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ALBERTO ELDUQUE

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SOME NEW SIMPLE MODULAR LIE SUPERALGEBRAS

ALBERTO ELDUQUE

Two new simple modular Lie superalgebras will be obtained in characteristics 3 and 5, which share the property that their even parts are orthogonal Lie algebras and the odd parts their spin modules. The characteristic 5 case will be shown to be related, by means of a construction of Tits, to the exceptional ten-dimensional Jordan superalgebra of Kac.

1. Introduction

There are well-known constructions of the exceptional simple Lie algebras of type E_8 and F_4 which go back to Witt [1941], as $\mathbb{Z}_2$-graded algebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with even part the orthogonal Lie algebras $\mathfrak{so}_{16}$ and $\mathfrak{so}_9$ respectively, and odd part given by their spin representations (see [Adams 1996]).

Brown [1982] found a new simple finite-dimensional Lie algebra over fields of characteristic 3 which presents the same pattern, but with $\mathfrak{g}_{\bar{0}} = \mathfrak{so}_7$.

Among the simple Lie superalgebras in Kac's classification [1977b], only the orthosymplectic Lie superalgebra $\mathfrak{osp}(1,4)$ presents the same pattern, since $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}_4$ in this case, and $\mathfrak{g}_{\bar{1}}$ is its natural four-dimensional module. But $\mathfrak{sp}_4$ is isomorphic to $\mathfrak{so}_5$, so $\mathfrak{g}_{\bar{1}}$ is its spin module.

In [Elduque 2006], we found another instance of this phenomenon. There exists a simple Lie superalgebra over fields of characteristic 3 with even part isomorphic to $\mathfrak{so}_{12}$ and odd part its spin module.

This paper is devoted to settling the question of which other simple $\mathbb{Z}_2$-graded Lie algebras or Lie superalgebras display this pattern: the even part being an orthogonal Lie algebra and the odd part its spin module.

It turns out that, besides the previously mentioned examples and the example of $\mathfrak{so}_9$, which is the direct sum of $\mathfrak{so}_8$ and its natural module, but where, because of triality, this natural module can be replaced by the spin module, there appear exactly two other possibilities for Lie superalgebras, one in characteristic 3 with

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even part isomorphic to $\mathfrak{so}_{13}$, and the other in characteristic 5, with even part isomorphic to $\mathfrak{so}_{11}$. These simple Lie superalgebras seem to appear here for the first time.

The characteristic-5 case will be shown to be strongly related to the ten-dimensional simple exceptional Kac Jordan superalgebra, by means of a construction due to Tits. As was proved by McCrimmon [2005], and indirectly hinted in [Elduque and Okubo 2000], the Grassmann envelope of this Jordan superalgebra satisfies the Cayley–Hamilton equation of degree 3 and hence, as shown in [Benkart and Zelmanov 1996] and [Benkart and Elduque 2003], this Jordan superalgebra $\mathcal{J}$ can be plugged into the second component of the Tits construction [1966], the first component being a Cayley algebra. The even part of the resulting Lie superalgebra is then isomorphic to $\mathfrak{so}_{11}$ and the odd part turns out to be its spin module.

The characteristic-3 case is related to the six-dimensional composition superalgebra $B(4,2)$ (see [Elduque and Okubo 2002; Shestakov 1997]) and, therefore, to the exceptional Jordan superalgebra of 3×3 hermitian matrices $H_3(B(4,2))$. This will be further discussed in [Cunha and Elduque 2006], where an extended Freudenthal magic square in characteristic 3 is considered.

Convention. *Throughout the paper*, k *will always denote an algebraically closed field of characteristic* $\neq 2$.

Overview. Section 2 reviews the basic properties of the orthogonal Lie algebras, associated Clifford algebras and spin modules in a way suitable to our purposes. In Section 3 we determine the simple $\mathbb{Z}_2$-graded Lie algebras and the simple Lie superalgebras whose even part is an orthogonal Lie algebra of type B and its odd part its spin module. The two new simple Lie superalgebras mentioned above appear here. Section 4 is devoted to type D, and here the objects that appear are either classical or a Lie superalgebra in characteristic 3 with even part $\mathfrak{so}_{12}$, which appeared for the first time in [Elduque 2006] related to a Freudenthal triple system, which in turn is constructed in terms of the Jordan algebra of the hermitian 3×3 matrices over a quaternion algebra. Finally, Section 5 is devoted to study the relationship of the exceptional Lie superalgebra that has appeared in characteristic 5, with even part isomorphic to $\mathfrak{so}_{11}$, to the Lie superalgebra obtained by means of Tits construction in terms of the Cayley algebra and of the exceptional ten-dimensional Jordan superalgebra of Kac.

2. Spin modules

Let V be a vector space of dimension $l\geq 1$ over the field k, let V^* be its dual vector space, and consider the $(2l+1)$-dimensional vector space $W=ku\oplus V\oplus V^*$, with the regular quadratic form q given by

$$q(\alpha u+v+f)=-\alpha^2+f(v), \tag{2.1}$$

for any $\alpha\in k$, $v\in V$ and $f\in V^*$.

Let $\mathfrak{Cl}(V\oplus V^*,q)$ be the Clifford algebra of the restriction of q to $V\oplus V^*$, and let $\mathfrak{Cl}_{\bar 0}(W,q)$ be the even Clifford algebra of q. As a general rule, the multiplication in Clifford algebras will be denoted by a dot: $x\cdot y$. The linear map

$$V\oplus V^*\to\mathfrak{Cl}_{\bar 0}(W,q):\ x\mapsto u\cdot x=\tfrac{1}{2}(u\cdot x-x\cdot u)=\tfrac{1}{2}[u,x]^{\cdot}$$

extends to an algebra isomorphism

$$\Psi:\mathfrak{Cl}(V\oplus V^*,q)\to\mathfrak{Cl}_{\bar 0}(W,q). \tag{2.2}$$

Let τ be the involution of $\mathfrak{Cl}(W,q)$ such that $\tau(w)=w$ for any $w\in W$, and let $\tau_{\bar 0}$ be its restriction to $\mathfrak{Cl}_{\bar 0}(W,q)$. Let τ' be the involution of $\mathfrak{Cl}(V\oplus V^*,q)$ such that $\tau'(x)=-x$ for any $x\in V\oplus V^*$. Then, for any $x\in V\oplus V^*$,

$$\tau_{\bar 0}\big(\Psi(x)\big)=\tau_{\bar 0}(u\cdot x)=\tau(x)\cdot\tau(u)=x\cdot u=-u\cdot x=u\cdot\tau'(x)=\Psi\big(\tau'(x)\big),$$

so Ψ in (2.2) is actually an isomorphism of algebras with involution:

$$\Psi:\big(\mathfrak{Cl}(V\oplus V^*,q),\tau'\big)\to\big(\mathfrak{Cl}_{\bar 0}(W,q),\tau_{\bar 0}\big). \tag{2.3}$$

Now consider the exterior algebra $\bigwedge V$. Multiplication here will be denoted by juxtaposition. This conveys a natural grading over $\mathbb{Z}_2$: $\bigwedge V=\bigwedge_{\bar 0}V\oplus\bigwedge_{\bar 1}V$. In other words, like Clifford algebras, $\bigwedge V$ is an associative superalgebra. For any $f\in V^*$, let $df:\bigwedge V\to\bigwedge V$ be the unique odd superderivation such that $(df)(v)=f(v)$ for any $v\in V\subseteq\bigwedge V$ (see, for instance, [Knus et al. 1998, §8]). Note that $(df)^2=0$.

Also, for any $v\in V$, the left multiplication by v gives an odd linear map $l_v:\bigwedge V\to\bigwedge V:x\mapsto vx$. Again $l_v^2=0$, and for any $v\in V$ and $f\in V^*$,

$$(l_v+df)^2=l_vdf+dfl_v=l_{(df)(v)}=f(v)\,\mathrm{id}=q(v+f)\,\mathrm{id}. \tag{2.4}$$

The linear map $V\oplus V^*\to\mathrm{End}_k(\bigwedge V)$ defined by $v+f\mapsto l_v+df$ then induces an isomorphism

$$\Lambda:\mathfrak{Cl}(V\oplus V^*,q)\to\mathrm{End}_k(\textstyle\bigwedge V). \tag{2.5}$$

Let $\bar{\ }:\bigwedge V\to\bigwedge V$ be the involution such that $\bar v=-v$ for any $v\in V$. Fix a basis $\{v_1,\dots,v_l\}$ of V, and let $\{f_1,\dots,f_l\}$ be its dual basis ($f_i(v_j)=\delta_{ij}$ for any $i,j=1,\dots,l$). Let $\Phi:\bigwedge V\to k$ be the linear function such that

$$\begin{aligned}&\Phi(v_1\cdots v_l)=1,\\ &\Phi(v_{i_1}\cdots v_{i_r})=0\quad\text{for any } r<l \text{ and } 1\le i_1<\cdots<i_r\le l,\end{aligned} \tag{2.6}$$

that is, Φ is a determinant, and consider the bilinear form

$$\begin{aligned} b : \textstyle\bigwedge V \times \bigwedge V &\to k \\ (s,t) \quad &\mapsto \Phi(\bar{s}t). \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} \Phi(\overline{v_1\cdots v_l}) &= (-1)^l\Phi(v_l\cdots v_1) = (-1)^l(-1)^{\binom{l}{2}}\Phi(v_1\cdots v_l) \\ &= (-1)^{\binom{l+1}{2}}\Phi(v_1\cdots v_l) = (-1)^{\binom{l+1}{2}}, \end{aligned}$$

it follows that, for any $s,t\in\bigwedge V$,

$$b(t,s) = \Phi(\bar{t}s) = \Phi(\overline{\bar{s}t}) = (-1)^{\binom{l+1}{2}}\Phi(\bar{s}t) = (-1)^{\binom{l+1}{2}}b(s,t).$$

Hence,

$$\begin{aligned} &b \text{ is symmetric if and only if } l\equiv 0 \text{ or } 3 \pmod 4, \\ &b \text{ is skew-symmetric if and only if } l\equiv 1 \text{ or } 2 \pmod 4. \end{aligned} \tag{2.8}$$

