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# A VOLUMISH THEOREM FOR THE JONES POLYNOMIAL OF ALTERNATING KNOTS

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**The Volume Conjecture claims that the hyperbolic volume of a knot is determined by the colored Jones polynomial.**

**Here we prove a "Volumish Theorem" for alternating knots in terms of the Jones polynomial, rather than the colored Jones polynomial: The ratio of the volume and certain sums of coefficients of the Jones polynomial is bounded from above and from below by constants.**

**Furthermore, we give experimental data on the relation of the growths of the hyperbolic volume and the coefficients of the Jones polynomial, both for alternating and nonalternating knots.**

## 1. Introduction

Since the introduction of the Jones polynomial, there has been a strong desire to have a geometrical or topological interpretation for it rather than a combinatorial definition.

The first major success in this direction was arguably the proof of the Melvin–Morton Conjecture by Bar-Natan and Garoufalidis [1996] (see [Vaintrob 1997; Chmutov 1998; Lin and Wang 2001; Rozansky 1997] for different proofs): The Alexander polynomial is determined by the so-called colored Jones polynomial. For a knot $K$ the colored Jones polynomial is given by the Jones polynomial and the Jones polynomials of cablings of $K$.

The next major conjecture that relates the Jones polynomial and its offsprings to classical topology and geometry was the Volume Conjecture of Kashaev, Murakami and Murakami (see [Murakami and Murakami 2001], for instance). This conjecture states that the colored Jones polynomial determines the Gromov norm of the knot complement. For hyperbolic knots the Gromov norm is proportional to the hyperbolic volume.

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*Keywords:* Jones polynomial, hyperbolic volume, alternating knots, volume conjecture.

A proof of the Volume Conjecture for all knots would also imply that the colored Jones polynomial detects the unknot [Murakami and Murakami 2001]. This problem is still wide open; even for the Jones polynomial there is no counterexample known (see [Dasbach and Hougardy 1997], for example).

The purpose of this paper is to show a relation of the coefficients of the Jones polynomial and the hyperbolic volume of alternating knot complements. More specifically we prove:

**Volumish Theorem.** *For an alternating, prime, nontorus knot $K$ let*

$$V_K(t) = a_n t^n + \cdots + a_m t^m$$

*be the Jones polynomial of $K$. Then*

$$v_8(\max(|a_{m-1}|, |a_{n+1}|) - 1) \leq \mathrm{Vol}(S^3 - K) \leq 10 v_3(|a_{n+1}| + |a_{m-1}| - 1).$$

*Here, $v_3 \approx 1.01494$ is the volume of an ideal regular hyperbolic tetrahedron and $v_8 \approx 3.66386$ is the volume of an ideal regular hyperbolic octahedron.*

For the proof of this theorem we make use of a result from [Lackenby 2004] (with its proof), stating that the hyperbolic volume is linearly bounded from above and below by the twist number. Lackenby's upper bound was improved by Ian Agol and Dylan Thurston and his lower bound by Ian Agol, Peter Storm and Bill Thurston [Agol et al. 2005].

In an appendix we give some numerical data on the relation between other coefficients and the hyperbolic volume, for both alternating and nonalternating knots. These data gives some hope for a Volumish Theorem for nonalternating knots as well.

## 2. The Jones polynomial evaluation of the Tutte polynomial

Our goal is to relate the hyperbolic volume of alternating knot complements to the coefficients of the Jones polynomial. We make use of the computation of the Jones polynomial of alternating links via the Tutte polynomial.

***Notation.*** Our objects are multigraphs, that is, graphs where parallel edges are allowed. Two edges are called parallel if they connect the same two vertices.

(a) A multigraph $G = (V, E)$ has a set $V$ of vertices and a set $E$ of edges.

(b) We denote by $\tilde{G} = (V, \tilde{E})$ a spanning subgraph of $G$ where parallel edges are deleted. See Figure 1. The set of vertices $V$ is the same.

(c) Each edge $e \in \tilde{E}$ in $\tilde{G}$ can be assigned a multiplicity $\mu(e)$, namely, the number of edges in $G$ that are parallel to $e$. For example, the graph in Figure 1 has one edge with multiplicity 2, one with multiplicity 3, and one with multiplicity 4. All other edges have multiplicity 1.

