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Dedicated to Alladi Sitaram on his sixtieth birthday

In 1978 M. Lassalle obtained an analogue of the Laurent series for holomorphic functions on the complexification of a compact symmetric space and proved a Plancherel type formula for such functions. In 2002 J. Faraut established such a formula, which he calls Gutzmer's formula, for all noncompact Riemannian symmetric spaces. This was immediately put into use by B. Krotz, G. Olafsson and R. Stanton to characterise the image of the heat kernel transform. In this article we prove an analogue of Gutzmer's formula for the Heisenberg motion group and use it to characterise Poisson integrals associated to the sublaplacian. We also use the Gutzmer's formula to study twisted Bergman spaces.

1. Introduction

Consider the Laplace–Beltrami operator on a Riemannian manifold M and the associated heat semigroup T_t. The problem of characterising the image of $L^2(M)$ under T_t has received considerable attention starting with Bargmann [1961]. He treated the case of $\mathbb{R}^n$ and showed that the image is a weighted Bergman space of entire functions. Similar results were obtained for compact Lie groups by Hall [1994] and for compact symmetric spaces by Stenzel [1999]. For the case of Hermite, Laguerre and Jacobi expansions see [Karp 2005] and the references there.

Contrary to the general expectation, such a characterisation is not true for the Laplace operator on the Heisenberg group $\mathbb{H}^n$, as shown by Krötz, Thangavelu and Xu in [Krötz et al. 2005b]. Specifically, we proved there that the image of $L^2(\mathbb{H}^n)$ is not a weighted Bergman space but a sum of two such spaces defined by signed weights.

Recently Krötz, Olafsson and Stanton [Krötz et al. 2005a] showed that when X is a noncompact Riemannian symmetric space the image of $L^2(X)$ cannot be

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described as a weighted Bergman space. In the same paper a different characterisation of the image was obtained, using orbital integrals. In the case of noncompact Riemannian symmetric spaces the functions $T_t f$ do not extend as entire functions but only as holomorphic functions on a domain called the *complex crown*. The behaviour of $T_t f$ is therefore similar to that of Poisson integrals of L^2 functions on $\mathbb{R}^n$. Recall that the Poisson integrals extend only as holomorphic functions on a tube domain. The main ingredient used in [Krötz et al. 2005a] is a result of Faraut [2002] which he calls Gutzmer's formula. Our aim in this paper is to show that, using Gutzmer's formula, Poisson integrals can be characterised as certain spaces of holomorphic functions.

In Section 2 we treat the Poisson integrals on $\mathbb{R}^n$, where the results are easy to obtain. Section 3 recapitulates necessary results on special Hermite functions. In Section 4 we prove an analogue of Gutzmer's formula for the Heisenberg group, and we use this in Section 5 to give a characterisation of Poisson integrals on $\mathbb{H}^n$. Since Gutzmer's formula is available on all noncompact Riemannian symmetric spaces, a similar characterisation of Poisson integrals on them should be possible. In Section 6 we revisit twisted Bergman spaces and give a new proof of their characterisation as the image of $L^2(\mathbb{C}^n)$ under the special Hermite semigroup. (This was proved in [Krötz et al. 2005b] by a different method.)

2. Poisson integrals on Euclidean spaces

Throughout this paper x^2 stands for $|x|^2$, for $x \in \mathbb{R}^n$. Let

$$p_t(x) = c_n t (t^2 + x^2)^{-(n+1)/2}$$

be the Poisson kernel on $\mathbb{R}^n$, where c_n is a suitable constant. The Poisson integral of a function $f \in L^2(\mathbb{R}^n)$ is defined by

$$f * p_t(x) = \int_{\mathbb{R}^n} f(u) p_t(x-u)\, du,$$

which is also given in terms of the Fourier transform by

$$f * p_t(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) e^{-t|\xi|}\, d\xi.$$

From any of these expressions it is clear that the function $F(x) = f * p_t(x)$ extends to the tube domain

$$\Omega_t = \{z = x + iy \in \mathbb{C}^n : |y| < t\}$$

as a holomorphic function. We are interested in knowing exactly what kind of holomorphic functions F arise as Poisson integrals.

To state our result, we recall the definition of the spherical function φ_λ on $\mathbb{R}^n$. For each $\lambda \in \mathbb{C}$ we define

$$\varphi_\lambda(x) = \int_{S^{n-1}} e^{-i\lambda x\cdot\omega}\, d\omega.$$

It is clear that φ_λ extends to $\mathbb{C}^n$ as an entire function. Moreover, when λ is real or purely imaginary it is given in terms of the Bessel function of order $n/2-1$. More precisely,

$$\varphi_\lambda(x) = c_n J_{n/2-1}(\lambda|x|)(\lambda|x|)^{-(n/2-1)}.$$

Using the Plancherel theorem for the Fourier transform, it is easy to see that for any r with $0 \le r < t$,

$$\int_{S^{n-1}} \int_{\mathbb{R}^n} |f * p_t(x+ir\omega)|^2\, dx\, d\omega = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 e^{-2t|\xi|} \varphi_{2ir}(\xi)\, d\xi. \tag{2-1}$$

Following [Faraut 2002] we call this *Gutzmer's formula* for Euclidean spaces.

The right-hand side of the formula is finite even if $r = t$, as long as

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 e^{-2t|\xi|} \varphi_{2it}(\xi)\, d\xi < \infty.$$

This happens precisely when

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^{-(n-1)/2}\, d\xi < \infty$$

as can be seen using the asymptotic properties of the Bessel functions; see [Szegö 1967], for example. Recall that the Sobolev space $H^s(\mathbb{R}^n)$, for $s \in \mathbb{R}$, is the space of tempered distributions f for which

$$\|f\|_{(s)}^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s\, d\xi$$

is finite. We have the following characterisation of Poisson integrals on $\mathbb{R}^n$.

Theorem 2.1. *A holomorphic function F on the tube domain Ω_t is the Poisson integral of a function $f \in H^{-(n-1)/2}(\mathbb{R}^n)$ if and only if*

$$\lim_{r\to t} \int_{S^{n-1}} \int_{\mathbb{R}^n} |F(x+ir\omega)|^2\, dx\, d\omega$$

is finite. Moreover, the limit is equivalent to the norm of $f \in H^{-(n-1)/2}(\mathbb{R}^n)$.

When $n = 1$, this says that $F = f * p_t$ with $f \in L^2(\mathbb{R})$ if and only if both integrals $\int_{\mathbb{R}} |F(x+it)|^2\, dx$ and $\int_{\mathbb{R}} |F(x-it)|^2\, dx$ are finite. We are interested in finding an analogue of Theorem 2.1 for the Heisenberg group. This will be achieved via Gutzmer's formula for the Heisenberg motion group.

We can rewrite Gutzmer's formula (2-1) in terms of the group $M(n)$ of Euclidean motions, which is the semidirect product of $\mathrm{SO}(n)$ and $\mathbb{R}^n$. The action of $M(n)$ on $\mathbb{R}^n$ is given by $(\sigma, u)x = u+\sigma x$. This action has a natural extension to $\mathbb{C}^n$: simply define $(\sigma, u)(x+iy) = u+\sigma x+i\sigma y$. Gutzmer's formula then takes the following form.

