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The local image of each conformal Q-curvature operator on the sphere admits no scalar constraint, although identities of Kazdan–Warner type hold for its graph.

1. Introduction

Let (m, n) be positive integers such that $n > 1$, and $n \geq 2m$ in case n is even. We work on the standard n-sphere $(\mathbb{S}^n, g_0)$, with pointwise conformal metric $g_u = e^{2u} g_0$. (All objects will be taken to be smooth.)

We are interested in the structure near $u = 0$ of the image of the conformal $2m$-th order Q-curvature increment operator $u \mapsto \boldsymbol{Q}_{m,n}[u] = Q_{m,n}(g_u) - Q_{m,n}(g_0)$ (see Section 2), thus considering a local Nirenberg-type problem (Nirenberg's problem was for $m = 1$; see, for example, [Moser 1973; Kazdan and Warner 1974; 1975; Aubin 1982, p. 122]). At the infinitesimal level, the situation looks as follows (dropping henceforth the subscript (m, n)):

Lemma 1.1. *Let $L = d\,\boldsymbol{Q}[0]$ stand for the linearization at $u = 0$ of the conformal Q-curvature increment operator and Λ_1, for the $(n + 1)$-space of first spherical harmonics on $(\mathbb{S}^n, g_0)$. Then L is self-adjoint and* $\operatorname{Ker} L = \Lambda_1$.

Further, the graph $\Gamma(\boldsymbol{Q}) := \{(u, \boldsymbol{Q}[u]), u \in C^\infty(\mathbb{S}^n)\}$ of $\boldsymbol{Q}$ in $C^\infty(\mathbb{S}^n) \times C^\infty(\mathbb{S}^n)$ admits scalar constraints which are the analogue for $\boldsymbol{Q}$ of the so-called Kazdan–Warner identities for the conformal scalar curvature (i.e., the case $m = 1$); see [Kazdan and Warner 1974; 1975; Bourguignon and Ezin 1987]. Here, a scalar constraint means a real-valued submersion defined near $\Gamma(\boldsymbol{Q})$ in $C^\infty(\mathbb{S}^n) \times C^\infty(\mathbb{S}^n)$ and vanishing on $\Gamma(\boldsymbol{Q})$. Specifically:

Theorem 1.2. *For each $(u, q) \in C^\infty(\mathbb{S}^n) \times C^\infty(\mathbb{S}^n)$ and each conformal Killing vector field X on $(\mathbb{S}^n, g_0)$, the condition $(u, q) \in \Gamma(\boldsymbol{Q})$ implies the vanishing of the*

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integral $\int_{\mathbb{S}^n}(X\cdot q)\,d\mu_u$, *where* $d\mu_u = e^{nu}d\mu_0$ *stands for the Lebesgue measure of the metric* g_u. *In particular, there is no solution* $u \in C^\infty(\mathbb{S}^n)$ *to the equation*

$$Q(g_u) = z + \text{constant} \quad \textit{with } z \in \Lambda_1.$$

Due to the naturality of Q (Remark 3.1) and the self-adjointness of $d\boldsymbol{Q}[u]$ in $L^2(M_n, d\mu_u)$ (Remarks 3.2 and 3.3), this theorem holds as a particular case of the more general Theorem 2.1 below.

Can one do better than Theorem 1.2 and drop the u variable occurring in the constraints and find constraints bearing on the sole image of the operator $\boldsymbol{Q}$? Since L is self-adjoint in $L^2(\mathbb{S}^n, g_0)$ (see [Graham and Zworski 2003]), Lemma 1.1 shows that the map $u \mapsto \boldsymbol{Q}[u]$ misses infinitesimally at $u = 0$ a vector space of dimension $n+1$. How does this translate at the local level? Calling a real-valued map K a *scalar constraint for the local image of* $\boldsymbol{Q}$ *near* 0 if K is a submersion defined near 0 in $C^\infty(\mathbb{S}^n)$ such that $K \circ \boldsymbol{Q} = 0$ near 0 in $C^\infty(\mathbb{S}^n)$, a spherical symmetry argument as in [Delanoë 2003, Corollary 5] shows that if the local image of $\boldsymbol{Q}$ admits a scalar constraint near 0, it must admit $n+1$ independent such constraints, the maximal number to be expected. In this context, our main result is quite in contrast with Theorem 1.2:

Theorem 1.3. *The local image of* $\boldsymbol{Q}$ *near*, 0 *admits no scalar constraint.*

The picture about the local image of the Q-curvature increment operator on $(\mathbb{S}^n, g_0)$ is completed with a remark:

Remark 1.4. The local Nirenberg problem for $\boldsymbol{Q}$ near 0 is governed by the nonlinear Fredholm formula (9) below. Thus, as in [Delanoë 2003, Corollary 5], a local result of Moser type [1973] holds: If $f \in C^\infty(\mathbb{S}^n)$ is close enough to zero and invariant under a nontrivial group of isometries of $(\mathbb{S}^n, g_0)$ acting without fixed points,[1] then $\mathcal{D}(f) = 0$ in (9), so f lies in the local image of $\boldsymbol{Q}$.

