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JULIANA ERLIJMAN AND HANS WENZL

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We extend subfactor constructions originally defined for unitary braid representations to the setting of braided C^*-tensor categories. The categorical approach is then used to compute the principal graph of these subfactors. We also determine the dual principal graph for several important cases. Here invertibility of the so-called S-matrix of a subcategory and certain related group actions play an important role.

It was noted by Vaughan Jones that his examples of subfactors gave rise to unitary braid representations. By this we mean representations of the infinite braid group $\mathcal{B}_\infty$ defined by infinitely many generators $\sigma_1, \sigma_2, \ldots$ which satisfy the familiar braid relations. Subsequently, unitary braid representations were used by A. Ocneanu and by H. Wenzl to construct new examples of subfactors; here the subfactor is given by the subgroup $\mathcal{B}_{2,\infty}$ generated by $\sigma_2, \sigma_3, \ldots$. This construction was denoted as the one-sided subfactor construction by J. Erlijman, as opposed to her multisided subfactors. Here, for a given integer $s > 1$, the s-sided subfactor is obtained as a suitable inductive limit of the embeddings of the quotients of $\mathcal{B}_n^s = \mathcal{B}_n \times \cdots \times \mathcal{B}_n$ (s times) into $\mathcal{B}_{ns}$ for $n \to \infty$. She also computed the indices of these subfactors and their first relative commutants.

The main motivation for this paper was to calculate the higher relative commutants of Erlijman's subfactors. To do this it is convenient to generalize the above mentioned constructions to the setting of a braided C^*-tensor category $\mathcal{C}$ with only finitely many simple objects up to isomorphism. By definition of such a category, we obtain a unitary representation of $\mathcal{B}_n$ in $\mathrm{End}(X^{\otimes n})$ for any object X in $\mathcal{C}$. The constructions in our paper in the category setting follow closely the above-mentioned braid constructions. They reduce to them in case that $\mathrm{End}(X^{\otimes n})$ is generated by the quotients of $\mathcal{B}_n$ for all $n \in \mathbb{N}$, where X is a generating object of $\mathcal{C}$. However, the categorical setting makes it easier to calculate the higher relative commutants, and also contains new nontrivial examples.

The main results of our paper are as follows. We show that the first principal graph is given by the fusion graph of $(\mathcal{C}')^s$, where $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ depending on the tensor powers of X in which the trivial object appears. The fusion graph

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describes the decomposition of the tensor product of s simple objects of $\mathscr{C}'$ into irreducible ones; see Theorem 4.6 for details. The situation is more complicated for the dual (or second) principal graph. If a certain matrix depending on the braiding structure, called the S-matrix for the category $\mathscr{C}'$, is invertible, the dual principal graph coincides with the principal graph.

We do not have a general complete result in the case of a noninvertible S-matrix. It is known that in this case there is a canonical subcategory $\mathscr{T}$ of $\mathscr{C}'$ which is equivalent to the representation category of a finite group G. If G is abelian, we obtain an action of G on the set of irreducible objects of $\mathscr{C}$, which is given by a labeling set Λ. The dual principal graph can now be fairly precisely characterized in terms of the orbits of the action of a group G_1^s on Λ^s; see Theorem 5.9 for details and, for an example, Proposition 6.1.

The basic idea of our paper is that we explicitly construct a number of $\mathscr{A}$–$\mathscr{B}$ bimodules, with $\{\mathscr{A}, \mathscr{B}\} \subset \{\mathscr{N}, \mathscr{M}\}$ and with $\mathscr{N} \subset \mathscr{M}$ being our s-sided inclusion. We show that these examples of bimodules are closed under induction and restriction. One deduces from this that the induction-restriction graph for these bimodules must coincide with the principal or dual principal graph under some mild additional assumptions.

Our findings are related to a number of results by different authors. If $s = 2$, our subfactors correspond to the subfactors obtained from the asymptotic inclusion of certain one-sided subfactors. In this case, the orbifold phenomenon for the dual principal graph has first been observed by Ocneanu for the example of the Jones subfactors. Further results have been obtained in [Evans and Kawahigashi 1998] and [Izumi 2000]. In particular, some of our proofs have been inspired by these results. More recently, after hearing a talk on this paper, M. Asaeda [2006] obtained an analogue of the s-sided construction under more general conditions.

More or less the same combinatorics as in our paper also appears in the work of Feng Xu [2000] on subfactors of type III_1 factors related to disconnected intervals. In spite of the similarity of principal graphs and indices, his construction of these subfactors is completely different from ours and relies on Wassermann's loop group construction, which has not appeared yet in print for all Lie types.

Here is a more detailed description of the contents of this paper. In the first chapter we review some basic results on bimodules in the type II_1 setting. The second chapter contains definitions concerning braided C^* tensor categories. In the third chapter we present the generalization of previous subfactor constructions to the setting of braided C^* tensor categories, as well as additional technical results. This is used in the following section to construct certain bimodules and compute the principal graph of these subfactors. In the last section we prove the already mentioned results about the dual principal graph. We then discuss examples of our construction including the case of the Jones subfactors.

1. Bimodules

1A. *Definitions.*

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be type II_1 factors, and let H be a Hilbert space.

(i) H is a *left $\mathcal{A}$-module* if there exists an action of $\mathcal{A}$ on H determined by a normal unital morphism $\lambda : \mathcal{A} \to B(H)$, where $B(H)$ is the von Neumann algebra of all bounded linear operators on H.

(ii) A *right $\mathcal{B}$-module* H is a left $\mathcal{B}^{\mathrm{opp}}$-module (here, $\mathcal{B}^{\mathrm{opp}}$ denotes the opposite algebra of $\mathcal{B}$).

(iii) H is an *$\mathcal{A}$–$\mathcal{B}$ bimodule* if it is a left $\mathcal{A}$-module, a right $\mathcal{B}$-module, and if the left and right actions intertwine. That is, if $\lambda : \mathcal{A} \to B(H)$ is the left action, and if $\rho : \mathcal{B}^{\mathrm{opp}} \to B(H)$ is the right action, then we must have that $\lambda(a)\rho(b) = \rho(b)\lambda(a)$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

(iv) If H and K are $\mathcal{A}$–$\mathcal{B}$ bimodules, we define the space of *intertwiners*, denoted by $\mathrm{Hom}_{\mathcal{A},\mathcal{B}}(H, K)$, to be the set of linear bounded operators $T : H \to K$ such that they intertwine the actions, in the sense that $T\lambda_H(a) = \lambda_K(a)T$ for all $a \in \mathcal{A}$ and $T\rho_H(b) = \rho_K(b)T$ for all $b \in \mathcal{B}$.

(v) Two $\mathcal{A}$–$\mathcal{B}$ bimodules H and K are *equivalent* or *isomorphic* if there exists a unitary operator in $\mathrm{Hom}_{\mathcal{A},\mathcal{B}}(H, K)$.

Definition 1.2. Let H be an $\mathcal{A}$–$\mathcal{B}$ bimodule with left action λ and right action ρ. The *inclusion generated by H* is the inclusion of factors given by

$$\lambda(\mathcal{A}) \subset \rho(\mathcal{B})'.$$

The *dual inclusion generated by H* is the inclusion of factors given by

$$\rho(\mathcal{B}) \subset \lambda(\mathcal{A})'.$$

Remark 1.3. Similarly, if we have an inclusion of type II_1-factors $\mathcal{N} \subset \mathcal{M}$, we can make $L^2(\mathcal{M}, \mathrm{tr})$ into an $\mathcal{M}$–$\mathcal{M}$, $\mathcal{M}$–$\mathcal{N}$, $\mathcal{N}$–$\mathcal{M}$ or $\mathcal{N}$–$\mathcal{N}$-bimodule via usual left and right multiplication. If $\mathcal{N} \subset \mathcal{M}$ is a reducible inclusion, i.e., the relative commutant $\mathcal{N}' \cap \mathcal{M}$ is larger than $\mathbb{C}1$, then we obtain further examples by reducing by projections in the relative commutant. For example, if $p \in \mathcal{N}' \cap \mathcal{M}$, we obtain the $\mathcal{N}$–$\mathcal{M}$ bimodule $L^2(p\mathcal{M}, \mathrm{tr})$.

If $\phi_i : \mathcal{M} \to \mathcal{M}$ are endomorphisms for $i = 1, 2$, we can also define an $\mathcal{M}$–$\mathcal{M}$-bimodule structure on $L^2(\mathcal{M}, \mathrm{tr})$ by perturbing the right and left actions by these endomorphisms, that is, by defining the action by $m_1.\xi.m_2 = \phi_1(m_1)\xi\phi_2(m_2)$.