Let τ_b be the adjoint involution of $\bigwedge V$ relative to b. Then, for any $v\in V$ and $s,t\in\bigwedge V$,

$$b\big(l_v(s),t\big) = \Phi(\overline{vs}t) = \Phi(\bar{s}\bar{v}t) = -\Phi(\bar{s}vt) = -b\big(s,l_vt\big),$$

so $\tau_b(l_v) = -l_v$. Also, if $f\in V^*$ and $v\in V$,

$$(df)(\bar{v}) = -(df)(v) = -f(v) = -\overline{f(v)} = (-1)^{|v|}\overline{(df)(v)},$$

where $\bigwedge V = \oplus_{i=0}^l\bigwedge^i V$ is the natural $\mathbb{Z}$-grading of $\bigwedge V$ and $|s| = i$ for $s\in\bigwedge^i V$. Also, assuming $(df)(\bar{s}) = (-1)^{|s|}\overline{(df)(s)}$ and $(df)(\bar{t}) = (-1)^{|t|}\overline{(df)(t)}$ for homogeneous $s,t\in\bigwedge V$,

$$\begin{aligned} (df)(\overline{st}) &= (df)(\bar{t}\bar{s}) = (df)(\bar{t})\bar{s} + (-1)^{|t|}\bar{t}(df)(\bar{s}) \\ &= (-1)^{|t|}\overline{(df)(t)}\bar{s} + (-1)^{|s|+|t|}\overline{\bar{t}(df)(s)} \\ &= (-1)^{|s|+|t|}\overline{(df)(s)t + (-1)^{|s|}s(df)(t)} \\ &= (-1)^{|st|}\overline{(df)(st)}. \end{aligned}$$

Hence $(df)(\bar{s}) = (-1)^{|s|}\overline{(df)(s)}$ for any homogeneous $s\in\bigwedge V$. Thus, for any $f\in V^*$ and $s,t\in\bigwedge V$,

$$\begin{aligned} b\big((df)(s),t\big) &= \Phi\big(\overline{(df)(s)}t\big) = (-1)^{|s|}\Phi\big((df)(\bar{s})t\big) \\ &= -\Phi\big(\bar{s}(df)(t)\big) \qquad \text{since } \Phi\big((df)(\textstyle\bigwedge V)\big) = 0 \\ &= -b\big(s,(df)(t)\big) \end{aligned}$$

and, therefore, $\tau_b(df) = -df$. As a consequence, the isomorphism Λ in (2.5) is actually an isomorphism of algebras with involution:

$$\Lambda : \big(\mathfrak{Cl}(V \oplus V^*, q), \tau'\big) \to \big(\mathrm{End}_k(\textstyle\bigwedge V), \tau_b\big). \tag{2.9}$$

The orthogonal Lie algebra $\mathfrak{so}_{2l+1} = \mathfrak{so}(W, q)$ is spanned by the linear maps

$$\sigma_{w_1,w_2} = q(w_1, \cdot\,)w_2 - q(w_2, \cdot\,)w_1 \tag{2.10}$$

where $q(w_1, w_2) = q(w_1+w_2) - q(w_1) - q(w_2)$ is the associated symmetric bilinear form.

But for any $w_1, w_2, w_3 \in W$, inside $\mathfrak{Cl}(W, q)$ one has

$$\begin{aligned}
[[w_1, w_2]^{\cdot}, w_3]^{\cdot} &= (w_1 \cdot w_2 - w_2 \cdot w_1) \cdot w_3 - w_3 \cdot (w_1 \cdot w_2 - w_2 \cdot w_1) \\
&= q(w_2, w_3)w_1 - w_1 \cdot w_3 \cdot w_2 - q(w_1, w_3)w_2 + w_2 \cdot w_3 \cdot w_1 \\
&\quad - q(w_1, w_3)w_2 + w_1 \cdot w_3 \cdot w_2 + q(w_2, w_3)w_1 - w_2 \cdot w_3 \cdot w_1 \\
&= -2\sigma_{w_1,w_2}(w_3).
\end{aligned}$$

Therefore, $\mathfrak{so}_{2l+1}$ embeds in $\mathfrak{Cl}_{\bar 0}(W, q)$ by means of $\sigma_{w_1,w_2} \mapsto -\frac{1}{2}[w_1, w_2]^{\cdot}$, so $\mathfrak{so}_{2l+1}$ can be identified with the subspace $[W, W]^{\cdot}$ in $\mathfrak{Cl}_{\bar 0}(W, q)$.

Under this identification, the action of $\mathfrak{so}_{2l+1} = \mathfrak{so}(W, q)$ on its natural module W corresponds to the adjoint action of $[W, W]^{\cdot}$ on W inside $\mathfrak{Cl}(W, q)$. Note that for any $x, y \in V \oplus V^*$,

$$\begin{aligned}
\Psi\big([x, y]^{\cdot}\big) = [u \cdot x, u \cdot y]^{\cdot} &= u \cdot x \cdot u \cdot y - u \cdot y \cdot u \cdot x \\
&= -u \cdot u \cdot (x \cdot y - y \cdot x) \\
&= x \cdot y - y \cdot x = [x, y]^{\cdot},
\end{aligned}$$

so Ψ acts "identically" on $\mathfrak{so}_{2l} = [V \oplus V^*, V \oplus V^*]^{\cdot} \subseteq \mathfrak{Cl}(V \oplus V^*, q)$.

The subspace $\mathfrak{h} = \mathrm{span}\,\{[v_i, f_i]^{\cdot} : i = 1, \ldots, l\}$ is a Cartan subalgebra of $\mathfrak{so}_{2l+1} \simeq [W, W]^{\cdot}$. Besides,

$$\begin{aligned}
[[v_i, f_i]^{\cdot}, u]^{\cdot} &= 0, \\
[[v_i, f_i]^{\cdot}, v_j]^{\cdot} &= 2\delta_{ij} v_j, \\
[[v_i, f_i]^{\cdot}, f_j]^{\cdot} &= -2\delta_{ij} f_j.
\end{aligned}$$

Hence, if $\epsilon_i : \mathfrak{h} \to k$ denotes the linear map with $\epsilon_i\big([v_j, f_j]^{\cdot}\big) = 2\delta_{ij}$, the weights of the natural module W relative to $\mathfrak{h}$ are 0 and $\pm\epsilon_i$, $i = 1, \ldots, l$, all of them of multiplicity 1; while there appears a root space decomposition

$$\mathfrak{so}_{2l+1} \simeq [W, W]^{\cdot} = \mathfrak{h} \oplus \big(\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha\big),$$

where

$$\Delta = \{\pm(\epsilon_i + \epsilon_j) : 1 \le i < j \le l\} \cup \{\pm\epsilon_i : 1 \le i \le l\} \cup \{\pm(\epsilon_i - \epsilon_j) : 1 \le i < j \le l\}.$$

Here $\mathfrak{g}_{\epsilon_i+\epsilon_j} = k[v_i, v_j]^{\cdot}$, $\mathfrak{g}_{-(\epsilon_i+\epsilon_j)} = k[f_i, f_j]^{\cdot}$, $\mathfrak{g}_{\epsilon_i-\epsilon_j} = k[v_i, f_j]^{\cdot}$, $\mathfrak{g}_{\epsilon_i} = k[u, v_i]^{\cdot}$, and $\mathfrak{g}_{-\epsilon_i} = k[u, f_i]^{\cdot}$, for any $i \neq j$. This root space decomposition induces a triangular decomposition

$$\mathfrak{so}_{2l+1} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_\pm = \oplus_{\alpha\in\Delta^\pm}\mathfrak{g}_\alpha$, with $\Delta^+ = \{\epsilon_i+\epsilon_j : 1 \le i < j \le l\} \cup \{\epsilon_i : 1 \le i \le l\} \cup \{\epsilon_i - \epsilon_j : 1 \le i < j \le l\}$, and $\Delta^- = -\Delta^+$.

The spin representation of $\mathfrak{so}_{2l+1}$ is given by the composition

$$\mathfrak{so}_{2l+1} \hookrightarrow \mathfrak{Cl}_{\bar{0}}(W, q) \xrightarrow{\Psi^{-1}} \mathfrak{Cl}(V \oplus V^*, q) \xrightarrow{\Lambda} \mathrm{End}_k(\textstyle\bigwedge V).$$

Denote this composition by

$$\rho = \Lambda \circ \Psi^{-1}|_{\mathfrak{so}_{2l+1}}, \tag{2.11}$$

and denote by $S = \bigwedge V$ the spin module. Note that for any $1 \le i \le l$ and any $1 \le i_1 < \cdots < i_r \le l$

$$\rho\big([v_i, f_i]^{\cdot}\big)(v_{i_1} \cdots v_{ir}) = \begin{cases} v_{i_1} \cdots v_{i_r} & \text{if } i = i_j \text{ for some } j, \\ -v_{i_1} \cdots v_{i_r} & \text{otherwise.} \end{cases}$$

Thus, $v_{i_1} \cdots v_{i_r}$ is a weight vector relative to $\mathfrak{h}$, with weight

$$\frac{1}{2}\Bigg(\sum_{i\in\{i_1,\dots,i_r\}} \epsilon_i - \sum_{j\notin\{i_1,\dots,i_r\}} \epsilon_j\Bigg),$$

and hence all the weights of the spin module have multiplicity 1.

Proposition 2.12. *Up to scalars, there is a unique $\mathfrak{so}_{2l+1}$-invariant bilinear map $S \times S \to \mathfrak{so}_{2l+1}$, $(s, t) \mapsto [s, t]$. This map is given by the formula*

$$\tfrac{1}{2}\,\mathrm{tr}\big(\sigma[s, t]\big) = b\big(\rho(\sigma)(s), t\big), \tag{2.13}$$

for any $\sigma \in \mathfrak{so}_{2l+1}$ and $s, t \in S$, where tr *denotes the trace of the natural representation of $\mathfrak{so}_{2l+1}$.*

Moreover, this bilinear map $[\,\cdot\,,\cdot\,]$ is symmetric if and only if l is congruent to 1 *or* 2 *modulo* 4. *Otherwise, it is skew-symmetric.*

Proof. First note that the trace form tr is $\mathfrak{so}_{2l+1}$-invariant, and so is b because Ψ and Λ in (2.3) and (2.9) are isomorphisms of algebras with involutions. Since both tr and b are nondegenerate, $[\,\cdot\,,\cdot\,]$ is well defined and $\mathfrak{so}_{2l+1}$-invariant. Now, the space of $\mathfrak{so}_{2l+1}$-invariant bilinear maps $S \times S \to \mathfrak{so}_{2l+1}$ is isomorphic to $\mathrm{Hom}_{\mathfrak{so}_{2l+1}}(S \otimes S, \mathfrak{so}_{2l+1})$ (all the tensor products are considered over the ground field k), or to the space of those tensors in $S \otimes S \otimes (\mathfrak{so}_{2l+1})^* \simeq S \otimes S \otimes \mathfrak{so}_{2l+1} \simeq \mathfrak{so}_{2l+1} \otimes S \otimes S^*$ (tr and b are nondegenerate) annihilated by $\mathfrak{so}_{2l+1}$, and hence to $\mathrm{Hom}_{\mathfrak{so}_{2l+1}}(\mathfrak{so}_{2l+1} \otimes S, S)$.