The outline of the paper is as follows. In Section 2 we present an independent account on general Kazdan–Warner type identities, implying Theorem 1.2. Then we focus on Theorem 1.3: we recall basic facts for the Q-curvature operators on spheres in Section 3 and sketch the proof of Theorem 1.3 in Section 4, relying on [Delanoë 2003] and reducing it to Lemma 1.1 and another key lemma. In the last two sections we carry out the proofs of these lemmas, deferring to an Appendix some eigenvalues calculations.

2. General identities of Kazdan–Warner type

The following statement is essentially due to Jean–Pierre Bourguignon [1986]:

[1] This condition is more general than a free action.

Theorem 2.1. *Let M_n be a compact n-manifold and $g \mapsto D(g) \in C^\infty(M)$ be a scalar differential operator defined on the open cone of Riemannian metrics on M_n, and natural in the sense of* [Stredder 1975] (see (5) below). *Given a conformal class $\boldsymbol{c}$ and a Riemannian metric $g_0 \in \boldsymbol{c}$, sticking to the notation $g_u = e^{2u} g_0$ for $u \in C^\infty(M)$, consider the operator $u \mapsto \boldsymbol{D}[u] := D(g_u)$ and its linearization $L_u = d\boldsymbol{D}[u]$ at u. Assume that, for each $u \in C^\infty(M)$, the linear differential operator L_u is formally self-adjoint in $L^2(M, d\mu_u)$, where $d\mu_u = e^{nu} d\mu_0$ stands for the Lebesgue measure of g_u. Then, for any conformal Killing vector field X on $(M_n, \boldsymbol{c})$ and any $u \in C^\infty(M)$, we have*

$$\int_M X \cdot \boldsymbol{D}[u] \, d\mu_u = 0.$$

In particular, if $(M_n, \boldsymbol{c})$ is equal to $\mathbb{S}^n$ equipped with its standard conformal class, there is no solution $u \in C^\infty(\mathbb{S}^n)$ to the equation

$$\boldsymbol{D}[u] = z + \text{constant} \quad \textit{with } z \in \Lambda_1$$

(*a first spherical harmonic*).

Proof. We rely on Bourguignon's functional integral invariants approach and follow the proof of [Bourguignon 1986, Proposition 3] (using freely notations from p. 101 of the same paper), presenting its functional geometric framework with some care. We consider the affine Fréchet manifold Γ whose generic point is the volume form (possibly of odd type in case M is not orientable [de Rham 1960]) of a Riemannian metric $g \in \boldsymbol{c}$; we denote by ω_g the volume form of a metric g (recall the tensor ω_g is natural [Stredder 1975, Definition 2.1]). The metric $g_0 \in \boldsymbol{c}$ yields a global chart of Γ defined by

$$\omega_g \in \Gamma \quad \mapsto \quad u := \frac{1}{n} \log \frac{d\omega_g}{d\omega_{g_0}} \in C^\infty(M_n)$$

(viewing volume-forms like measures and using the Radon–Nikodym derivative) — in other words, such that $\omega_g = e^{nu} \omega_{g_0}$; changes of such charts are indeed affine (and pure translations). It will be easier, though, to avoid the use of charts on Γ, except for proving that a 1-form is closed (see below). The tangent bundle to Γ is trivial, equal to $T\Gamma = \Gamma \times \Omega^n(M_n)$ (setting $\Omega^k(A)$ for the k-forms on a manifold A), and there is a canonical Riemannian metric on Γ of Fischer type [Friedrich 1991], given at $\omega_g \in \Gamma$ by

$$\langle v, w \rangle := \int_M \frac{dv}{d\omega_g} \frac{dw}{d\omega_g} \omega_g \quad \text{for } (v, w) \in T_{\omega_g}\Gamma.$$

From Riesz's theorem, a tangent covector $a \in T^*_{\omega_g}\Gamma$ may thus be identified with a tangent vector $a^\sharp \in \Omega^n(M_n)$ or else with the function $da^\sharp/d\omega_g =: \rho_g(a) \in C^\infty(M_n)$

such that

$$a(\varpi) = \int_M \rho_g(a)\varpi \quad \text{for } \varpi \in T_{\omega_g}\Gamma. \tag{1}$$

