbb{Z} / p}(Y)_{p}^{\wedge}$ is $K(n)_{*}$-local by (5-2), and so there is a map

$$
g: L_{K(n)_{*}}(Y) \rightarrow L_{\Sigma^{n} B \mathbb{Z} / p}(Y)_{p}^{\wedge}
$$

with $g \kappa_{Y} \simeq\left(\phi_{Y}\right)_{p}^{\wedge}$. From the universality of the localizations we see that $L_{K(n)_{*}}(Y)$ is mod $p$ homotopy equivalent to $L_{\Sigma^{n} B \mathbb{Z} / p}(Y)$. This completes the proof.

To prove Theorem C, we need a lemma:
Lemma 5.4. Let $p$ be an odd prime. Assume that $X$ is a connected $H$-space whose $\bmod p$ cohomology $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathbb{Z} / p$. If $x \in Q H^{2 p^{t}}(X ; \mathbb{Z} / p)$ is a generator of infinite height with $t \geq 2$, then $\mathscr{P}^{1} \beta(x) \neq 0$ in $Q H^{2 p^{t}+2 p-1}(X ; \mathbb{Z} / p)$ or there is a generator $y \in Q H^{2 p^{t-1}+1}(X ; \mathbb{Z} / p)$ with $\mathscr{P} p^{t-1}(y)=\beta(x) \neq 0$ in $Q H^{2 p^{t}+1}(X ; \mathbb{Z} / p)$.
Proof. Let $\tilde{X}$ be the universal covering of $X$. Then there is an $H$-fibration

$$
\tilde{X} \xrightarrow{\iota} X \xrightarrow{\rho} K\left(\pi_{1}(X), 1\right),
$$

where $K\left(\pi_{1}(X), 1\right)$ has the $\bmod p$ homotopy type of a finite product of $\left(S^{1}\right) \wedge_{p} \mathrm{~s}$ and $B \mathbb{Z} / p^{i}$ s for $i \geq 1$. According to [Browder 1959], the $\bmod p$ cohomology $H^{*}(\widetilde{X} ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathbb{Z} / p$ and

$$
\begin{equation*}
\iota^{*}: Q H^{s}(X ; \mathbb{Z} / p) \rightarrow Q H^{s}(\tilde{X} ; \mathbb{Z} / p) \tag{5-3}
\end{equation*}
$$

is an isomorphism if $s \neq 2,2 p^{j}-1$ for $j \geq 1$.
Recall that the $\bmod p$ cohomology of $B^{2} \mathbb{Z} / p$ is given by

$$
\begin{aligned}
H^{*}\left(B^{2} \mathbb{Z} / p ; \mathbb{Z} / p\right) \cong \mathbb{Z} / p\left[u, \beta \mathscr{P}^{1} \beta(u), \ldots,\right. & \left.\beta \mathscr{P}^{\Delta_{t}} \beta(u), \ldots\right] \\
& \otimes \Lambda\left(\beta(u), \mathscr{P}^{1} \beta(u), \ldots, \mathscr{P}^{\Delta_{t}} \beta(u), \ldots\right),
\end{aligned}
$$

where $u \in Q H^{2}\left(B^{2} \mathbb{Z} / p ; \mathbb{Z} / p\right)$ denotes the generator and $\mathscr{P}^{\Delta_{t}}=\mathscr{P} p^{t} \ldots \mathscr{P}^{1}$ for $t \geq$ 0 . Let $\tilde{x}=\iota^{*}(x) \in Q H^{2 p^{t}}(\tilde{X} ; \mathbb{Z} / p)$. By [Crespo 2001, Theorem 2.10, Proposition 5.7], there is an $H$-space $Y$ and an $H$-fibration

$$
\begin{equation*}
\tilde{X} \longrightarrow Y \longrightarrow B^{2} \mathbb{Z} / p \tag{5-4}
\end{equation*}
$$