All the examples of bimodules encountered in this paper are of one of these types or tensor products or direct summands of them.

Definition 1.4. Let $\mathscr{A}$ and $\mathscr{B}_i$ be type II_1 factors for $i = 1, 2$. Let H_i be $\mathscr{A}$–$\mathscr{B}_i$ bimodules with left actions λ_i and right actions ρ_i, respectively, for $i = 1, 2$, and assume that $\dim_{\mathscr{A}}(H_2) \leq \dim_{\mathscr{A}}(H_1) < \infty$. Then we say that H_2 is *(left)-weakly reduced* or a *(left)-weak reduction* of H_2 if there exists a nonzero projection $p \in \mathscr{B}_1$ and an isomorphism $\Psi : \mathscr{B}_2 \cong p\mathscr{B}_1 p$ such that $H_1 p := \rho_1(b)H_1$ and H_2 are isomorphic $\mathscr{A} - \mathscr{B}_2$-bimodules; here the $\mathscr{A} - \mathscr{B}_2$-bimodule structure on $\mathscr{H}_1 p$ is defined by $a.\xi.b = \lambda_1(a)\xi\rho_1(\Psi(b))$ for $a \in A$, $b \in B_2$ and $\xi \in H_1 p$.

Remark 1.5. (1) Since right multiplication by p commutes with the left action of $\mathscr{A}$ and also with the commutant of the right action of $\mathscr{B}_1$, we obtain isomorphic inclusions $\lambda_1(\mathscr{A}) \subset \rho_1(\mathscr{B}_1)'$ and $\lambda_1(\mathscr{A})p \subset \rho_1(\mathscr{B}_1)'p$. It follows from this and the fact that isomorphic bimodules define isomorphic inclusions that a left weak reduction of a bimodule yields an isomorphic inclusion.

(2) If we perturb the right-action on an $\mathscr{A}$–$\mathscr{B}$ bimodule H by an outer automorphism α of $\mathscr{B}$, the resulting bimodule H_α is not isomorphic to H. However, it is a left weak reduction of H.

(3) One can similarly define a notion of (right)-weak reduction. We shall mostly be concerned with (left)-weak reduction, and will usually just call it weak reduction. Also, we shall often suppress the notation λ and ρ if it is clear from which side the algebras act.

1B. *Tensor products.* Tensor products of bimodules have been defined by Connes and Sauvageot. A good review with results for our paper can be found in [Bisch 1997].

Proposition 1.6. *Let H_i be $\mathscr{A}$–$\mathscr{B}_i$ bimodules for $i = 1, 2$, and let $\mathscr{D}$ be a type II_1 factor. If H_2 is weakly reduced from H_1, then also $L \otimes_{\mathscr{A}} H_2$ is weakly reduced from $L \otimes_{\mathscr{A}} H_1$, for any $\mathscr{D}$–$\mathscr{A}$ bimodule L.*

Proof. By definition, since H_2 is weakly reduced from H_2, there must exist a projection $p \in \mathscr{B}_1$ such that $H_1 p$ and H_2 are isomorphic as $\mathscr{A}$–$\mathscr{B}_2$ bimodules, assuming $\dim_{\mathscr{A}}(H_1) \geq \dim_{\mathscr{A}}(H_2)$. This isomorphism extends in an obvious way to a spatial isomorphism between $L \otimes_{\mathscr{A}} H_1 p = (L \otimes_{\mathscr{A}} H_1)(1 \otimes p)$ and $L \otimes_{\mathscr{A}} H_2$. □

1C. *Higher relative commutants.* Let $\mathscr{N} \subset \mathscr{M}$ be type II_1 factors with normalized trace tr. There exists a canonical extension $\mathscr{M}_1 \supset \mathscr{M}$, called Jones' basic construction for $\mathscr{N} \subset \mathscr{M}$, which is the von Neumann algebra generated by $\mathscr{M}$ acting via left multiplication on $L^2(\mathscr{M}, \mathrm{tr})$ and by the orthogonal projection $e_{\mathscr{N}}$ onto the subspace $L^2(\mathscr{N}, \mathrm{tr}) \subset L^2(\mathscr{M}, \mathrm{tr})$. It is well-known that the Jones index $[\mathscr{M} : \mathscr{N}]$ is finite if and only if $\mathscr{M}_1$ is again a type II_1 factor; it is given by $[\mathscr{M} : \mathscr{N}] = 1/\mathrm{tr}(e_{\mathscr{N}})$, with tr denoting the unique normalized trace on $\mathscr{M}_1$. In this case, we can apply the basic construction again for $\mathscr{M} \subset \mathscr{M}_1$ to obtain an extension $\mathscr{M}_2 \supset \mathscr{M}_1$. Iterating this construction, we obtain a sequence of II_1 factors $\mathcal{N} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$. We obtain important invariants of the original inclusion $\mathcal{N} \subset \mathcal{M}$ via the so-called higher relative commutants $\mathcal{N}' \cap \mathcal{M}_k$ and $\mathcal{M}' \cap \mathcal{M}_k$. These are finite-dimensional C*-algebras. If there exists a uniform bound for the dimensions of the centers of the relative commutants, the subfactor $\mathcal{N} \subset \mathcal{M}$ is called a *finite depth subfactor*. In this case, the inclusion diagram for $\mathcal{N}' \cap \mathcal{M}_{2k} \subset \mathcal{N}' \cap \mathcal{M}_{2k+1}$ does not depend on k for k sufficiently large; the corresponding graph is called the *principal graph* of $\mathcal{N} \subset \mathcal{M}$. Similarly, one defines the *dual principal graph* from the inclusion of $\mathcal{M}' \cap \mathcal{M}_{2k} \subset \mathcal{M}' \cap \mathcal{M}_{2k+1}$ for k sufficiently large. These graphs are important invariants for the inclusion $\mathcal{N} \subset \mathcal{M}$.

The following results are presented in [Bisch 1997] in great detail and with precise references to the original sources.

Proposition 1.7. *Let $\mathcal{N} \subset \mathcal{M}$ be a finite depth subfactor with finite index. Then*

(a) *The inclusions $\mathcal{N} \subset \mathcal{M}_{2k+1}$, $\mathcal{N} \subset \mathcal{M}_{2k}$, $\mathcal{M} \subset \mathcal{M}_{2k+1}$, $\mathcal{M} \subset \mathcal{M}_{2k}$ are given by the bimodule $\mathcal{M}^{\otimes k} = \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M} \otimes_{\mathcal{N}} \cdots \otimes_{\mathcal{N}} \mathcal{M}$ (k times), viewed, respectively, as an $\mathcal{N}$–$\mathcal{N}$, $\mathcal{N}$–$\mathcal{M}$, $\mathcal{M}$–$\mathcal{N}$ and $\mathcal{M}$–$\mathcal{M}$ bimodule.*

(b) *The embedding of $\mathcal{N}' \cap \mathcal{M}_k \subset \mathcal{N}' \cap \mathcal{M}_{k+1}$ coincides with the embedding of the algebras $\mathrm{End}_{\mathcal{M}-\mathcal{N}}(\mathcal{M}^{\otimes k}) \subset \mathrm{End}_{\mathcal{N}-\mathcal{N}}(\mathcal{M}^{\otimes k})$ for k even. If k is odd, the embedding of $\mathcal{N}' \cap \mathcal{M}_k \subset \mathcal{N}' \cap \mathcal{M}_{k+1}$ coincides with the embedding of $\mathrm{End}_{\mathcal{N}-\mathcal{N}}(\mathcal{M}^{\otimes k}) \subset \mathrm{End}_{\mathcal{M}-\mathcal{N}}(\mathcal{M}^{\otimes k+1})$, given by $x \in \mathrm{End}_{\mathcal{N}-\mathcal{N}}(\mathcal{M}^{\otimes k}) \to 1_{\mathcal{M}} \otimes x$.*

(c) *Analogous statements hold for the embedding of $\mathcal{M}' \cap \mathcal{M}_k \subset \mathcal{M}' \cap \mathcal{M}_{k+1}$; we only need to replace $\mathrm{Hom}_{\mathcal{X}-\mathcal{N}}$ by $\mathrm{Hom}_{\mathcal{X}-\mathcal{M}}$ in all the statements in (b), with $\mathcal{X} \in \{\mathcal{M}, \mathcal{N}\}$.*

Proof. Statement (a) is shown in [Bisch 1997], Proposition 3.2. Statement (b) can be found in [Bisch 1997], Corollaries 4.2 and 4.4 (with tensoring from the right instead of tensoring from the left, as we have chosen here). Statement (c) follows from (b) and (a). □