We also consider the Lie group G of conformal maps on $(M_n, \boldsymbol{c})$, acting on the manifold Γ by

$$(\varphi, \omega_g) \in G \times \Gamma \to \varphi^*\omega_g \in \Gamma$$

(indeed, we have $\varphi^*\omega_g = \omega_{\varphi^*g}$ by naturality and $\varphi \in G \Rightarrow \varphi^*g \in \boldsymbol{c}$). For each conformal Killing field X on $(M_n, \boldsymbol{c})$, the flow of X as a map $t \in \mathbb{R} \to \varphi_t \in G$ yields a vector field $\bar{X}$ on Γ defined by

$$\omega_g \mapsto \bar{X}(\omega_g) := \frac{d}{dt}\left(\varphi_t^*\omega_g\right)_{t=0} \equiv L_X\omega_g$$

(L_X standing here for the Lie derivative on M_n). In this context, regardless of any Banach completion, one may define the (global) flow $t \in \mathbb{R} \to \bar{\varphi}_t \in \mathrm{Diff}(\Gamma)$ of $\bar{X}$ on the Fréchet manifold Γ by setting

$$\bar{\varphi}_t(\omega_g) := \varphi_t^*\omega_g \quad \text{for } \omega_g \in \Gamma;$$

indeed, the latter satisfies

$$\frac{d}{dt}\left(\varphi_t^*\omega_g\right) = \varphi_t^*(L_X\omega_g) \equiv L_X(\varphi_t^*\omega_g) = \bar{X}\left[\bar{\varphi}_t(\omega_g)\right]$$

(see [Kobayashi and Nomizu 1963, p. 33], for example). With the flow $(\bar{\varphi}_t)_{t\in\mathbb{R}}$ at hand, we can define the Lie derivative $L_{\bar{X}}$ of forms on Γ as usual, by setting $L_{\bar{X}}a := (d/dt)\,(\bar{\varphi}_t{}^*a)_{t=0}$. Finally, one can check Cartan's formula for $\bar{X}$, namely

$$L_{\bar{X}} = i_{\bar{X}}d + di_{\bar{X}}, \tag{2}$$

where $i_{\bar{X}}$ denotes the interior product with $\bar{X}$, by verifying it for a generic function f on Γ and for its exterior derivative df (with d defined as in [Lang 1962]).

Following [Bourguignon 1986], and using our global chart $\omega_g \mapsto u$, we apply (2) to the 1-form σ on Γ defined at ω_g by the function $\rho_g(\sigma) := \boldsymbol{D}[u]$; see (1). Arguing as on p. 102 of the same reference, one readily verifies in the chart u (and using constant local vector fields on Γ) that the 1-form σ is closed due to the self-adjointness of the linearized operator L_u in $L^2(M_n, d\mu_u)$; furthermore (dropping the chart u), one derives at once the G-invariance of σ from the naturality of $g \mapsto D(g)$. We thus have $d\sigma = 0$ and $L_{\bar{X}}\sigma = 0$, hence $d(i_{\bar{X}}\sigma) = 0$ by (2). So the function $i_{\bar{X}}\sigma$ is constant on Γ; in other words, $\int_M \boldsymbol{D}[u]\; L_X\omega_u$ is independent of u, or else, integrating by parts, so is $\int_M X \cdot \boldsymbol{D}[u]\, d\mu_u$ (where $X\cdot$ stands for X acting as a derivation on real-valued functions on M_n).

To complete the proof of the first part of Theorem 2.1, we show that the integrand $X \cdot D(g_0)$ of the latter expression at $u = 0$ vanishes for a suitable choice of the metric

g_0 in the conformal class $\boldsymbol{c}$. We recall the Ferrand–Obata theorem [Lelong-Ferrand 1969; Obata 1971/72], according to which either the conformal group G is compact or $(M_n, \boldsymbol{c})$ is equal to $\mathbb{S}^n$ equipped with its standard conformal class. In the former case, averaging on G, we may pick $g_0 \in \boldsymbol{c}$ invariant under the action of G: with this choice, $D(g_0)$ is also G-invariant by naturality, hence $X \cdot D(g_0) \equiv 0$ as needed. In the latter case, as observed in the proof of Proposition 4.2 below, $D(g_0)$ is constant on $\mathbb{S}^n$, and again the desired result follows.