such that $\tau(\tilde{x})=\mathscr{P}^{\Delta_{t-1}} \beta(u) \in H^{2 p^{t}+1}\left(B^{2} \mathbb{Z} / p ; \mathbb{Z} / p\right)$ and $\tau(\beta(\tilde{x}))=\beta \mathscr{P}^{\Delta_{t-1}} \beta(u) \in$ $H^{2 p^{t}+2}\left(B^{2} \mathbb{Z} / p ; \mathbb{Z} / p\right)$ in the spectral sequence associated to the $H$-fibration (5-4), where $\tau: Q H^{s}(\widetilde{X} ; \mathbb{Z} / p) \rightarrow H^{s+1}\left(B^{2} \mathbb{Z} / p ; \mathbb{Z} / p\right)$ denotes the transgression of the spectral sequence for $s \geq 2$. Then by [Crespo 2001, Theorem 1.5], there is a generator $\tilde{y} \in Q H^{2 p^{t-1}+1}(\tilde{X} ; \mathbb{Z} / p)$ with

$$
\begin{equation*}
\tau(\tilde{y})=\beta \mathscr{P}^{\Delta_{t-2}} \beta(u) \in H^{2 p^{t-1}+2}\left(B^{2} \mathbb{Z} / p ; \mathbb{Z} / p\right) \tag{5-5}
\end{equation*}
$$

or there is a generator $\tilde{z} \in Q H^{2 p^{t}+2 p-1}(\tilde{X} ; \mathbb{Z} / p)$ with

$$
\begin{equation*}
\tau(\tilde{z})=\left(\beta \mathscr{P}^{\Delta_{t-2}} \beta(u)\right)^{p} \in H^{2 p^{t}+2 p}\left(B^{2} \mathbb{Z} / p ; \mathbb{Z} / p\right) \tag{5-6}
\end{equation*}
$$

If we have (5-5), then it follows that $\mathscr{P} p^{t-1}(\tilde{y})=\beta(\tilde{x})$ in $Q H^{2 p^{t}+1}(\tilde{X} ; \mathbb{Z} / p)$ by the choice of the generators in [Crespo 2001, p. 126] since $\mathscr{P}^{p^{t-1}}\left(\beta \mathscr{P}^{\Delta_{t-2}} \beta(u)\right)=$ $\beta \mathscr{P}^{\Delta_{t-1}} \beta(u)$ by Lemma 3.2 of the same reference. Choose $y \in Q H^{2 p^{t-1}+1}(X ; \mathbb{Z} / p)$ with $\iota^{*}(y)=\tilde{y}$. Then $\mathscr{P} p^{t-1}(y)=\beta(x)$ in $Q H^{2 p^{t}+1}(X ; \mathbb{Z} / p)$ by (5-3).

In the case of (5-6), since $\mathscr{P}^{1}\left(\beta \mathscr{P}^{\Delta_{t-1}} \beta(u)\right)=\left(\beta \mathscr{P}^{\Delta_{t-2}} \beta(u)\right)^{p}$ by [Crespo 2001, Lemma 3.3], we have $\mathscr{P}^{1} \beta(\tilde{x})=\tilde{z}$ in $Q H^{2 p^{t}+2 p-1}(\tilde{X} ; \mathbb{Z} / p)$. Then by (5-3), we have $\mathscr{P}^{1} \beta(x) \neq 0$ in $Q H^{2 p^{t}+2 p-1}(X ; \mathbb{Z} / p)$. This completes the proof.

Proof of Theorem C. By Theorem A, [Kawamoto 2002, Theorem 2.1(1)] and Theorem 5.1, we have that $L_{B \mathbb{Z} / p}(X)$ is a connected $\mathbb{Z} / p$-finite $A_{p}$-space admitting an $A C_{n}$-form with $n>(p-1) / 2$.

If $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathscr{A}_{p}^{*}$ and the operations $\mathscr{P}^{j}$ act on $Q H^{*}(X ; \mathbb{Z} / p)$ trivially for $j \geq 1$, then we see that $H^{*}(X ; \mathbb{Z} / p)$ is finitely generated as an algebra over $\mathbb{Z} / p$, and so by Remark 5.2, $F\left(\phi_{X}\right)$ is $\bmod p$ homotopy equivalent to a finite product of $\left(\mathbb{C} P^{\infty}\right)_{p}^{\wedge} \mathrm{s}$ and $B \mathbb{Z} / p^{i} \mathrm{~s}$ for $i \geq 1$.