Proposition 2.2. *Let F be holomorphic in Ω_t and let the restriction of F to $\mathbb{R}^n$ be such that $\hat{F}$ is square integrable with respect to the measure $\varphi_{2it}(\xi)\,d\xi$. Then*

$$\lim_{y^2\to t^2}\int_{M(n)} |F(g.(x+iy))|^2\,dg = \int_{\mathbb{R}^n} |\hat{F}(\xi)|^2\varphi_{2it}(\xi)\,d\xi.$$

To see why this is true, we observe that

$$\int_{M(n)} |F(g.(x+iy))|^2\,dg = \int_{\mathrm{SO}(n)}\int_{\mathbb{R}^n} |F(u+\sigma x+i\sigma y)|^2 du\,d\sigma.$$

Plancherel theorem for the Fourier transform shows that the $\mathbb{R}^n$ integral is

$$\int_{\mathbb{R}^n} |\hat{F}(\xi)|^2 e^{-2\sigma y\cdot\xi}\,d\xi.$$

The proposition follows since

$$\int_{\mathrm{SO}(n)} e^{-2\sigma y\cdot\xi}\,d\sigma = c_n\varphi_{2i|y|}(\xi).$$

In Section 4 we prove an analogue of this proposition for the Heisenberg motion group, which is then used in Section 5 to characterise Poisson integrals on the Heisenberg group.

3. Some results on special Hermite functions

We collect here relevant information about special Hermite functions and prove some estimates required in the next section. We closely follow the notations used in [Thangavelu 1998; 2004]; see those monographs for more details.

We will denote by Φ_α, $\alpha\in\mathbb{N}^n$, the Hermite functions on $\mathbb{R}^n$, normalised so that their L^2 norms are 1. On finite linear combinations of such functions we can define certain operators $\pi(z, w)$, where $z, w\in\mathbb{C}^n$, by setting

$$\pi(z, w)\Phi_\alpha(\xi) = e^{i(z\cdot\xi+(z/2)\cdot w)}\Phi_\alpha(\xi+w),$$

where $\cdot$ denotes the Euclidean inner product. Note that $\Phi_\alpha(\xi) = H_\alpha(\xi)e^{-|\xi|^2/2}$, where H_α is a polynomial on $\mathbb{R}^n$ and for $z\in\mathbb{C}^n$ we define $\Phi_\alpha(z)$ to be $H_\alpha(z)e^{-z^2/2}$, where $z^2 = z\cdot z$. The special Hermite functions $\Phi_{\alpha,\beta}(z, w)$ are then defined by

$$\Phi_{\alpha,\beta}(z, w) = (2\pi)^{-n/2}(\pi(z, w)\Phi_\alpha, \Phi_\beta). \tag{3-1}$$

The restrictions of $\Phi_{\alpha,\beta}(z, w)$ to $\mathbb{R}^n \times \mathbb{R}^n$ are usually called *special Hermite functions*. The family $\{\Phi_{\alpha,\beta}(x, u) : \alpha, \beta \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. The functions $\Phi_{\alpha,\beta}(z, w)$ are just the holomorphic extensions of $\Phi_{\alpha,\beta}(x, u)$ to $\mathbb{C}^n \times \mathbb{C}^n$. An easy calculation shows that

$$(\pi(z, w)\Phi_\alpha, \Phi_\beta) = (\Phi_\alpha, \pi(-\bar{z}, -\bar{w})\Phi_\beta). \tag{3-2}$$

This means that for $x, u \in \mathbb{R}^n$ the operators $\pi(x, u)$ are unitary on $L^2(\mathbb{R}^n)$ (related to the Schrödinger representation π_1 of the Heisenberg group) and

$$(\pi(ix, iu)\Phi_\alpha, \Phi_\beta) = (\Phi_\alpha, \pi(ix, iu)\Phi_\beta).$$

We can also verify that

$$\pi(z, w)\pi(z', w') = e^{(i/2)(z'\cdot w - z\cdot w')}\pi(z + z', w + w'). \tag{3-3}$$

For $x, u \in \mathbb{R}^n$ this gives

$$(\pi(2ix, 2iu)\Phi_\alpha, \Phi_\alpha) = \|\pi(ix, iu)\Phi_\alpha\|_2^2. \tag{3-4}$$

Let L_k^{n-1} be the Laguerre polynomials of type $n-1$ and define the Laguerre functions φ_k by

$$\varphi_k(x, u) := L_k^{n-1}\big(\tfrac{1}{2}(x^2 + u^2)\big)e^{-(x^2+u^2)/4} = \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(x, u),$$

where the second equality is classical. The Laguerre functions have a natural holomorphic extension to $\mathbb{C}^n \times \mathbb{C}^n$, which we denote by the same symbol:

$$\varphi_k(z, w) = \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(z, w). \tag{3-5}$$

From this expression we obtain the following estimate for the complexified Laguerre functions.

Proposition 3.1. *For $z, w \in \mathbb{C}^n$ and $k \in \mathbb{N}$,*

$$|\varphi_k(z, w)|^2 \le C\frac{(k+n-1)!}{k!\,(n-1)!}e^{(u\cdot y - v\cdot x)}\varphi_k(2iy, 2iv),$$

where $z = x + iy$ and $w = u + iv$.

Proof. We have

$$\Phi_{\alpha,\alpha}(x + iy, u + iv) = (2\pi)^{-n/2}(\pi(x + iy, u + iv)\Phi_\alpha, \Phi_\alpha)$$

and this gives in view of (3-3)

$$\Phi_{\alpha,\alpha}(x + iy, u + iv) = (2\pi)^{-n/2}e^{(u\cdot y - v\cdot x)/2}(\pi(iy, iv)\Phi_\alpha, \pi(-x, -u)\Phi_\alpha).$$

Since $\pi(-x,-u)$ is unitary we get the estimate

$$|\Phi_{\alpha,\alpha}(x+iy,u+iv)| \leq (2\pi)^{-n/2} e^{(u\cdot y - v\cdot x)/2} \|\pi(iy,iv)\Phi_\alpha\|_2.$$

Applying the Cauchy–Schwarz inequality in (3-5), recalling that

$$\sum_{|\alpha|=k} 1 = \frac{(k+n-1)!}{k!\,(n-1)!},$$

and using (3-4) we get the proposition. □

The functions $\varphi_k(z,w)$ are members of the twisted Bergman spaces $\mathcal{B}_t$ studied in [Krötz et al. 2005b]. Since evaluations are continuous functionals on reproducing kernel Hilbert spaces of holomorphic functions and as $\mathcal{B}_t$ is one such space we get the estimate

$$|\varphi_k(z,w)| \leq C_t \|\varphi_k\|_{\mathcal{B}_t} \leq C_t \left(\frac{(k+n-1)!}{k!\,(n-1)!}\right)^{1/2} e^{(2k+n)t}.$$

However, we can greatly improve this estimate using the proposition above.

Proposition 3.2. *For each $r > 0$ we have the uniform estimates*

$$|\varphi_k(z,w)| \leq C_r e^{(u\cdot y - v\cdot x)/2} \left(\frac{(k+n-1)!}{k!\,(n-1)!}\right)^{1/2} k^{n/4-3/8} \exp\big(r(2k+n)^{1/2}\big),$$

valid for $y^2+v^2 \leq r^2$.

