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JULIA WEBER

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For discrete groups G, we introduce equivariant Nielsen invariants. They are equivariant analogs of the Nielsen number and give lower bounds for the number of fixed point orbits in the G-homotopy class of an equivariant endomorphism $f : X \to X$. Under mild hypotheses, these lower bounds are sharp.

We use the equivariant Nielsen invariants to show that a G-equivariant endomorphism f is G-homotopic to a fixed point free G-map if the generalized equivariant Lefschetz invariant $\lambda_G(f)$ is zero. Finally, we prove a converse of the equivariant Lefschetz fixed point theorem.

1. Introduction

The Lefschetz number is a classical invariant in algebraic topology. If the Lefschetz number $L(f)$ of an endomorphism $f : X \to X$ of a compact CW-complex is nonzero, then f has a fixed point. This is the famous Lefschetz fixed point theorem. The converse does not hold: If the Lefschetz number of f is zero, we cannot conclude f to be fixed point free.

A more refined invariant which allows to state the converse is the Nielsen number: The Nielsen number $N(f)$ is zero if and only if f is homotopic to a fixed point free map. More generally, the Nielsen number is used to give precise minimal bounds for the number of fixed points of maps homotopic to f. Its development was started by Nielsen [1920]; a comprehensive treatment can be found in [Jiang 1983].

We are interested in the equivariant generalization of these results. Given a discrete group G and a G-equivariant endomorphism $f : X \to X$ of a finite proper G-CW-complex, we introduce equivariant Nielsen invariants called $N_G(f)$ and $N^G(f)$. They are equivariant analogs of the Nielsen number and are derived from the generalized equivariant Lefschetz invariant $\lambda_G(f)$ [Weber 2005, Definition 5.13].

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We proceed to show that these Nielsen invariants give minimal bounds for the number of orbits of fixed points in the G-homotopy class of f. One even obtains results concerning the type and "location" (connected component of the relevant fixed point set) of these fixed point orbits. These lower bounds are sharp if X is a cocompact proper smooth G-manifold satisfying the standard gap hypotheses.

Finally, we prove a converse of the equivariant Lefschetz fixed point theorem: If X is a G-Jiang space as defined in Definition 5.2, then $L_G(f) = 0$ implies that f is G-homotopic to a fixed point free map. Here, $L_G(f)$ is the equivariant Lefschetz class [Lück and Rosenberg 2003a, Definition 3.6], the equivariant analog of the Lefschetz number.

These results were motivated by work of Lück and Rosenberg [2003a; 2003b]. For G a discrete group and an endomorphism f of a cocompact proper smooth G-manifold M, they prove an equivariant Lefschetz fixed point theorem [2003a, Theorem 0.2]. The converse of that theorem is proven here.

Another motivation for the present article is the fact that the algebraic approach to the equivariant Reidemeister trace provides a good framework for computation. The connection to the machinery used in the study of transformation groups [Lück 1989; tom Dieck 1987] allows results to translate more readily from transformation groups to geometric equivariant topology and vice-versa.

When G is a compact Lie group, Wong [1993] obtains results on equivariant Nielsen numbers which strongly influenced us. The main difference between our work and Wong's is that we treat possibly infinite discrete groups. Another difference is that our approach is more structural. We can read off the equivariant Nielsen invariants from the generalized equivariant Lefschetz invariant $\lambda_G(f)$.

In case G is a finite group, Ferrario [2003] studies a collection of generalized Lefschetz numbers which can be thought of as an equivariant generalized Lefschetz number. In contrast to the generalized equivariant Lefschetz invariant $\lambda_G(f)$ these do not incorporate the WH-action on the fixed point set X^H. A generalized Lefschetz trace for equivariant maps has also been defined by Wong, as mentioned in [Hart 1999].

For compact Lie groups, earlier definitions of generalized Lefschetz numbers for equivariant maps were made in [Wilczyński 1984; Fadell and Wong 1988]. These authors used the collection of generalized Lefschetz numbers of the maps f^H, for $H < G$. In general, these numbers are not sufficient since they do not take the equivariance into account adequately. For further reading on equivariant fixed point theory, see [Ferrario 2005], where an extensive list of references is given.

Organization of paper. In Section 2, we introduce the generalized equivariant Lefschetz invariant. We briefly assemble the concepts and definitions which are needed for the definition of equivariant Nielsen invariants in Section 3.

The equivariant Nielsen invariants give lower bounds for the number of fixed point orbits of f. This is shown in Section 4. The standard gap hypotheses are introduced, and it is shown that under these hypotheses these lower bounds are sharp.

In the nonequivariant case, we know that the generalized Lefschetz invariant is the right element when looking for a precise count of fixed points. We read off the Nielsen number from this invariant. In general, the Lefschetz number contains too little information.

But under certain conditions, we can conclude facts about the Nielsen number from the Lefschetz number directly. These are called the Jiang conditions [1983, Definition II.4.1] (see also [Brown 1971, Chapter VII]). In Section 5, we introduce the equivariant version of these conditions. We also give examples of G-Jiang spaces.

In Section 6, we derive equivariant analogs of statements about Nielsen numbers found in [Jiang 1983], generalizing results from [Wong 1993] to infinite discrete groups. In particular, if X is a G-Jiang space, the converse of the equivariant Lefschetz fixed point theorem holds.

2. The generalized equivariant Lefschetz invariant

Classically, the Nielsen number is defined geometrically by counting essential fixed point classes [Brown 1971, Chapter VI; Jiang 1983, Definition I.4.1]. Alternatively one defines it using the generalized Lefschetz invariant.

Let X be a finite CW-complex, let $f : X \to X$ be an endomorphism, let x be a basepoint of X, and let

$$\lambda(f) = \sum_{\overline{\alpha} \in \pi_1(X,x)_\phi} n_{\overline{\alpha}} \cdot \overline{\alpha} \in \mathbb{Z}\pi_1(X,x)_\phi$$

be the *generalized Lefschetz invariant* associated to f [Reidemeister 1936; Wecken 1941], where

$$\mathbb{Z}\pi_1(X,x)_\phi := \mathbb{Z}\pi_1(X,x)/\phi(\gamma)\alpha\gamma^{-1} \sim \alpha, \text{ with } \gamma, \alpha \in \pi_1(X,x).$$

Here ϕ is the map induced by f on the fundamental group $\pi_1(X,x)$. We have $\phi(\gamma) = wf(\gamma)w^{-1}$, where w is a path from x to $f(x)$. The generalized Lefschetz invariant is also called *Reidemeister trace* in the literature, which goes back to the original name "Reidemeistersche Spureninvariante" used by Wecken.

The set $\pi_1(X,x)_\phi$ is often denoted by $\mathcal{R}(f)$ and called set of *Reidemeister classes* of f. Since we will introduce a variation of this set in Definition 2.3, we prefer to stick with the notation used in [Weber 2005].