Finally, the last assertion of the theorem (consistently with Proposition 4.2 below and the Fredholm theorem if L_0 is elliptic) follows from the first one, by taking for the vector field X the gradient of z with respect to the standard metric of $\mathbb{S}^n$, which is known to be conformal Killing. □

3. Back to Q-curvatures on spheres: basic facts recalled

The special case n = 2m. Here we will consider the Q-curvature increment operator given by $\boldsymbol{Q}[u] = Q(g_u) - Q_0$, with

$$Q(g_u) = e^{-2mu}(Q_0 + P_0[u]) \tag{3}$$

where $Q_0 = Q(g_0)$ is equal to $Q_0 = (2m-1)!$ on $(\mathbb{S}^n, g_0)$, and where

$$P_0 = \prod_{k=1}^{m} \big(\Delta_0 + (m-k)(m+k-1)\big) \tag{4}$$

(see [Branson 1987; Beckner 1993]), Δ_0 denoting the positive laplacian relative to g_0. We call P_0 the Paneitz–Branson operator of the metric g_0.

Remark 3.1. Following [Branson 1995], one can define a Paneitz–Branson operator P_0 for *any* metric g_0 (given by a formula more general than (4) of course), and a Q-curvature $Q(g_0)$ transforming like (3) under the conformal change of metrics $g_u = e^{2u} g_0$. Importantly then, the map $g \mapsto Q(g) \in C^\infty(\mathbb{S}^n)$ is natural, meaning (see [Stredder 1975, Definition 2.1], for instance) that any diffeomorphism ψ satisfies

$$\psi^* Q(g) = Q(\psi^* g). \tag{5}$$

Remark 3.2. From (3) and the formal self-adjointness of P_0 in $L^2(\mathbb{S}^n, d\mu_0)$ [Graham and Zworski 2003, p. 91], one readily verifies that, for each $u \in C^\infty(\mathbb{S}^n)$, the linear differential operator $d\,\boldsymbol{Q}[u]$ is formally self-adjoint in $L^2(\mathbb{S}^n, d\mu_u)$.

The case n ≠ 2m. The expression of the Paneitz–Branson operator on $(\mathbb{S}^n, g_0)$ becomes

$$P_0 = \prod_{k=1}^{m} \left(\Delta_0 + \left(\tfrac{1}{2}n - k\right)\left(\tfrac{1}{2}n + k - 1\right)\right) \tag{6}$$

(see [Guillarmou and Naud 2006, Proposition 2.2]), while that for the metric $g_u = e^{2u} g_0$ is given by

$$P_u(\,\cdot\,) = e^{-(\frac{1}{2}n+m)u} P_0\big(e^{(\frac{1}{2}n-m)u}\,\cdot\,\big), \tag{7}$$

with the Q-curvature of g_u given accordingly by $\big(\frac{1}{2}n-m\big)\,Q(g_u) = P_u(1)$. The analogue of Remark 3.1 still holds (now see [Graham et al. 1992; Graham and Zworski 2003]). We will consider the (renormalized) Q-curvature increment operator $\boldsymbol{Q}[u] = \big(\frac{1}{2}n-m\big)\,(Q(g_u)-Q_0)$, now with

$$\big(\tfrac{1}{2}n-m\big)\,Q_0 = \big(\tfrac{1}{2}n-m\big)\,Q(g_0) = P_0(1) = \prod_{k=0}^{2m-1}\big(k+\tfrac{1}{2}n-m\big). \tag{8}$$

Remark 3.3. Finally, we note again that the linearized operator $d\,\boldsymbol{Q}[u]$ is formally self-adjoint in $L^2(\mathbb{S}^n, d\mu_u)$. Indeed, a straightforward calculation yields

$$d\,\boldsymbol{Q}[u](v) = \big(\tfrac{1}{2}n-m\big)\,P_u(v) - \big(\tfrac{1}{2}n+m\big)\,P_u(1)\,v,$$

and the Paneitz–Branson operator P_u is known to be self-adjoint in $L^2(\mathbb{S}^n, d\mu_u)$ [Graham and Zworski 2003, p. 91].

For later use, and in all the cases for (m,n), we will set p_0 for the degree m polynomial such that $P_0 = p_0(\Delta_0)$.

4. Proof of Theorem 1.3

The case $m=1$ was settled in [Delanoë 2003] with a proof robust enough to be followed again. For completeness, let us recall how it goes (see [Delanoë 2003] for details).

If $\mathcal{P}_1$ stands for the orthogonal projection of $L^2(\mathbb{S}^n, g_0)$ onto Λ_1, Lemma 1.1 and the self-adjointness of L imply [Delanoë 2003, Theorem 7] that the modified operator

$$u \mapsto \boldsymbol{Q}[u] + \mathcal{P}_1 u$$

is a local diffeomorphism of a neighborhood of 0 in $C^\infty(\mathbb{S}^n)$ onto another one: set $\mathcal{S}$ for its inverse and $\mathcal{D} = \mathcal{P}_1 \circ \mathcal{S}$ (defect map). Then $u = \mathcal{S} f$ satisfies the local nonlinear Fredholm-like equation

$$\boldsymbol{Q}[u] = f - \mathcal{D}(f). \tag{9}$$

By [Delanoë 2003, Theorem 2], if a local constraint exists for $\boldsymbol{Q}$ at 0, then $\mathcal{D}\circ\boldsymbol{Q} = 0$ (recalling the symmetry fact above). Fixing $z \in \Lambda_1$, we will prove Theorem 1.3 by showing that $\mathcal{D} \circ \boldsymbol{Q}[tz] \neq 0$ for small $t \in \mathbb{R}$; here is how. On the one hand, setting

$$u_t = \mathcal{S} \circ \boldsymbol{Q}[tz] := tu_1 + t^2 u_2 + t^3 u_3 + O(t^4),$$