Proof. In view of Proposition 3.1 it is enough to estimate $\varphi_k(2iy,2iv)$. In the region $\delta \leq y^2+v^2 \leq r^2$, with $\delta > 0$, the required inequality follows using Perron's estimate [Szegö 1967, Theorem 8.22.3]:

$$L_k^\alpha(s) = \tfrac{1}{2}\pi^{-1/2} e^{s/2} (-s)^{-\alpha/2-1/4} k^{\alpha/2-1/4} e^{2(-ks)^{1/2}} \big(1+O(k^{-1/2})\big),$$

valid for s in the complex plane cut along the positive real axis (we require the estimate when $s < 0$). We now use the representation

$$L_k^\alpha(s) = \frac{(-1)^k \pi^{-1/2}\Gamma(k+\alpha+1)}{\Gamma(\alpha+\frac{1}{2})(2k)!} \int_{-1}^{1} (1-t^2)^{\alpha-1/2} H_{2k}(s^{1/2}t)\,dt$$

for the Laguerre polynomials in terms of Hermite polynomials, along with the estimates given in [Szegö 1967, Theorem 8.22.6] for Hermite polynomials, to get the uniform estimates even when $\delta = 0$. □

We need one more result on special Hermite functions. If $F(z,w)$ is a function on $\mathbb{C}^n \times \mathbb{C}^n$ we define its twisted translation by

$$\tau(z',w')F(z,w) = e^{-(i/2)(z'\cdot w - z\cdot w')} F(z-z', w-w'). \tag{3-6}$$

In view of (3-3) we have

$$\tau(z', w')\Phi_{\alpha,\alpha}(z, w) = (2\pi)^{-n/2}(\pi(z, w)\pi(-z', -w')\Phi_\alpha, \Phi_\alpha).$$

Using (3-2) this can be written as

$$\tau(z', w')\Phi_{\alpha,\alpha}(z, w) = (2\pi)^{-n/2}(\pi(-z', -w')\Phi_\alpha, \pi(-\bar{z}, -\bar{w})\Phi_\alpha).$$

Expanding $\pi(-z', -w')\Phi_\alpha$ and $\pi(-\bar{z}, -\bar{w})\Phi_\alpha$ in terms of Hermite functions we obtain

$$\tau(z', w')\Phi_{\alpha,\alpha}(z, w) = (2\pi)^{n/2}\sum_\beta \Phi_{\alpha,\beta}(-z', -w')\Phi_{\beta,\alpha}(z, w). \tag{3-7}$$

By taking $z = -z' = iy$ and $w = -w' = iv$ we obtain

$$\Phi_{\alpha,\alpha}(2iy, 2iv) = (2\pi)^{n/2}\sum_\beta |\Phi_{\alpha,\beta}(iy, iv)|^2, \tag{3-8}$$

since we have the relation

$$\Phi_{\beta,\alpha}(iy, iv) = (2\pi)^{-n/2}(\pi(iy, iv)\Phi_\beta, \Phi_\alpha) = \overline{\Phi_{\alpha,\beta}(iy, iv)}.$$

Proposition 3.3. *For* $y, v \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |\varphi_k(x+iy, u+iv)|^2 e^{-(u\cdot y - v\cdot x)}\, dx\, du = (2\pi)^{n/2}\varphi_k(2iy, 2iv).$$

Proof. In view of (3-8) we have

$$\varphi_k(2iy, 2iv) = (2\pi)^{-n/2}\sum_{|\alpha|=k}\sum_\beta |\Phi_{\alpha,\beta}(iy, iv)|^2.$$

On the other hand from (3-6) we get

$$\Phi_{\alpha,\alpha}(x+iy, u+iv)e^{-(u\cdot y - v\cdot x)/2} = \tau(-iy, -iv)\Phi_{\alpha,\alpha}(x, u),$$

which gives, in view of (3-7),

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |\tau(-iy, -iv)\Phi_{\alpha,\alpha}(x, u)|^2 dxdu = (2\pi)^n \sum_\beta |\Phi_{\alpha,\beta}(iy, iv)|^2.$$

Summing over all α with $|\alpha| = k$ we get the proposition. □

4. Gutzmer's formula on the Heisenberg group

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with multiplication defined by

$$(z, t)(z', t') = \left(z + z',\ t + t' + \tfrac{1}{2}\,\mathrm{Im}(z\cdot\bar{z}')\right).$$

More often we write (x, u, t) in place of (z, t) and the group law takes the form

$$(x, u, t)(x', u', t') = \big(x + x',\ u + u',\ t + t' + \tfrac{1}{2}(u \cdot x' - x \cdot u')\big),$$

where $x, u, x', u' \in \mathbb{R}^n$. For a function f on $\mathbb{H}^n$ we define

$$f^\lambda(z) = \int_{\mathbb{R}} f(z, t) e^{i\lambda t}\, dt.$$

For each nonzero $\lambda \in \mathbb{R}$ the Schrödinger representation π_λ of $\mathbb{H}^n$ is defined by

$$\pi_\lambda(x, u, t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + x\cdot u/2)} \varphi(\xi + u),$$

where $\varphi \in L^2(\mathbb{R}^n)$. We define $\Phi^\lambda_\alpha(x) = |\lambda|^{n/4}\Phi(|\lambda|^{1/2}x)$ and

$$E^\lambda_{\alpha,\beta}(x, u, t) = (2\pi)^{-n/2}(\pi_\lambda(x, u, t)\Phi^\lambda_\alpha, \Phi^\lambda_\beta).$$

Note that

$$E^\lambda_{\alpha,\beta}(x, u, t) = e^{i\lambda t}\Phi^\lambda_{\alpha,\beta}(x, u),$$

where $\Phi^\lambda_{\alpha,\beta}(x, u) = E^\lambda_{\alpha,\beta}(x, u, 0)$. We write $\varphi^\lambda_k(x, u) = \varphi_k(|\lambda|^{1/2}(x, u))$, so in view of (3-5) we have

$$\varphi^\lambda_k(x, u) = \sum_{|\alpha|=k} \Phi^\lambda_{\alpha,\alpha}(x, u) = \varphi_k(|\lambda|^{1/2}(x, u))$$

and let $e^\lambda_k(x, u, t) = e^{i\lambda t}\varphi^\lambda_k(x, u)$. The results proved in the previous section are all valid for these functions for every nonzero $\lambda \in \mathbb{R}$.

The inversion formula for the group Fourier transform on $\mathbb{H}^n$ can be written in the form

$$f(x, u, t) = \int_{-\infty}^{\infty} \bigg(\sum_{k=0}^{\infty} f * e^\lambda_k(x, u, t)\bigg)\, d\mu(\lambda), \tag{4-1}$$

where $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n\, d\lambda$ is the Plancherel measure on the Heisenberg group. Define the λ-twisted convolution of two functions F and G on $\mathbb{C}^n$ by

$$F *_\lambda G(x, u) = \int_{\mathbb{R}^{2n}} F(x', u')G(x - x', u - u')e^{-(i/2)\lambda(u\cdot x' - x\cdot u')}\, dx'\, du'.$$

Then (4-1) takes the form

$$f(x, u, t) = \int_{-\infty}^{\infty} e^{-i\lambda t}\bigg(\sum_{k=0}^{\infty} f^\lambda *_\lambda \varphi^\lambda_k(x, u)\bigg)\, d\mu(\lambda).$$

We would like to rewrite this inversion formula in terms of certain representations of the Heisenberg motion group, whose definition we recall. The unitary group $U(n)$ acts on the Heisenberg group as automorphisms, the action being defined by $\sigma(z,t) = (\sigma.z, t)$, where $\sigma \in U(n)$. The Heisenberg motion group G_n is the semidirect product of $U(n)$ and $\mathbb{H}^n$ with group law

$$(\sigma, z, t)(\tau, w, s) = (\sigma\tau, (z,t)(\sigma.w, s)).$$

Functions on $\mathbb{H}^n$ can be considered as right $U(n)$-invariant functions on G_n. Hence the inversion formula for such functions on G_n will involve only certain class-one representations of G_n. We now proceed to describe the relevant representations.