Definition 2.1. The *Nielsen number* of f is defined by

$$N(f) := \#\{\overline{\alpha} \mid n_{\overline{\alpha}} \neq 0\}.$$

The Nielsen number is the number of classes in $\pi_1(X, x)_\phi$ with nonzero coefficients. A class α with nonzero coefficient corresponds to an essential fixed point class in the geometric sense.

In the equivariant setting, the fundamental category replaces the fundamental group. The fundamental category of a topological space X with an action of a discrete group G is defined as follows [Lück 1989, Definition 8.15].

Definition 2.2. Let G be a discrete group, and let X be a G-space. Then the *fundamental category* $\Pi(G, X)$ is the following category:

- The objects $\mathrm{Ob}(\Pi(G, X))$ are G-maps $x : G/H \to X$, where the $H \leq G$ are subgroups.
- The morphisms $\mathrm{Mor}(x(H), y(K))$ are pairs $(\sigma, [w])$, where
  - σ is a G-map $\sigma : G/H \to G/K$
  - $[w]$ is a homotopy class of G-maps $w : G/H \times I \to X$ relative $G/H \times \partial I$ such that $w_1 = x$ and $w_0 = y \circ \sigma$.

The fundamental category is a combination of the orbit category of G and the fundamental groupoid of X. If X is a point, then the fundamental category is just the orbit category of G, whereas when G is the trivial group, the definition reduces to the definition of the fundamental groupoid of X.

We often view x as the point $x(1H)$ in the fixed point set X^H. We call $X^H(x)$ the connected component of X^H containing $x(1H)$. We also consider the relative fixed point set, the pair $(X^H(x), X^{>H}(x))$. Here $X^{>H}(x) = \{z \in X^H(x) \mid G_z \neq H\}$ is the singular set, where G_z denotes the isotropy group of z. In order to simplify notation, we use $f^H(x)$ to denote $f|_{X^H(x)}$, and we use $f_H(x)$ instead of $f|_{(X^H(x), X^{>H}(x))}$.

Fixed points of f can only exist in $X^H(x)$ when $X^H(f(x)) = X^H(x)$, i.e, when the points $f(x)$ and x lie in the same connected component of X^H.

Definition 2.3. For $x \in \mathrm{Ob}\,\Pi(G, X)$ with $X^H(f(x)) = X^H(x)$ and a morphism $v = (\mathrm{id}, [w]) \in \mathrm{Mor}(f(x), x)$, set

$$\mathbb{Z}\pi_1(X^H(x), x)_{\phi'} := \mathbb{Z}\pi_1(X^H(x), x)/\phi(\gamma)\alpha\gamma^{-1} \sim \alpha,$$

where $\alpha \in \pi_1(X^H(x), x)$, $\gamma \in \mathrm{Aut}(x)$ and $\phi(\gamma) = v\phi(\gamma)v^{-1} \in \mathrm{Aut}(x)$.

The automorphism group $\mathrm{Aut}(x)$ of the object x in the category $\Pi(G, X)$ is a group extension lying in the short exact sequence

$$1 \to \pi_1(X^H(x), x) \to \mathrm{Aut}(x) \to WH_x \to 1.$$

Here $WH := N_G H/H$ is the Weyl group of H, it acts on X^H. We call WH_x the subgroup of WH which fixes the connected component $X^H(x)$.

Groups obtained from different choices of the path w and of the point x in its isomorphism class $\bar{x}$ are canonically isomorphic, so those choices do not play a role. The group $\mathbb{Z}\pi_1\big(X^H(x), x\big)_{\phi'}$ generalizes the group $\mathbb{Z}\pi_1(X, x)_\phi$ defined above. So it can be seen as the free abelian group generated by equivariant Reidemeister classes of $f^H(x)$ with respect to the action of WH_x on $X^H(x)$.

A map $(\sigma, [w]) \in \operatorname{Mor}(x, y)$ induces a group homomorphism

$$(\sigma, [w])^* : \mathbb{Z}\pi_1\big(X^K(y), y\big)_{\phi'} \to \mathbb{Z}\pi_1\big(X^H(x), x\big)_{\phi'}$$

by twisted conjugation, and we know that the induced group homomorphism is the same for every map in $\operatorname{Mor}(x, y)$ [Weber 2005, Lemma 5.2].

The *generalized equivariant Lefschetz invariant* [Weber 2005, Definition 5.13], $\lambda_G(f)$, is an element in the group

$$\Lambda_G(X, f) := \bigoplus_{\substack{\bar{x}\in \operatorname{Is}\Pi(G,X),\\ X^H(f(x))=X^H(x)}} \mathbb{Z}\pi_1(X^H(x), x)_{\phi'}.$$

Here $\operatorname{Is}\Pi(G, X)$ denotes the set of isomorphism classes of the category $\Pi(G, X)$. Geometrically, it corresponds to the set of WH-orbits of connected components $X^H(x)$ of the fixed point sets X^H, for $(H) \in \operatorname{consub}(G)$, i.e., for a set of representatives of conjugacy classes of subgroups of G. There is a bijection $\operatorname{Is}\Pi(G, X) \xrightarrow{\cong} \coprod_{(H)\in\operatorname{consub}(G)} WH \setminus \pi_0(X^H)$ which sends $x : G/H \to X$ to the orbit under the WH-action on $\pi_0(X^H)$ of the component $X^H(x)$ of X^H which contains the point $x(1H)$ [Lück and Rosenberg 2003a, Equation 3.3].

Let $\widetilde{f^H(x)}$ and $\widetilde{f^{>H}(x)}$ denote the lift of $f^H(x)$ to the universal covering space $\widetilde{X^H(x)}$ and to the subset $\widetilde{X^{>H}(x)} \subseteq \widetilde{X^H(x)}$ that projects to $X^{>H}(x)$ under the covering map.

At the summand indexed by $\bar{x}$, the generalized equivariant Lefschetz invariant is given by

$$\lambda_G(f)_{\bar{x}} := L^{\mathbb{Z}\operatorname{Aut}(x)}\big(\widetilde{f^H(x)}, \widetilde{f^{>H}(x)}\big) \in \mathbb{Z}\pi_1(X^H(x), x)_{\phi'},$$

where the *refined equivariant Lefschetz number* [Weber 2005, Definition 5.7] appears on the right hand side. It is defined by

$$L^{\mathbb{Z}\operatorname{Aut}(x)}\big(\widetilde{f^H(x)}, \widetilde{f^{>H}(x)}\big) := \sum_{p\geq 0}(-1)^p \operatorname{tr}_{\mathbb{Z}\operatorname{Aut}(x)}(C_p^c(\widetilde{f^H(x)}, \widetilde{f^{>H}(x)})),$$

where the trace map $\operatorname{tr}_{\mathbb{Z}\operatorname{Aut}(x)}$ [Weber 2005, Definition 5.4] is induced by the projection $\mathbb{Z}\operatorname{Aut}(x) \to \mathbb{Z}\pi_1(X^H(x), x)_{\phi'}$, $\sum_{g\in\operatorname{Aut}(x)} r_g \cdot g \mapsto \sum_{g\in\pi_1(X^H(x),x)} r_g \cdot \bar{g}$. Instead of $\mathbb{Z}$, other rings can be used.