Lemma 1.1 yields $u_1 = 0$; also, as and easily verified general fact, we have

$$\boldsymbol{Q}[u_t] + \mathscr{P}_1 u_t = t^2 (L + \mathscr{P}_1) u_2 + t^3 (L + \mathscr{P}_1) u_3 + O(t^4). \tag{10}$$

On the other hand, consider the expansion of $\boldsymbol{Q}[tz]$:

$$\boldsymbol{Q}[tz] = t^2 c_2[z] + t^3 c_3[z] + O(t^4), \tag{11}$$

and focus on its third order coefficient $c_3[z]$, for which we will prove:

Lemma 4.1. *Let (m, n) be positive integers such that $n > 1$ and $n \geq 2m$ in case n is even. Then*

$$\int_{\mathbb{S}^n} z\, c_3[z]\, d\mu_0 \neq 0.$$

Granted this lemma, we are done: indeed, the equality

$$\boldsymbol{Q}[u_t] + \mathscr{P}_1 u_t = \boldsymbol{Q}[tz],$$

combined with (10)–(11), yields

$$(L + \mathscr{P}_1) u_3 = c_3[z],$$

which, integrated against z, implies that

$$\int_{\mathbb{S}^n} z \mathscr{P}_1 u_3\, d\mu_0 \neq 0$$

(recalling that L is self-adjoint and $z \in \operatorname{Ker} L$ by Lemma 1.1). Therefore $\mathscr{P}_1 u_3 \neq 0$, hence also $\mathscr{D} \circ \boldsymbol{Q}[tz] \neq 0$.

Thus we have reduced the proof of Theorem 1.3 to that of Lemmas 1.1 and 4.1, which we now present.

Proof of Lemma 1.1. (1) *Proof of the inclusion* $\Lambda_1 \subset \operatorname{Ker} L$. We need neither ellipticity nor conformal covariance for this inclusion to hold; the naturality property (5) suffices. We state a general result that implies at once what we need:

Proposition 4.2. *Let $g \mapsto D(g)$ be any scalar natural differential operator on $\mathbb{S}^n$, defined on the open cone of Riemannian metrics, valued in $C^\infty(\mathbb{S}^n)$. For each $u \in C^\infty(\mathbb{S}^n)$, set $\boldsymbol{D}[u] = D(g_u) - D(g_0)$ and $L = d\boldsymbol{D}[0]$, where $g_u = e^{2u} g_0$. Then $\Lambda_1 \subset \operatorname{Ker} L$.*

Proof. Let us first observe that $D(g_0)$ must be constant. Indeed, for each isometry ψ of $(\mathbb{S}^n, g_0)$, the naturality of D implies $\psi^* D(g_0) \equiv D(g_0)$; so the result follows because the group of such isometries acts transitively on $\mathbb{S}^n$. Morally, since g_0 has constant curvature, this result is also expectable from the theory of Riemannian invariants (see [Stredder 1975] and references therein), here though, without any regularity (or polynomiality) assumption.

Given an arbitrary nonzero $z \in \Lambda_1$, let $S = S(z) \in \mathbb{S}^n$ stand for its corresponding south pole (where $z(S) = -M$ is minimum) and, for each small real t, let ψ_t denote the conformal diffeomorphism of $\mathbb{S}^n$ fixing S and composed elsewhere of: Ster_S, the stereographic projection with pole S, the dilation $X \in \mathbb{R}^n \mapsto e^{Mt} X \in \mathbb{R}^n$, and the inverse of Ster_S. As t varies, the family ψ_t satisfies

$$\psi_0 = I, \qquad \frac{d}{dt}(\psi_t)_{t=0} = -\nabla_0 z,$$

where ∇_0 denotes the gradient relative to g_0. If we set $e^{2u_t} g_0 = \psi_t^* g_0$, we get

$$\frac{d}{dt}(u_t)_{t=0} \equiv z.$$

Recalling that $D(g_0)$ is constant, the naturality of D implies

$$\boldsymbol{D}[u_t] = \psi_t^* D(g_0) - D(g_0) = 0;$$

in particular, differentiating this equation at $t = 0$ yields $Lz = 0$ hence we conclude that $\Lambda_1 \subset \operatorname{Ker} L$.