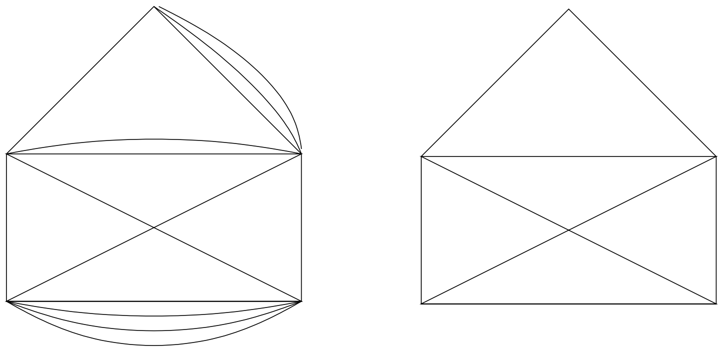

**Figure 1.** A multigraph $G$ and its spanning subgraph $\tilde{G}$.

(d) We define $n(j)$ to be the number of edges $e \in \tilde{E}$ with $\mu(j) \geq j$. In particular $n(1) = |\tilde{E}|$. Thus the graph in Figure 1 has $n(2) = 3$, $n(3) = 2$, $n(4) = 1$.

(e) The number of components of a graph $G$ is $k(G)$. If $V$ is apparent from the context and $G = (V, E)$, we set $k(E) := k(G)$.

(f) The Tutte polynomial of a multigraph $G$ (see [Bollobás 1998], for example) is

$$T_G(x, y) := \sum_{F \subseteq E} (x-1)^{k(F)-k(E)} (y-1)^{|F|-|V|+k(F)}.$$

***The Tutte polynomial and the Jones polynomial for alternating links.*** Let $K$ be an alternating link with an alternating plane projection $P(K)$. The region of the projection can be colored with two colors, say, purple and gold, such that two adjacent faces have different colors.

Two graphs are assigned to the projection, one corresponding to the purple regions and one to the golden regions. Every region gives rise to a vertex in the graph and two vertices are connected by an edge if the corresponding regions are adjacent to a common crossing. Such graphs are called checkerboard graphs.

Each edge comes with a sign as in Figure 2.

For an alternating link $K$ all edges are either positive or negative. Thus we have a positive checkerboard graph and a negative checkerboard graph. These two graphs

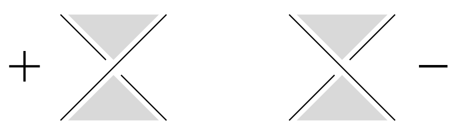


**Figure 2.** A positive and a negative sign for the shaded region in the checkerboard graph.

are dual to each other. Let $G$ be the positive checkerboard graph, $a$ be the number of vertices in $G$ and $b$ be the number of vertices in the negative checkerboard graph.

The Jones Polynomial of an alternating link $K$ with positive checkerboard graph $G$ satisfies

$$V_K(t) = (-1)^w t^{(b-a+3w)/4} T_G(-t, -1/t);$$

see [Bollobás 1998], for instance. Here $w$ is the writhe number, that is, the algebraic crossing number of the link projection.

Since we are interested in the absolute values of the Jones coefficients, all information relevant to us is contained in the evaluation $T_G(-t, -1/t)$ of the Tutte polynomial.

***Reduction of multiple edges to simple edges.*** Our first step is to reduce the computation of the Tutte polynomial of a multigraph to the computation of a weighted Tutte polynomial of a spanning simple graph.

If $G = (V, E)$ is a connected graph without vertices of valence 1 (that is, without loops) and $\tilde{G} = (V, \tilde{E})$ is a spanning simple graph for it, we have