This trace map generalizes the trace map used in [Lück and Rosenberg 2003a], and the refined equivariant Lefschetz number is a generalization of the orbifold Lefschetz number (Definition 1.4 of that reference).

The refined equivariant Lefschetz number $L^{\mathbb{Q}\operatorname{Aut}(x)}\big(\widetilde{f^H(x)}\big)$ will be particularly important to us, so we give some formulas describing it. For a finite proper G-CW-complex X we have [Weber 2005, Lemma 5.9]

$$L^{\mathbb{Q}\operatorname{Aut}(x)}\big(\widetilde{f^H(x)}\big)=\sum_{p\geq 0}(-1)^p\sum_{G\cdot e\in G\backslash I_p(X)}|G_e|^{-1}\cdot \operatorname{inc}_\phi(f,e)\in \mathbb{Q}\pi_1(X^H(x),x)_{\phi'}.$$

Here $I_p(X)$ denotes the set of p-cells of X, e runs through the equivariant cells of X, and G_e is its isotropy group. The *refined incidence number* [Weber 2005, Definition 5.8] $\operatorname{inc}_\phi(f,e)\in\mathbb{Z}\pi_1(X^H(x),x)_{\phi'}$ for a p-cell $e\in I_p(X)$ is defined to be the "degree" of the composition

$$\bar{e}/\partial e \xrightarrow{i_e} \bigvee_{e'\in I_p(X)} \overline{e'}/\partial e' \xrightarrow{h\sim} X_p/X_{p-1} \xrightarrow{f} X_p/X_{p-1} \xrightarrow{h^{-1}\sim} \bigvee_{e'\in I_p(X)} \overline{e'}/\partial e'$$
$$\xrightarrow{\operatorname{pr}_{\pi\cdot\bar{e}/\partial e}} \pi\cdot\bar{e}/\partial e \xrightarrow{\bar{\cdot}} \pi_{\phi'}\cdot\bar{e}/\partial e.$$

Here $\bar{e}$ is the closure of the open p-cell e and $\partial e=\bar{e}\backslash e$. The map i_e is the inclusion, h is a homeomorphism and $\operatorname{pr}_{\pi\cdot\bar{e}/\partial e}$ is the projection.

If $X=M$ is a cocompact proper G-manifold, we have [Weber 2005, Theorem 6.6]

$$L^{\mathbb{Q}\operatorname{Aut}(x)}\big(\widetilde{f^H(x)}\big)=\sum_{\substack{WH_x\cdot z\in\\ WH_x\backslash\operatorname{Fix}(f^H(x))}}\big|(WH_x)_z\big|^{-1}\deg\Big(\big(\operatorname{id}_{T_zM^H(x)}-T_z(f^H(x))\big)^c\Big)\cdot\overline{\alpha_z}.$$

Here the map on the tangent space is extended to the one-point compactification $\big(T_zM^H(x)\big)^c$. The relative versions of these formulas also hold.

We have $L^{\mathbb{Q}\operatorname{Aut}(x)}\big(\widetilde{f^H(x)}\big)=\operatorname{ch}_G(X,f)\big(\lambda_G(f)\big)_{\bar{x}}$ [Weber 2005, Lemma 6.4], where $\operatorname{ch}_G(X,f):\Lambda_G(X,f)\to\bigoplus_{\bar{y}\in\operatorname{Is}\Pi(G,X)}\mathbb{Q}\pi_1(X^K(y),y)_{\phi'}$ is the character map [Weber 2005, Definition 6.2]. So we can derive $L^{\mathbb{Q}\operatorname{Aut}(x)}\big(\widetilde{f^H(x)}\big)$ from $\lambda_G(f)$.

The equivariant analog of the Lefschetz number is the *equivariant Lefschetz class* $L_G(f)\in\bigoplus_{\bar{x}\in\operatorname{Is}\Pi(G,X),X^H(f(x))=X^H(x)}\mathbb{Z}$, whose value at $\bar{x}$ is

$$L_G(f)_{\bar{x}}=L^{\mathbb{Z}WH_x}\big(f^H(x),f^{>H}(x)\big)$$

[Lück and Rosenberg 2003a, Definition 3.6]. The projection of $\pi_1(X^H(x),x)_{\phi'}$ to the trivial group $\{1\}$ induces an augmentation map sending $\lambda_G(f)$ to $L_G(f)$:

$$s:\bigoplus_{\substack{\bar{x}\in\operatorname{Is}\Pi(G,X),\\ X^H(f(x))=X^H(x)}}\mathbb{Z}\pi_1(X^H(x),x)_{\phi'}\;\to\;\bigoplus_{\substack{\bar{x}\in\operatorname{Is}\Pi(G,X),\\ X^H(f(x))=X^H(x)}}\mathbb{Z}.$$

3. Equivariant Nielsen invariants

Given an element $\sum_{\overline{\alpha}} n_{\overline{\alpha}} \cdot \overline{\alpha} \in \mathbb{Z}\pi_1(X^H(x), x)_{\phi'}$, we call a class $\overline{\alpha} \in \pi_1(X^H(x), x)_{\phi'}$ *essential* if the coefficient $n_{\overline{\alpha}}$ is nonzero.

Let G be a discrete group and let X be a cocompact proper smooth G-manifold. Let $f : X \to X$ be a smooth G-equivariant map such that $\mathrm{Fix}(f) \cap \partial X = \varnothing$ and such that for every $z \in \mathrm{Fix}(f)$ the determinant of the map $(\mathrm{id}_{T_z X} - T_z f)$ is different from zero. One can always find a representative in the G-homotopy class of f which satisfies this assumption. Since the generalized equivariant Lefschetz invariant is G-homotopy invariant, we can replace f by this representative if necessary.

Definition 3.1. The *equivariant Nielsen class* of f is

$$\nu_G(f) = \sum_{Gz \in G \backslash \mathrm{Fix}(f)} \frac{\det\bigl(\mathrm{id}_{T_z X} - T_z(f)\bigr)}{\bigl|\det\bigl(\mathrm{id}_{T_z X} - T_z(f)\bigr)\bigr|} \cdot \overline{\alpha_z}.$$

Here $\alpha_z \in \pi_1(X^{G_z}(z), x)$ is the loop given by $[t * f(t)^{-1} * w]$, where x is a basepoint in $X^{G_z}(z)$, t is a path from x to z and w is a path from $f(x)$ to x. The basepoint x may differ from z, e.g., if we have more than one fixed point in a connected component of X^{G_z}. If $x = z$, we may choose t and w to be constant. The equivalence relation assures that this definition is independent of the choices involved.