Consider the spectral sequence associated to the $H$-fibration (5-1) whose $E_{2}$ term is given by

$$
E_{2}^{*, *} \cong H^{*}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right) \otimes H^{*}\left(F\left(\phi_{X}\right) ; \mathbb{Z} / p\right)
$$

Let us show that the spectral sequence collapses. If $w \in Q H^{1}\left(F\left(\phi_{X}\right) ; \mathbb{Z} / p\right)$ is a generator, then $d_{2}(w) \in P H^{2}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right)$ by [Kane 1988, §1-6], where $P A$ denotes the primitive module of $A$. By [Kane 1988, §12-3, Corollary B], $H^{*}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right)$ is an exterior algebra generated by odd dimensional generators. Since $P H^{2}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right)=0$, we have $d_{2}(w)=0$. Assume that there is a generator $u \in Q H^{2}\left(F\left(\phi_{X}\right) ; \mathbb{Z} / p\right)$ with $d_{3}(u)=v \neq 0$ in $Q H^{3}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right)$. Then $d_{3}\left(u^{p}\right)=\mathscr{P}^{1}(v) \neq 0$ in $Q H^{2 p+1}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right)$ by [Hemmi and Kawamoto 2007, Theorem $\mathrm{A}(2)$ ], and so by computing the spectral sequence, we have a generator $x \in Q H^{2 p^{t}}(X ; \mathbb{Z} / p)$ with $t \geq 2$. Since the operations $\mathscr{P}^{j}$ act on $Q H^{*}(X ; \mathbb{Z} / p)$ trivially for $j \geq 1$, we have a contradiction by Lemma 5.4. Then the spectral sequence collapses, and so we have

$$
H^{*}(X ; \mathbb{Z} / p) \cong H^{*}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right) \otimes H^{*}\left(F\left(\phi_{X}\right) ; \mathbb{Z} / p\right)
$$

Since the operations $\mathscr{P}^{j}$ act on $Q H^{*}(X ; \mathbb{Z} / p)$ trivially for $j \geq 1$, they also act on $Q H^{*}\left(L_{B \mathbb{Z} / p}(X) ; \mathbb{Z} / p\right)$ trivially, which implies that $L_{B \mathbb{Z} / p}(X)$ has the mod $p$ homotopy type of a torus by [Hemmi and Kawamoto 2007, Theorem B] and Remark 5.5. Then there is a map $\zeta: L_{B \mathbb{Z} / p}(X) \times F\left(\phi_{X}\right) \rightarrow X$ which induces an
isomorphism on the mod $p$ cohomology, and so $\zeta$ is a $\bmod p$ homotopy equivalence; compare [Mimura and Toda 1991, p. 157, Corollary 1.6]. This completes the proof of Theorem C.
Remark 5.5. In [Hemmi and Kawamoto 2007], all spaces are assumed to be localized at $p$ in the sense of [Bousfield and Kan 1972]. However, the proof of Theorem B in our paper with Hemmi is also available for $\mathbb{Z} / p$-finite $A_{p}$-spaces, even if they are not localized at $p$.