Let $\mathcal{N}, \mathcal{M}, \mathcal{B}$ be type II_1 factors with $\mathcal{N} \subset \mathcal{M}$ a subfactor of finite index. Let $\{H_\lambda\}_\lambda$ and $\{K_\nu\}_\nu$ be a collection of mutually nonisomorphic irreducible $\mathcal{N}$–$\mathcal{B}$ and $\mathcal{M}$–$\mathcal{B}$ bimodules, respectively. Observe that $\mathcal{M} \otimes_{\mathcal{N}} H_\lambda$ is an $\mathcal{M}$–$\mathcal{B}$ bimodule for any $\mathcal{N}$–$\mathcal{B}$ bimodule H_λ. Similarly, we can view any $\mathcal{M}$–$\mathcal{B}$ bimodule K_ν as an $\mathcal{N}$–$\mathcal{B}$ bimodule by restricting the left action to $\mathcal{N}$. We say that the system of bimodules $(\{H_\lambda\}_\lambda, \{K_\nu\}_\nu)$ is *closed under induction and restriction* if

- for each $\mathcal{N}$–$\mathcal{B}$ bimodule H_λ the induced $\mathcal{M}$–$\mathcal{B}$ bimodule $\mathcal{M} \otimes_{\mathcal{N}} H_\lambda$ is isomorphic to a direct sum of irreducible $\mathcal{M}$–$\mathcal{B}$ bimodules each of which is isomorphic to an element in $\{K_\nu\}_\nu$,
- for each $\mathcal{M}$–$\mathcal{B}$ bimodule K_ν the $\mathcal{N}$–$\mathcal{B}$ bimodule $\mathcal{M} \otimes_{\mathcal{M}} K_\nu$ obtained from K_ν by restricting the left action to $\mathcal{N}$ is isomorphic to a direct sum of irreducible $\mathcal{N}$–$\mathcal{B}$ bimodules each of which is isomorphic to an element in $\{H_\lambda\}_\lambda$.

The *induction-restriction graph* for our system of bimodules is the bipartite graph whose (say) odd vertices are labeled by the elements in $\{H_\lambda\}_\lambda$ and whose even vertices are labeled by the elements in $\{K_\nu\}_\nu$. A vertex labeled by H_λ is connected with a vertex labeled by K_ν by L_λ^ν edges, where L_λ^ν is the multiplicity of H_λ in K_ν, viewed as an $\mathcal{N}$–$\mathcal{B}$ bimodule. By Frobenius reciprocity (see [Bisch 1997, Theorem 1.18], for example), this number coincides with the multiplicity of K_ν in $\mathcal{M} \otimes_{\mathcal{N}} H_\lambda$.

Proposition 1.8. *Let* $(\{H_\lambda\}_\lambda, \{K_\nu\}_\nu)$ *be a system of* $\mathcal{N}$–$\mathcal{B}$- *and* $\mathcal{M}$–$\mathcal{B}$-*bimodules which is closed under induction and restriction.*

(a) *If* $\{H_\lambda\}_\lambda$ *contains a bimodule* H_0 *which is weakly reduced from the trivial* $\mathcal{N}$–$\mathcal{N}$-*bimodule* $\mathcal{N}$, *then the principal graph for* $\mathcal{N} \subset \mathcal{M}$ *is given by the connected component of the induction- restriction graph for* $(\{H_\lambda\}_\lambda, \{K_\nu\}_\nu)$ *which contains* H_0.

(b) *If* $\{K_\nu\}_\nu$ *contains a bimodule* K_0 *which is weakly reduced from the trivial* $\mathcal{M}$–$\mathcal{M}$-*bimodule* $\mathcal{M}$, *then the dual principal graph for* $\mathcal{N} \subset \mathcal{M}$ *is given by the connected component of the induction- restriction graph for* $(\{H_\lambda\}_\lambda, \{K_\nu\}_\nu)$ *which contains* K_0.

(c) *If* $\phi : \mathcal{B} \to \mathcal{B}$ *is an endomorphism of the* II_1 *factor* $\mathcal{B}$ *and* $p \in \phi(\mathcal{B})' \cap \mathcal{B}$ *such that* $[p\mathcal{B}p : p\phi(\mathcal{B})] = 1$, *then* $L^2(\mathcal{B}, \mathrm{tr})p$, *with action* $b_1.\xi p.b_2 = b_1 \xi p \phi(b_2)$ *for* $b_1, b_2 \in \mathcal{B}$, $\xi \in L^2(\mathcal{B}, \mathrm{tr})$ *is weakly reduced from the trivial* $\mathcal{B}$–$\mathcal{B}$ *bimodule* $L^2(\mathcal{B}, \mathrm{tr})$.

Proof. Part (a) follows from Proposition 1.6 and Proposition 1.7(b). Similarly, part (b) follows from Proposition 1.6 and Proposition 1.7(c). Part (c) follows almost immediately from Definition 1.4, using the fact that $p\mathcal{B}p = \phi(\mathcal{B})p$. □

Remark 1.9. In the setting of Proposition 1.8(a), there may be more than one bimodule H_λ which is weakly reduced from the trivial $\mathcal{N}$–$\mathcal{N}$-bimodule $\mathcal{N}$. Which of those will correspond to the trivial $\mathcal{N}$–$\mathcal{N}$-bimodule $\mathcal{N}$ will depend on the choice of the automorphism between $p\mathcal{N}p$ and $\mathcal{B}$. The resulting graph will be independent of this choice. A similar phenomenon may also occur in part (b).

Let H be an $\mathcal{A}$–$\mathcal{B}$ bimodule. We define $\mathrm{ind}(H)$ to be equal to the index $[\rho(\mathcal{B})' : \lambda(\mathcal{A})] = [\lambda(\mathcal{A})' : \rho(\mathcal{B})]$. In the following lemma, $(H_\lambda)_\lambda$ and $(K_\nu)_\nu$ are bimodules as in the last proposition, where we now assume for simplicity that they only denote the bimodules which label the vertices of a given principal graph. Moreover, we also assume the subfactor to be of finite depth, meaning that both sets only contain finitely many bimodules.

Lemma 1.10. *With notations as above, we have*:

(a) $\sum_\nu \mathrm{ind}(K_\nu) = \sum_\lambda \mathrm{ind}(H_\lambda)$.

(b) *Assume that the $\mathscr{A}$–$\mathscr{B}$-bimodule H decomposes as $H = \bigoplus m_i H_i$, with H_i irreducible $\mathscr{A}$–$\mathscr{B}$-bimodules, and let $l = \dim(\mathrm{End}_{\mathscr{A},\mathscr{B}}(H)) = \sum_i m_i^2$. Then we have $\sum_i \mathrm{ind}(H_i) \geq \mathrm{ind}(H)/l$, with equality only if $\dim_{\mathscr{A}}(H_i) = m_i \dim_{\mathscr{A}}(H)/l$ for all i.*

Proof. It is well-known that the inclusion of higher relative commutants $\mathcal{M}' \cap \mathcal{M}_k \subset \mathcal{N}' \cap \mathcal{M}_k$ defines periodic commuting squares which generate in the limit a subfactor of index $[\mathcal{M} : \mathcal{N}]$. Hence we can use the results of [Wenzl 1988], Theorem 1.5(iii). It follows that the index is equal to the quotient of the l^2-norms of the weight vectors of $\mathcal{M}' \cap \mathcal{M}_k$ and $\mathcal{N}' \cap \mathcal{M}_k$ for k sufficiently large. Let p_λ and p_μ be minimal idempotents in $\mathcal{M}' \cap \mathcal{M}_k$ and $\mathcal{N}' \cap \mathcal{M}_k$ respectively. Then we have $\mathrm{ind}(p_\nu \mathcal{M}_k) = \mathrm{tr}(p_\nu)^2 [\mathcal{M} : \mathcal{N}]^k$ and $\mathrm{ind}(p_\lambda \mathcal{M}_k) = \mathrm{tr}(p_\lambda)^2 [\mathcal{M} : \mathcal{N}]^{k+1}$. Solving for $\mathrm{tr}(p_\lambda)^2$ and $\mathrm{tr}(p_\nu)^2$, we obtain

$$[\mathcal{M} : \mathcal{N}] = \frac{\sum_\nu \mathrm{ind}(p_\nu \mathcal{M}_k)/[\mathcal{M} : \mathcal{N}]^k}{\sum_\lambda \mathrm{ind}(p_\lambda \mathcal{M}_k)/[\mathcal{M} : \mathcal{N}]^{k+1}}.$$

The claimed formula follows from this in the case that our system of bimodules labels the vertices of the principal graph. One obtains the claim for the dual principal graph by the same proof applied to the inclusion $\mathcal{M} \subset \mathcal{M}_1$.