We can also derive the equivariant Nielsen class $\nu_G(f)$ from the generalized equivariant Lefschetz invariant $\lambda_G(f)$.

Lemma 3.2. *The invariant $\nu(f)$ is the image of the generalized equivariant Lefschetz invariant $\lambda(f)$ under the quotient map where we divide out the images of nonisomorphisms*:

$$\nu_G(f) = \overline{\lambda_G(f)} \in \bigoplus_{\substack{\bar{x} \in \mathrm{Is}\,\Pi(G,X), \\ X^H(f(x)) = X^H(x)}} \mathbb{Z}\pi_1(X^H(x), x)_{\phi'} / \{\mathrm{Im}(\sigma, [w])^* \mid \sigma \textit{ nonisom.}\}$$

Proof. We consider the equation obtained in the refined equivariant Lefschetz fixed point theorem [Weber 2005, Theorem 0.2]. We have

$$\lambda_G(f) = \sum_{Gz \in G \backslash \mathrm{Fix}(f)} \Lambda_G(z, f) \circ \mathrm{ind}_{G_z \subseteq G}(\mathrm{Deg}_0^{G_z}((\mathrm{id}_{T_z X} - T_z f)^c)).$$

Here $\mathrm{Deg}_0^{G_z}$ is the equivariant degree [Lück and Rosenberg 2003a], it has values in the Burnside ring $A(G_z)$. On basis elements $[G_z/L] \in A(G_z)$, the map $\Lambda_G(z, f) \circ \mathrm{ind}_{G_z \subseteq G}$ is given by

$$\Lambda_G(z, f) \circ \mathrm{ind}_{G_z \subseteq G}([G_z/L]) = (\mathrm{pr}, [\mathrm{cst}])^* \overline{\alpha_z},$$

where cst denotes the constant map and

$$(\mathrm{pr}, [\mathrm{cst}])^* : \mathbb{Z}\pi_1(X^{G_z}(z), x)_{\phi'} \to \mathbb{Z}\pi_1(X^L(z \circ \mathrm{pr}), x \circ \mathrm{pr})_{\phi'}$$

is the map induced by the projection $\mathrm{pr} : G_z/L \to G_z/G_z$.

We know that $\mathrm{Deg}_0^{G_z}((\mathrm{id}_{T_zX} - T_z f)^c)$ is a unit of the Burnside ring $A(G_z)$ since $\big(\mathrm{Deg}_0^{G_z}((\mathrm{id}_{T_zX} - T_z f)^c)\big)^2 = 1$ [Lück and Rosenberg 2003a, Example 4.7]. In general a unit of the Burnside ring $A(G_z)$ may consist of more than one summand [tom Dieck 1979]. The summand $[G_z/G_z]$ is always included with a coefficient $+1$ or -1, but there might be summands $[G_z/L]$ for $L < G_z$ appearing. So one fixed point might give more than one class with nonzero coefficients.

If we divide out the images of nonisomorphisms, then we divide out the image of $(\mathrm{pr}, [\mathrm{cst}])^*$ for all $L \neq G_z$. We are left with the summand $\pm\overline{\alpha_z}$ coming from $\pm 1[G_z/G_z]$. This cannot lie in the image of any nonisomorphism. So each fixed point leads to exactly one summand. The sign is the sign of the determinant $\det\big(\mathrm{id}_{T_zX} - T_z(f)\big)$, so the claim follows. □

We set

$$\mathbb{Z}\pi_1(X^H(x), x)_{\phi''} := \mathbb{Z}\pi_1(X^H(x), x)_{\phi'} / \{\mathrm{Im}(\sigma, [w])^* \mid \sigma \text{ nonisom.}\}.$$

We use the equation established in Lemma 3.2 to define $\nu_G(f)$ directly for all endomorphisms of finite proper G-CW-complexes.

Definition 3.3. Let X be a finite proper G-CW-complex, and let $f : X \to X$ be an equivariant endomorphism. Then the *equivariant Nielsen class* of f is

$$\nu_G(f) := \overline{\lambda_G(f)} \in \bigoplus_{\substack{\bar{x} \in \mathrm{Is}\,\Pi(G,X), \\ X^H(f(x)) = X^H(x)}} \mathbb{Z}\pi_1(X^H(x), x)_{\phi''}.$$

We define equivariant Nielsen invariants by counting the essential classes $\overline{\alpha}$ of $\nu_G(f)_{\bar{x}}$ in $\mathbb{Z}\pi_1(X^H(x), x)_{\phi''}$ and of $L^{\mathbb{Q}\,\mathrm{Aut}(x)}\big(\widetilde{f^H(x)}\big)$ in $\mathbb{Q}\pi_1(X^H(x), x)_{\phi'}$.

Definition 3.4. Let G be a discrete group, let X be a finite proper G-CW-complex, and let $f : X \to X$ be a G-equivariant map.

Then the *equivariant Nielsen invariants* of f are elements

$$N_G(f), N^G(f) \in \bigoplus_{\bar{x} \in \mathrm{Is}\,\Pi(G,X)} \mathbb{Z}$$

defined for $\bar{x}$ with $X^H(f(x)) = X^H(x)$ by

$$N_G(f)_{\bar{x}} := \#\big\{\text{essential classes of } \nu_G(f)_{\bar{x}}\big\},$$

$$N^G(f)_{\bar{x}} := \min\Big\{\#\mathscr{C} \;\Big|\; \mathscr{C} \subseteq \bigcup_{y \geq x} \pi_1(X^K(y), y)_{\phi'} \text{ such that for all } \bar{z} \geq \bar{x} \text{ and}$$
$$\text{for all essential classes } \overline{\alpha} \text{ of } L^{\mathbb{Q}\,\mathrm{Aut}(z)}\big(\widetilde{f^{G_z}(z)}\big) \text{ there are}$$
$$\overline{\beta} \in \mathscr{C} \text{ and } (\sigma, [t]) \in \mathrm{Mor}(z, y_{\overline{\beta}}) \text{ such that } (\sigma, [t])^*(\overline{\beta}) = \alpha\Big\}.$$

We continue them by 0 to $\bar{x} \in \mathrm{Is}\,\Pi(G, X)$ with $X^H(f(x)) \neq X^H(x)$.

Note that $N_G(f)_{\bar{x}} = N_G(f_H(x))$ and $N^G(f)_{\bar{x}} = NO_G(f^H(x))$ in the notation of [Wong 1993]. Thus the invariants defined here using the algebraic approach are equivalent to the invariants defined using the classical covering space approach of Wong.

An essential class $\overline{\alpha}$ of $L^{\mathbb{Q}\,\mathrm{Aut}(x)}\big(\widetilde{f^H(x)}\big)$ corresponds to an essential fixed point class of $f^H(x)$, a WH_x-orbit of fixed points which one cannot get rid of under any G-homotopy, as can be seen from the refined orbifold Lefschetz fixed point theorem [Weber 2005, Theorem 6.6]. An essential class $\overline{\alpha}$ of $\nu_G(f)_{\bar{x}}$ corresponds to an essential fixed point class of $f_H(x)$, an orbit of fixed points on $X^H(x) \setminus X^{>H}(x)$ that cannot be moved into $X^{>H}(x)$. Counting the essential classes will give us information on the number of fixed points and fixed point orbits.