## References

[Aguadé and Smith 1986] J. Aguadé and L. Smith, "On the mod $p$ torus theorem of John Hubbuck", Math. Z. 191:2 (1986), 325-326. MR 87e:57044 Zbl 0591.55003
[Bousfield 1994] A. K. Bousfield, "Localization and periodicity in unstable homotopy theory", $J$. Amer. Math. Soc. 7:4 (1994), 831-873. MR 95c:55010 Zbl 0839.55008
[Bousfield 2001] A. K. Bousfield, "On the telescopic homotopy theory of spaces", Trans. Amer. Math. Soc. 353:6 (2001), 2391-2426. MR 2001k:55030 Zbl 0971.55016
[Bousfield and Kan 1972] A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Math. 304, Springer, Berlin, 1972. MR 51 \#1825 Zbl 0259.55004
[Broto et al. 2001] C. Broto, J. A. Crespo, and L. Saumell, "Non-simply connected $H$-spaces with finiteness conditions", Math. Proc. Cambridge Philos. Soc. 130 (2001), 475-488. MR 2002b:55015 Zbl 1002.55007
[Browder 1959] W. Browder, "The cohomology of covering spaces of $H$-spaces", Bull. Amer. Math. Soc. 65 (1959), 140-141. MR 22 \#1891 Zbl 0088.38802
[Castellana et al. 2006] N. Castellana, J. A. Crespo, and J. Scherer, "On the cohomology of highly connected covers of finite complexes", preprint, 2006. math.AT/0607201
[Castellana et al. 2007] N. Castellana, J. A. Crespo, and J. Scherer, "Deconstructing Hopf spaces", Invent. Math. 167:1 (2007), 1-18. MR MR2264802 Zbl 05118881
[Chachólski 1996] W. Chachólski, "On the functors $C W_{A}$ and $P_{A}$ ", Duke Math. J. 84:3 (1996), 599-631. MR 97i:55023 Zbl 0873.55014
[Crespo 2001] J. A. Crespo, "Structure of mod $p H$-spaces with finiteness conditions", pp. 103-130 in Cohomological methods in homotopy theory (Bellaterra, 1998), edited by J. Aguadé et al., Progr. Math. 196, Birkhäuser, Basel, 2001. MR 2002g:55016 Zbl 0990.55004
[Farjoun 1996] E. D. Farjoun, Cellular spaces, null spaces and homotopy localization, Lecture Notes in Math. 1622, Springer, Berlin, 1996. MR 98f:55010 Zbl 0842.55001
[Hemmi 1991] Y. Hemmi, "Higher homotopy commutativity of $H$-spaces and the mod $p$ torus theorem", Pacific J. Math. 149:1 (1991), 95-111. MR 92a:55010 Zbl 0691.55007
[Hemmi and Kawamoto 2004] Y. Hemmi and Y. Kawamoto, "Higher homotopy commutativity of $H$-spaces and the permuto-associahedra", Trans. Amer. Math. Soc. 356:10 (2004), 3823-3839. MR 2005c:55020 Zbl 1064.55005
[Hemmi and Kawamoto 2007] Y. Hemmi and Y. Kawamoto, "Higher homotopy commutativity and cohomology of finite $H$-spaces", pp. 167-186 in Proceedings of the Nishida Fest (Kinosaki, 2003), edited by M. Ando et al., Geom. Topol. Monogr. 10, Mathematical Science Publishers, Berkeley, CA, 2007.
[Hubbuck 1969] J. R. Hubbuck, "On homotopy commutative $H$-spaces", Topology 8 (1969), 119126. MR 38 \#6592 Zbl 0176.21301
[Iriye and Kono 1985] K. Iriye and A. Kono, "Mod $p$ retracts of $G$-product spaces", Math. Z. 190:3 (1985), 357-363. MR 88a:55017 Zbl 0554.55004
[Iwase and Mimura 1989] N. Iwase and M. Mimura, "Higher homotopy associativity", pp. 193-220 in Algebraic topology (Arcata, CA, 1986), edited by G. Carlsson et al., Lecture Notes in Math. 1370, Springer, Berlin, 1989. MR 90f:55017 Zbl 0677.55010
[Kane 1988] R. M. Kane, The homology of Hopf spaces, Math. Library 40, North-Holland, Amsterdam, 1988. MR 90f:55018 Zbl 0651.55001
[Kapranov 1993] M. M. Kapranov, "The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation", J. Pure Appl. Algebra 85:2 (1993), 119-142. MR 94b:52017 Zbl 0812.18003