But $\mathfrak{so}_{2l+1} \otimes S$ is generated, as a module for $\mathfrak{so}_{2l+1}$, by the tensor product of any nonzero element (like $[v_1, v_2]^\cdot$) in the root space $(\mathfrak{so}_{2l+1})_{\epsilon_1+\epsilon_2}$ ($(\mathfrak{so}_3)_{\epsilon_1}$ if $l = 1$), and any nonzero element (like 1) in the weight space $S_{-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_l)}$. (Note that $\epsilon_1 + \epsilon_2$ is the longest root in the lexicographic order given by $\epsilon_1 > \cdots > \epsilon_l > 0$, while $-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_l)$ is the lowest weight in S.) The image of this basic tensor under any homomorphism of $\mathfrak{so}_{2l+1}$-modules lies in the weight space of weight $\frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \cdots - \epsilon_l)$, which is one-dimensional. Hence, $\dim_k \operatorname{Hom}_{\mathfrak{so}_{2l+1}}(\mathfrak{so}_{2l+1} \otimes S, S) = 1$, as required.

The last part of the Proposition follows from (2.8). $\square$

For future use, note that for any $w_1, w_2, w_3, w_4 \in W$,

$$\operatorname{tr}\big(\sigma_{w_1,w_2}\sigma_{w_3,w_4}\big) = 2\big(q(w_1, w_4)q(w_2, w_3) - q(w_1, w_3)q(w_2, w_4)\big),$$

and hence, under the identification $\mathfrak{so}_{2l+1} \simeq [W, W]^\cdot$ ($\sigma_{w_1,w_2} \mapsto -\frac{1}{2}[w_1, w_2]^\cdot$),

$$\frac{1}{2}\operatorname{tr}\big([w_1, w_2]^\cdot[w_3, w_4]^\cdot\big) = 4\big(q(w_1, w_4)q(w_2, w_3) - q(w_1, w_3)q(w_2, w_4)\big). \tag{2.14}$$

To deal with the Lie algebras $\mathfrak{so}_{2l}$ (type D), $l \geq 2$, consider the involution of $\mathfrak{Cl}(V \oplus V^*, q)$, which will be denoted by τ too, which is the identity on $V \oplus V^*$. Also consider the involution $\hat{}: \bigwedge V \to \bigwedge V$ such that $\hat{v} = v$ for any $v \in V \subseteq \bigwedge V$, and the nondegenerate bilinear form

$$\begin{aligned} \hat{b} : \textstyle\bigwedge V \times \bigwedge V &\to k \\ (s, t) \quad &\mapsto \Phi(\hat{s}t), \end{aligned} \tag{2.15}$$

where Φ is as in (2.6). Here, with the same arguments as for (2.8),

$$\begin{aligned} &\hat{b} \text{ is symmetric if and only if } l \equiv 0 \text{ or } 1 \pmod 4, \\ &\hat{b} \text{ is skew-symmetric if and only if } l \equiv 2 \text{ or } 3 \pmod 4. \end{aligned} \tag{2.16}$$

Moreover, if l is even, then $\hat{b}\big(\bigwedge_{\bar{0}} V, \bigwedge_{\bar{1}} V\big) = 0$, so the restrictions of $\hat{b}$ to $S^+ = \bigwedge_{\bar{0}} V$ and $S^- = \bigwedge_{\bar{1}} V$ are nondegenerate. However, if l is odd, then both S^+ and S^- are isotropic subspaces relative to $\hat{b}$.

The nondegenerate bilinear form $\hat{b}$ induces the adjoint involution $\tau_{\hat{b}}$ on $\bigwedge V$ and, as before, the isomorphism Λ in (2.5) becomes an isomorphism of algebras with involution:

$$\Lambda : \big(\mathfrak{Cl}(V \oplus V^*, q), \tau\big) \to \big(\operatorname{End}_k(\textstyle\bigwedge V), \tau_{\hat{b}}\big). \tag{2.17}$$

Under this isomorphism, the even Clifford algebra $\mathfrak{Cl}_{\bar{0}}(V \oplus V^*, q)$ maps onto $\operatorname{End}_k(\bigwedge_{\bar{0}} V) \oplus \operatorname{End}_k(\bigwedge_{\bar{1}} V)$.

Also, as before, $\mathfrak{so}_{2l} = \mathfrak{so}(V \oplus V^*, q)$ can be identified with the subspace $[V \oplus V^*, V \oplus V^*]^\cdot$ of $\mathfrak{Cl}_{\bar{0}}(V \oplus V^*, q)$, $\mathfrak{h} = \operatorname{span}\{[v_i, f_i]^\cdot : i = 1, \ldots, l\}$ is a Cartan subalgebra, the roots are $\{\pm\epsilon_i \pm \epsilon_j : 1 \le i < j \le l\}$, the set of weights of the natural module $V \oplus V^*$ are $\{\pm\epsilon_i : 1 \le i \le l\}$, all the weights appear with multiplicity one, and the composition

$$\mathfrak{so}_{2l} \hookrightarrow \mathfrak{Cl}_{\bar{0}}(V \oplus V^*, q) \xrightarrow{\Lambda} \operatorname{End}_k(\textstyle\bigwedge_{\bar{0}} V) \oplus \operatorname{End}_k(\textstyle\bigwedge_{\bar{1}} V)$$

gives two representations

$$\rho^+ : \mathfrak{so}_{2l} \to \operatorname{End}_k(\textstyle\bigwedge_{\bar{0}} V) \quad \text{and} \quad \rho^- : \mathfrak{so}_{2l} \to \operatorname{End}_k(\textstyle\bigwedge_{\bar{1}} V), \tag{2.18}$$

called the half-spin representations. The weights in $S^+ = \bigwedge_{\bar{0}} V$ (respectively $S^- = \bigwedge_{\bar{1}} V$) are the weights $\frac{1}{2}(\pm\epsilon_1 \pm \cdots \pm \epsilon_l)$, with an even (respectively odd) number of $+$ signs.

Proposition 2.19. (i) *If l is odd, $l \ge 3$, there is no nonzero $\mathfrak{so}_{2l}$-invariant bilinear map $S^+ \times S^+ \to \mathfrak{so}_{2l}$.*

(ii) *If l is even, there is a unique, up to scalars, such bilinear map, which is given by the formula*

$$\tfrac{1}{2}\operatorname{tr}\bigl(\sigma[s,t]\bigr) = \hat{b}\bigl(\rho^+(\sigma)(s), t\bigr), \tag{2.20}$$

for any $\sigma \in \mathfrak{so}_{2l}$ and $s, t \in S^+$. Moreover, this bilinear map $[\cdot,\cdot]$ is symmetric if and only if l is congruent to 2 or 3 modulo 4, and it is skew-symmetric otherwise.

Proof. If l is odd ($l \ge 3$), then $S^+ \otimes S^+$ is generated, as a module for $\mathfrak{so}_{2l}$, by $v_1 \cdots v_{l-1} \otimes v_{l-1} v_l$ (the tensor product or a nonzero highest weight vector and a nonzero lowest weight vector), and its image under any nonzero $\mathfrak{so}_{2l}$-invariant linear map $S^+ \otimes S^+ \to \mathfrak{so}_{2l}$ lies in the root space of root $\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{l-1} - \epsilon_l) + \frac{1}{2}(-\epsilon_1 - \cdots - \epsilon_{l-2} + \epsilon_{l-1} + \epsilon_l) = \epsilon_{l-1}$. But ϵ_{l-1} is not a root, so its image must vanish.

For l even $\hat{b}$ is nondegenerate on S^+ and, as in Proposition 2.12, it is enough to compute $\dim_k \operatorname{Hom}_{\mathfrak{so}_{2l}}(\mathfrak{so}_{2l} \otimes S^+, S^+)$, which is proven to be 1 with the same arguments given there. $\square$

Remark 2.21. In $\mathfrak{Cl}_{\bar{1}}(V \oplus V^*, q)$ there are invertible elements a such that $a^{\cdot 2} \in k1$ and $a \cdot (V \oplus V^*) \cdot a^{-1} \subseteq V \oplus V^*$. For instance, one can take the element $a = [v_1, f_1]^{\cdot} \cdots [v_{l-1}, f_{l-1}]^{\cdot} \cdot (v_l + f_l)$, which satisfies $a^{\cdot 2} = (-1)^{l-1}$. (Note that $[v_i, f_i]^{\cdot 2} = v_i \cdot f_i \cdot v_i \cdot f_i + f_i \cdot v_i \cdot f_i \cdot v_i = v_i \cdot (1 - v_i \cdot f_i) \cdot f_i + f_i \cdot (1 - f_i \cdot v_i) \cdot v_i = v_i \cdot f_i + f_i \cdot v_i = f_i(v_i) = 1$ and $(v_i + f_i) \cdot (v_i + f_i) = v_i \cdot f_i + f_i \cdot v_i = 1$.) Consider the linear isomorphism $\phi_a : S^- = \bigwedge_{\bar{1}} V \to S^+ = \bigwedge_{\bar{0}} V$, $s \mapsto \rho(a)(s)$, and the order two automorphism $\operatorname{Ad}_a : \mathfrak{Cl}(V \oplus V^*, q) \to \mathfrak{Cl}(V \oplus V^*, q)$, $x \mapsto a \cdot x \cdot a^{-1}$. Ad_a preserves $V \oplus V^*$, and hence also $\mathfrak{so}_{2l} \simeq [V \oplus V^*, V \oplus V^*]^{\cdot}$. Then, for any $\mathfrak{so}_{2l}$-invariant bilinear map $[\cdot,\cdot] : S^+ \times S^+ \to \mathfrak{so}_{2l}$, one gets a $\mathfrak{so}_{2l}$-invariant bilinear map $[\,\cdot\,,\cdot\,]^- : S^- \times S^- \to \mathfrak{so}_{2l}$ at once by

$$[s,t]^- = \mathrm{Ad}_a\bigl([\phi_a(s), \phi_a(t)]\bigr).$$

Therefore, it is enough to deal with the half spin representation S^+.

3. Type *B*

Let $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ be either a simple $\mathbb{Z}_2$-graded Lie algebra or a simple Lie superalgebra with $\mathfrak{g}_{\bar 0} = \mathfrak{so}_{2l+1}$ and $\mathfrak{g}_{\bar 1} = S$ (its spin module).

Because of Proposition 2.12, the product of two odd elements can be assumed to be given by the bilinear map $[s,t]$ in (2.13). Therefore, the possibilities for such a $\mathfrak{g}$ are given precisely by the values of l such that the product $[s,t]$ satisfies the Jacobi identity

$$J(s_1,s_2,s_3) = \rho([s_1,s_2])(s_3) + \rho([s_2,s_3])(s_1) + \rho([s_3,s_1])(s_2) = 0,$$

for any $s_1, s_2, s_3 \in S$. As in Section 2, ρ denotes the spin representation of $\mathfrak{so}_{2l+1}$.