The equivariant Nielsen invariants are G-homotopy invariant since they are derived from $\lambda_G(f)$, which is itself G-homotopy invariant.

Proposition 3.5. *Given a G-homotopy $f \simeq_G f'$, we have*

$$N_G(f) = N_G(f'), \qquad N^G(f) = N^G(f').$$

Proof. If $f \simeq_G f'$, with a homotopy $H : X \times I \to X$ such that $H_0 = f$ and $H_1 = f'$, then by invariance under homotopy equivalence [Weber 2005, Theorem 5.14] we have an isomorphism $\Lambda_G(i_1)^{-1}\Lambda_G(i_0) : \Lambda_G(X, f) \xrightarrow{\sim} \Lambda_G(X, f')$ which sends $\lambda_G(f)$ to $\lambda_G(f')$. The isomorphisms $\Lambda_G(i_1)$ and $\Lambda_G(i_0)$ are given by composition of maps, so they do not change the number of essential classes. They also do not change the property of a class to lie in the image of a nonisomorphism. So we have $N_G(f) = N_G(f')$.

An isomorphism $i_{0*} : \mathbb{Q}\Pi(G, X)_{\phi,\bar{y}} \to \mathbb{Q}\Pi(G, X \times I)_{\Phi,\overline{i_0(y)}}$ is induced by the inclusion i_0, and analogously i_1 induces an isomorphism. These isomorphisms do not change the number of essential classes. We have $\mathrm{ch}_G(X, f)(\lambda_G(f)) = (i_{0*})^{-1} i_{1*}\, \mathrm{ch}_G(X, f')(\lambda_G(f'))$, so $N^G(f) = N^G(f')$. □

4. Lower bound property

The equivariant Nielsen invariants give a lower bound for the number of fixed point orbits on $X^H(x) \setminus X^{>H}(x)$ and on $X^H(x)$, for maps lying in the G-homotopy class of f. Under mild hypotheses, this is even a sharp lower bound.

Definition 4.1. Let G be a discrete group, let X be a finite proper G-CW-complex, and let $f : X \to X$ be a G-equivariant map. For every $\bar{x} \in \operatorname{Is}\Pi(G, X)$, with $x : G/H \to X$, we set

$$M_G(f)_{\bar{x}} := \min\bigl\{\#\text{ fixed point orbits of } \varphi_H(x) \bigm| \varphi \simeq_G f\bigr\},$$
$$M^G(f)_{\bar{x}} := \min\bigl\{\#\text{ fixed point orbits of } \varphi^H(x) \bigm| \varphi \simeq_G f\bigr\}.$$

When speaking of fixed point orbits of $f^H(x)$, we can either look at the WH_x-orbits $WH_x \cdot z \subseteq X^H(x)$ or at the G-orbits $G \cdot z \subseteq X^{(H)}(x)$, for a fixed point z in $X^H(x)$. These two notions are of course equivalent.

We now proceed to show the first important property of the equivariant Nielsen invariants, the lower bound property.

Proposition 4.2. *For every $\bar{x} \in \operatorname{Is}\Pi(G, X)$ we have*

$$N_G(f)_{\bar{x}} \le M_G(f)_{\bar{x}}, \qquad N^G(f)_{\bar{x}} \le M^G(f)_{\bar{x}}.$$

Proof. (1) If $\overline{\alpha} \in \pi_1(X^H(x), x)_{\phi''}$ is an essential class of $\nu_G(f)_{\bar{x}}$, there has to be at least one fixed point orbit in $X^H(x) \setminus X^{>H}(x)$ that corresponds to $\overline{\alpha}$ and that cannot be moved into $X^{>H}(x)$. So, for any $\varphi \simeq_G f$, the restriction φ_H must have at least $N_G(f)_{\bar{x}}$ fixed point orbits in $X^H(x) \setminus X^{>H}(x)$. We arrive at $N_G(f)_{\bar{x}} \le$ {#fixed point orbits of φ_H} for all $\varphi \simeq_G f$, so $N_G(f)_{\bar{x}} \le M_G(f)_{\bar{x}}$.

(2) Let $\bar{x} \in \operatorname{Is}\Pi(G, X)$. Suppose that $\varphi \simeq_G f$ such that $\varphi^H(x)$ has $M^G(f)_{\bar{x}}$ fixed point orbits in $X^H(x)$. Let $\mathscr{C} \subseteq \bigcup_{\bar{x} \le \bar{y}} \pi_1(X^K(y), y)_{\phi'}$ such that $N^G(\varphi)_{\bar{x}} = N^G(f)_{\bar{x}} = \#\mathscr{C}$. If there were less than $\#\mathscr{C}$ fixed point orbits in $X^H(x)$, there would be less that $\#\mathscr{C}$ essential classes and we could have chosen a smaller $\mathscr{C}$. So there are at least $\#\mathscr{C}$ essential classes, and thus $\varphi^H(x)$ has at least $\#\mathscr{C}$ fixed point orbits. □

To prove the sharpness of this lower bound, we need certain hypotheses, which are usually introduced when dealing with these problems. Such conditions were first used in [Fadell and Wong 1988]. Some authors treat slightly weakened assumptions [Ferrario 1999; Ferrario 2003; Jezierski 1995; Wilczyński 1984]. We do not weaken the standard gap hypotheses in the context of functorial equivariant Lefschetz invariants since the standard gap hypotheses are not homotopy invariant. So an analog of Theorem 6.3 would not hold.

Definition 4.3. Let G be a discrete group and let X be a cocompact smooth G-manifold. We say that X satisfies the *standard gap hypotheses* if for each $\bar{x} \in \operatorname{Is}\Pi(G, X)$, with $x : G/H \to X$, the inequalities $\dim X^H(x) \ge 3$ and $\dim X^H(x) - \dim X^{>H}(x) \ge 2$ hold.

Under these hypotheses, we can use an equivariant analog of Wecken's classical method [1941] to coalesce fixed points.