$$\begin{aligned}
T_G\Big(-t, -\frac{1}{t}\Big) &= \sum_{F \subseteq E} (-t-1)^{k(F)-1} \Big(-\frac{1}{t} - 1\Big)^{|F|-|V|+k(F)} \\
&= \sum_{\tilde{F} \subseteq \tilde{E}} (-t-1)^{k(\tilde{F})-1} \Big(-\frac{1}{t} - 1\Big)^{-|V|+k(\tilde{F})} \\
&\quad \times \Bigg( \sum_{\substack{r(e_1)=1,\dots,r(e_j)=1 \\ e_1,\dots,e_j \in \tilde{F}}}^{\mu(e_1),\dots,\mu(e_j)} \binom{\mu(e_1)}{r(e_1)} \cdots \binom{\mu(e_j)}{r(e_j)} \Big(-\frac{1}{t} - 1\Big)^{r(e_1)+\cdots+r(e_j)} \Bigg) \\
&= \sum_{\tilde{F} \subseteq \tilde{E}} \Bigg( (-t-1)^{k(\tilde{F})-1} \Big(-\frac{1}{t} - 1\Big)^{-|V|+k(\tilde{F})} \prod_{e \in \tilde{F}} \Big( \Big(-\frac{1}{t}\Big)^{\mu(e)} - 1 \Big) \Bigg).
\end{aligned}$$

Setting $P(m) := \dfrac{(-1/t)^m - 1}{-1/t - 1} = 1 - t^{-1} + t^{-2} - \cdots \pm t^{-m+1}$, we have

$$T_G\Big(-t, -\frac{1}{t}\Big) = \sum_{\tilde{F} \subseteq \tilde{E}} \Bigg( (-t-1)^{k(\tilde{F})-1} \Big(-\frac{1}{t} - 1\Big)^{|\tilde{F}|-|V|+k(\tilde{F})} \prod_{e \in \tilde{F}} P(\mu(e)) \Bigg). \tag{1}$$

**Proposition 2.1** (Highest Tutte coefficients). *Let $G = (V, E)$ be a planar multigraph with spanning simple graph $\tilde{G} = (V, \tilde{E})$. Let the Tutte polynomial evaluate to*

$$T_G(-t, -1/t) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_{m-1} t^{m-1} + a_m t^m,$$

*for suitable $n$ and $m$.*

*Then the coefficients of the highest degree terms of $T_G(-t, -1/t)$ are*:

(a) *The highest degree term* $t^m$ *of* $T_G(-t,-1/t)$ *in* $t$ *is* $t^{|V|-1}$ *with coefficient*

$$a_m = (-1)^{|V|-1}.$$

(b) *The second highest degree term is* $t^{|V|-2}$, *with coefficient*

$$a_{m-1} = (-1)^{|V|-1}\big(|V|-1-|\tilde{E}|\big).$$

*Note that* $|a_{m-1}| = |\tilde{E}|+1-|V|$.

(c) *The third highest degree term is* $t^{|V|-3}$, *with coefficient*

$$(-1)^{|V|}\left(-\binom{|V|-1}{2}+(|V|-2)|\tilde{E}|-n(2)-\binom{|\tilde{E}|}{2}+\mathrm{tri}\right),$$

*where* tri *is the number of triangles in* $\tilde{E}$. *This term equals*

$$a_{m-2} = (-1)^{|V|}\left(-\binom{|a_{m-1}|+1}{2}-n(2)+\mathrm{tri}\right).$$

*Proof.* It is easy to see that $|\tilde{F}|-|V|+k(F)\geq 0$ for all $F$. Therefore,

$$\left(-\frac{1}{t}-1\right)^{|\tilde{F}|-|V|+k(F)}\prod_{e\in\tilde{F}}P(\mu(e)) = \pm 1 + \text{ higher terms in } t^{-1}$$

This means that to determine the highest terms of $T_G(-t,-1/t)$ we have to analyze terms where $k(\tilde{F})$ is large.

*Case* $k(\tilde{F})=|V|$: this means that $|\tilde{F}|=0$. Thus the contribution in the sum in (1) is

$$(-t-1)^{|V|-1} = (-1)^{|V|-1}\left(t^{|V|-1}+(|V|-1)\,t^{|V|-2}+\binom{|V|-1}{2}t^{|V|-3}+\cdots+1\right).$$

*Case* $k(\tilde{F})=|V|-1$: this means that $|\tilde{F}|=1$. Thus the contribution is

$$(-t-1)^{|V|-2}\sum_{e\in\tilde{E}}P(\mu(e)).$$

Recalling that $n(j)$ is the number of edges in $\tilde{E}$ of multiplicity $\geq j$, we have

$$\sum_{e\in\tilde{E}}P(\mu(e)) = |\tilde{E}|-n(2)t^{-1}+n(3)t^{-2}-n(4)t^{-3}+\cdots.$$