But $S\otimes S\otimes S$ is generated, as a module for $\mathfrak{so}_{2l+1}$, by the elements $1\otimes v_1\cdots v_l \otimes v_{i_1}\cdots v_{i_r}$, where $0 \le r \le l$ and $1 \le i_1 < \cdots < i_r \le l$. As in Section 2, $\{v_1,\ldots,v_l\}$ denotes a fixed basis of V and $\{f_1,\ldots,f_l\}$ the corresponding dual basis in V^*. The trilinear map $S\times S\times S \to S$, $(s_1,s_2,s_3) \mapsto J(s_1,s_2,s_3)$ is $\mathfrak{so}_{2l+1}$-invariant, so it is enough to check for which values of l the Jacobian

$$J(1, v_1\cdots v_l, v_{i_1}\cdots v_{i_r})$$

is 0 for any $0 \le r \le l$ and $1 \le i_1 < \cdots < i_r \le l$. By symmetry, it is enough to check the Jacobians

$$J(1,, v_1\cdots v_l, v_1\cdots v_r)$$

for $0 \le r \le l$.

Theorem 3.1. *Let $l \in \mathbb{N}$ and let $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ be the $\mathbb{Z}_2$-graded algebra with $\mathfrak{g}_{\bar 0} = \mathfrak{so}_{2l+1}$, $\mathfrak{g}_{\bar 1} = S$ (its spin module), and multiplication given by the Lie bracket of elements in $\mathfrak{so}_{2l+1}$, and by*

$$[\sigma, s] = -[s,\sigma] = \rho(\sigma)(s), \qquad \rho \text{ as in (2.11)},$$
$$[s,t] \quad \text{given by (2.13)}.$$

for any $\sigma \in \mathfrak{g}_{\bar 0}$ and $s,t \in \mathfrak{g}_{\bar 1}$. Then:

(i) $\mathfrak{g}$ *is a Lie algebra if and only if either*
- $l = 3$ *and the characteristic of k is* 3, *and then $\mathfrak{g}$ is isomorphic to the* 29*-dimensional simple Lie algebra discovered by Brown* [1982], *or*
- $l = 4$, *and then $\mathfrak{g}$ is isomorphic to the simple Lie algebra of type F_4.*

(ii) $\mathfrak{g}$ *is a Lie superalgebra if and only if either*

- $l = 1$, *and then* $\mathfrak{g}$ *is isomorphic to the orthosymplectic Lie superalgebra* $\mathfrak{osp}(1, 2)$, *or*
- $l = 2$, *and then* $\mathfrak{g}$ *is isomorphic to the orthosymplectic Lie superalgebra* $\mathfrak{osp}(1, 4)$, *or*
- $l = 5$ *and the characteristic of* k *is* 5, *or*
- $l = 6$ *and the characteristic of* k *is* 3.

Proof. With the same notations as in Section 2, note that $v_1 \cdots v_r$ is a weight vector relative to $\mathfrak{h}$, of weight $\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_r - \epsilon_{r+1} - \cdots - \epsilon_l)$ for any $0 \le r \le l$. Hence $[1, v_1 \cdots v_r] \in (\mathfrak{so}_{2l+1})_{-(\epsilon_{r+1}+\cdots+\epsilon_l)}$. In the same vein, $[v_1 \cdots v_l, v_1 \cdots v_r] \in (\mathfrak{so}_{2l+1})_{\epsilon_1+\cdots\epsilon_r}$. In particular,

$$[1, v_1 \cdots v_r] = 0 \text{ if } 0 \le r \le l-3; \quad [v_1 \cdots v_l, v_1 \cdots v_r] = 0 \text{ if } 3 \le r \le l. \tag{3.2}$$

Also, $[1, v_1 \cdots v_l] \in \mathfrak{h}$, so that $[1, v_1 \cdots v_l] = \sum_{i=1}^{l} \alpha_i [v_i, f_i]^{\cdot}$ for some $\alpha_1, \ldots, \alpha_l \in k$. By (2.13)

$$\frac{1}{2}\sum_{i=1}^{l} \alpha_i \operatorname{tr}\bigl(\sigma[v_i, f_i]^{\cdot}\bigr) = b\bigl(\rho(\sigma)(1), v_1 \cdots v_l\bigr). \tag{3.3}$$

Let $\sigma = [v_j, f_j]^{\cdot}$, then by (2.14)

$$\frac{1}{2}\sum_{i=1}^{l} \alpha_i \operatorname{tr}\bigl([v_j, f_j]^{\cdot}[v_i, f_i]^{\cdot}\bigr) = 4 f_i(v_j) f_j(v_i) = 4\delta_{ij},$$

while $\rho\bigl([v_j, f_j]^{\cdot}\bigr)(1) = [l_{v_j}, df_j](1) = -(df_j)(v_j) = -1$. Thus, (3.3) gives $4\alpha_j = -1$ for any $j = 1, \ldots, l$, so

$$[1, v_1 \cdots v_l] = -\frac{1}{4}\sum_{i=1}^{l} [v_i, f_i]^{\cdot}. \tag{3.4}$$

In the same vein, $[1, v_1 \cdots v_{l-1}] \in (\mathfrak{so}_{2l+1})_{-\epsilon_l}$, so $[1, v_1 \cdots v_{l-1}] = \alpha[u, f_l]^{\cdot}$ for some $\alpha \in k$, and by (2.13)

$$\tfrac{1}{2}\alpha \operatorname{tr}\bigl([u, v_l]^{\cdot}[u, f_l]^{\cdot}\bigr) = b\bigl(\rho([u, v_l]^{\cdot}(1), v_1 \cdots v_{l-1}\bigr). \tag{3.5}$$

The left-hand side is $4\alpha\bigl(-q(u, u)q(v_l, f_l)\bigr) = 8\alpha$, while on the right-hand side $\rho\bigl([u, v_l]^{\cdot}\bigr) = 2\rho\bigl(\Psi(v_l)\bigr) = 2\Lambda(v_l)$, so this side becomes

$$\begin{aligned} 2b(v_l, v_1 \cdots v_{l-1}) &= 2\Phi(\bar{v}_l v_1 \cdots v_{l-1}) = -2\Phi(v_l v_1 \cdots v_{l-1}) \\ &= 2(-1)^l \Phi(v_1 \cdots v_l) = 2(-1)^l. \end{aligned}$$

Therefore $\alpha = \frac{1}{4}(-1)^l$ and

$$[1, v_1 \cdots v_{l-1}] = \tfrac{1}{4}(-1)^l [u, f_l]^{\cdot}. \tag{3.6}$$

Similar arguments, which are left to the reader, give

$$[1, v_1 \cdots v_{l-2}] = \tfrac{1}{2}[f_{l-1}, f_l]^{\cdot}, \text{ if } l \geq 2, \tag{3.7}$$

$$[v_1 \cdots v_l, v_1] = -\tfrac{1}{4}(-1)^{\binom{l+1}{2}}[u, v_1]^{\cdot}, \tag{3.8}$$

$$[v_1 \cdots v_l, v_1 v_2] = -\tfrac{1}{2}(-1)^{\binom{l+1}{2}}[v_1, v_2]^{\cdot}. \tag{3.9}$$

Now, if $l \geq 7$ and $3 \leq r \leq l-3$,

$$\begin{aligned} J(1, v_1 \cdots v_l, v_1 \cdots v_r) &= [[1, v_1 \cdots v_l], v_1 \cdots v_r] && \text{(by (3.2))} \\ &= -\tfrac{1}{4} \sum_{i=1}^{l} [[v_i, f_i]^{\cdot}, v_1 \cdots v_r] && \text{(by (3.4))} \\ &= -\tfrac{1}{4} \sum_{i=1}^{l} \rho\big([v_i, f_i]^{\cdot}\big)(v_1 \cdots v_r) \\ &= -\tfrac{1}{4}(r-(l-r))v_1 \cdots v_r = \tfrac{1}{4}(l-2r)v_1 \cdots v_r. \end{aligned}$$

With $r = \frac{1}{2}(l-2)$ if l is even or $\frac{1}{2}(l-1)$ if l is odd, $l-2r$ $(= 1$ or $2) \neq 0$, so the Jacobi identity is not satisfied.

Assume now that $l = 6$, so $[s, t]$ is symmetric in $s, t \in S$ by Proposition 2.12. Then

$$\begin{aligned} J(1, v_1 \cdots v_6, 1) &= 2[[1, v_1 \cdots v_6], 1] && \text{(because } [1,1]=0\text{; see (3.2))} \\ &= -\tfrac{1}{2} \sum_{i=1}^{6} [[v_i, f_i]^{\cdot}, 1] && \text{(by (3.4))} \\ &= -\tfrac{1}{2} \sum_{i=1}^{6} (-1) = 3, \end{aligned}$$

so the characteristic of k must be 3. Assuming this is so, it is easily checked that $J(1, v_1 \cdots v_6, v_1 \cdots v_r) = 0$ for any $0 \leq r \leq 6$.

For $l = 5$, the product $[s, t]$ is also symmetric (Proposition 2.12) and, as before,

$$J(1, v_1 \cdots v_5, 1) = 2[[1, v_1 \cdots v_5], 1] = -\tfrac{1}{2} \sum_{i=1}^{5} (-1) = \tfrac{5}{2},$$

so the characteristic of k must be 5, and then $J(1, v_1 \cdots v_5, v_1 \cdots v_r) = 0$ for any $0 \leq r \leq 5$.

For $l = 4$, the product $[s, t]$ is skew-symmetric, by Proposition 2.12. Therefore $J(1, v_1 \cdots v_4, 1) = J(1, v_1 \cdots v_4, v_1 \cdots v_4) = 0$ by skew-symmetry. The other instances of $J(1, v_1 \cdots v_4, v_1 \cdots v_r)$, $1 \leq r \leq 3$, are also checked to be trivial.

With $l=3$, the product $[s,t]$ is skew-symmetric too. Hence, by (3.4), (3.6), and (3.7),

$$\begin{aligned} J(1, v_1v_2v_3, v_1) &= [[1, v_1v_2v_3], v_1] + [[v_1v_2v_3, v_1], 1] + [[v_1, 1], v_1v_2v_3] \\ &= -\tfrac{1}{4}\sum_{i=1}^{3}[[v_i, f_i]^{\cdot}, v_1] - \tfrac{1}{4}[[u, v_1]^{\cdot}, 1] - \tfrac{1}{2}[[f_2, f_3]^{\cdot}, v_1v_2v_3] \\ &= -\tfrac{1}{4}(1-1-1)v_1 - \tfrac{1}{4}(2v_1) - \tfrac{1}{2}(-1-1)v_1 = \tfrac{3}{4}v_1, \end{aligned}$$

and hence the characteristic must be 3. The other instance of the Jacobi identity to be checked: $J(1, v_1v_2v_3, v_1v_2) = 0$, also holds easily.

For $l=1$ or $l=2$, the Jacobi identity is satisfied too.