Lemma 4.4. *Let G be a discrete group and let X be a cocompact proper smooth G-manifold satisfying the standard gap hypotheses. Let $f : X \to X$ be a G-equivariant map. Let $\mathbb{O}_1 = Gx_1$ and $\mathbb{O}_2 = Gx_2$ be two distinct isolated G-fixed point orbits, where $x_1 : G/H \to X$ and $x_2 : G/K \to X$ with $x_1 \le x_2$. Suppose that there are paths $(\sigma_1, [t_1]) \in \mathrm{Mor}(x, x_1)$ and $(\sigma_2, [t_2]) \in \mathrm{Mor}(x, x_2)$ for an $\bar{x} \in \Pi(G, X)$, with $x : G/H \to X$, such that $(\sigma_1, [t_1])^* \overline{1_{x_1}} = \overline{\alpha} = (\sigma_2, [t_2])^* \overline{1_{x_2}}$, i.e., that the fixed point orbits induce the same $\overline{\alpha} \in \pi_1(X^H(x), x)_{\phi'}$. Then there exists a G-homotopy $\{f_t\}$ relative to $X^{>(H)}$ such that $f_0 = f$ and $\mathrm{Fix}\, f_1 = \mathrm{Fix}\, f_0 - G\mathbb{O}_1$.*

Proof. Suppose first that $\overline{x_1} < \overline{x_2}$. Then $\mathrm{Mor}(x_1, x_2) \neq \varnothing$. By replacing x_1 and x_2 with other points in the orbit if necessary, we can suppose that there exists a morphism $(\tau, [v]) \in \mathrm{Mor}(x_1, x_2)$, where v is a path in $X^H(x)$ with $v_1 = x_1$ and $v_0 = x_2 \circ \tau$ and $\tau : G/H \to G/K$ is a projection. We know that $v \simeq f^H \circ v$ (relative endpoints). (This is an equivalent characterization of x_1 and x_2 belonging to the same fixed point class [Jiang 1983, I.1.10].) Since $x_1 \in X^H(x) \setminus X^{>H}(x)$ and $x_2 \in X^{>H}(x)$ and $\dim X^H(x) - \dim X^{>H}(x) \ge 2$, we may assume that v can be chosen such that $v((0, 1]) \subseteq X^H(x) \setminus X^{>H}(x)$. We coalesce x_1 and x_2 along v as in [Wong 1991b, 1.1] and [Schirmer 1986, 6.1]. We can do this by only changing f in a (cone-shaped) neighborhood $U(v)$ of v. Because of the proper action of G on X and the free action of WH on $X^H \setminus X^{>H}$, this neighborhood $U(v)$ can be chosen such that in $X^H \setminus X^{>H}$ it does not intersect its g-translates for $g \notin H \le G$. Taking the G-translates of $U(v)$, we move $\mathbb{O}_1$ to $\mathbb{O}_2$ along the paths Gv in $GU(v)$, not changing the map f outside $GU(v)$.

Now suppose $\overline{x_1} = \overline{x_2}$. In this case, the result follows from [Wong 1991a, 5.4], since $X^H(x) \setminus X^{>H}(x)$ is a free and proper WH_x-space, where again the proper action of G on X ensures that we can find a neighborhood of a path from x_1 to x_2 such that the G/H-translates do not intersect. □

From Lemma 4.4, we can conclude the sharpness of the lower bound given by the equivariant Nielsen invariants.

Theorem 4.5. *Let G be a discrete group. Let X be a cocompact proper smooth G-manifold satisfying the standard gap hypotheses. Let $f : X \to X$ be a G-equivariant endomorphism. Then*

$$M_G(f)_{\bar{x}} = N_G(f)_{\bar{x}}, \qquad M^G(f)_{\bar{x}} = N^G(f)_{\bar{x}}$$

for all $\bar{x} \in \mathrm{Is}\,\Pi(G, X)$.

Proof. (1) Since X is a cocompact smooth G-manifold, there is a G-map f' which is G-homotopic to f and which has only finitely many fixed point orbits. We apply Lemma 4.4 to f' to coalesce fixed point orbits in $X^H(x) \setminus X^{>H}(x)$ with others of the same class $\overline{\alpha} \in \mathbb{Z}\pi_1(X^H(x), x)_{\phi'}$. We move them into $X^{>H}(x)$ whenever possible. (We might need to create a fixed point orbit in the inessential fixed point class beforehand; see [Wong 1991b, 1.1].) We remove the inessential fixed point orbits. We arrive at a map $h \simeq_G f$ such that $N_G(f)_{\bar{x}} = \#\{\text{fixed point orbits of } h_H(x)\} \geq M_G(f)_{\bar{x}}$. Using Proposition 4.2, we obtain equality.

(2) Since X is a cocompact smooth G-manifold, there is a map f' which is G-homotopic to f and which only has finitely many fixed point orbits. We have a partial ordering on the $\bar{y} \geq \bar{x}$ given by $\bar{y} \geq \bar{z} \Leftrightarrow \mathrm{Mor}(z, y) \neq \varnothing$. We apply Lemma 4.4 to f' to coalesce fixed point orbits of the same class, starting from the top. Note that when we remove fixed point orbits, we can only move them up in this partial ordering. That is why the definition has to be so complicated. We remove the inessential fixed point orbits. We are left with one fixed point orbit for every essential class.

We now look at a class $\mathscr{C}$ such that $N^G(f)_{\bar{x}} = \#\mathscr{C}$, and we coalesce the essential fixed point orbits with the corresponding classes appearing in $\mathscr{C}$. (If the corresponding class in $\mathscr{C}$ is inessential, we might need to create a fixed point orbit in this inessential fixed point class beforehand.) We obtain a map $h \simeq_G f$ which has exactly $\#\mathscr{C}$ fixed point orbits. Hence

$$N^G(f)_{\bar{x}} = \#\{\text{fixed point orbits of } h^H(x)\} \geq M^G(f)_{\bar{x}}.$$

Using Proposition 4.2, we obtain equality. □

In general, it is not possible to find a map $h \simeq_G f$ realizing all minima simultaneously. As an example, one can take $G = \mathbb{Z}/2$ acting on $X = S^4$ as an involution so that $X^{\mathbb{Z}/2} = S^3$. One obtains $M_G(\mathrm{id}_{S^4})_{\bar{x}} = 0$ for all $\bar{x} \in \mathrm{Is}\,\Pi(\mathbb{Z}/2, S^4)$, but the minimal number of fixed points in the G-homotopy class of the identity id_{S^4} is equal to 1 [Wong 1993, Remark 3.4]. In this example, the standard gap hypotheses are not satisfied. Other examples where the standard gap hypotheses do not hold and where the converse of the equivariant Lefschetz theorem is false are given in [Ferrario 1999, Section 5].

5. The *G*-Jiang condition

In the nonequivariant case, we know that the generalized Lefschetz invariant is the right element when looking for a precise count of fixed points. We read off the Nielsen numbers from this invariant. In general, the Lefschetz number contains too little information. But under certain conditions, we can conclude facts about the Nielsen numbers from the Lefschetz numbers directly, and thus obtain a converse of the Lefschetz fixed point theorem.

These conditions are called Jiang conditions. See [Jiang 1983, Definition II.4.1], where one can find a thorough treatment, and [Brown 1971, Chapter VII]. The Jiang group is a subgroup of $\pi_1(X, f(x))$ [Jiang 1983, Definition II.3.5]. We generalize its definition to the equivariant case.