*Case* $k(\tilde{F}) = |V| - 2$: this means that $|\tilde{F}|$ equals 2 or that $\tilde{F}$ is a triangle and $|\tilde{F}|$ equals 3. Thus the contribution is

$$\sum_{e,f\in\tilde{E}} (-t-1)^{|V|-3}\, P(\mu(e))\, P(\mu(f)) \\ + \sum_{\substack{e,f,g\in\tilde{E}\\ (e,f,g)\text{ triangle}}} (-t-1)^{|V|-3}\Big(-\frac{1}{t}-1\Big) P(\mu(e))P(\mu(f))P(\mu(g)).$$

By combining these computations we get the result. □

## 3. An algebraic point of view

It is interesting to formulate the results of Proposition 2.1 in a purely algebraic way, as follows.

Let $G$ be a multigraph and $A$ its $N \times N$ adjacency matrix, so in particular $n(2)$ equals half the number of entries in $A$ that exceed 1. Let $\tilde{A}$ be the matrix obtianed from $A$ by replacing every nonzero entry $A$ by 1. Thus, $\tilde{A}$ has only 1 and 0 as entries; further, the trace of $\tilde{A}^2$ is twice the number of edges of $\tilde{G}$ and the trace of $\tilde{A}^3$ is six times the number edges in $\tilde{G}$ (see [Biggs 1993], for example). Combining this with Proposition 2.1 immediately yields:

**Corollary 3.1.** *Let*

$$T_G(-t,-1/t) = a_n t^n + a_{n+1}t^{n+1} + \cdots + a_{m-1}t^{m-1} + a_m t^m$$

*be the Jones evaluation of the Tutte Polynomial of a planar graph $G$.*

$$\begin{aligned} |a_m| &= 1, \\ |a_{m-1}| &= \tfrac{1}{2}\text{trace } \tilde{A}^2 - 1 - N, \\ |a_{m-2}| &= \binom{|a_{m-1}|+1}{2} + n(2) - \tfrac{1}{6}\text{trace } \tilde{A}^3. \end{aligned}$$

## 4. The twist number and the volume of an hyperbolic alternating knot

(For information on hyperbolic structures on knot complements see [Callahan and Reid 1998], for instance.)

The figure-eight knot has minimal volume among all hyperbolic knot complements [Cao and Meyerhoff 2001]. For a hyperbolic knot $K$ with crossing number $c > 4$, by a result of Colin Adams quoted in [Callahan and Reid 1998], the hyperbolic volume of the complement satisfies

$$\text{Vol}(S^3 - K) \le (4c - 16)v_3,$$

where $v_3$ is the volume of a regular ideal hyperbolic tetrahedron.

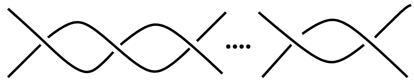

**Figure 3.** A twist in a diagram of a knot .

For alternating knot complements a better general upper bound is known in terms of the twist number. As shown by Bill Menasco [1984], a nontorus alternating knot is hyperbolic.

The twist number of a diagram of an alternating knot is the minimal number of twists (see Figure 3) in it. Here, a twist can consist of a single crossing. The knot diagram shown on page 287 has twist number 8.

A twist corresponds to parallel edges in one of the checkerboard graphs. Let $D$ be a diagram for an alternating knot $K$, and let $G = (V, E)$ and $G^* = (V^*, E^*)$ be the two checkerboard graphs, which are dual to one another, so $|E| = |E^*|$. We can now define the twist number by

$$\begin{aligned} T(K) &:= |E| - (|E| - |\tilde{E}|) - (|E^*| - |\tilde{E}^*|) \\ &= |E| - (|E| - |\tilde{E}|) - (|E| - |\tilde{E}^*|) = |\tilde{E}| + |\tilde{E}^*| - |E|. \end{aligned} \tag{2}$$

It is an easy exercise to see that

(a) $T(K)$ is indeed realized as the twist number of a diagram of $K$, and

(b) $T(K)$ is an invariant of all alternating projections of $K$. This follows from the Tait–Menasco–Thistlethwaite flyping theorem [Menasco and Thistlethwaite 1993]. Below we will give a different argument for it.