(2) *Proof of the reverse inclusion* $\ker L \subset \Lambda_1$. For a contradiction, assume the existence of a nonzero $v \in \Lambda_1^\perp \cap \operatorname{Ker} L$. If $\mathcal{B}$ is an orthonormal basis of eigenfunctions of Δ_0 in $L^2(\mathbb{S}^n, d\mu_0)$, there exists an integer $i \neq 1$ and a function $\varphi_i \in \Lambda_i \cap \mathcal{B}$ (where Λ_i henceforth denotes the space of i-th spherical harmonics) such that

$$\int_{\mathbb{S}^n} \varphi_i v \, d\mu_0 \neq 0$$

(actually $i \neq 0$, due to $\int_{\mathbb{S}^n} v \, d\mu_0 = 0$, obtained just by averaging $Lv = 0$ on $\mathbb{S}^n$). By the self-adjointness of L, we may write

$$0 = \int_{\mathbb{S}^n} \varphi_i L v \, d\mu_0 = \int_{\mathbb{S}^n} v L \varphi_i \, d\mu_0,$$

then infer (see below) that

$$0 = \big(p_0(\lambda_i) - p_0(\lambda_1)\big) \int_{\mathbb{S}^n} \varphi_i v \, d\mu_0,$$

and finally get the desired contradiction, because $p_0(\lambda_i) \neq p_0(\lambda_1)$ for $i \neq 1$ (see the Appendix). Here, we used the following auxiliary facts, obtained by differentiating (3) or (7) at $u = 0$ in the direction of $w \in C^\infty(\mathbb{S}^n)$:

$$n = 2m \;\Rightarrow\; Lw = P_0(w) - n!\, w,$$
$$n \neq 2m \;\Rightarrow\; Lw = \big(\tfrac{1}{2}n - m\big)\, P_0(w) - \big(\tfrac{1}{2}n + m\big)\, p_0(\lambda_0) w.$$

From $\Lambda_1 \subset \operatorname{Ker} L$, we get, taking $w = z \in \Lambda_1$:

$$\begin{aligned} &n = 2m \;\Rightarrow\; p_0(\lambda_1) - n! = 0, \\ &n \neq 2m \;\Rightarrow\; \left(\tfrac{1}{2}n - m\right) p_0(\lambda_1) - \left(\tfrac{1}{2}n + m\right) p_0(\lambda_0) = 0. \end{aligned} \tag{12}$$

Moreover, taking $w = \varphi_i \in \Lambda_i$, we then have

$$\begin{aligned} &n = 2m \;\Rightarrow\; L\varphi_i = \big(p_0(\lambda_i) - p_0(\lambda_1)\big)\varphi_i, \\ &n \neq 2m \;\Rightarrow\; L\varphi_i = \left(\tfrac{1}{2}n - m\right)\big(p_0(\lambda_i) - p_0(\lambda_1)\big)\varphi_i. \end{aligned}$$ □

Proof of Lemma 4.1. (1) *The case $m = 2n$.* For fixed $z \in \Lambda_1$ and for $t \in \mathbb{R}$ close to 0, we compute the third order expansion of $\boldsymbol{Q}[tz]$. By Lemma 1.1 it vanishes up to first order. Noting the identity $\boldsymbol{Q}[v]/Q_0 \equiv e^{-nv}(1 + nv) - 1$, valid for all $v \in \Lambda_1$, we find at once

$$\frac{\boldsymbol{Q}[tz]}{Q_0} = -2m^2t^2z^2 + \tfrac{8}{3}m^3t^3z^3 + O(t^4);$$

in particular (with the notation of Section 1), we have $c_3[z] = \frac{8}{3}m^3Q_0z^3$, and Lemma 4.1 holds trivially.

(2) *The case $m \neq 2n$.* In this case, calculations are drastically simplified by picking the nonlinear argument of P_0 in $P_u(1)$, namely $w := \exp\!\big((\frac{1}{2}n - m)u\big)$ (see (7)), as new parameter for the local image of the conformal curvature-increment operator. Since w is close to 1, we further set $w = 1 + v$, so the conformal factor becomes

$$e^{2u} = (1 + v)^{4/(n-2m)}$$

and the renormalized Q-curvature increment operator accordingly becomes

$$\boldsymbol{Q}[u] \equiv \tilde{Q}[v] := (1 + v)^{1-2^\star} P_0(1 + v) - \left(\tfrac{1}{2}n - m\right) Q_0 \tag{13}$$

where $2^\star$ stands for $2n/(n - 2m)$ in our context (admittedly a loose notation, customary for critical Sobolev exponents). Of course, Lemma 1.1 still holds for the operator $\tilde{Q}$ (with $\tilde{L} := d\tilde{Q}[0] \equiv (2^\star/n)L$) and proving Theorem 1.3 for $\tilde{Q}$ is equivalent to proving it for $\boldsymbol{Q}$. Altogether, we may thus focus on the proof of Lemma 4.1 for $\tilde{Q}$ instead of $\boldsymbol{Q}$. (The reader can instead prove Lemma 4.1 directly for $\boldsymbol{Q}$, but it takes a few pages.)