The assertions about which Lie algebras or superalgebras appear follows at once, since all the algebras and superalgebras mentioned in the statement of the Theorem satisfy the hypotheses. (For $\mathfrak{osp}(1,4)$, the even part is isomorphic to the symplectic Lie algebra $\mathfrak{sp}_4$, and the odd part is its natural 4-dimensional module. However $\mathfrak{sp}_4$ is isomorphic to $\mathfrak{so}_5$, and viewed like this, the 4-dimensional module is the spin module. The same happens for $\mathfrak{osp}(1,2)$.) $\square$

Remark 3.10. Up to our knowledge, the modular Lie superalgebras that occur for $l=5$ and $l=6$ have not appeared previously in the literature. Note that the simplicity of $\mathfrak{so}_{2l+1}$ and the irreducibility of its spin module imply that these superalgebras are simple.

4. Type *D*

In this section the situation in which $\mathfrak{g}_{\bar 0} = \mathfrak{so}_{2l}$ ($l \geq 2$), and $\mathfrak{g}_{\bar 1} = S^+$ (half-spin module) will be considered. First note that it does not matter which half-spin representation is used (Remark 2.21). By Proposition 2.19, it is enough to deal with even values of l.

Theorem 4.1. *Let l be an even positive integer, and let $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ be the $\mathbb{Z}_2$-graded algebra with $\mathfrak{g}_{\bar 0} = \mathfrak{so}_{2l}$, $\mathfrak{g}_{\bar 1} = S^+$, and multiplication given by the Lie bracket of elements in $\mathfrak{so}_{2l}$, and by*

$$[\sigma, s] = -[s,\sigma] = \rho^+(\sigma)(s), \quad \rho^+ \text{ as in (2.18)},$$
$$[s,t] \quad \text{given by (2.20)}.$$

for any $\sigma \in \mathfrak{g}_{\bar 0}$ and $s,t \in \mathfrak{g}_{\bar 1}$. Then:

(i) *$\mathfrak{g}$ is a Lie algebra if and only if either*
- *$l=8$ and then $\mathfrak{g}$ is isomorphic to the simple Lie algebra of type E_8, or*
- *$l=4$, and then $\mathfrak{g}$ is isomorphic to the simple Lie algebra $\mathfrak{so}_9$ (of type B_4).*

(ii) *$\mathfrak{g}$ is a Lie superalgebra if and only if either*

- $l = 6$, *and the characteristic of* k *is* 3, *and then* $\mathfrak{g}$ *is isomorphic to the Lie superalgebra in [Elduque 2006, Theorem 3.2(v)], or*
- $l = 2$, *and then* $\mathfrak{g}$ *is isomorphic to the direct sum* $\mathfrak{osp}(1, 2) \oplus \mathfrak{sl}_2$, *of the orthosymplectic Lie superalgebra* $\mathfrak{osp}(1, 2)$ *and the three-dimensional simple Lie algebra.*

Proof. The restriction to $S^+ = \bigwedge_{\bar{0}} V$ of the bilinear form $\hat{b}$ in (2.15) coincides with the restriction of the bilinear form b in (2.7). Hence, as in the proof of Theorem 3.1, the equations (3.2), (3.4), (3.7), and (3.9) are all valid here.

If $l \geq 10$ and $4 \leq r \leq l-4$, r even,

$$J(1, v_1 \cdots v_l, v_1 \cdots v_r) = [[1, v_1 \cdots v_l], v_1 \cdots v_r] = -\tfrac{1}{4} \sum_{i=1}^{l} [[v_i, f_i]^{\cdot}, v_1 \cdots v_r]$$
$$= -\tfrac{1}{4}(r - (l-r)) v_1 \cdots v_r = \tfrac{1}{4}(l - 2r) v_1 \cdots v_r,$$

so, with $r = \frac{1}{2}(l-2)$ if l is congruent to 2 modulo 4, or $r = \frac{1}{4}(l-4)$ otherwise, $l - 2r$ equals 2 or 4, and the Jacobi identity is not satisfied.

For $l = 8$, $[s, t]$ is skew-symmetric and it is enough to check that the Jacobian $J(1, v_1 \cdots v_8, v_1 \cdots v_r)$ is 0 for $r = 2$, 4 or 6, which is straightforward.

For $l = 6$, $[s, t]$ is symmetric and

$$J(1, v_1 \cdots v_6, 1) = 2[[1, v_1 \cdots v_6], 1] = -\tfrac{1}{2} \sum_{i=1}^{6} [[v_i, f_i]^{\cdot}, 1] = -\tfrac{1}{2} \sum_{i=1}^{6} (-1) = 3,$$

so the characteristic of k must be 3 and then all the other instances of the Jacobi identity hold.

For $l = 4$, $[s, t]$ is skew-symmetric, and thus it is enough to deal with

$$\begin{aligned} &J(1, v_1 v_2 v_3 v_4, v_1 v_2) \\ &\qquad = [[1, v_1 v_2 v_3 v_4], v_1 v_2] + [[v_1 v_2 v_3 v_4, v_1 v_2], 1] + [[v_1 v_2, 1], v_1 v_2 v_3 v_4] \\ &\qquad = -\tfrac{1}{4} \sum_{i=1}^{4} [[v_i, f_i]^{\cdot}, v_1 v_2] - \tfrac{1}{2}[[v_1, v_2]^{\cdot}, 1] - \tfrac{1}{2}[[f_1, f_2]^{\cdot}, v_1 v_2 v_3 v_4] \\ &\qquad = -\tfrac{1}{4}(1 + 1 - 1 - 1) v_1 v_2 - v_1 v_2 + v_1 v_2 = 0. \end{aligned}$$

It is well-known that $\mathfrak{g} = \mathfrak{so}_9$ is $\mathbb{Z}_2$-graded with $\mathfrak{g}_{\bar{0}} = \mathfrak{so}_8$ and $\mathfrak{g}_{\bar{1}}$ the natural module for $\mathfrak{so}_8$. But here, the triality automorphism permutes the natural and the two half-spin modules, so one can replace the natural module by any of its half-spin modules. Therefore, the Lie algebra that appears is isomorphic to $\mathfrak{so}_9$.

Finally, for $l = 2$, $[s, t]$ is symmetric and the Jacobi identity is easily checked to hold. Since $\mathfrak{so}_4$ is isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and the two half-spin representations are the two natural (two-dimensional) modules for each of the two copies of $\mathfrak{sl}_2$. It follows then that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $(\mathfrak{g}_1)_{\bar{0}} \simeq \mathfrak{sl}_2$ and $(\mathfrak{g}_1)_{\bar{1}}$ the natural module for $\mathfrak{sl}_2$ (and hence $\mathfrak{g}_1 \simeq \mathfrak{osp}(1,2)$), while $\mathfrak{g}_2 = (\mathfrak{g}_2)_{\bar{0}} = \mathfrak{sl}_2$. Alternatively, the subspaces span $\{[v_1, v_2]^{\cdot}, [f_1, f_2]^{\cdot}, [v_1, f_1]^{\cdot} + [v_2, f_2]^{\cdot}, 1, v_1 v_2\}$ and span $\{[v_1, f_2]^{\cdot}, [v_2, f_1]^{\cdot} [v_1, f_1]^{\cdot} - [v_2, f_2]^{\cdot}\}$ are ideals of $\mathfrak{g}$, the first one being isomorphic to $\mathfrak{osp}(1,2)$, and the second one to $\mathfrak{sl}_2$. $\square$

5. The Kac Jordan superalgebra and the Tits construction

The aim of this section is to show that the Lie superalgebra in Theorem 3.1 for $l = 5$ (and characteristic 5) is related to a well-known construction by Tits, applied to the Cayley algebra and the ten-dimensional Kac Jordan superalgebra over k.

This last superalgebra is easily described in terms of the smaller Kaplansky superalgebra. The tiny Kaplansky superalgebra is the three-dimensional Jordan superalgebra $K = K_{\bar{0}} \oplus K_{\bar{1}}$, with $K_{\bar{0}} = ke$ and $K_{\bar{1}} = U$, a two-dimensional vector space endowed with a nonzero alternating bilinear form $(\cdot | \cdot)$, and multiplication given by

$$e^2 = e, \quad ex = xe = \tfrac{1}{2}x, \quad xy = (x|y)e,$$

for any $x, y \in U$. The bilinear form $(\cdot | \cdot)$ can be extended to a supersymmetric bilinear form by means of $(e|e) = \frac{1}{2}$ and $(K_{\bar{0}}|K_{\bar{1}}) = 0$.

For any homogeneous $u, v \in K$, we know from [Benkart and Elduque 2002, (1.61)] that

$$[L_u, L_v] = L_u L_v - (-1)^{\bar{u}\bar{v}} L_v L_u = \tfrac{1}{2}\big(u(v|\cdot) - (-1)^{\bar{u}\bar{v}} v(u|\cdot)\big), \tag{5.1}$$

where L_x denotes the left multiplication by x, $\bar{x}$ being the degree of the homogeneous element x. Moreover, the Lie superalgebra of derivations of K is (see [Benkart and Elduque 2002])

$$\mathfrak{Der}\, K = [L_K, L_K] = \mathfrak{osp}(K) \ \big(\simeq \mathfrak{osp}(1,2)\big).$$

The Kac Jordan superalgebra is

$$\mathcal{J} = k1 \oplus (K \otimes K),$$

with unit element 1 and product determined by

$$(a \otimes b)(c \otimes d) = (-1)^{\bar{b}\bar{c}}\big(ac \otimes bd - \tfrac{3}{4}(a|c)(b|d)1\big), \tag{5.2}$$

for homogeneous elements $a, b, c, d \in K$ (see [Benkart and Elduque 2002, (2.1)]). By Proposition 2.7 and Theorem 2.8 of the same reference, the superspace spanned by the associators $(x, y, z) = (xy)z - x(yz) = (-1)^{\bar{y}\bar{z}}[L_x, L_z](y)$ is $(\mathcal{J}, \mathcal{J}, \mathcal{J}) = K \otimes K$, and the Lie superalgebra of derivations of $\mathcal{J}$ is $\mathfrak{Der}\, \mathcal{J} = [L_{\mathcal{J}}, L_{\mathcal{J}}]$, which acts trivially on 1 and leaves invariant $(\mathcal{J}, \mathcal{J}, \mathcal{J}) = K \otimes K$. Considered then as subspaces of $\mathrm{End}_k(K \otimes K)$

$$\mathfrak{Der}\, J = (\mathfrak{Der}\, K \otimes \mathrm{id}) \oplus (\mathrm{id} \otimes \mathfrak{Der}\, K) \ \big(\simeq \mathfrak{osp}(1,2) \oplus \mathfrak{osp}(1,2)\big). \tag{5.3}$$