**Theorem 4.1** (Lackenby [2004], Agol, D. Thurston).

$$v_3(T(K) - 2) \leq \mathrm{Vol}(S^3 - K) < 10 v_3(T(K) - 1),$$

*where* $\mathrm{Vol}(S^3 - K)$ *is the hyperbolic volume and* $v_3$ *is the volume of an ideal regular hyperbolic tetrahedron.*

Using work of Perelman the lower bound was improved by Agol, Storm and W. Thurston [Agol et al. 2005] to

$$\tfrac{1}{2} v_8(T(K) - 2) \leq \mathrm{Vol}(S^3 - K),$$

where $v_8 \approx 3.66386$ is the volume of an ideal regular hyperbolic octahedron.

## 5. Coefficients of the Jones polynomial

Let $K$ be an alternating knot with reduced alternating diagram $D$ having $c$ crossings. From [Thistlethwaite 1987; Kauffman 1987; Murasugi 1987] we know that:

(a) the span of the Jones polynomial is $c$;

(b) the signs of the coefficients are alternating;

(c) the absolute values of the highest and lowest coefficients are 1.

Proposition 2.1 immediately leads to:

**Theorem 5.1.** *Let $V_K(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_m t^m$ be the Jones polynomial of an alternating knot $K$ and let $G = (V, E)$ be a checkerboard graph of a reduced alternating projection of $K$. Then*:

(a) $|a_n| = |a_m| = 1$.

(b) $|a_{n+1}| + |a_{m-1}| = T(K)$.

(c) $|a_{n+2}| + |a_{m-2}| + |a_{m-1}||a_{n+1}| = \dfrac{T(K)+T(K)^2}{2} + n(2) + n^*(2) - \mathrm{tri} - \mathrm{tri}^*$,

*where $n(2)$ is the number of edges in $\tilde{E}$ of multiplicity $> 1$ and $n^*(2)$ the corresponding number in the dual checkerboard graph.*

*The number* tri *is the number of triangles in the graph $\tilde{G} = (V, \tilde{E})$ and* tri$^*$ *corresponds to* tri *in the dual graph.*

(d) *In particular*, *the twist number is an invariant of reduced alternating projections of the knot.*

*Proof.* Let $K$ be as in the statement, and let $G^* = (V^*, E^*)$ be the checkerboard graph dual to $G(V, E)$. We have $|E| = |E^*|$ and $|V| + |V^*| = |E| + 2$. Next recall from Equation (2) the definition of $T(K)$, which leads to

$$T(K) = (|\tilde{E}| - |V| + 1) + (|\tilde{E^*}| - |V^*| + 1).$$

The identities in the theorem then follow from Proposition 2.1. □

**Volumish Theorem.** *For an alternating*, *prime*, *nontorus knot $K$ let*

$$V_K(t) = a_n t^n + \cdots + a_m t^m$$

*be the Jones polynomial of $K$. Then*

$$v_8(\max(|a_{m-1}|, |a_{n+1}|) - 1) \le \mathrm{Vol}(S^3 - K) \le 10\, v_3(|a_{n+1}| + |a_{m-1}| - 1).$$

*Here*, $v_3 \approx 1.01494$ *is the volume of an ideal regular hyperbolic tetrahedron and* $v_8 \approx 3.66386$ *is the volume of an ideal regular hyperbolic octahedron.*

*Proof.* The upper bound follows from Theorem 4.1 and 5.1. For the lower bound we need a closer look at [Lackenby 2004].

We can suppose that $K$ admits a diagram such that both checkerboard graphs are imbedded so that every pair of edges connecting the same two vertices are adjacent to each other in the plane. This can be done through flypes.

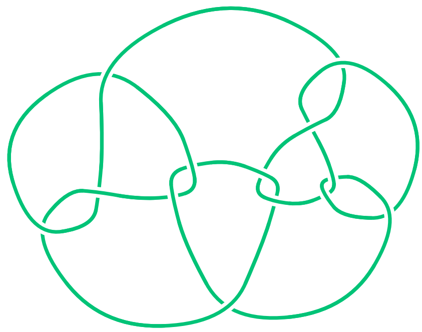

**Figure 4.** The alternating knot 13.123 in the Knotscape Census.

Suppose $G_p = (V_p, E_p)$ is the positive (colored in purple) and $G_g = (V_g, E_g)$ is the negative (colored in gold) checkerboard graph. Since $G_p^* = G_g$ we have $|E_p| = |E_g|$ and