For each $k \in \mathbb{N}$ and nonzero $\lambda \in \mathbb{R}$ let $\mathcal{H}_k^\lambda$ be the Hilbert space for which the functions $E_{\alpha,\beta}^\lambda$, with $\alpha, \beta \in \mathbb{N}^n$, $|\alpha| = k$, form an orthonormal basis. The inner product in $\mathcal{H}_k^\lambda$ is defined by

$$(F, G) = |\lambda|^n \int_{\mathbb{C}^n} F(z, 0)\overline{G(z, 0)}\, dz.$$

On this Hilbert space we define a representation ρ_k^λ of the Heisenberg motion group by

$$\rho_k^\lambda(\sigma, z, t)F(w, s) = F((\sigma, z, t)^{-1}(w, s)).$$

Then it is known (from [Ratnakumar et al. 1997], for example) that ρ_k^λ is an irreducible unitary representation of G_n. As $(G_n, U(n))$ is a Gelfand pair, ρ_k^λ has a unique $U(n)$ fixed vector, which is none other than e_k^λ.

Given $f \in L^1(\mathbb{H}^n)$ we can define its group Fourier transform by

$$\rho_k^\lambda(f) = \int_{G_n} f(z, t)\rho_k^\lambda(\sigma, z, t)\, d\sigma\, dz\, dt,$$

which is a bounded operator acting on $\mathcal{H}_k^\lambda$. In this integral $d\sigma$ stands for the normalised Haar measure on $U(n)$. From calculations done in [Thangavelu 1998, Chapter 3] we infer that

$$\operatorname{tr}(\rho_k^\lambda(\sigma, z, t)^*\rho_k^\lambda(f)) = \frac{k!(n-1)!}{(k+n-1)!} f * e_k^\lambda(z, t)$$

and the inversion formula for a right $U(n)$-invariant function on G_n takes the form

$$f(z, t) = (2\pi)^{-n-1}\int_{-\infty}^{\infty}\biggl(\sum_{k=0}^{\infty}\operatorname{tr}(\rho_k^\lambda(\sigma, z, t)^*\rho_k^\lambda(f))\frac{(k+n-1)!}{k!(n-1)!}\biggr)|\lambda|^n\, d\lambda.$$

Also the Plancherel theorem can be written as

$$\int_{\mathbb{H}^n}|f(z, t)|^2\, dz\, dt = (2\pi)^{-n-1}\int_{-\infty}^{\infty}\biggl(\sum_{k=0}^{\infty}\|\rho_k^\lambda(f)\|_{HS}^2\frac{(k+n-1)!}{k!(n-1)!}\biggr)|\lambda|^n\, d\lambda.$$

From this we can read off the Plancherel measure for G_n when dealing with right $U(n)$-invariant functions.

It can be shown that the operator $\rho_k^\lambda(f)$ is of rank one. Indeed, $\rho_k^\lambda(f)E_{\alpha,\beta}=0$ if $\alpha\neq\beta$ and

$$\rho_k^\lambda(f)E_{\alpha,\alpha}^\lambda=\frac{k!(n-1)!}{(k+n-1)!}f*e_k^\lambda.$$

From this we infer that $\rho_k^\lambda(f)e_k^\lambda=f*e_k^\lambda$. Hence it follows that for any $F\in\mathcal{H}_k^\lambda$ we have

$$\rho_k^\lambda(f)F=\frac{k!(n-1)!}{(k+n-1)!}(F,e_k^\lambda)\rho_k^\lambda(f)e_k^\lambda.$$

Proposition 4.1. *For every Schwartz class function f on $\mathbb{H}^n$ the inversion formula*

$$f(z,t)=(2\pi)^{n/2}\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\left(\rho_k^\lambda(f)e_k^\lambda,\,\rho_k^\lambda(1,z,t)\frac{k!(n-1)!}{(k+n-1)!}e_k^\lambda\right)\frac{(k+n-1)!}{k!(n-1)!}\right)d\mu(\lambda)$$

holds, where 1 *stands for the identity matrix in $U(n)$.*

Proof. In view of the inversion formula (4-1) it is enough to show that

$$\left(\rho_k^\lambda(f)e_k^\lambda,\,\rho_k^\lambda(1,z,t)e_k^\lambda\right)=f*e_k^\lambda(z,t).$$

As $e_k^\lambda(z,t)=\sum_{|\alpha|=k}E_{\alpha,\alpha}^\lambda(z,t)$ we consider

$$\rho_k^\lambda(1,z,t)E_{\alpha,\alpha}^\lambda(w,s)=E_{\alpha,\alpha}^\lambda((-z,-t)(w,s)).$$

Recall that

$$E_{\alpha,\alpha}^\lambda((-z,-t)(w,s))=(2\pi)^{-n/2}(\pi_\lambda(-z,-t)\pi_\lambda(w,s)\Phi_\alpha^\lambda,\,\Phi_\alpha^\lambda).$$

Expanding $\pi_\lambda(w,s)\Phi_\alpha^\lambda$ and $\pi_\lambda(z,t))\Phi_\alpha^\lambda$ in terms of Φ_β^λ we get

$$\rho_k^\lambda(1,z,t)E_{\alpha,\alpha}^\lambda(w,s)=(2\pi)^{n/2}\sum_\beta E_{\alpha,\beta}^\lambda(w,s)\,\overline{E_{\alpha,\beta}^\lambda(z,t)}.$$

Since $\rho_k^\lambda(f)e_k^\lambda=f*e_k^\lambda$ and $\{E_{\alpha,\beta}^\lambda(w,s):\alpha,\beta\in\mathbb{N}^n,\,|\alpha|=k\}$ is an orthogonal basis for $\mathcal{H}_k^\lambda$ we get the proposition. □

From now on we identify $\mathbb{H}^n$ with $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ and use the notation (x,u,t) rather than $(x+iu,t)$ to denote elements of $\mathbb{H}^n$. The action of $U(n)$ on $\mathbb{H}^n$ then takes the form $\sigma.(x,u,t)=(a.x-b.u,\,b.x+a.u,\,t)$, where a and b are the real and imaginary parts of σ. This action has a natural extension to $\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}$ given by $\sigma.(z,w,\zeta)=(a.z-b.w,\,b.z+a.w,\,\zeta)$. With this definition we can extend the action of G_n on $\mathbb{H}^n$ to $\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}$:

$$(a+ib,x',u',t')(z,w,\zeta)=(x',u',t')(a.z-b.w,\,b.z+a.w,\,\zeta).$$

This action is then extended to functions defined on $\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}$:

$$\rho(g)f(z,w,\zeta)=f(g^{-1}.(z,w,\zeta)),\quad g\in G_n.$$

We are now ready to prove Gutzmer's formula for the Heisenberg group. Suppose f is a Schwartz class function on $\mathbb{H}^n$ such that $f^\lambda = 0$ for all $|\lambda| > A$ and $\rho_k^\lambda(f) = 0$ for all λ, k such that $(2k+n)|\lambda| > B$. We say that the Fourier transform of f is compactly supported if this condition is satisfied for some A and B. Now the inversion formula