Picking z and t as above, plugging $v = tz$ in (13), and using the equality

$$P_0(z) = p_0(\lambda_1)z \equiv (2^\star - 1)\left(\tfrac{1}{2}n - m\right) Q_0 z,$$

obtained from (12), we readily calculate the expansion

$$\frac{1}{\left(\frac{1}{2}n - m\right) Q_0}\,\tilde{Q}[tz] = -\tfrac{1}{2}(2^\star - 2)(2^\star - 1)\,t^2z^2 + \tfrac{1}{3}(2^\star - 2)(2^\star - 1)2^\star\,t^3z^3 + O(t^4),$$

thus finding for its third order coefficient

$$\frac{1}{\left(\frac{1}{2}n-m\right)Q_0}\,\tilde{c}_3[z]=\tfrac{1}{3}(2^\star-2)(2^\star-1)2^\star\,z^3.$$

So Lemma 4.1 obviously holds, and with it Theorem 1.3. □

Appendix: Eigenvalue calculations

As well known (see [Berger et al. 1971], for instance), for each $i\in\mathbb{N}$, the i-th eigenvalue of Δ_0 on $\mathbb{S}^n$ equals $\lambda_i=i(i+n-1)$. Recalling (6), we must calculate

$$p_0(\lambda_i)=\prod_{k=1}^{m}\left(\lambda_i+\left(\tfrac{1}{2}n-k\right)\left(\tfrac{1}{2}n+k-1\right)\right).$$

Setting provisionally

$$r=\tfrac{1}{2}(n-1),\qquad s_k=k-\tfrac{1}{2},$$

so that $\frac{1}{2}n-k=r-s_k$, $\frac{1}{2}n+k-1=r+s_k$ and $\lambda_i=i^2+2ir$, we can rewrite

$$\begin{aligned}p_0(\lambda_i)&=\prod_{k=1}^{m}\left((i+r)^2-s_k^2\right)\\&=\prod_{k=1}^{m}\left(\tfrac{1}{2}+i+r-k\right)\left(\tfrac{1}{2}+i+r+k-1\right)\equiv\prod_{k=0}^{2m-1}\left(\tfrac{1}{2}+i+r-m+k\right),\end{aligned}$$

getting (back to m, n and k only)

$$p_0(\lambda_i)=\prod_{k=0}^{2m-1}\left(i+\tfrac{1}{2}n-m+k\right).$$

In particular,

$$P_0(1)\equiv p_0(\lambda_0)=\left(\tfrac{1}{2}n-m\right)\prod_{k=1}^{2m-1}\left(\tfrac{1}{2}n-m+k\right)$$

as asserted in (8) (and consistently there with the value of Q_0 in case $n=2m$). An easy induction argument yields

$$p_0(\lambda_{i+1})=\frac{\left(\frac{1}{2}n+m+i\right)}{\left(\frac{1}{2}n-m+i\right)}\,p_0(\lambda_i)\quad\text{for all } i\in\mathbb{N}$$

(consistently when $i=0$ with (12)), which implies that $|p_0(\lambda_{i+1})|>|p_0(\lambda_i)|$ for all $i\in\mathbb{N}$, hence in particular $p_0(\lambda_i)\neq p_0(\lambda_1)$ for $i>1$, as required in the proof of Lemma 1.1. This also implies the final formula

$$p_0(\lambda_i)=\frac{\left(\frac{1}{2}n+m\right)\dots\left(\frac{1}{2}n+m+i-1\right)}{\left(\frac{1}{2}n-m\right)\dots\left(\frac{1}{2}n-m+i-1\right)}\,p_0(\lambda_0)\quad\text{for all } i\geq 1.$$

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References

[Aubin 1982] T. Aubin, *Nonlinear analysis on manifolds: Monge–Ampère equations*, Grundlehren der Math. Wissenschaften **252**, Springer, New York, 1982. MR 85j:58002 Zbl 0512.53044

[Beckner 1993] W. Beckner, "Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality", *Ann. of Math.* (2) **138**:1 (1993), 213–242. MR 94m:58232 Zbl 0826.58042