More precisely (see [Benkart and Elduque 2002, (2.4)]), for any homogeneous $a, b, c, d \in K$,

$$[L_{a\otimes b}, L_{c\otimes d}] = (-1)^{\bar{b}\bar{c}}\big([L_a, L_c] \otimes (b|d)\,\mathrm{id} + (a|c)\,\mathrm{id} \otimes [L_b, L_d]\big), \tag{5.4}$$

as endomorphisms of $K \otimes K$. (Note that, with the usual conventions for superalgebras, $\mathrm{id} \otimes \varphi$ acts on $a \otimes b$ as $(-1)^{\bar{\varphi}\bar{a}} a \otimes \varphi(b)$ for homogeneous φ and a, b.) In particular,

$$(\mathfrak{Der}\, \mathcal{J})_{\bar{0}} = \big((\mathfrak{Der}\, K)_{\bar{0}} \otimes \mathrm{id}\big) \oplus \big(\mathrm{id} \otimes (\mathfrak{Der}\, K)_{\bar{0}})\big),$$

and $(\mathfrak{Der}\, K)_{\bar{0}}$ is isomorphic to $\mathfrak{sp}(U) = \mathfrak{sp}_2 = \mathfrak{sl}_2$ (acting trivially on the idempotent e). The restriction of $(\mathfrak{Der}\, \mathcal{J})_{\bar{0}}$ to the subspace $K_{\bar{1}} \otimes K_{\bar{1}} = U \otimes U$ of $K \otimes K$ then gives an isomorphism

$$(\mathfrak{Der}\, \mathcal{J})_{\bar{0}} \cong \mathfrak{so}(U \otimes U) \ \big(= (\mathfrak{sp}(U) \otimes \mathrm{id}) \oplus (\mathrm{id} \otimes \mathfrak{sp}(U))\big),$$

where $U \otimes U$ is endowed with the symmetric bilinear form given by

$$(u_1 \otimes u_2 | v_1 \otimes v_2) = (u_1|v_1)(u_2|v_2),$$

for any $u_1, u_2, v_1, v_2 \in U$.

Assume now that the characteristic of k is $\neq 2, 3$. Let (C, n) be a unital composition algebra over k with norm n. That is, n is a regular quadratic form satisfying $n(ab) = n(a)n(b)$ for any $a, b \in C$. (See [Schafer 1966, Chapter III] for basic facts about these algebras.)

Since the field k is assumed to be algebraically closed, C is isomorphic to either k, $k \times k$, $\mathrm{Mat}_2(k)$ or the Cayley algebra $\mathscr{C}$ over k.

The map $D_{a,b} : C \to C$ given by

$$D_{a,b}(c) = [[a, b], c] - 3(a, b, c) \tag{5.5}$$

is the inner derivation determined by $a, b \in C$, and the Lie algebra $\mathfrak{Der}\, C$ is spanned by these derivations. The subspace $C^0 = \{a \in C : n(1, a) = 0\}$ orthogonal to 1 is invariant under $\mathfrak{Der}\, C$.

For later use, let us state some properties of the inner derivations of Cayley algebras. For any a, let $\mathrm{ad}_a = L_a - R_a$ (L_a and R_a denote, respectively, the left and right multiplication by the element a), and consider $\mathrm{ad}_C = \{\mathrm{ad}_a : a \in C\} = \{\mathrm{ad}_a : a \in C^0\}$.

Lemma 5.6. *Let $\mathscr{C}$ be the Cayley algebra over k* ($\mathrm{char}\, k \neq 2, 3$). *Then*:

(i) $\mathscr{C}^0$ *is invariant under* $\mathfrak{Der}\, C$ *and* $\mathrm{ad}_{\mathscr{C}}$, *both of which annihilate* $k1$. *Moreover, as subspaces of* $\mathrm{End}_k(\mathscr{C}^0)$, $\mathfrak{so}(\mathscr{C}^0, n) = \mathfrak{Der}\,\mathscr{C} \oplus \mathrm{ad}_{\mathscr{C}}$.

(ii) $[\mathrm{ad}_a, \mathrm{ad}_b] = 2D_{a,b} - \mathrm{ad}_{[a,b]}$ *for any* $a, b \in \mathscr{C}$.

(iii) $D_{a,b} + \frac{1}{2}\,\mathrm{ad}_{[a,b]} = 3\big(n(a,\cdot)b - n(b,\cdot)a\big)$ *for any* $a, b \in \mathscr{C}^0$.

Proof. First, in [Schafer 1966, Chapter III, §8] it is proved that $\mathfrak{Der}\, C$ leaves invariant $\mathscr{C}^0$ and, as a subspace of $\mathrm{End}_k(\mathscr{C}^0)$, it is contained in the orthogonal Lie algebra $\mathfrak{so}(\mathscr{C}^0, n)$. The same happens for $\mathrm{ad}_{\mathscr{C}} = \mathrm{ad}_{\mathscr{C}^0}$, and $\mathfrak{Der}\,\mathscr{C} \cap \mathrm{ad}_{\mathscr{C}} = 0$. Hence, by dimension count, $\mathfrak{so}(\mathscr{C}^0, n) = \mathfrak{Der}\,\mathscr{C} \oplus \mathrm{ad}_{\mathscr{C}}$, which gives (i).

Now, $\mathscr{C}$ is an alternative algebra. That is, the associator $(a, b, c) = (ab)c - a(bc)$ is alternating on its arguments. Hence, for any $a, b, c \in \mathscr{C}$,

$$\big(L_{ab} - L_a L_b\big)(c) = (a, b, c) = -(a, c, b) = [L_a, R_b](c).$$

Interchange a and b and subtract to get $L_{[a,b]} - [L_a, L_b] = 2[L_a, R_b]$. Similarly,

$$\begin{aligned} &R_{[a,b]} + [R_a, R_b] = -2[L_a, R_b],\\ &\mathrm{ad}_{[a,b]} = [L_a, L_b] + [R_a, R_b] + 4[L_a, R_b]. \end{aligned}$$

Hence

$$\begin{aligned} D_{a,b} &= \mathrm{ad}_{[a,b]} - 3(a, b, \cdot) = \mathrm{ad}_{[a,b]} + 3(a, \cdot, b)\\ &= \mathrm{ad}_{[a,b]} - 3[L_a, R_b] = [L_a, L_b] + [R_a, R_b] + [L_a, R_b] \end{aligned} \tag{5.7}$$

and $[\mathrm{ad}_a, \mathrm{ad}_b] = [L_a - R_a, L_b - R_b] = [L_a, L_b] + [R_a, R_b] - 2[L_a, R_b] = D_{a,b} - 3[L_a, R_b] = 2D_{a,b} - \mathrm{ad}_{[a,b]}$, which yields (ii).

Now, for any $a \in \mathscr{C}$, we have (see [Schafer 1966, Chapter III, §4])

$$a^2 - n(1, a)a + n(a)1 = 0,$$

so for any $a \in \mathscr{C}^0$, $a^2 = -n(a)1$ and hence

$$ab + ba = -n(a, b)1, \tag{5.8}$$

for any $a, b \in \mathscr{C}^0$. Therefore, for any $a, b, c \in \mathscr{C}^0$,

$$\begin{aligned} &2\big(n(a, c)b - n(b, c)a\big)\\ &\qquad = -(ac + ca)b - b(ac + ca) + (bc + cb)a - a(bc + cb)\\ &\qquad = -(a, c, b) + (b, c, a) - (ca)b + (cb)a - b(ac) + a(bc)\\ &\qquad = \big(2[L_a, R_b] + [R_a, R_b] + [L_a, L_b]\big)(c)\\ &\qquad = \big(D_{a,b} + [L_a, R_b]\big)(c) = \big(\tfrac{2}{3} D_{a,b} + \tfrac{1}{3}\,\mathrm{ad}_{[a,b]}\big)(c), \end{aligned}$$

because of (5.7), which gives (iii). $\square$

Let $J = J_{\bar 0} \oplus J_{\bar 1}$ be now a unital Jordan superalgebra with a normalized trace $t : J \to k$. That is, t is a linear map such that $t(1) = 1$, and $t(J_{\bar 1}) = 0 = t\big((J, J, J)\big)$ (see [Benkart and Elduque 2003, §1]). Then $J = k1 \oplus J^0$, where $J^0 = \{x \in J : t(x) = 0\}$, which contains $J_{\bar 1}$. For $x, y \in J^0$, $xy = t(xy)1 + x * y$, where $x * y = xy - t(xy)1$ is a supercommutative multiplication on J^0. Since $(J, J, J) = [L_J, L_J](J)$ is contained in J^0, the subspace J^0 is invariant under $\mathfrak{inder}\, J = [L_J, L_J]$ (the Lie superalgebra of inner derivations).

Given a unital composition algebra C and a unital Jordan superalgebra with a normalized trace J, consider the superspace

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C^0 \otimes J^0) \oplus \mathfrak{inder}\, J,$$

with the superanticommutative product $[\,\cdot\,,\cdot\,]$ specified by the following conditions (see [Benkart and Elduque 2003]):

- $\mathfrak{der}\, C$ is a Lie subalgebra and $\mathfrak{inder}\, J$ a Lie subsuperalgebra of $\mathcal{T}(C, J)$,
- $[\mathfrak{der}\, C, \mathfrak{inder}\, J] = 0$,
- $[D, a \otimes x] = D(a) \otimes x$, $[d, a \otimes x] = a \otimes d(x)$,
- $[a \otimes x, b \otimes y] = t(xy)D_{a,b} + [a, b] \otimes x * y - 2n(a, b)[L_x, L_y]$,

for all $D \in \mathfrak{der}\, C$, $d \in \mathfrak{inder}\, J$, $a, b \in C^0$ and $x, y \in J^0$.

If the Grassmann envelope $G(J)$ satisfies the Cayley–Hamilton equation

$$\mathrm{ch}_3(x) = 0$$

for 3×3-matrices, where

$$\mathrm{ch}_3(x) = x^3 - 3t(x)x^2 + \big(\tfrac{9}{2}t(x)^2 - \tfrac{3}{2}t(x^2)\big)x - \big(t(x^3) - \tfrac{9}{2}t(x^2)t(x) + \tfrac{9}{2}t(x)^3\big)1,$$

then $\mathcal{T}(C, J)$ is a Lie superalgebra; see [Benkart and Elduque 2003, Sections 3 and 4].

This construction, for algebras, was considered by Tits [1966], who used it to give a unified construction of the exceptional simple Lie algebras. In the above terms, it was considered in [Benkart and Zelmanov 1996; Benkart and Elduque 2003].