$$f(g.(x, u, \xi)) = \int_{-A}^{A} \sum_{(2k+n)|\lambda| \le B} \big(\rho_k^\lambda(f)e_k^\lambda, \rho_k^\lambda(g)\rho_k^\lambda(1, x, u, \xi)e_k^\lambda\big)\, d\mu(\lambda)$$

is valid for any $g \in G_n$. Moreover, as each of $\rho_k^\lambda(1, x, u, \xi)e_k^\lambda$ extends to $\mathbb{C}^{2n+1}$ as an entire function, the same is true of $f(g.(x, u, \xi))$ and we have

$$f(g.(z, w, \zeta)) = \int_{-A}^{A} e^{\lambda\eta} \sum_{(2k+n)|\lambda| \le B} \big(\rho_k^\lambda(f)e_k^\lambda, \rho_k^\lambda(g)\rho_k^\lambda(1, x, u, \xi)e_k^\lambda\big)\, d\mu(\lambda),$$

where $\zeta = \xi + i\eta$. We then have the following Gutzmer's formula for the action of Heisenberg motion group on $\mathbb{C}^{2n+1}$, which is the complexification of $\mathbb{H}^n$.

Theorem 4.2. *Let f be Schwartz function whose Fourier transform is compactly supported in the sense above. Then f extends to $\mathbb{C}^{2n+1}$ as an entire function and we have the identity*

$$\int_{G_n} \big|f(g.(z, w, \zeta))\big|^2 dg =$$

$$(2\pi)^{n/2} \int_{-\infty}^{\infty} e^{2\lambda\eta} e^{-\lambda(u\cdot y - v\cdot x)} \left(\sum_{k=0}^{\infty} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^\lambda(2iy, 2iv) \right) d\mu(\lambda),$$

where $\|f^\lambda *_\lambda \varphi_k^\lambda\|_2$ is the $L^2(\mathbb{C}^n)$ norm of $f^\lambda *_\lambda \varphi_k^\lambda$.

We will prove this theorem by appealing to a general result on locally compact unimodular groups. Gutzmer's formula in the case of the circle group S^1 is just the Plancherel formula for the Fourier series applied to the Laurent series expansion of a function holomorphic in an annulus containing S^1. Lassalle [1978] has made an extensive study of Laurent series expansion for functions holomorphic in certain domains D contained in the complexification $X_{\mathbb{C}}$ of compact Riemannian symmetric spaces X. He obtained a Plancherel formula for such a series, which he later used in studying analogues of Hardy spaces over tube domains associated to compact symmetric spaces; see [Lassalle 1985]. Faraut [2002] treated the case of noncompact Riemannian symmetric spaces and proved a formula which he calls Gutzmer's formula. We recall the general setup for the reader's convenience.

Let G be a locally compact unimodular group with unitary dual $\hat{G}$. Let Λ be a Borel subset of $\hat{G}$ and let $d\mu$ be the Plancherel measure. For each $\lambda \in \Lambda$ choose a unitary representation (π_λ, V_λ) of class λ. Let $A(\lambda)$ be a family of trace class operators for which $\int_\Lambda \operatorname{tr}(|A(\lambda)|)\,d\mu$ and $\int_\Lambda \operatorname{tr}(A(\lambda)^*A(\lambda))\,d\mu$ are finite. Define a function f on G by

$$f(g) = \int_\Lambda \operatorname{tr}(A(\lambda)\pi_\lambda(g^{-1}))\,d\mu.$$

Then f lies in $L^2(G)$, and by the Plancherel Theorem,

$$\int_G |f(g)|^2\,dg = \int_\Lambda \operatorname{tr}(A(\lambda)^*A(\lambda))\,d\mu.$$

This sets up a one-to-one correspondence between subspaces of $L^2(G)$ whose Fourier transforms are supported on Λ and families $(A(\lambda))$ of Hilbert–Schmidt operators satisfying

$$\int_\Lambda \operatorname{tr}(A(\lambda)^*A(\lambda))\,d\mu < \infty.$$

The isometry just described takes a particularly simple form when $(A(\lambda))$ is a family of rank-one operators. Let $A(\lambda)v = (v, \eta(\lambda))\xi(\lambda)$, where $v \in V_\lambda$ and $\eta(\lambda), \xi(\lambda)$ are measurable functions taking values in V_λ. We then have the following result, whose proof can be found in [Faraut 2002; 2003].

Proposition 4.3. *Assume that $\eta(\lambda)$ and $\xi(\lambda)$ satisfy*

$$\int_\Lambda \|\eta(\lambda)\|^2\|\xi(\lambda)\|^2\,d\mu < \infty.$$

Then the function f defined by $f(g) = \int_\Lambda (\pi_\lambda(g^{-1})\xi(\lambda), \eta(\lambda))\,d\mu$ belongs to $L^2(G)$ and satisfies

$$\int_G |f(g)|^2\,dg = \int_\Lambda \|\eta(\lambda)\|^2\|\xi(\lambda)\|^2\,d\mu.$$

Proof of Theorem 4.2. Take $G = G_n$ and $\Lambda = \mathbb{R}\times\mathbb{N}$. The relevant representations are ρ_k^λ acting on the Hilbert spaces $\mathcal{H}_k^\lambda$. As already seen, the operators $\rho_k^\lambda(f)$ have rank one. We take $\xi(\gamma) := \rho_k^\lambda(f)e_k^\lambda$ and

$$\eta(\gamma) := \frac{k!\,(n-1)!}{(k+n-1)!}\,\rho_k^\lambda(1, z, w, \zeta)\,e_k^\lambda$$

when $\gamma = (\lambda, k) \in \Lambda$. (The first factor on the right is used because its reciprocal appears in the Plancherel measure for G_n.) We wish to appeal to Proposition 4.3 to complete the proof of Theorem 4.2.

We are therefore left with proving the equality

$$\|\rho_k^\lambda(1, z, w, \zeta)e_k^\lambda\|^2 = e^{2\lambda\eta}e^{-\lambda(u\cdot y - v\cdot x)}\varphi_k^\lambda(2iy, 2iv).$$

Recall that $e_k^\lambda(x', u', t') = e^{i\lambda t'}\varphi_k^\lambda(x, u)$ and this can be extended to $\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}$ as a holomorphic function. The action of $\rho_k^\lambda(1, x, u, t)$ on e_k^λ is given by

$$\rho_k^\lambda(1, x, u, t)e_k^\lambda(x', u', t') = e_k^\lambda((x, u, t)^{-1}(x', u', t'))$$

which reduces to

$$e^{i\lambda(t'-t-(u\cdot x'-x\cdot u')/2)}\varphi_k^\lambda(x'-x,u'-u).$$

The holomorphic extension of this is given by

$$\rho_k^\lambda(1,z,w,\zeta)e_k^\lambda(x',u',t')=e^{i\lambda(t'-\zeta-(w\cdot x'-z\cdot u')/2)}\varphi_k^\lambda(x'-z,u'-w).$$