Part (b) is proved using Lagrange multipliers as follows: Let $x_i = \dim_{\mathscr{A}}(H_i)$ and let $d = \dim_{\mathscr{A}} H$. Then the minimum of the function $f(x_1, \ldots, x_r) = \sum x_i^2$ subject to the condition $\sum m_i x_i = d$ is obtained for $2x_i = \lambda m_i$, and we deduce from the constraint that $d = \frac{\lambda}{2} \sum m_i^2 = l\lambda/2$. Hence $x_i = m_i d/l$ and

$$\sum_i (\dim_{\mathscr{A}} H_i)^2 \geq \frac{d^2}{l^2} \sum_i m_i^2 = d^2/l. \qquad (*)$$

Now observe that if p_i is the projection onto the submodule $H_i \subset H$, we have $\mathrm{tr}(p_i) = \dim_{\mathscr{A}}(H_i)/\dim_{\mathscr{A}}(H)$ and $\mathrm{ind}(H_i) = \mathrm{tr}(p_i)^2 \mathrm{ind}(H)$ (again see [Wenzl 1988], Theorem 1.5(iii)). The claim follows from this after multiplying $(*)$ by $\mathrm{ind}(H)/d^2$. □

2. Categories

In this section we deal with categories which can be considered as generalizations of the representation categories of finite groups. This allows us to deal simultaneously with categories of bimodules of von Neumann factors, fusion categories (which can be constructed using quantum groups or loop groups) and categories obtained from unitary braid representations. For more details, we refer to [Mac Lane 1998], [Freyd 1964] for general categorical notions, and to [Kassel 1995], [Turaev 1994] for tensor categories; our treatment of traces also uses results from [Longo and Roberts 1997].

2A. ***General definitions.*** We recall some basic definitions and set up notations.

In the following, $\mathscr{C}$ will always denote a strict monoidal complex tensor category with unit $\mathbb{1}$. This means that $\mathscr{C}$ is a category with a functor $\otimes : \mathscr{C}\times\mathscr{C} \to \mathscr{C}$ called the *tensor product* which satisfies certain associativity conditions such as the *Pentagon Axiom*. There are similar axioms involving the morphisms $l_X : \mathbb{1}\otimes X \to X$ and $r_X : X\otimes\mathbb{1} \to X$ called the left and right *unit constraints*. Moreover, $\mathscr{C}$ being a complex category just means that the homomorphisms $\mathrm{Hom}(X, Y)$ form a complex vector space for any objects X and Y in $\mathscr{C}$.

The complex tensor category $\mathscr{C}$ is called a $*$ tensor category if there exists a contragredient complex conjugate functor $* : \mathscr{C} \to \mathscr{C}$ which is compatible with $\otimes$. This means in detail that:

- if $f \in \mathrm{Hom}(X, Y)$, then $f^* \in \mathrm{Hom}(Y, X)$,
- $(\alpha f)^* = \bar{\alpha} f^*$ for all $\alpha \in \mathbb{C}$ and $f \in \mathrm{Hom}(X, Y)$,
- $(fg)^* = g^* f^*$ for $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(U, X)$,
- $(f\otimes g)^* = f^*\otimes g^*$ for $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(U, V)$,
- $1_X^* = 1_X$ for the identity morphism 1_X for any object X in $\mathscr{C}$.

2B. ***Duality and Frobenius reciprocity.*** An object X in a strict monoidal category $\mathscr{C}$ is called *left rigid* if there exists an object $\bar{X} \in \mathscr{C}$ and a pair of morphisms $i_X : \mathbb{1} \to X\otimes\bar{X}$ and $d_X : \bar{X}\otimes X \to \mathbb{1}$ such that the maps $(1_X \otimes d_X)(i_X\otimes 1_X) : X \to X$ and $(d_X\otimes 1_{\bar{X}})(1_X \otimes i_{\bar{X}}) : \bar{X} \to \bar{X}$ are 1_X and $1_{\bar{X}}$. An object X is called right rigid if we can find an object $\bar{X}'$ and morphisms $i'_X : \mathbb{1} \to \bar{X}'\otimes X$ and $d'_X : X\otimes\bar{X}' \to \mathbb{1}$ satisfying analogous identities. It is easy to check that in a $*$ category any left rigid object is also right rigid, with $\bar{X}' = \bar{X}$, $i'_X = d_X^*$ and $d'_X = i_X^*$. Hence we will in the following only talk about rigid objects. A category $\mathscr{C}$ is called rigid if every object of $\mathscr{C}$ is rigid.

With this notion of duality, we also have the usual Frobenius reciprocity isomorphism between $\mathrm{Hom}(V, W\otimes\bar{X})$ and $\mathrm{Hom}(V\otimes X, W)$ for any objects V, W in $\mathscr{C}$. One checks easily that these isomorphisms are given by the maps

$$a \mapsto (1_W \otimes d_X)\circ(a\otimes 1_X) \quad \text{and} \quad b \mapsto (b\otimes 1_Y)\circ(1_V \otimes i_X)$$

for $a \in \mathrm{Hom}(V, W\otimes X)$ and $b \in \mathrm{Hom}(V\otimes Y, W)$. In particular, one obtains as a special case that $\dim\mathrm{Hom}(\mathbb{1}, X\otimes\bar{X}) = \dim\mathrm{End}(X) = 1$ if X is a simple object. Hence the morphisms i_X and d_X are unique up to scalar multiples for X simple. We shall say that the rigidity morphisms i_X and d_X are *normalized* if $i_X^* i_X = d_X d_X^* *$.

2C. ***Dimension, trace and conditional expectation.*** In the following we always assume the rigidity morphisms i_X and d_X to be normalized for any object X. If X is simple, this can always be assumed after some rescaling in view of the discussion in the last section. For normalized rigidity morphisms, we can now define the dimension of a simple object X to be equal to the scalar

$$\dim(X) = i_X^* i_X = d_X d_X^*.$$

Of course, we would like the dimension to be additive with respect to a decomposition $X = \bigoplus W_i$, with the W_i being simple objects. To do so, we define morphisms $\phi_i : W_i \to X$ such that $\phi_i^* \phi_j = \delta_{ij} 1_{W_i}$ and $\sum_i \phi_i \phi_i^* = 1_X$, and we define

$$i_X = \sum (\phi_i \otimes \bar{\phi}_i) i_{W_i}, \quad d_X = \sum d_{W_i} (\bar{\phi}_i^* \otimes \phi_i^*), \tag{2-1}$$

where the $\bar{\phi}_i$ are the analogous morphisms for the decomposition of the dual $\bar{X} = \sum \bigoplus_i \bar{W}_i$. Then it is easy to check that these morphisms satisfy the rigidity axiom, and they are normalized if the ϕ_i are so. Moreover, one also checks that these morphisms yield the desired additivity property of the dimension function.

Additionally, the dimension function should be multiplicative with respect to the tensor product. If $X \otimes Y$ is a tensor product of simple objects X and Y, we obtain normalized rigidity morphisms

$$i_{X\otimes Y} = (1_X \otimes i_Y \otimes 1_{\bar{X}}) i_X, \quad d_{X\otimes Y} = d_Y (1_{\bar{Y}} \otimes d_X \otimes 1_X).$$

It can be shown that these rigidity morphisms define the same dimension as the one we obtain from the decomposition $X \otimes Y \cong \bigoplus_i W_i$, with W_i simple and with rigidity morphisms as defined in the last paragraph. It will be convenient to represent the rigidity morphisms i_X and d_X, by the following pictures:

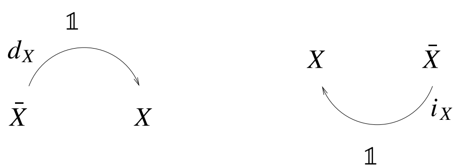


Figure 1. Rigidity morphisms.