$$|V_p| - |E_p| + |V_g| = 2 = |V_p| - |E_g| + |V_g|.$$

Let $r_p$ and $r_g$ be the number of vertices in $G_p$ and $G_g$ having valence at least 3. It is proved in [Lackenby 2004] (and the bound was improved in [Agol et al. 2005]) that

$$\mathrm{Vol}(S^3 - K) \geq v_8(\max(r_p, r_g) - 2).$$

If $\tilde{G}_p = (V_p, \tilde{E}_p)$ and $\tilde{G}_g = (V_g, \tilde{E}_g)$ are the reduced graphs of $G_p$ and $G_g$ than it is easy to see that

$$r_p = |V_p| - (|E_g| - |\tilde{E}_g|) = 2 - |V_g| + |\tilde{E}_g| = |a_{n+1}| + 1.$$

Similarly, $r_g = |a_{m-1}| + 1$ and the lower bound follows. □

***Example.*** The checkerboard graph $G$ of the knot in Figure 4 has $|V| = 8$ vertices, $|\tilde{E}| = 11$, $n(2) = 2$ and $\mathrm{tri} = 1$.

Its dual has $|V^*| = 7$ vertices, $|\tilde{E}^*| = 10$, $n^*(2) = 3$ and $\mathrm{tri}^* = 2$. Therefore, with the preceding notation for the coefficients of the Jones polynomial,

$$\begin{aligned}
|a_n| &= 1, \\
|a_{n+1}| &= |\tilde{E}| + 1 - |V| = 4, \\
|a_{n+2}| &= \binom{|a_{n+1}|+1}{2} + n(2) - \mathrm{tri} = 10 + 2 - 1 = 11, \\
|a_m| &= 1, \\
|a_{m-1}| &= |\tilde{E}^*| + 1 - |V^*| = 4, \\
|a_{m-2}| &= \binom{|a_{m-1}|+1}{2} + n^*(2) - \mathrm{tri}^* = 10 + 3 - 2 = 11.
\end{aligned}$$

The complete Jones polynomial of the knot is, according to Knotscape,

$$V_{13.121}(t) = t^{-12} - 4t^{-11} + 11t^{-10} - 23t^{-9} + 35t^{-8} - 47t^{-7} + 53t^{-6} \\ - 52t^{-5} + 47t^{-4} - 34t^{-3} + 22t^{-2} - 11t^{-1} + 4 - t,$$

and the hyperbolic volume is

$$\mathrm{Vol}(S^3 - K) \approx 21.1052106828.$$

## Appendix

***Higher twist numbers of prime alternating knots on* 14 *crossings.*** Here we give experimental data on the relationship between the twist number, as computed using the Jones polynomial, and the hyperbolic volume of knots. All data are taken from Knotscape, written by Jim Hoste, Morwen Thistlethwaite and Jeff Weeks [Hoste et al. 1998]. We confined ourselves to knots with crossing number 14. As before, let $V_K(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_m t^m$ be the Jones polynomial of an alternating prime knot $K$.

As shown, the twist number is $T(K) = |a_{n+1}| + |a_{m-1}|$. We call $T_i(K) = |a_{n+i}| + |a_{m-i}|$ the higher twist numbers. In particular, $T(L) = T_1(L)$.