In terms of real and imaginary parts of $z=x+iy$, $w=u+iv$, and $\zeta=\xi+i\eta$, this becomes

$$\rho_k^\lambda(1,z,w,\zeta)e_k^\lambda(x',u',t')=e^{\lambda\eta}e^{i\lambda(t'-\xi-(u\cdot x'-x\cdot u')/2)}e^{-\lambda(u'\cdot y-x'\cdot v)/2}\varphi_k^\lambda(x'-z,u'-w).$$

From the definition of the Hilbert space $\mathcal{H}_k^\lambda$, the norm of $\rho_k^\lambda(1,z,w,\zeta)e_k^\lambda$ in $\mathcal{H}_k^\lambda$ is

$$\|\rho_k^\lambda(1,z,w,\zeta)e_k^\lambda\|^2=|\lambda|^n e^{2\lambda\eta}\int_{\mathbb{R}^n\times\mathbb{R}^n}e^{-\lambda(u'\cdot y-x'\cdot v)}|\varphi_k^\lambda(x'-z,u'-w)|^2\,dx'\,du'.$$

Without loss of generality we can assume $\lambda>0$. By a change of variables the integral reduces to

$$e^{-\lambda(u\cdot y-x\cdot v)}\int_{\mathbb{R}^n\times\mathbb{R}^n}e^{-\lambda^{1/2}(u'\cdot y-x'\cdot v)}\left|\varphi_k(x'-i\lambda^{1/2}y,u'-i\lambda^{1/2}v)\right|^2dx'\,du'.$$

Using Proposition 3.3 we see that the integral is equal to $e^{-\lambda(u\cdot y-x\cdot v)}\varphi_k^\lambda(2iy,2iv)$, as required. □

Remark. We have stated Gutzmer's formula for functions whose Fourier transforms are compactly supported. This condition is not necessary for the validity of the formula. Consider the inversion formula stated in Proposition 4.1, namely

$$f(z,w,\zeta)=(2\pi)^{n/2}\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\left(\rho_k^\lambda(f)e_k^\lambda,\ \rho_k^\lambda(1,z,w,\zeta)e_k^\lambda\right)\right)d\mu(\lambda).$$

In view of the calculations made above the series converges as long as

$$\sum_{k=0}^{\infty}\|f^\lambda*_\lambda\varphi_k^\lambda\|_2\,e^{-\lambda(u\cdot y-v\cdot x)/2}(\varphi_k^\lambda(2iy,2iv))^{1/2}<\infty.$$

If we assume that $f^\lambda(x,u)$ is compactly supported in λ and the norms $\|f^\lambda*_\lambda\varphi_k^\lambda\|_2$ have enough decay, then the inversion formula is valid and $f(z,w,\zeta)$ is holomorphic. For example, when f belongs to the image of $L^2(\mathbb{H}^n)$ under the heat semigroup associated to the full Laplacian then $\|f^\lambda*_\lambda\varphi_k^\lambda\|_2\le Ce^{-t(2k+n)|\lambda|}$ and f extends as an entire function (see [Krötz et al. 2005b]). For such functions Gutzmer's formula is valid, as can be proved by using a density argument. We refer to [Faraut 2003] for some details in the case of noncompact Riemannian symmetric spaces.

5. Poisson integrals on the Heisenberg group

On the Heisenberg group we consider the sublaplacian $\mathscr{L}$ defined by

$$\mathscr{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2),$$

where $X_1, \dots, X_n, Y_1, \dots, Y_n$ together with $T = \partial/\partial t$ form a basis for the Heisenberg Lie algebra. (See [Thangavelu 1998; 2004] for explicit expressions for these vector fields.) The operator $\mathscr{L}$ is nonnegative, so using the spectral theorem we can define the Poisson semigroup $e^{-a(\mathscr{L})^{1/2}}$, $a > 0$. This is explicitly given by the spectral representation

$$e^{-a(\mathscr{L})^{1/2}} f(x, u, t) = \int_{-\infty}^{\infty} e^{-i\lambda t}\left(\sum_{k=0}^{\infty} e^{-(2k+n)^{1/2}|\lambda|^{1/2}a} f^\lambda *_\lambda \varphi_k^\lambda(x, u)\right) d\mu(\lambda).$$

We may suppose that f is a Schwartz class function whose Fourier transform is compactly supported in the sense defined in the previous section. For such functions the series above converges pointwise. We denote $\exp(-a(\mathscr{L})^{1/2}) f$ by $P_a f$ and call it the *Poisson integral of f*. For each $r > 0$, define

$$\Omega_r = \{(x+iy, u+iv, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} : y^2 + v^2 < r^2\}.$$

Theorem 5.1. *Let f be a Schwartz class function on $\mathbb{H}^n$ and assume that $f^\lambda(x, u)$ is compactly supported in $(-b, b)$ as a function of λ. Then for $0 < r < a$, we can extend $P_a f(x, u, t)$ to Ω_r as a holomorphic function of (z, w, ζ), and*

$$\int_{G_n} |P_a f(g.(z, w, \zeta))|^2 \, dg = c_n \times$$
$$\int_{-b}^{b} e^{2\lambda\eta} e^{-\lambda(u\cdot y - v\cdot x)}\left(\sum_{k=0}^{\infty} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 e^{-2(2k+n)^{1/2}|\lambda|^{1/2}a} \frac{k!\,(n-1)!}{(k+n-1)!} \varphi_k^\lambda(2iy, 2iv)\right) d\mu(\lambda).$$

Proof. Once we show that $P_a f$ extends as a holomorphic function of (z, w, ζ) on Ω_r the theorem will follow from Gutzmer's formula. Consider now the expansion

$$P_a f(x, u, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t}\left(\sum_{k=0}^{\infty} f^\lambda *_\lambda \varphi_k^\lambda(x, u) e^{-(2k+n)^{1/2}|\lambda|^{1/2}a}\right) |\lambda|^n \, d\lambda$$

and define the Poisson kernel by

$$P_a^\lambda(x, u) = \sum_{k=0}^{\infty} \varphi_k^\lambda(x, u) e^{-(2k+n)^{1/2}|\lambda|^{1/2}a}.$$

Then we have

$$P_a f(x,u,t) \\ = \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} e^{-i\lambda t} f^\lambda(x',u') P_a^\lambda(x-x',u-u') e^{-i(\lambda/2)(x\cdot u'-u\cdot x')}\,dx'\,du'\,d\lambda.$$

It is therefore enough to show that $P_a^\lambda(x,u)$ extends as a holomorphic function of (z,w). Fix λ. In view of Proposition 3.1, the series

$$\sum_{k=0}^\infty \varphi_k^\lambda(x+iy,u+iv)e^{-(2k+n)^{1/2}|\lambda|^{1/2}a}$$

is bounded by a constant times

$$\biggl(\sum_{k=0}^\infty \biggl(\frac{(k+n-1)!}{k!(n-1)!}\biggr)^{1/2} (\varphi_k^\lambda(2iy,2iv))^{1/2} e^{-(2k+n)^{1/2}|\lambda|^{1/2}a}\biggr) e^{|\lambda|(u\cdot y-v\cdot x)/2}.$$

For any fixed (y,v) with $y^2+v^2\le r^2<a^2$, Perron's formula gives the estimate

$$(\varphi_k^\lambda(2iy,2iv))^{1/2}\le C_r k^{(n-1)/4-1/8} e^{(2k+n)^{1/2}|\lambda|^{1/2}r}$$

and hence the series is dominated by

$$\sum_{k=1}^\infty k^{(n-1)/2}k^{(n-1)/4-1/8}e^{-(2k+n)^{1/2}|\lambda|^{1/2}(a-r)}$$

which certainly converges as long as $r<a$.