In a $*$ tensor category we define the *categorical trace* of an endomorphism $f \in \mathrm{End}(X)$ by

$$\mathrm{Tr}_X(f) = i_X^* \circ (f \otimes 1_{\bar{X}}) \circ i_X \in \mathrm{End}(\mathbb{1}). \tag{2-2}$$

If $Z = \bigoplus m_i X_i$, where X_i is a simple object, and m_i is the multiplicity of X_i in Z, we can write an element $f \in \mathrm{End}(Z)$ in the form $f = \bigoplus f_i$, where $f_i \in \mathrm{End}(m_i X_i)$ can be viewed as an $m_i \times m_i$ matrix. Defining rigidity morphisms i_Z, d_Z with respect to this decomposition, and using (2-1), one checks easily that

$$\mathrm{Tr}_Z(f) = \sum \dim(X_i)\, \mathrm{Tr}(f_i),$$

where $\mathrm{Tr}(f_i)$ is the usual trace of a matrix. This shows that we obtain a well-defined trace for $\mathrm{End}(Z)$ for any object Z, and that $\mathrm{Tr}_Z(fg) = \mathrm{Tr}_Z(gf)$ for any $f, g \in \mathrm{End}(Z)$. Moreover, using this formula, one shows as well that we can define the trace also by

$$\mathrm{Tr}_X(f) = i^*_{\bar{X}} \circ (1_{\bar{X}} \otimes f) \circ i_{\bar{X}} \in \mathrm{End}(\mathbb{1}).$$

This shows that $*$-categories satisfy the axioms of a spherical category (see [Barrett and Westbury 1999]).

The *normalized trace* tr_X on $\mathrm{End}(X)$ is defined by $\mathrm{tr}_X(f) = \mathrm{Tr}_X(f)/(\dim X)$. In the following we will often just write Tr, tr for the trace or normalized trace when it is clear for which object it is defined.

Conditional expectations can also be very naturally defined using our categorical definitions. Let X be an object. Let $A = \mathrm{End}(X) \cong A \otimes 1_V \subset B = \mathrm{End}(X \otimes V)$. We define the map E_A from B onto A by

$$E_A(b) = \frac{1}{\dim V}(1_X \otimes i^*_V)(b \otimes 1_{\bar{V}})(1_X \otimes i_V);$$

in the tangle picture, $E_A(b)$ is obtained from b by closing up the tangle with color V and renormalizing by $1/\dim V$.

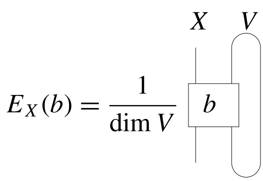


Figure 2. Conditional expectation.

It is known and easy to check that this definition of conditional expectation coincides with the usual definition of conditional expectation in operator algebras (see [Orellana and Wenzl 2002, Proposition 1.4], for instance). Actually, one can show more: Let X_1, X_2, X_3 be objects in our $*$ tensor category $\mathscr{C}$. Define the algebras $A = \mathrm{End}(X_2)$, $B = \mathrm{End}(X_1 \otimes X_2)$, $C = \mathrm{End}(X_2 \otimes X_3)$ and $D = \mathrm{End}(X_1 \otimes X_2 \otimes X_3)$. We can consider all these algebras as subalgebras of D, say by identifying A with $1_{X_1} \otimes \mathrm{End}(X_2) \otimes 1_{X_3}$. The next proposition now follows immediately from the graphical description of the conditional expectations.

Proposition 2.1. *The algebras A, B, C, D form a commuting square; that is,* $E_B E_C = E_A = E_C E_B$.

2D. *Braided tensor categories.* A strict monoidal category $\mathscr{C}$ is called *braided* if, for any objects X, Y in $\mathscr{C}$, there exists a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$ called the *braiding* such that:

$$c_{X,Y\otimes Z} = (1_Y \otimes c_{X,Z})(c_{X,Y} \otimes 1_Z)$$

and

$$c_{X\otimes Y,Z} = (c_{X,Z} \otimes 1_Y)(1_X \otimes c_{Y,Z}).$$

Naturality means that for any morphisms $f : X \to X'$ and $g : Y \to Y'$

$$(g \otimes f) \circ c_{X,Y} = c_{X',Y'} \circ (f \otimes g).$$

Finally, we also require that $c_{\mathbb{1},X} = 1_X = c_{X,\mathbb{1}}$ under the isomorphisms $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$.

2E. *C^* tensor categories.* We call a complex $*$ tensor category a C^* tensor category if

(a) for any objects X, Y in $\mathscr{C}$ the space $\mathrm{Hom}(X, Y)$ is a Hilbert space with inner product $(a, b) = \mathrm{Tr}(b^*a)$ for $a, b \in \mathrm{Hom}(X, Y)$,

(b) for any objects X, Y in $\mathscr{C}$ the algebra $\mathrm{End}(Y)$ is a C^*-algebra acting on the Hilbert space $\mathrm{Hom}(X, Y)$.

Observe that these definitions imply that the dimensions of all objects are positive. A *braided C^* tensor category* is a C^* tensor category with a braiding for which all its braiding morphisms are unitary operators. For examples of C^*-tensor categories, see Section 6A.

3. The multisided construction

3A. *Categorical setting.* We shall use the following conventions: Let $\mathscr{C}$ be a finite braided C^* tensor category, where finite means that we only have finitely many equivalence classes of simple objects in $\mathscr{C}$. Let $\{X_\lambda, \lambda \in \Lambda\}$ be a set of representative nonequivalent simple objects, indexed by some labeling set Λ. We define d_λ to be the dimension of X_λ. We shall also assume that the category $\mathscr{C}$ is generated by an object X, so any simple object appears in some tensor power of X. We define algebras $A_n = \mathrm{End}(X^{\otimes n}) = \mathrm{End}_{\mathscr{C}}(X^{\otimes n})$. By the definition of A_n, the simple components of A_n are labeled by the equivalence classes of simple objects which appear in the n-th tensor power of X, i.e., by a certain subset Λ_n of Λ. We define the embeddings $\iota_r : a \in A_n \to a \otimes 1_r \in A_{n+r}$, where we will often omit the subscript r. It follows from the definitions that the vertices of the inclusion diagram for $\iota : A_n \to A_{n+1}$ are labeled by the elements of Λ_n and Λ_{n+1} respectively; the vertex labeled by $\lambda \in \Lambda_n$ is connected with the one labeled by μ by L_λ^μ edges, where L_λ^μ is the multiplicity of the object X_μ in $X_\lambda \otimes X$. We have the commuting diagram of embeddings

$$\begin{array}{ccc} 1_m \otimes A_n & \subset & A_{n+m} \\ {\scriptstyle 1\otimes\iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ 1_m \otimes A_{n+1} & \subset & A_{n+m+1} \end{array} \tag{3-1}$$

We will also assume that the Bratteli diagram for the algebras (A_n) is *strongly connected*. This means that for any X_λ, there exists an r such that $X_\lambda \otimes X^{\otimes r}$ contains all irreducible representations which appear in $X^{\otimes|\lambda|+r}$, where $|\lambda|$ is the smallest integer such that $X_\lambda \in X^{\otimes|\lambda|}$. Equivalently, it means that for any projection $p \in A_n$ there exists an r such that the central support of p in A_{n+r} is 1. We define $k = k(X) = gcd\{n, \mathbb{1} \subset X^{\otimes n}\}$. Let $\mathscr{C}'$ be the subcategory of $\mathscr{C}$ generated by the simple objects in $X^{\otimes mk}$, $m \in \mathbb{N}$.

Lemma 3.1. *Let $\mathscr{C}$ be a finite C^*-tensor category, not necessarily braided. Then we have*

(a) $\Lambda_n = \Lambda_{n+k}$ *for n sufficiently large and* $\Lambda_n \cap \Lambda_m = \varnothing$ *if* $|n-m| < k$; *in particular* $\Lambda' := \Lambda_{nk}$ *for n sufficiently large labels the simple objects of* $\mathscr{C}'$.

(b) *The weight vector for the trace on the algebra A_n is* $\vec{v}_n = (d_\lambda/(\dim X)^n)_{\lambda\in\Lambda_n}$.

(c) *The inductive limit of* $(1_m \otimes A_n \subset A_{n+m})$, *for* $n \to \infty$, *defines an inclusion* $B \subset A$ *of hyperfinite II_1 factors with index* $(\dim X)^{2m}$.