[Berger et al. 1971] M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variété riemannienne*, Lecture Notes in Math. **194**, Springer, Berlin, 1971. MR 43 #8025 Zbl 0223.53034

[Bourguignon 1986] J.-P. Bourguignon, "Invariants intégraux fonctionnels pour des équations aux dérivées partielles d'origine géométrique", pp. 100–108 in *Differential geometry* (Peñíscola 1985), edited by A. M. Naveira et al., Lecture Notes in Math. **1209**, Springer, Berlin, 1986. MR 88b:58145 Zbl 0623.53003

[Bourguignon and Ezin 1987] J.-P. Bourguignon and J.-P. Ezin, "Scalar curvature functions in a conformal class of metrics and conformal transformations", *Trans. Amer. Math. Soc.* **301**:2 (1987), 723–736. MR 88e:53054 Zbl 0622.53023

[Branson 1987] T. P. Branson, "Group representations arising from Lorentz conformal geometry", *J. Funct. Anal.* **74**:2 (1987), 199–291. MR 90b:22016 Zbl 0643.58036

[Branson 1995] T. P. Branson, "Sharp inequalities, the functional determinant, and the complementary series", *Trans. Amer. Math. Soc.* **347**:10 (1995), 3671–3742. MR 96e:58162 Zbl 0848.58047

[Delanoë 2003] P. Delanoë, "Local solvability of elliptic, and curvature, equations on compact manifolds", *J. Reine Angew. Math.* **558** (2003), 23–45. MR 2004e:53054 Zbl 1040.53047

[Friedrich 1991] T. Friedrich, "Die Fisher-Information und symplektische Strukturen", *Math. Nachr.* **153** (1991), 273–296. MR 93e:58008 Zbl 0792.62003

[Graham and Zworski 2003] C. R. Graham and M. Zworski, "Scattering matrix in conformal geometry", *Invent. Math.* **152**:1 (2003), 89–118. MR 2004c:58064 Zbl 1030.58022

[Graham et al. 1992] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, "Conformally invariant powers of the Laplacian. I. Existence", *J. London Math. Soc.* (2) **46**:3 (1992), 557–565. MR 94c:58226 Zbl 0726.53010

[Guillarmou and Naud 2006] C. Guillarmou and F. Naud, "Wave 0-trace and length spectrum on convex co-compact hyperbolic manifolds", *Comm. Anal. Geom.* **14**:5 (2006), 945–967. MR 2287151 Zbl 05141318

[Kazdan and Warner 1974] J. L. Kazdan and F. W. Warner, "Curvature functions for compact 2-manifolds", *Ann. of Math.* (2) **99** (1974), 14–47. MR 49 #7949 Zbl 0273.53034

[Kazdan and Warner 1975] J. L. Kazdan and F. W. Warner, "Scalar curvature and conformal deformation of Riemannian structure", *J. Differential Geometry* **10** (1975), 113–134. MR 51 #1661 Zbl 0296.53037

[Kobayashi and Nomizu 1963] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. I, Wiley-Interscience, New York and London, 1963. MR 27 #2945 Zbl 0119.37502

[Lang 1962] S. Lang, *Introduction to differentiable manifolds*, Wiley, New York, 1962. MR 27 #5192 Zbl 0103.15101

[Lelong-Ferrand 1969] J. Lelong-Ferrand, "Transformations conformes et quasiconformes des variétés riemanniennes; application à la démonstration d'une conjecture de A. Lichnerowicz", *C. R. Acad. Sci. Paris Sér. A-B* **269** (1969), A583–A586. MR 40 #7989 Zbl 0201.09701

[Moser 1973] J. Moser, "On a nonlinear problem in differential geometry", pp. 273–280 in *Dynamical systems* (Salvador, 1971), edited by M. Peixoto, Academic Press, New York, 1973. MR 49 #4018 Zbl 0275.53027

[Obata 1971/72] M. Obata, "The conjectures on conformal transformations of Riemannian manifolds", *J. Differential Geometry* **6** (1971/72), 247–258. MR 46 #2601 Zbl 0236.53042

[de Rham 1960] G. de Rham, *Variétés différentiables: formes, courants, formes harmoniques*, 2nd ed., ActualitÃ©s scientifiques et industrielles **1222**, Hermann, Paris, 1960. Translated as *Differentiable manifolds: forms, currents, harmonic forms*, Grundlehren der math. Wissenschaften **266**, Springer, New York, 1984. Zbl 0089.08105

[Stredder 1975] P. Stredder, "Natural differential operators on Riemannian manifolds and representations of the orthogonal and special orthogonal groups", *J. Differential Geometry* **10**:4 (1975), 647–660. MR 54 #3772 Zbl 0318.53046

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