Definition 5.1. Let G be a discrete group, let X be a finite proper G-CW-complex, and let $f : X \to X$ be a G-equivariant endomorphism. Then a G-equivariant self-homotopy $h : f \simeq_G f$ of f determines a path $h(x, -) \in \pi_1(X^H(x), f(x))$ for every $\bar{x} \in \operatorname{Is}\Pi(G, X)$, with $x : G/H \to X$. Define the *G-Jiang group* of (X, f) to be

$$J_G(X, f) := \left\{ \sum_{\bar{x}\in\Pi(G,X)} \big[h(x, -)\big] \,\middle|\, h : f \simeq_G f \ G\text{-equivariant self-homotopy} \right\}$$
$$\leq \bigoplus_{\bar{x}\in\Pi(G,X)} \pi_1\big(X^H(x), f(x)\big),$$

and define the *G-Jiang group* of X to be

$$J_G(X) := \left\{ \sum_{\bar{x}\in\Pi(G,X)} \big[h(x, -)\big] \,\middle|\, h : \mathrm{id} \simeq_G \mathrm{id} \ \ G\text{-equivariant self-homotopy} \right\}$$
$$\leq \bigoplus_{\bar{x}\in\Pi(G,X)} \pi_1\big(X^H(x), x\big).$$

In the nonequivariant case, we know that the Jiang group $J(X, f, x)$ is a subgroup of the centralizer of $\pi_1(f, x)\big(\pi_1(X, x)\big)$ in $\pi_1\big(X, f(x)\big)$. In particular,

$$J(X) \leq Z\big(\pi_1(X, x)\big),$$

where $Z\big(\pi_1(X, x)\big)$ denotes the center of $\pi_1(X, x)$ [Jiang 1983, Lemma II.3.7]. Furthermore, the isomorphism $(f \circ w)_* : \pi_1\big(X, f(x_1)\big) \to \pi_1\big(X, f(x_0)\big)$ induced by a path w from x_0 to x_1 induces an isomorphism $(f \circ w)_* : J(X, f, x_1) \to J(X, f, x_0)$ which does not depend on the choice of w. So the definition does not depend on the choice of the basepoint [Jiang 1983, Lemma II.3.9]. It is also known that $J(X) \leq J(X, f) \leq \pi_1(X)$ for all f [Jiang 1983, Lemma II.3.8]. This leads to the consideration of spaces with $J(X) = \pi_1(X)$ in the definition of a Jiang space. All these lemmata also make sense in the equivariant case. Thus we make the following definition.

Definition 5.2. Let G be a discrete group and let X be a cocompact G-CW-complex. Then X is called a *G-Jiang space* if for all $\bar{x} \in \operatorname{Is}\Pi(G, X)$ we have

$$J_G(X)_{\bar{x}} = \pi_1\big(X^H(x), x\big).$$

The group $J_G(X, f)$ acts on $\Lambda_G(X, f)$ as follows: If $X^H(f(x)) = X^H(x)$ and $X^H(f(x)) = X^H(x)$, then $J_G(X, f)_{\bar{x}}$ acts on $\mathbb{Z}\pi_1(X^H(x), x)$. The element $u = [h(x, -)] \in J_G(X, f)_{\bar{x}}$ acts as composition with $[wuw^{-1}]$, where $v = (\mathrm{id}, [w]) \in \operatorname{Mor}(f(x), x)$.

Since $J_G(X, f)_{\bar{x}}$ is contained in the centralizer of $\pi_1(f^H(x), x)\big(\pi_1(X^H(x), x)\big)$ in $\pi_1\big(X^H(x), f(x)\big)$, this action induces an action on $\mathbb{Z}\pi_1(X^H(x), x)_{\phi'}$ by composition, whence on $\Lambda_G(X, f)_{\bar{x}}$. Thus $J_G(X, f)$ acts on $\Lambda_G(X, f)$, and by invariance of $\lambda_G(f)$ under homotopy equivalence, we see that

$$\lambda_G(f) \in \big(\Lambda_G(X, f)\big)^{J_G(X,f)}.$$

Examples of G-Jiang spaces can be obtained from Jiang spaces. It is known [Jiang 1983, Theorem II.3.11] that the class of Jiang spaces is closed under homotopy equivalence and the topological product operation and contains

- simply connected spaces,
- generalized lens spaces,
- H-spaces,
- homogeneous spaces of the form A/A_0 where A is a topological group and A_0 is a subgroup which is a connected compact Lie group.

Hence we obtain many examples of G-Jiang spaces using the following proposition, analogous to [Wong 1993, Proposition 4.9].

Proposition 5.3. *Let G be a discrete group, and let X be a free cocompact connected proper G-space. If X/G is a Jiang space, then X is a G-Jiang space.*

Proof. Since X is connected and free, the set $\operatorname{Is}\Pi(G, X)$ consists of one element. Let x be a basepoint of X. We need to check that $J_G(X)_{\bar{x}} = \pi_1(X, x)$. Let $X \xrightarrow{p} X/G$ be the projection. The Jiang subgroup of X/G is given by

$$\begin{aligned} J(X/G) &:= \Big\{\big[h(p(x), -)\big] \,\Big|\, h : \mathrm{id}_{X/G} \simeq \mathrm{id}_{X/G} \text{ self-homotopy}\Big\} \\ &\leq \pi_1\big(X/G, p(x)\big). \end{aligned}$$

Let $\alpha \in \pi_1(X, x)$. Since $X \xrightarrow{p} X/G$ is a discrete cover, $\widetilde{X} = \widetilde{X/G}$. There is a map $p_\# : \pi_1(X, x) \to \pi_1(X/G, p(x))$ induced by the projection. Since X/G is a Jiang space, $J(X/G) = \pi_1(X/G, p(x))$, so there is a homotopy $h : \mathrm{id}_{X/G} \simeq \mathrm{id}_{X/G}$ such that $p_\#(\alpha) = [h(p(x), -)]$. Because of the free and proper action of G on X, this homotopy h can be lifted to a G-equivariant homotopy $h' : \mathrm{id}_X \simeq_G \mathrm{id}_X$ such that $\alpha = [h'(x, -)]$. Thereby $\alpha \in J_G(X)$. $\square$

6. The converse of the equivariant Lefschetz Fixed Point Theorem

One can derive equivariant analogs of statements about Nielsen numbers found in [Jiang 1983], generalizing results from [Wong 1993] to infinite discrete groups. In particular, if X is a G-Jiang space, the converse of the equivariant Lefschetz fixed point theorem holds. The next theorem can be compared with [Jiang 1983, Theorem II.4.1].