(d) $\sum_{\lambda\in\Lambda_n} d_\lambda^2 = \frac{1}{k}\sum_{\lambda\in\Lambda} d_\lambda^2$ *for n sufficiently large.*

Proof. If the trivial object $\mathbb{1}$ appears in the r-th tensor power of X and $X_\lambda \subset X^{\otimes n}$, then we have

$$X_\lambda \cong X_\lambda \otimes \mathbb{1} \subset X_\lambda \otimes X^{\otimes r} \subset X^{\otimes n+r}.$$

Hence $\Lambda_n \subset \Lambda_{n+r}$ for all $n \in \mathbb{N}$. As Λ is finite, these inclusions become equalities for n sufficiently large. Applying this to any r such that $\mathbb{1} \subset X^{\otimes r}$, we can similarly prove $\Lambda_n = \Lambda_{n+k}$ for k the gcd of all such r and n sufficiently large. Finally, if $0 < m-n = k' < k$ and $\lambda \in \Lambda_n \cap \Lambda_m$, then we also have $\nu \in \Lambda_{n+r} \cap \Lambda_{m+r} = \Lambda_{n+k'+r}$ for any $X_\nu \subset X_\lambda \otimes X^{\otimes r}$ and $r \in \mathbb{N}$. As the Bratteli diagram for (A_n) is strongly connected, we obtain $\Lambda_{n+r} = \Lambda_{n+r+k'}$ for r sufficiently large. Using the convention $X_0 = \mathbb{1}$, we can find r such that $0 \in \Lambda_{n+r} = \Lambda_{n+r+k'}$, contradicting the definition of k. This shows (a).

Statement (b) follows from the fact that the value of the normalized trace of a projection p_λ corresponding to a simple object $X_\lambda \subset X^{\otimes n}$ is given by $\mathrm{tr}(p_\lambda) = d_\lambda/(\dim X)^n$.

For statement (c) observe that Diagram (3-1) defines a commuting square by Proposition 2.1. Moreover, the sequence of algebras as in the statement has a k-periodic pattern: By part (a), we have the same labeling sets for the algebras in Diagram (3-1) if we substitute n by $n+k$ everywhere, for n sufficiently large. Moreover, also the inclusion pattern remains the same by the discussion before Diagram (3-1). It follows from [Wenzl 1988], Theorem 1.5(iii), that the index $[A : B]$ is given by the ratio $\|\vec{v}_n\|^2/\|\vec{v}_{n+1}\|^2$, for n large enough. As this holds for any sufficiently large n, we have

$$[A:B]^k = \prod_{i=1}^{k} \frac{\|\vec{v}_{n-1+i}\|^2}{\|\vec{v}_{n+i}\|^2} = \frac{\|\vec{v}_n\|^2}{\|\vec{v}_{n+k}\|^2}.$$

The claim now follows from the fact that $\vec{v}_n = (\dim X)^k \vec{v}_{n+k}$, by (a) and (b). Finally observe that $(\dim X)^2 \|\vec{v}_{n+1}\|^2 = \|\vec{v}_n\|^2$ implies $\sum_{\lambda\in\Lambda_n}(d_\lambda)^2 = \sum_{\mu\in\Lambda_{n+1}}(d_\mu)^2$ for all n sufficiently large. As $\Lambda_n \cap \Lambda_m = \varnothing$ whenever $|n-m| < k$, we obtain Statement (d). □

3B. *Multisided construction.* The subfactors constructed in the last section will sometimes be denoted as one-sided subfactors. We will now generalize the construction in [Erlijman 2001] to the setting of braided C^*-tensor categories, which we call multisided subfactors in analogy to the notation in [Erlijman 2001]. We will fix a positive integer s. For the s-sided construction, we will have to define an embedding of algebras $A_n^{\otimes s} \subset A_{ns}$ such that we will obtain a subfactor if we consider the inductive limit over n.

We shall need special braids $\gamma_n \in \mathcal{B}_{sn}$, which can be defined inductively by $\gamma_1 = 1_s$ and by Figure 3.

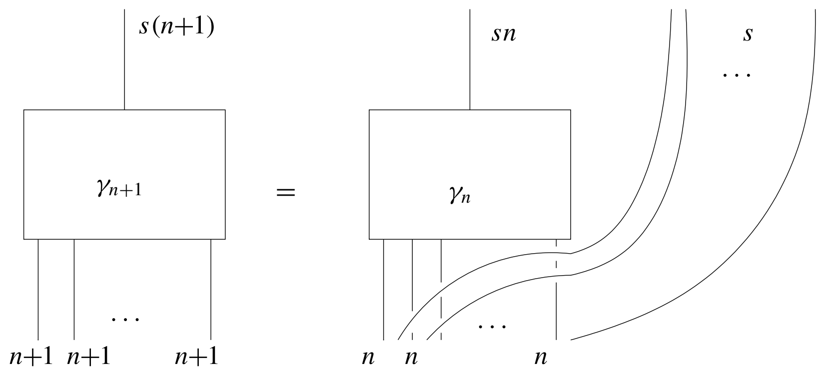


Figure 3. Inductive property of intertwining braids.

Alternatively, the braid γ_n can be described as follows: arrange the points labeled by the numbers 1 up to ns in a rectangular pattern with height n and width s. Now we can numerate the points either by first going down the columns, or by first going to the right in each row. This defines a permutation π mapping the i-th point in the column-first count to the i-th point in the row-first count. The braid γ_n is now defined by this permutation where the i-th lower point is connected with the $\pi(i)$-th upper point and where we assume all crossings to be positive (i.e., the strand going from southwest to northeast crosses over the one going from southeast to northwest). A picture for this braid can be found in [Erlijman 2003, p. 83].

Let $c = c_{X,X}$ be the braiding morphism for X. By definition, we obtain a unitary representation ρ of the braid group $\mathcal{B}_n$ into A_n by mapping the generator σ_i to $c_i = 1_{i-1} \otimes c \otimes 1_{n-1-i}$. We define the unitary $u_n = u_n^{(s)} = \rho(\gamma_n)$, with γ_n defined as in Figure 3. Finally, the embedding from $A_n^{\otimes s}$ into A_{ns} is given by first identifying $A_n^{\otimes s}$ with $\mathrm{End}(X^{\otimes n})^{\otimes s} \subset \mathrm{End}(X^{\otimes ns}) = A_{ns}$ and by then conjugating this with u_n, i.e., by

$$(a_1 \otimes \cdots \otimes a_s) \stackrel{\hat{u}_n}{\longmapsto} u_n(a_1 \otimes \cdots \otimes a_s)u_n^*;$$

throughout this paper, $\hat{u}$ will denote the inner automorphism given by conjugation via the unitary u unless stated otherwise. We now obtain the following diagram of maps, where the vertical arrows are labeled by $\iota^{\otimes s} = \iota_1^{\otimes s}$ and $\iota = \iota_s$ respectively:

(3-2)
$$\begin{array}{ccc} A_n^{\otimes s} & \xrightarrow{\ \hat{u}_n\ } & A_{ns} \\ \downarrow & & \downarrow \\ A_{n+1}^{\otimes s} & \xrightarrow[\ \hat{u}_{n+1}\ ]{} & A_{(n+1)s} \end{array}$$

Then we have the following lemma which has essentially already been proved in [Erlijman 2001], Section 3.2; the case proved there would correspond to the special case in which A_n is generated by the image of $\mathcal{B}_n$.

Lemma 3.2. *The diagram (3-2) above commutes and also forms a commuting square. Moreover, the inclusion pattern is k-periodic.*

Proof. We check first that Diagram (3-2) is a commuting diagram: This is most easily seen by the following pictures (these proofs by pictures contain all the necessary details and translate faithfully to the algebraic proofs by simply rewriting the definitions already included in this article). We take $s = 3$ for simplicity. For $b \in \mathcal{A}_n^{\otimes s}$, we have:

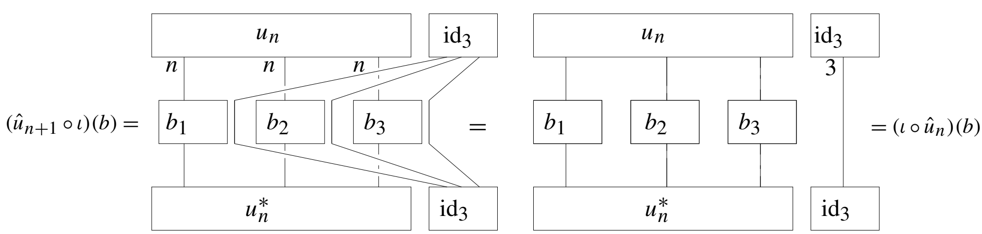


Figure 4. Diagram (3-2) is a commuting diagram.