The Kac Jordan superalgebra $\mathcal{J}$ is endowed with a unique normalized trace, given necessarily by $t(1) = 1$ and $t(K \otimes K) = 0$. Note that if $f = f^2$ is an idempotent linearly independent to 1 in a unital Jordan superalgebra with a normalized trace t, and if the Grassmann envelope satisfies the Cayley–Hamilton equation $\mathrm{ch}_3(x) = 0$, in particular $\mathrm{ch}_3(f) = 0$ so, by linear independence, $t(f) - \frac{9}{2}t(f)^2 + \frac{9}{2}t(f)^3 = 0$, and $t(f)$ is either 0, $\frac{1}{3}$ or $\frac{2}{3}$. But, if $t(f)$ were 0, $0 = \mathrm{ch}_3(f)$ would be equal to f^3, hence to f, a contradiction. Hence $t(f) = \frac{1}{3}$ or $\frac{2}{3}$. In the Kac superalgebra $\mathcal{J}$, the element $f = -\frac{1}{2} + 2e \otimes e$ is an idempotent with $t(f) = -\frac{1}{2}$. Hence the Grassmann envelope of $\mathcal{J}$ cannot satisfy the Cayley–Hamilton equation of degree 3 unless, $-\frac{1}{2} = \frac{1}{3}$ or $-\frac{1}{2} = \frac{2}{3}$, that is, unless the characteristic of k be 5 or 7. Actually, McCrimmon [2005] has shown that the Grassmann envelope $G(\mathcal{J})$ satisfies this Cayley–Hamilton equation if and only if the characteristic is 5. In retrospect, this explains the appearance of the nine-dimensional pseudocomposition superalgebras over fields of characteristic 5 (and only over these fields) in [Elduque and Okubo 2000, Example 9, Theorem 14 and concluding notes].

Assume from now on that the characteristic of the ground field k is 5.

Then, if C is a unital composition algebra, then $\mathcal{T}(C, \mathcal{J})$ is always a Lie superalgebra. Obviously $\mathcal{T}(k, \mathcal{J}) = \mathfrak{inder}\,\mathcal{J} = \mathfrak{der}\,\mathcal{J}$, which is isomorphic to

$$\mathfrak{osp}(1, 2) \oplus \mathfrak{osp}(1, 2)$$

(see (5.3)), and $\mathcal{T}(k \times k, \mathcal{J})$ is naturally isomorphic to $L_{\mathcal{J}^0} \oplus \mathfrak{der}\,\mathcal{J}$ which, in turn, is isomorphic to $\mathfrak{osp}(K \oplus K) \cong \mathfrak{osp}(2, 4)$ (see [Benkart and Elduque 2002, Theorem 2.13]). Also, it is well-known that $\mathcal{T}(\mathrm{Mat}_2(k), \mathcal{J})$ is isomorphic to the Tits–Kantor–Koecher Lie superalgebra of $\mathcal{J}$, which is isomorphic to the exceptional Lie superalgebra of type $F(4)$. (This was used by Kac [1977a] in his classification of the complex finite-dimensional simple Jordan superalgebras.)

Our final result shows that the Lie superalgebra $\mathcal{T}(\mathcal{C}, \mathcal{J})$ is, up to isomorphism, the simple Lie superalgebra in Theorem 3.1 for $l = 5$.

Theorem 5.9. *Let $\mathcal{C}$ be the Cayley algebra and let $\mathcal{J}$ be the Kac Jordan superalgebra over an algebraically closed field k of characteristic* 5. *Then*:

(i) $\mathcal{T}(\mathcal{C}, \mathcal{J})_{\bar{0}}$ *is isomorphic to the orthogonal Lie algebra* $\mathfrak{so}_{11}$.

(ii) $\mathcal{T}(\mathcal{C}, \mathcal{J})_{\bar{1}}$ *is isomorphic to the spin module for* $\mathcal{T}(\mathcal{C}, \mathcal{J})_{\bar{0}}$.

Proof. For (i) consider the vector space

$$M = \mathcal{C}^0 \oplus (U \otimes U)$$

(recall that $U = K_{\bar{1}}$), endowed with the symmetric bilinear form Q such that

$$Q(\mathcal{C}^0, U \otimes U) = 0,$$
$$Q(x) = -n(x),$$
$$Q(u_1 \otimes u_2, v_1 \otimes v_2) = -(u_1|v_1)(u_2|v_2),$$

for $x \in \mathcal{C}^0$ and $u_1, u_2, v_1, v_2 \in U$. It will be shown that $\mathcal{T}(\mathcal{C}, \mathcal{J})_{\bar{0}}$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(M, Q)$.

This last orthogonal Lie algebra is spanned by the maps

$$\sigma^Q_{x,y} = Q(x, \cdot)y - Q(y, \cdot)x$$

for $x, y \in M$, and for any $\sigma \in \mathfrak{so}(M, Q)$,

$$[\sigma, \sigma^Q_{x,y}] = \sigma^Q_{\sigma(x),y} + \sigma^Q_{x,\sigma(y)}.$$

Moreover, since $\mathscr{C}^0$ and $U \otimes U$ are orthogonal relative to Q,

$$\mathfrak{so}(M, Q) = \Big(\mathfrak{so}\big(\mathscr{C}^0, Q|_{\mathscr{C}^0}\big) \oplus \mathfrak{so}\big(U \otimes U, Q|_{U\otimes U}\big)\Big) \oplus \sigma^Q_{\mathscr{C}^0, U\otimes U} \tag{5.10}$$

(which gives a $\mathbb{Z}_2$-grading of $\mathfrak{so}(M, Q)$), where $\mathfrak{so}(\mathscr{C}^0, Q|_{\mathscr{C}^0})$ and $\mathfrak{so}(U\otimes U, Q|_{U\otimes U})$ are embedded in $\mathfrak{so}(M, Q)$ in a natural way, and $\mathfrak{so}(M, Q)$ is generated, as a Lie algebra, by $\sigma^Q_{\mathscr{C}^0, U\otimes U}$. Besides, for any $a, b \in \mathscr{C}^0$, and $u_1, u_2, v_1, v_2 \in U$,

$$[\sigma^Q_{a,u_1\otimes u_2}, \sigma^Q_{b,v_1\otimes v_2}] = Q(u_1 \otimes u_2, v_1 \otimes v_2)\sigma^Q_{a,b} + Q(a, b)\sigma^Q_{u_1\otimes u_2, v_1\otimes v_2}.$$

Also, by Lemma 5.6(iii), for any $a, b \in \mathscr{C}^0$,

$$\sigma^Q_{a,b} = -2D_{a,b} - \mathrm{ad}_{[a,b]}\,. \tag{5.11}$$

Now, the multiplication in $\mathcal{T}(\mathscr{C}, \mathscr{J})$ gives, for any $a, b \in \mathscr{C}^0$, $u, u_1, u_2, v, v_1, v_2 \in U$, $D \in \mathfrak{der}\,\mathscr{C}$ and $d \in (\mathfrak{der}\,\mathscr{J})_{\bar{0}}$:

$$[D, a \otimes (u \otimes v)] = D(a) \otimes (u \otimes v), \tag{5.12a}$$

$$[a \otimes (e \otimes e), b \otimes (u \otimes v)] = \tfrac{1}{4}[a, b] \otimes (u \otimes v) = -\,\mathrm{ad}_a(b) \otimes (u \otimes v), \tag{5.12b}$$

$$[d, a \otimes (u \otimes v)] = a \otimes d(u \otimes v), \tag{5.12c}$$

$$[D, d] = 0 = [d, a \otimes (e \otimes e)], \tag{5.12d}$$

$$[L_{u_1\otimes u_2}, L_{v_1\otimes v_2}]|_{U\otimes U} = \tfrac{1}{2}\sigma^Q_{u_1\otimes u_2, v_1\otimes v_2}|_{U\otimes U}, \tag{5.12e}$$

since

$$\begin{aligned}(u_1 \otimes u_2)\big((v_1 \otimes v_2)(w_1 \otimes w_2)\big) &= Q(v_1 \otimes v_2, w_1 \otimes w_2)(u_1 \otimes u_2)(e \otimes e - \tfrac{3}{4}1)\\ &= -\tfrac{1}{2}Q(v_1 \otimes v_2, w_1 \otimes w_2)(u_1 \otimes u_2),\end{aligned}$$

for any $u_1, u_2, v_1, v_2, w_1, w_2 \in U$.

Moreover, for any $a, b \in \mathscr{C}^0$ and $u_1, u_2, v_1, v_2 \in U$,

$$\begin{aligned}&[a\otimes(u_1 \otimes u_2), b \otimes (v_1 \otimes v_2)]\\ &\quad= t\big(u_1 \otimes u_2)(v_1 \otimes v_2)\big)D_{a,b} + [a, b] \otimes \big((u_1 \otimes u_2) * (v_1 \otimes v_2)\big)\\ &\qquad\qquad - 2n(a, b)[L_{u_1\otimes u_2}, L_{v_1\otimes v_2}]\\ &\quad= -2Q(u_1 \otimes u_2, v_1 \otimes v_2)D_{a,b} + [a, b] \otimes \big(Q(u_1 \otimes u_2, v_1 \otimes v_2)(e \otimes e)\big)\\ &\qquad\qquad - n(a, b)\big(2[L_{u_1\otimes u_2}, L_{v_1\otimes v_2}]\big)\\ &\quad= Q(u_1 \otimes u_2, v_1 \otimes v_2)\big(-2D_{a,b} + [a, b] \otimes (e \otimes e)\\ &\qquad\qquad + Q(a, b)\big(2[L_{u_1\otimes u_2}, L_{v_1\otimes v_2}]\big).\end{aligned}$$

Now, the equations in Lemma 5.6 and equations (5.11) and (5.12) prove that the linear map

$$\Phi_{\bar 0}:\mathcal{T}(\mathcal{C},\mathcal{J})_{\bar 0}=\mathfrak{der}\,C\oplus\bigl(\mathcal{C}^0\otimes(k(e\otimes e)\oplus U\otimes U)\bigr)\oplus(\mathfrak{der}\,\mathcal{J})_{\bar 0}\to\mathfrak{so}(M,Q),$$

such that

- $\Phi_{\bar 0}(D)=D$ for any $D\in\mathfrak{der}\,\mathcal{C}$ $\bigl(\subseteq\mathfrak{so}(\mathcal{C}^0,n)=\mathfrak{so}(\mathcal{C}^0,-n)\subseteq\mathfrak{so}(M,Q)\bigr)$, for any $D\in\mathfrak{der}\,\mathcal{C}$,
- $\Phi_{\bar 0}(d)=d|_{U\otimes U}$ $\bigl(\in\mathfrak{so}(U\otimes U,Q)\subseteq\mathfrak{so}(M,Q)\bigr)$, for any $d\in(\mathfrak{der}\,\mathcal{J})_{\bar 0}$,
- $\Phi_{\bar 0}\bigl(a\otimes(e\otimes e)\bigr)=-\operatorname{ad}_a$ $\bigl(\in\mathfrak{so}(\mathcal{C}^0,-n)\subseteq\mathfrak{so}(M,Q)\bigr)$, for any $a\in\mathcal{C}^0$,
- $\Phi_{\bar 0}\bigl(a\otimes(u\otimes v)\bigr)=\sigma^Q_{a,u\otimes v}$, for any $a\in\mathcal{C}^0$ and $u,v\in U$,

is an isomorphism of Lie algebras. This proves the first part of the Theorem.