Moreover, using the asymptotic estimates given by Perron's formula the integral

$$\int_{-\infty}^\infty (\varphi_k^\lambda(2iy,2iv))^{1/2}e^{-(2k+n)^{1/2}|\lambda|^{1/2}a}|\lambda|^n\,d\lambda$$

is bounded by a constant multiple of

$$k^{(n-1)/4-1/8}\int_0^\infty e^{-(2k+n)^{1/2}t(a-r)}t^{(6n+9)/4-1}\,dt,$$

so for $y^2+v^2\le r^2$ and $|\lambda|\le b$, the sum

$$\sum_{k=0}^\infty\int_{-\infty}^\infty |\varphi_k^\lambda(x+iy,u+iv)|e^{-(2k+n)^{1/2}|\lambda|^{1/2}a}e^{|\lambda|(u\cdot y-v\cdot x)/2}|\lambda|^n\,d\lambda$$

is dominated by a constant times

$$e^{b|u\cdot y-v\cdot x|}\sum_{k=1}^\infty k^{(n-1)/2}k^{(n-1)/4-1/8}k^{-(6n+9)/8},$$

which is finite. Hence standard arguments show that $P_a^\lambda(x,u)$ extends to Ω_r as a holomorphic function. $\square$

We now use Theorem 5.1 to get a characterisation of functions that arise as Poisson integrals. Let $\mathcal{G}(\Omega_a)$ be the space of functions $F(z,w,\zeta)$ holomorphic on Ω_a and such that $F^\lambda(x,u)$ is compactly supported in λ and

$$\|F\|^2_{\Omega_a} = \lim_{y^2+v^2\to a^2} \int_{G_n} |F(g.(iy,iv,t))|^2\,dg < \infty.$$

We define $\mathcal{B}(\Omega_a)$ to be its completion. For $F\in\mathcal{G}(\Omega_a)$, Gutzmer's formula shows that the restriction of F to $\mathbb{H}^n$ satisfies

$$\int_{-\infty}^{\infty}\sum_{k=0}^{\infty}\left(\|F^\lambda *_\lambda \varphi_k^\lambda\|_2^2 \frac{k!(n-1)!}{(k+n-1)!}\varphi_k^\lambda(2iy,2iv)\right)|\lambda|^n\,d\lambda < \infty$$

whenever $y^2+v^2\le a^2$.

We also need to recall the definition of the Sobolev spaces $H^s(\mathbb{H}^n)$. This is the space of all tempered distributions for which $(1+\mathcal{L})^{s/2}f\in L^2(\mathbb{H}^n)$. The norm is given by the expression

$$\|f\|^2_{(s)} = c_n\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2\left((2k+n)|\lambda|\right)^s\right)|\lambda|^n\,d\lambda.$$

The asymptotic formula we have used for $\varphi_k^\lambda(2iy,2iv)$ reads as

$$\frac{k!(n-1)!}{(k+n-1)!}\varphi_k^\lambda(2iy,2iv) \le C((2k+n)|\lambda|)^{-(2n-1)/4}e^{-2(2k+n)^{1/2}|\lambda|^{1/2}r},$$

and therefore f lies in $H^{-(2n-1)/4}(\mathbb{H}^n)$ precisely when

$$\int_{-\infty}^{\infty}\sum_{k=0}^{\infty}\left(\|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 \frac{k!(n-1)!}{(k+n-1)!}\varphi_k^\lambda(2iy,2iv)\right)|\lambda|^n\,d\lambda < \infty$$

for all y, v with $y^2+v^2\le a^2$.

Theorem 5.2. *A function F lies in $\mathcal{B}(\Omega_a)$ if and only if $F=P_af$ for some $f\in H^{-(2n-1)/4}(\mathbb{H}^n)$. The norm of f in $H^{-(2n-1)/4}(\mathbb{H}^n)$ is equivalent to*

$$\lim_{y^2+v^2\to a^2}\int_{G_n}|F(g.(iy,iv,t))|^2\,dg.$$

Proof. If $f\in H^{-(2n-1)/4}(\mathbb{H}^n)$ then f can be approximated by a sequence f_n of functions whose Fourier transforms in the central variable are compactly supported. For such functions we have verified that P_af_n extends to a function in $\mathcal{G}(\Omega_a)$. This proves half the theorem. Since all the steps in our calculations are reversible, the converse also follows in light of the remarks preceding the theorem. $\square$

6. Revisiting twisted Bergman spaces

Consider the full Laplacian $\Delta = \mathscr{L} - T^2$ on the Heisenberg group. In [Krötz et al. 2005b] we studied the problem of characterising the image of $L^2(\mathbb{H}^n)$ under the semigroup $e^{-t\Delta}$ as a weighted Bergman space over the domain $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$, which is just the complexification of $\mathbb{H}^n$. It turned out that the image is not a weighted Bergman space, contrary to expectation. What is true is that the image is a direct integral of certain weighted Bergman spaces $\mathscr{B}_t^\lambda$, which are the images of $L^2(\mathbb{C}^n)$ under the semigroup e^{-tL_λ} generated by the special Hermite operators L_λ, which are related to $\mathscr{L}$. In this section we give a different proof of this characterisation of $\mathscr{B}_t^\lambda$ using Gutzmer's formula on the Heisenberg group.

We briefly recall the definitions of L_λ and the twisted Bergman spaces $\mathscr{B}_t^\lambda$, referring to [Krötz et al. 2005b] for more details. For each $\lambda \neq 0$, the operator L_λ is defined by $\mathscr{L}(e^{-i\lambda t} f(z)) = e^{-i\lambda t} L_\lambda f(z)$. The spectral decomposition of L_λ is given by

$$L_\lambda f(x,u) = (2\pi)^{-n} \sum_{k=0}^{\infty} (2k+n)|\lambda| f *_\lambda \varphi_k^\lambda(x,u).$$

The operator L_λ generates a diffusion semigroup e^{-tL_λ} given by twisted convolution with the kernel

$$p_t^\lambda(x,u) = c_n \left(\frac{\lambda}{\sinh(\lambda t)}\right)^n e^{-\lambda \coth(\lambda t)(x^2+u^2)/4}.$$

For each $f \in L^2(\mathbb{C}^n)$ the function

$$f *_\lambda p_t^\lambda(x,u) = \int_{\mathbb{R}^{2n}} f(x-x', u-u') p_t^\lambda(x',u') e^{i/2\lambda(u\cdot x' - x\cdot u')} \, dx' \, du'$$

can be extended to $\mathbb{C}^n \times \mathbb{C}^n$ as an entire function. We let

$$W_t^\lambda(x+iy, u+iv) = 4^n e^{\lambda(u\cdot y - v\cdot x)} p_{2t}^\lambda(2y, 2v)$$

and define $\mathscr{B}_t^\lambda$ to be the space of all entire functions on $\mathbb{C}^n \times \mathbb{C}^n$ that are square integrable with respect to the weight function $W_t^\lambda(z,w)$.