Now we check that Diagram (3-2) is a commuting square, i.e., that $(E_{A_{sn}} \circ \hat{u}_{n+1})(b) = (\hat{u}_n \circ E_{A_n^{\otimes s}})(b)$ for $b \in A_{n+1}^{\otimes s}$. We use the categorical definition for a conditional expectation as described in Section 2C, Figure 2. For $b = b_1 \otimes \cdots \otimes b_s \in A_{n+1}^{\otimes s}$, we have

$(E_{A_{sn}} \circ \hat{u}_{n+1})(b) =$

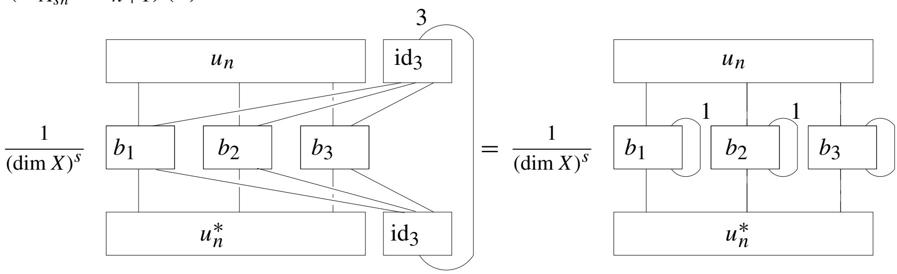


Figure 5. Diagram (3-2) is a commuting square.

which in turn equals $(\hat{u}_n \circ E_{A_n^{\otimes s}})(b)$. To show that the inclusion diagrams are k-periodic for large n, observe that Lemma 3.1(a) implies that we have a one-to-one correspondence between the labeling sets of simple components of $A_n^{\otimes s}$ and $A_{n+k}^{\otimes s}$ as well as between the components of A_{ns} and $A_{(n+1)s}$. This identification of edges is compatible with the number of edges between them, which again is just given by tensor product multiplicities. □

Theorem 3.3. *Fix $s \in \mathbb{N}$, $s > 1$. There is an embedding of the factor $\mathcal{N} :=$ $\lim\operatorname{ind} A_n^{\otimes s}$ (inductive limit) in $\mathcal{M} := \lim\operatorname{ind} A_{ns}$ given by $\hat{u} := \lim\operatorname{ind} \hat{u}_n$, with u_n as above. The index of the resulting inclusion is*

$$\Big(\sum_{\lambda\in\Lambda'} d_\lambda^2\Big)^{s-1},$$

where Λ' is an indexing set for the simple objects of the subcategory $\mathscr{C}'$ as defined at the beginning of this subsection and $d_\lambda = \dim(X_\lambda)$.

Proof. This was done in [Erlijman 2001] in the case that the A_n's are generated by braid elements only. By Lemma 3.2, Diagram (3-2) is a periodic commuting square for large n. Thus, by [Wenzl 1988], Theorem 1.5(iii), $\hat{u} : \mathcal{N} \hookrightarrow \mathcal{M}$ is an inclusion of hyperfinite II_1 factors with index given by $\|\vec{t}_n\|^2/\|\vec{v}_n\|^2$ for n sufficiently large, where $\vec{t}_n$ and $\vec{v}_n$ are the trace vectors for the trace in $\mathcal{M}$ restricted to the finite-dimensional approximants $A_n^{\otimes s}$ and A_{ns}, respectively. For this observe that if $k|n$ the dimension vectors for $A_n^{\otimes s}$ and A_{ns} are given by $\vec{t}_{ns} = (d_{\vec{\lambda}}/(\dim X)^{ns})_{\vec{\lambda}}$ and $\vec{v}_{ns} = (d_\nu/(\dim X)^{ns})_\nu$, with $\vec{\lambda} \in (\Lambda')^s$ and $\nu \in \Lambda'$; here $d_{\vec{\lambda}} = \prod_{i=1}^s d_{\lambda_i}$. Hence we obtain

$$[\mathcal{M} : \mathcal{N}] = \frac{\|\vec{t}_n\|^2}{\|\vec{v}_n\|^2} = \frac{\sum_{\vec{\lambda}\in(\Lambda')^s} d_{\vec{\lambda}}^2}{\sum_{\nu\in\Lambda'} d_\nu^2} = \Big(\sum_{\lambda\in\Lambda'} d_\lambda^2\Big)^{s-1}. \qquad \square$$

3C. ***More embeddings.*** We shall need a variation of the embeddings in the last section for the construction of certain bimodules.

Lemma 3.4. *Let $\vec{m} = (m_1, \dots, m_s)$, where $m_i \in \mathbb{Z}_{\geq 0}$, and $m_1 \geq m_2 \geq \cdots \geq m_s$, and let $|\vec{m}| = \sum_i |m_i|$. Then there exist unitaries $u_{\vec{m},n} = u_{\vec{m},n}(s) \in A_{|\vec{m}|+sn}$ such that we obtain k periodic commuting squares*

$$\begin{array}{ccc} A_{n+m_1} \otimes \cdots \otimes A_{n+m_s} & \xrightarrow{\hat{u}_{\vec{m},n}} & A_{|\vec{m}|+ns} \\ \downarrow & & \downarrow \\ A_{n+1+m_1} \otimes \cdots \otimes A_{n+1+m_s} & \xrightarrow[\hat{u}_{\vec{m},n+1}]{} & A_{|\vec{m}|+(n+1)s} \end{array} \tag{3-3}$$

which produce an inclusion $\hat{u}_{\vec{m}} : \mathcal{N} \to \mathcal{M}$ isomorphic to the map $\hat{u} : \mathcal{N} \to \mathcal{M}$ of Theorem 3.3.

Proof. We shall give diagrammatic representations of the unitaries $u_{\vec{m},n} = u_{\vec{m},n}(s) \in A_{|\vec{m}|+sn}$ as follows. Let $t_{\vec{m}} = t_{\vec{m}}(s)$ be the unitary in $A_{|\vec{m}|}$ given by

$t_{\vec{m}} =$

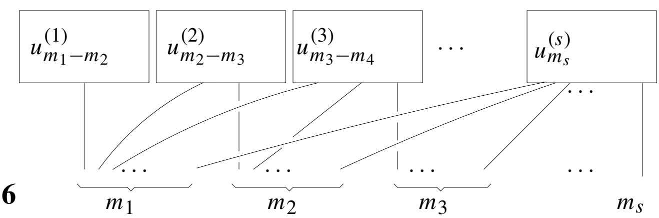


Figure 6

where the unitary $u_r^{(s)}$ is given by Figure 3 for $s > 1$ (with $n+1$ replaced by r) and is equal to id_r for $s = 1$, with any positive integer r.

The unitary $u_{\vec{m},n}$ is then defined from $t_{\vec{m}}$ and $u_n^{(s)}$ as in Figure 7.

$u^{(s)}_{\vec{m},n} =$

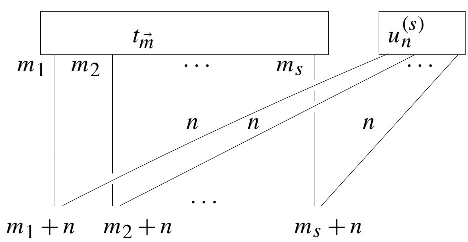


Figure 7

We proceed as in Lemma 3.2 to show that Diagram (3-3) is a commuting square. First we check that our diagram is a commuting diagram; we shall denote the vertical arrows by $\iota^{\otimes s}$ and ι respectively. Assume $s = 3$ again for simplicity. For $b \in A_{n+m_1} \otimes \cdots \otimes A_{n+m_s}$, we have $(\hat{u}_{\vec{m},n+1} \circ \iota^{\otimes s})(b) = = (\iota \circ \hat{u}_{\vec{m},n})(b)$. The commuting square property as well as k periodicity is shown in the same way as in Lemma 3.2.

It remains to show that the subfactor constructed in this lemma is conjugate to the one in Theorem 3.3. We define an automorphism Φ of the factor $\mathcal{M} =$

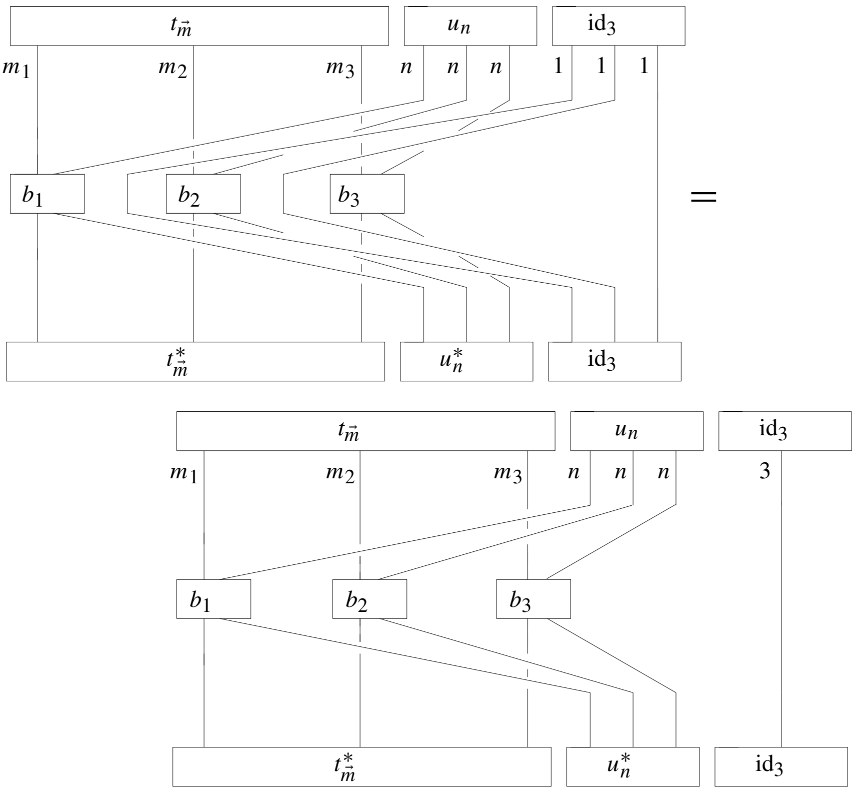


Figure 8. Diagram (3-3) is a commuting diagram.