[Kawamoto 1999] Y. Kawamoto, "Loop spaces of $H$-spaces with finitely generated cohomology", Pacific J. Math. 190:2 (1999), 311-328. MR 2000i:55027 Zbl 1011.55006
[Kawamoto 2002] Y. Kawamoto, "Higher homotopy commutativity of $H$-spaces with finitely generated cohomology", Pacific J. Math. 204:1 (2002), 145-161. MR 2003d:55010 Zbl 1053.55005
[Lin 1985] J. P. Lin, "A cohomological proof of the torus theorem", Math. Z. 190:4 (1985), 469-476. MR 87c:55009 Zbl 0563.55007
[Lin 1994] J. P. Lin, "Loops of $H$-spaces with finitely generated cohomology rings", Topology Appl. 60:2 (1994), 131-152. MR 95m:55022 Zbl 0837.55012
[Lin 1995] J. P. Lin, " $H$-spaces with finiteness conditions", pp. 1095-1141 in Handbook of algebraic topology, North-Holland, Amsterdam, 1995. MR 97c:55017 Zbl 0941.55002
[Lin and Williams 1991] J. P. Lin and F. Williams, "Homotopy-commutative $H$-spaces", Proc. Amer. Math. Soc. 113:3 (1991), 857-865. MR 92b:55011 Zbl 0758.55008
[McGibbon 1984] C. A. McGibbon, "Homotopy commutativity in localized groups", Amer. J. Math. 106:3 (1984), 665-687. MR 86a:55011 Zbl 0574.55004
[McGibbon 1989] C. A. McGibbon, "Higher forms of homotopy commutativity and finite loop spaces", Math. Z. 201:3 (1989), 363-374. MR 90f:55019 Zbl 0682.55006
[McGibbon and Neisendorfer 1984] C. A. McGibbon and J. A. Neisendorfer, "On the homotopy groups of a finite-dimensional space", Comment. Math. Helv. 59:2 (1984), 253-257. MR 86b:55015 Zbl 0538.55010
[Mimura and Toda 1991] M. Mimura and H. Toda, Topology of Lie groups, vol. I, Transl. Math. Monographs 91, Amer. Math. Soc., Providence, RI, 1991. MR 92h:55001 Zbl 0757.57001
[Reiner and Ziegler 1994] V. Reiner and G. M. Ziegler, "Coxeter-associahedra", Mathematika 41:2 (1994), 364-393. MR 95m:52023 Zbl 0822.52007
[Slack 1991] M. Slack, A classification theorem for homotopy commutative $H$-spaces with finitely generated mod 2 cohomology rings, Mem. Amer. Math. Soc. 449, Amer. Math. Soc., Providence, RI, 1991. MR 92k:55015 Zbl 0755.55003
[Stasheff 1963] J. D. Stasheff, "Homotopy associativity of $H$-spaces, I, II", Trans. Amer. Math. Soc. 108 (1963), 275-292, 293-312. MR 28 \#1623 Zbl 0114.39402
[Stasheff 1970] J. Stasheff, H-spaces from a homotopy point of view, Lecture Notes in Math. 161, Springer, Berlin, 1970. MR 42 \#5261 Zbl 0205.27701
[Sugawara 1957] M. Sugawara, "A condition that a space is group-like", Math. J. Okayama Univ. 7 (1957), 123-149. MR 20 \#3546 Zbl 0091.37201
[Sugawara 1960] M. Sugawara, "On the homotopy-commutativity of groups and loop spaces", Mem. Coll. Sci. Univ. Kyoto Ser. A Math. 33 (1960), 257-269. MR 22 \#11394 Zbl 0113.16903
[Williams 1969] F. D. Williams, "Higher homotopy-commutativity", Trans. Amer. Math. Soc. 139 (1969), 191-206. MR 39 \#2163 Zbl 0185.27103
[Zabrodsky 1976] A. Zabrodsky, Hopf spaces, Math. Studies 22, North-Holland, Amsterdam, 1976. MR 55 \#13416 Zbl 0339.55009
[Ziegler 1995] G. M. Ziegler, Lectures on polytopes, Graduate Texts in Math. 152, Springer, New York, 1995. MR 96a:52011 Zbl 0823.52002

Received January 11, 2006. Revised December 17, 2006.
Yusuke Kawamoto
Department of Mathematics
National Defense Academy
YOKOSUKA 239-8686
JAPAN
yusuke@nda.ac.jp


[^0]:    MSC2000: primary 55P45, 55P48; secondary 55P15, 55P65.
    Keywords: higher homotopy commutativity, homotopy localizations, $H$-spaces, $A_{n}$-spaces, $A C_{n}$-spaces.