Theorem 6.1 [Krötz et al. 2005b]. *An entire function F on $\mathbb{C}^n \times \mathbb{C}^n$ belongs to $\mathscr{B}_t^\lambda$ if and only if $F(x,u) = f *_\lambda p_t^\lambda(x,u)$ for some $f \in L^2(\mathbb{C}^n)$.*

In this section we give a different and more transparent proof that $f *_\lambda p_t^\lambda(z,w)$ belongs to $\mathscr{B}_t^\lambda$, based on Gutzmer's formula for special Hermite expansions. In proving this we assume $\lambda = 1$ and simply write p_t in place of p_t^1.

Consider the reduced Heisenberg group (or Heisenberg group with compact center) $\mathbb{H}^n_{\text{red}}$ defined to be $\mathbb{C}^n \times S^1$ with group law

$$(z, e^{it})(w, e^{is}) = (z+w, e^{i(t+s+\frac{1}{2}\operatorname{Im}(z\cdot\bar{w}))}).$$

The infinite-dimensional irreducible unitary representations of $\mathbb{H}^n_{\text{red}}$ that are nontrivial at the center are given (up to unitary equivalence) by the Schrödinger representations π_j, where now j is a nonzero integer. The case $j = 0$ corresponds to the one-dimensional representations that factor through characters of $\mathbb{C}^n$. For functions f on $\mathbb{H}^n_{\text{red}}$ having mean value zero, i.e., $\int_{S^1} f(z, e^{it})\,dt = 0$, the relevant representations are just π_j with $j \neq 0$. Let G_n^{red} be the Heisenberg motion group formed with $\mathbb{H}^n_{\text{red}}$ in place of $\mathbb{H}^n$. Then with the same notations as in Section 4 we have the following result. Let Ω_r be defined as before.

Theorem 6.2. *Let f be a function on $\mathbb{H}^n_{\text{red}}$ having mean value zero and satisfying the conditions stated in Theorem 4.2. For all $(z, w) \in \Omega_r$, we have*

$$\int_{G_n^{\text{red}}} |f(g.(iy, iv, e^{it}))|^2\,dg = (2\pi)^{-n-1} \sum_{j\neq 0} \left(\sum_{k=0}^{\infty} \|f^j *_j \varphi_k^j\|_2^2 \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^j(2iy, 2iv) \right) |j|^n.$$

When we take functions of the form $f(x, u)e^{-it}$, exactly one term (corresponding to $j = 1$) survives in this sum over j (since f^j is the $(-j)$-th Fourier coefficient of f), and we get

$$\int_{\mathbb{R}^{2n}} |f(x+iy, u+iv)|^2 e^{(u\cdot y - v\cdot x)}\,dx\,du = c_n \sum_{k=0}^{\infty} \|f \times \varphi_k\|_2^2 \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(2iy, 2iv),$$

which we refer to as *Gutzmer's formula for special Hermite expansions*. Here $f \times \varphi_k$ is the twisted convolution, which is just the λ-twisted convolution when $\lambda = 1$. The equality is valid for all functions f for which the right-hand side converges. This is so if the norms of the projections $f \times \varphi_k$ decay fast enough. In particular, the preceding formula is valid if f is replaced by $e^{-tL}f$ with f in $L^2(\mathbb{C}^n)$.

Applying Gutzmer's formula to the function $F(z, w) = e^{-tL} f(z, w)$ we obtain

$$\int_{\mathbb{R}^{2n}} |F(x+iy, u+iv)|^2 e^{(u\cdot y - v\cdot x)}\,dx\,du = c_n \sum_{k=0}^{\infty} \|f \times \varphi_k\|_2^2 e^{-2(2k+n)t} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(2iy, 2iv).$$

If we can show that

$$\int_{\mathbb{R}^{2n}} \varphi_k(2iy, 2iv)\, p_{2t}(2y, 2v)\,dy\,dv = \frac{(k+n-1)!}{k!(n-1)!} e^{2(2k+n)t},$$

we can integrate Gutzmer's formula against $p_{2t}(2y, 2v)\,dy\,dv$ to get

$$\int_{\mathbb{C}^{2n}} |F(z, w)|^2 W_t^1(z, w)\,dz\,dw = c_n \int_{\mathbb{R}^{2n}} |f(x, u)|^2\,dx\,du,$$

which will prove our claim and hence Theorem 6.1. So it remains to prove the following lemma.

Lemma 6.3. $\displaystyle\int_{\mathbb{R}^{2n}} \varphi_k(iy, iv) p_t(y, v)\, dy\, dv = \frac{(k+n-1)!}{k!(n-1)!} e^{(2k+n)t}.$

Before proving the lemma we make some remarks. Since the heat kernel p_t is given by the expansion

$$p_t(y, v) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} \varphi_k(y, v)$$

it follows, in view of the orthogonality properties of φ_k, that

$$\int_{\mathbb{R}^{2n}} p_t(y, v) \varphi_k(y, v)\, dy\, dv = \frac{(k+n-1)!}{k!(n-1)!} e^{-(2k+n)t}. \tag{6-1}$$

As φ_k are the spherical functions associated to the Gelfand pair $(G_n, U(n))$, the formula in the lemma is the analogue of the formula

$$\int_{\mathbb{R}^n} \varphi_\lambda(iy) e^{-y^2/(4t)}\, dy = c_n e^{t\lambda^2},$$

where φ_λ are the spherical functions on $\mathbb{R}^n$, namely, the Bessel functions. This was the key formula used in characterising Bergman spaces associated to the Laplacian on $\mathbb{R}^n$.

Proof of the lemma. Recall from [Szegö 1967] that

$$L_k^{n-1}(s) = \sum_{j=0}^{k} c_{k,j} (-s)^j,$$

where the $c_{k,j}$ are constants whose exact values are immaterial. Equation (6-1) now reads as

$$(\sinh t)^{-n} \int_{\mathbb{R}^{2n}} \sum_{j=0}^{k} c_{k,j} (-1)^j 2^{-j} (y^2+v^2)^j e^{-(1+\coth t)(y^2+v^2)/4}\, dy\, dv \\ = (2\pi)^n \frac{(k+n-1)!}{k!(n-1)!} e^{-(2k+n)t}.$$

This can be rewritten as

$$(\cosh t)^{-n} \sum_{j=0}^{k} c_{k,j} (-1)^j 2^{-j} a_j (\tanh t)^j (1+\tanh t)^{-j-n} \\ = (2\pi)^n \frac{(k+n-1)!}{k!(n-1)!} e^{-(2k+n)t}, \tag{6-2}$$

where the a_j are constants defined by

$$a_j = \int_{\mathbb{R}^{2n}} (y^2+v^2)^j e^{-(y^2+v^2)/4}\, dy\, dv.$$

Now both sides of (6-2) are holomorphic functions of t in a strip containing the real line, so the equation is true for t negative as well. This leads to

$$(\cosh t)^{-n}\sum_{j=0}^{k} c_{k,j}2^{-j}a_j(\tanh t)^j(1-\tanh t)^{-j-n}=(2\pi)^n\frac{(k+n-1)!}{k!(n-1)!}e^{(2k+n)t}.$$

The left-hand side now is simply

$$(\sinh t)^{-n}\int_{\mathbb{R}^{2n}}\sum_{j=0}^{k} c_{k,j}2^{-j}(y^2+v^2)^j e^{-(-1+\coth t)(y^2+v^2)/4}\,dy\,dv,$$

which is the same as

$$(2\pi)^n\int_{\mathbb{R}^{2n}}\varphi_k(iy,iv)p_t(y,v)\,dy\,dv.$$

This completes the proof of the lemma and hence of Theorem 6.1. □

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