Now, let us consider the linear map

$$\begin{aligned}\Psi: M&\longrightarrow \operatorname{End}_k\bigl(\mathcal{C}\otimes(U\oplus U)\bigr)\\ a\in\mathcal{C}^0&\mapsto\quad L_a\otimes\begin{pmatrix}-\mathrm{id}&0\\0&\mathrm{id}\end{pmatrix},\\ u_1\otimes u_2&\mapsto \mathrm{id}\otimes\begin{pmatrix}0&(u_2|\cdot)u_1\\(u_1|\cdot)u_2&0\end{pmatrix}.\end{aligned}$$

(The elements in $U\oplus U$ are written as $\binom{u_1}{u_2}$, and then $\operatorname{End}_k(U\oplus U)$ is identified with $\mathrm{Mat}_2\bigl(\operatorname{End}_k(U)\bigr)$.) For any $a\in\mathcal{C}^0$ and $u_1,u_2,v_1,v_2\in U$,

$$\Psi(a)\Psi(u_1\otimes u_2)+\Psi(u_1\otimes u_2)\Psi(a)=0,$$

$$\Psi(a)^2=-n(a)\,\mathrm{id}=Q(a)\,\mathrm{id}\quad(\text{as } a(ab)=a^2b=-n(a)b \text{ since } \mathcal{C} \text{ is alternative}),$$

$$\begin{aligned}&\Psi(u_1\otimes u_2)\Psi(v_1\otimes v_2)+\Psi(v_1\otimes v_2)\Psi(u_1\otimes u_2)\\ &\quad=\mathrm{id}\otimes\begin{pmatrix}(u_2|v_2)(v_1|\cdot)u_1+(v_2|u_2)(u_1|\cdot)v_1&0\\0&(u_1|v_1)(v_2|\cdot)u_2+(v_1|u_1)(u_2|\cdot)v_2\end{pmatrix}\\ &\quad=\mathrm{id}\otimes\begin{pmatrix}-(u_1|v_1)(u_2|v_2)\,\mathrm{id}&0\\0&-(u_1|v_1)(u_2|v_2)\,\mathrm{id}\end{pmatrix}\\ &\quad=Q(u_1\otimes u_2,v_1\otimes v_2)\,\mathrm{id},\end{aligned}$$

since $(u_2|v_2)\bigl((v_1|w_1)u_1+(w_1|u_1)v_1\bigr)=-(u_2|v_2)(u_1|v_1)w_1$, because $(u_1|v_1)w_1+(v_1|w_1)u_1+(w_1|u_1)v_1=0$, as this is an alternating trilinear map on a two-dimensional vector space.

Therefore, Ψ induces an algebra homomorphism

$$\mathfrak{Cl}(M, Q) \to \operatorname{End}_k\big(\mathscr{C} \otimes (U \oplus U)\big),$$

whose restriction $\Psi : \mathfrak{Cl}_{\bar 0}(M, Q) \to \operatorname{End}_k\big(\mathscr{C} \otimes (U \oplus U)\big)$ is an isomorphism, by dimension count. Therefore, $\mathscr{C} \otimes (U \oplus U)$ is the spin module for $\mathfrak{so}(M, Q)$. Recall that $\mathfrak{so}(M, Q)$ embeds in $\mathfrak{Cl}_{\bar 0}(M.Q)$ by means of $\sigma^Q_{x,y} \mapsto -\frac{1}{2}[x, y]^{\cdot}$. Since $\mathfrak{so}(M, Q)$ is generated by the elements $\sigma^Q_{a,u_1\otimes u_2}$ ($a \in \mathscr{C}^0$, $u_1, u_2 \in U$), the spin representation is determined by

$$\begin{aligned}\rho\big(\sigma^Q_{a,u_1\otimes u_2}\big) &= -\tfrac{1}{2}\Psi\big([a, u_1 \otimes u_2]^{\cdot}\big) = -\tfrac{1}{2}[\Psi(a), \Psi(u_1 \otimes u_2)]\\ &= -\Psi(a)\Psi(u_1 \otimes u_2) = L_a \otimes \begin{pmatrix} 0 & (u_2|\cdot)u_1 \\ -(u_1|\cdot)u_2 & 0 \end{pmatrix}.\end{aligned}$$

Now identify $\mathscr{T}(\mathscr{C}, \mathscr{J})_{\bar 0}$ with $\mathfrak{so}(M, Q)$ through $\Phi_{\bar 0}$, and identify $\mathscr{T}(\mathscr{C}, \mathscr{J})_{\bar 1} = \mathscr{C}^0 \otimes \big((U \otimes e) \oplus (e \otimes U)\big) \oplus (\mathfrak{Der}\,\mathscr{J})_{\bar 1}$ with $\mathscr{C} \otimes (U \oplus U)$ by means of

$$\begin{aligned}\Phi_{\bar 1} : \mathscr{T}(\mathscr{C}, \mathscr{J})_{\bar 1} &\longrightarrow \mathscr{C} \otimes (U \oplus U)\\ a \otimes (u_1 \otimes e + e \otimes u_2) &\mapsto \quad a \otimes \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\\ [L_e, L_{u_1}] \otimes \mathrm{id} + \mathrm{id} \otimes [L_e, L_{u_2}] &\mapsto -\tfrac{1}{2}\Big(1 \otimes \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\Big),\end{aligned}$$

for $a \in \mathscr{C}^0$ and $u_1, u_2 \in U$.

In $\mathscr{T}(\mathscr{C}, \mathscr{J})$, for any $a, b \in \mathscr{C}^0$, $u_1, u_2, v_1, v_2 \in U$, using (5.4) we get

$$\begin{aligned}&[a \otimes (u_1 \otimes u_2), b \otimes (v_1 \otimes e + e \otimes v_2)]\\ &\quad= [a, b] \otimes \tfrac{1}{2}\big(u_1 \otimes (u_2|v_2)e - (u_1|v_1)e \otimes u_2\big)\\ &\qquad - 2n(a, b)[L_{u_1\otimes u_2}, L_{v_1\otimes e+e\otimes v_2}]\\ &\quad= \tfrac{1}{2}[a, b] \otimes \big((u_2|v_2)u_1 \otimes e - e \otimes (u_1|v_1)u_2\big)\\ &\qquad - 2n(a, b)\big(-\tfrac{1}{2}(u_1|v_1)\,\mathrm{id} \otimes [L_{u_2}, L_e] + \tfrac{1}{2}[L_{u_1}, L_e] \otimes (u_2|v_2)\,\mathrm{id}\big)\\ &\quad= \tfrac{1}{2}[a, b] \otimes \big((u_2|v_2)u_1 \otimes e - e \otimes (u_1|v_1)u_2\big)\\ &\qquad - \tfrac{1}{2}n(a, b)\big(-2\big([L_e, L_{(u_2|v_2)u_1}] \otimes \mathrm{id} - \mathrm{id} \otimes [L_e, L_{(u_1|v_1)v_2}]\big)\big).\end{aligned}$$

That is,

$$\Big[a \otimes (u_1 \otimes u_2), \Phi_{\bar 1}^{-1}\Big(b \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big)\Big] = \Phi_{\bar 1}^{-1}\Big(ab \otimes \begin{pmatrix} 0 & (u_2|\cdot)u_1 \\ -(u_1|\cdot)u_2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big),$$

or

$$\Big[\Phi_{\bar 0}^{-1}\big(\sigma^Q_{a,u_1\otimes u_2}\big), \Phi_{\bar 1}^{-1}\Big(b \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big)\Big] = \Phi_{\bar 1}^{-1}\Big(\rho\big(\sigma^Q_{a,u_1\otimes u_2}\big)\Big(b \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big)\Big),$$

because $ab = -\frac{1}{2}n(a,b) + \frac{1}{2}[a,b]$ for any $a, b \in \mathscr{C}^0$ by (5.8). But also,

$$\begin{aligned}\big[a \otimes (u_1 \otimes u_2), [L_e, L_{v_1}] \otimes \mathrm{id} + \mathrm{id} \otimes [L_e, L_{v_2}]\big] \\ &= a \otimes \big(-[L_e, L_{v_1}](u_1) \otimes u_2 + u_1 \otimes [L_e, L_{v_2}](u_2)\big) \\ &= a \otimes \big(\tfrac{1}{2}(u_1|v_1)e \otimes u_2 - \tfrac{1}{2}u_1 \otimes (u_2|v_2)e\big),\end{aligned}$$

or

$$\left[a \otimes (u_1 \otimes u_2), \Phi_{\bar 1}^{-1}\left(1 \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)\right] = \Phi_{\bar 1}^{-1}\left(a \otimes \begin{pmatrix} 0 & (u_2|\cdot)u_1 \\ -(u_1|\cdot)u_2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right),$$

that is,

$$\left[\Phi_{\bar 0}^{-1}\big(\sigma^Q_{a,u_1\otimes u_2}\big), \Phi_{\bar 1}^{-1}\left(1 \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)\right] = \Phi_{\bar 1}^{-1}\left(\rho\big(\sigma^Q_{a,u_1\otimes u_2}\big)\left(1 \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)\right),$$

and this shows that, if $\mathscr{T}(\mathscr{C}, \mathscr{J})_{\bar 0}$ is identified with $\mathfrak{so}(M, Q)$ by means of $\Phi_{\bar 0}$ and $\mathscr{T}(\mathscr{C}, \mathscr{J})_{\bar 1}$ with $\mathscr{C} \otimes (U \oplus U)$ by means of $\Phi_{\bar 1}$, the action of $\mathscr{T}(\mathscr{C}, \mathscr{J})_{\bar 0}$ on $\mathscr{T}(\mathscr{C}, \mathscr{J})_{\bar 1}$ is given, precisely, by the spin representation. □

The Lie superalgebra in Theorem 3.1 for $l = 6$ and characteristic 3, appears in the extended Freudenthal magic square in this characteristic [Cunha and Elduque 2006], as the Lie superalgebra $\mathfrak{g}\big(\overline{B(4,2)}, \overline{B(4,2)}\big)$, associated to two copies of the unique six-dimensional symmetric composition superalgebra. This is related to the six-dimensional simple alternative superalgebra $B(4,2)$ [Shestakov 1997], and hence to the exceptional Jordan superalgebra of 3×3 hermitian matrices over $B(4,2)$, which is exclusive of characteristic 3.

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ALBERTO ELDUQUE
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE ZARAGOZA
50009 ZARAGOZA
SPAIN

elduque@unizar.es
http://www.unizar.es/matematicas/algebra/elduque

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