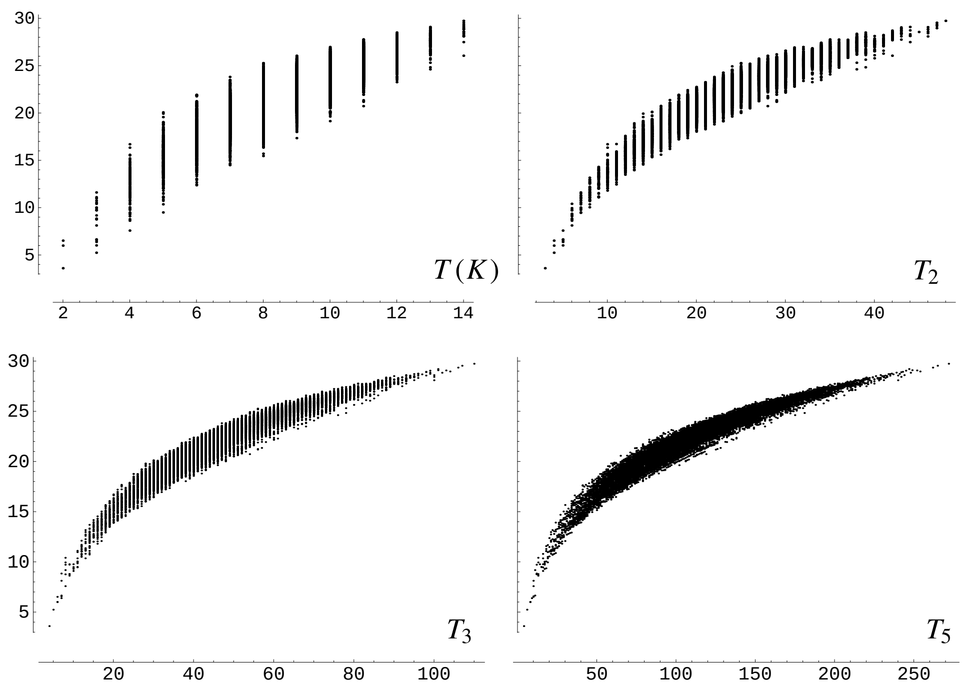


**Figure 5.** Twist numbers vs. volume correlation for 14-crossing alternating knows. Each dot stands for a knot in the census.

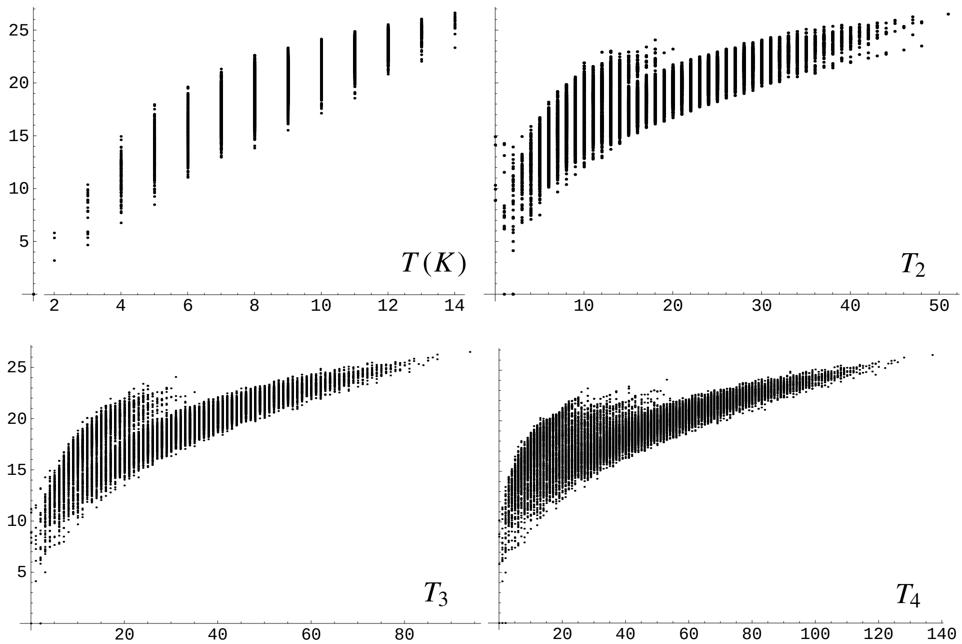


**Figure 6.** Twist numbers vs. volume correlation for 14-crossing nonalternating knots. Nonhyperbolic knots are assigned zero volume.

***Higher twist numbers of prime nonalternating knots on* 14 *crossings.*** For nonalternating knots we keep the notation, although there is no direct geometrical justification known:

Again, let $V_L(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_m t^m$ be the Jones polynomial of a nonalternating knot $L$.

Define the twist number as $T(L) = |a_{n+1}| + |a_{m-1}|$. As in the alternating case, we call $T_i(L) = |a_{n+i}| + |a_{m-i}|$ the higher twist numbers. In particular, $T(L) = T_1(L)$.

The pictures give, for nonalternating knots with crossing number 14, the relation between the twist number (or one of the higher twist numbers $T_2$, $T_3$, $T_4$) and the volume.

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