Theorem 6.1. *Let G be a discrete group, and let X be a finite proper G-CW-complex which is a G-Jiang space. Then for any G-map $f : X \to X$ and $\bar{x} \in \operatorname{Is}\Pi(G, X)$ with $x : G/H \to X$ we have*:

$$\begin{aligned} L_G(f)_{\bar{x}} = 0 &\implies \lambda_G(f)_{\bar{x}} = 0 \text{ and } N_G(f)_{\bar{x}} = 0, \\ L_G(f)_{\bar{x}} \neq 0 &\implies \lambda_G(f)_{\bar{x}} \neq 0 \text{ and } N_G(f)_{\bar{x}} = \#\{\pi_1(X^H(x), x)_{\phi''}\}. \end{aligned}$$

Here $L_G(f)$ is the equivariant Lefschetz class [Lück and Rosenberg 2003a, Definition 3.6], the equivariant analog of the Lefschetz number.

Proof. Since X is a G-Jiang space, the G-Jiang group $J_G(X)$ acts transitively on $\pi_1(X^H(x), x)$ for all $\bar{x} \in \operatorname{Is}\Pi(G, X)$. This implies that

$$\lambda_G(f)_{\bar{x}} = \sum_{\overline{\alpha}} n_{\overline{\alpha}} \cdot \overline{\alpha} = n \cdot \sum_{\overline{\alpha}} \overline{\alpha}$$

for some $n \in \mathbb{Z}$. This leads to $L_G(f)_{\bar{x}} = n \cdot \#\{\pi_1(X^H(x), x)_{\phi'}\}$ by the augmentation map. We see that

$$\begin{aligned} L_G(f)_{\bar{x}} = 0 &\implies n = 0 \\ &\implies \lambda_G(f)_{\bar{x}} = 0 \\ &\implies \nu_G(f)_{\bar{x}} = 0 \\ &\implies N_G(f)_{\bar{x}} = 0, \end{aligned}$$

$$\begin{aligned} L_G(f)_{\bar{x}} \neq 0 &\implies n \neq 0 \\ &\implies \lambda_G(f)_{\bar{x},\overline{\alpha}} \neq 0 \text{ for all } \overline{\alpha} \in \mathbb{Z}(\pi_1(X^H(x), x))_{\phi'} \\ &\implies \nu_G(f)_{\bar{x},\overline{\alpha}} \neq 0 \text{ for all } \overline{\alpha} \in \mathbb{Z}(\pi_1(X^H(x), x))_{\phi''} \\ &\implies N_G(f)_{\bar{x}} = \#\{\pi_1(X^H(x), x)_{\phi''}\}. \end{aligned}$$

□

The proof of Theorem 6.1 already works if $J_G(X, f)$ acts transitively on every summand of $\Lambda_G(X, f)$. We could have called X a G-Jiang space if the condition that $J_G(X, f)$ acts transitively on every summand of $\Lambda_G(X, f)$ is satisfied. But this condition is less tractable. It is implied by $J_G(X, f)_{\bar{x}} = \pi_1(X^H(x), f(x))$ for all $\bar{x}$, which is implied by $J_G(X)_{\bar{x}} = \pi_1(X^H(x), x)$ for all $\bar{x}$.

We now show that f is G-homotopic to a fixed point free G-map if the generalized equivariant Lefschetz invariant $\lambda_G(f)$ is zero.

Theorem 6.2. *Let G be a discrete group. Let X be a cocompact proper smooth G-manifold satisfying the standard gap hypotheses. Let $f : X \to X$ be a G-equivariant endomorphism. If $\lambda_G(f) = 0$, then f is G-homotopic to a fixed point free G-map.*

Proof. If $\lambda_G(f)=0$, then $\mathrm{ch}_G(X, f)(\lambda_G(f))=0$, and therefore we have $N^G(f)_{\bar{x}} = 0$ for all $\bar{x} \in \mathrm{Is}\,\Pi(G, X)$. We know from Theorem 4.5 that $N^G(f)_{\bar{x}} = M^G(f)_{\bar{x}} = \min\{\#$ fixed point orbits of $\varphi^H(x) \mid \varphi \simeq_G f\}$. In particular, for $x : G/\{1\} \to X$ we obtain a map φ such that $\varphi^{\{1\}}(x)$ is fixed point free and $\varphi \simeq_G f$. Thus we obtain our result on every connected component of X, and combining these we arrive at a map $h \simeq_G f$ which is fixed point free. □

These two theorems, Theorem 6.1 and Theorem 6.2, combine to give the main theorem of this paper, the converse of the equivariant Lefschetz fixed point theorem.

Theorem 6.3. *Let G be a discrete group. Let X be a cocompact proper smooth G-manifold satisfying the standard gap hypotheses which is a G-Jiang space. Let $f : X \to X$ be a G-equivariant endomorphism. Then the following holds*:

If $L_G(f) = 0$, then f is G-homotopic to a fixed point free G-map.

Proof. We know that $L_G(f) = 0$ means that $L_G(f)_{\bar{x}} = 0$ for all $\bar{x} \in \mathrm{Is}\,\Pi(G, X)$. Since X is a G-Jiang space, by Theorem 6.1 this implies that $\lambda_G(f)_{\bar{x}} = 0$ for all $\bar{x} \in \mathrm{Is}\,\Pi(G, X)$, so we have $\lambda_G(f) = 0$. We apply Theorem 6.2 to arrive at the desired result. □

Remark 6.4. As another corollary of Theorem 6.2, we obtain: If G is a discrete group and X is a cocompact proper smooth G-manifold satisfying the standard gap hypotheses, then $\chi^G(X) = 0$ implies that the identity id_X is G-homotopic to a fixed point free G-map. This was already stated in [Lück and Rosenberg 2003a, Remark 6.8]. Here $\chi^G(X)$ is the universal equivariant Euler characteristic of X [Lück and Rosenberg 2003a, Definition 6.1] defined by $\chi^G(X)_{\bar{x}} = \chi\big(WH_x \setminus X^H(x), WH_x \setminus X^{>H}(x)\big) \in \mathbb{Z}$, we have $\chi^G(X) = L_G(\mathrm{id}_X)$. We calculate that

$$\begin{aligned}\lambda_G(\mathrm{id}_X)_{\bar{x}} &= \sum_{p\geq 0}(-1)^p \sum_{\substack{\mathrm{Aut}(x)\cdot e\in \\ \mathrm{Aut}(x)\setminus I_p(\widetilde{X^H(x)},\widetilde{X^{>H}(x)})}} \mathrm{inc}_\phi\big(\mathrm{id}_{\widetilde{X^H(x)}}, e\big) \\ &= \chi\big(WH_x \setminus X^H(x), WH_x \setminus X^{>H}(x)\big)\cdot \bar{1} \in \mathbb{Z}\pi_1\big(X^H(x), x\big)_{\phi'}.\end{aligned}$$

So we have $\chi^G(X) = 0$ if and only if $\lambda_G(\mathrm{id}_X) = 0$, and with Theorem 6.2 we conclude that there is an endomorphism G-homotopic to the identity which is fixed point free.

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JULIA WEBER
MAX-PLANCK-INSTITUT FÜR MATHEMATIK
VIVATSGASSE 7
D-53111 BONN
GERMANY
jweber@mpim-bonn.mpg.de