$\lim \mathrm{ind}\, A_{sn+|\vec{m}|} = \lim \mathrm{ind}\, A_{s(n+m_1)}$ that will carry the subfactor defined here,

$$\hat{u}_{\vec{m}}(\mathcal{N}) = \lim \mathrm{ind}\, u_{\vec{m},n}(A_{n+m_1} \otimes \cdots \otimes A_{n+m_s})u^*_{\vec{m},n},$$

to the subfactor $\hat{u}(\mathcal{N}) = \lim \mathrm{ind}\, u_n A_n^{\otimes s} u_n^*$ from Theorem 3.3. Define Φ_n at the finite-dimensional level by

$$\begin{array}{ccc}
\mathcal{M} & & \mathcal{M} \\
\cup & \Phi_n & \cup \\
A_{sn+|\vec{m}|} & \longrightarrow & A_{s(n+m_1)} \\
\Cup & & \Cup \\
a & \longmapsto & u_{n+m_1} b_n \iota(u^*_{\vec{m},n} a u_{\vec{m},n}) b_n^* u^*_{n+m_1},
\end{array}$$

where $b_n \in A_{s(n+m_1)}$ is a unitary described in Figure 9 on the next page, and where $\iota : A_{sn+|\vec{m}|} \to A_{s(n+m_1)}$ is the usual inclusion (recall $m_1 \geq m_i$). Observe that

$$b_n(A_{n+m_1} \otimes \cdots \otimes A_{n+m_s} \otimes 1_{sm_1-|\vec{m}|})b_n^*$$

equals the image of the natural inclusion map $A_{n+m_1} \otimes \cdots \otimes A_{n+m_s} \to A_{n+m_1}^{\otimes s}$.

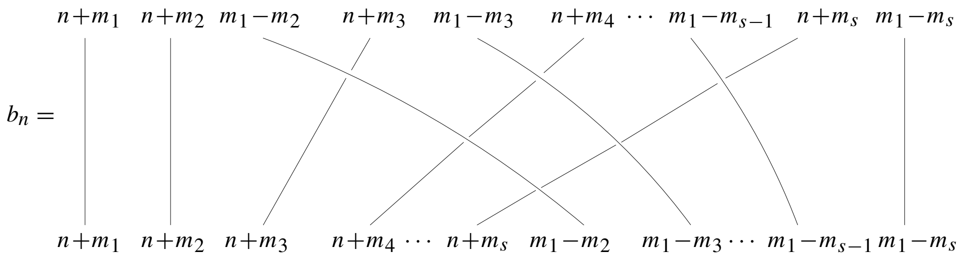


Figure 9. Pictorial description of $b_n \in A_{s(n+m_1)}$.

It is easy to check that the maps Φ_n are compatible with respect to n, and so we can define $\Phi : \mathcal{M} = \lim\operatorname{ind} A_{sn+|\vec{m}|} \to \mathcal{M} = \lim\operatorname{ind} A_{s(n+m_1)}$ by $\Phi := \lim\operatorname{ind} \Phi_n$. We observe that

$$u_{\vec{m},n}(a_1 \otimes \cdots \otimes a_s)u^*_{\vec{m},n} \overset{\Phi}{\mapsto} u_{n+m_1}(a_1 \otimes \cdots \otimes a_s)u^*_{n+m_1}$$

for $a_i \in A_{n+m_i}$, so Φ carries $\hat{u}_{\vec{m}}(\mathcal{N})$ to $\hat{u}(\mathcal{N})$. To check that Φ is an automorphism, one first observes that $\Phi = \lim\operatorname{ind} \Phi_n = \lim\operatorname{ind} \big(\hat{u}_{n+m_1} \circ \hat{b}_n \circ \iota \circ \hat{u}^*_{\vec{m},n}\big)$ (the "hat" morphisms denote the adjoint morphisms $\hat{w}(x) := wxw^*$). Then one checks that Φ has left inverse given by $\Phi_l^{-1} := \lim\operatorname{ind} \big(\hat{u}_{\vec{m},n+m_1} \circ \iota \circ \hat{b}^*_n \circ \hat{u}^*_{n+m_1}\big)$, where $\iota : A_{s(n+m_1)} \to A_{s(n+m_1)+m_1}$ also denotes the canonical inclusion, and a right inverse given by $\Phi_r^{-1} := \lim\operatorname{ind} \big(\hat{u}_{\vec{m},n} \circ \iota \circ b^*_{n-m_1} \circ \hat{u}^*_n\big)$, where $\iota : A_{sn} \to A_{sn+|\vec{m}|}$ again denotes the canonical inclusion, also observing that in the inductive limit the canonical inclusions turn out to be the identity map. □

3D. ***Endomorphisms.*** We now want to construct bimodules with respect to the just constructed factors $\mathcal{N}$ and $\mathcal{M}$ in the proof of the last theorem. This will be done according to the recipe described in Remark 1.3. To do so, we need to define the endomorphisms mentioned in the braid setting before, in the categorical setting.

Lemma 3.5. *Fix $m_i \in \mathbb{Z}_{\geq 0}$, $i = 1, 2, \ldots, s$, with $m_1 \geq m_2 \geq \cdots \geq m_s$.*

(a) *For $n \in \mathbb{N}$, the maps*

$$\begin{aligned} A_n^{\otimes s} &\to A_{m_1+n} \otimes \cdots \otimes A_{m_s+n} \\ a_1 \otimes \cdots \otimes a_s &\mapsto (1_{m_1} \otimes a_1) \otimes \cdots \otimes (1_{m_s} \otimes a_s) \end{aligned}$$

extend to an endomorphism $\mathrm{Shift}^{\mathcal{N}}_{\vec{m}} : \mathcal{N} \to \mathcal{N}$, where $\vec{m} := (m_1, \ldots, m_s)$.

(b) *Let $\hat{u}$ denote the embedding of $\mathcal{N} \hookrightarrow \mathcal{M}$. The endomorphism $\mathrm{Shift}^{\mathcal{N}}_{\vec{m}}$ extends to an endomorphism of $\mathcal{M}$, denoted by $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}$. In other words, we have a* *commuting diagram*

$$\begin{array}{ccc} \mathcal{N} & \xhookrightarrow{\hat{u}} & \mathcal{M} \\ {\scriptstyle \mathrm{Shift}^{\mathcal{N}}_{\vec{m}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Shift}^{\mathcal{M}}_{\vec{m}}} \\ \mathcal{N} & \xhookrightarrow[\hat{u}_{\vec{m}}]{} & \mathcal{M} \end{array}$$

(c) $(\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} \circ \hat{u})$ *only depends on the norm* $|\vec{m}|$ *of* $\vec{m}$, *and it is of the form*

$$\begin{aligned} A_n^{\otimes s} &\to A_{|\vec{m}|+sn} \\ (a_1 \otimes \cdots \otimes a_s) &\mapsto 1_{|\vec{m}|} \otimes u_n(a_1 \otimes \cdots \otimes a_s)u_n^*. \end{aligned}$$

Proof. (a) If $s = 1$, $\mathcal{N} = \mathcal{R} = \lim\operatorname{ind} A_n$, and we obtain the familiar one-sided shift Shift_m. For $s > 1$, $\mathcal{N} = \mathcal{R} \otimes \cdots \otimes \mathcal{R}$ (s factors) and $\mathrm{Shift}_{\vec{m}} = \mathrm{Shift}_{m_1} \otimes \cdots \otimes \mathrm{Shift}_{m_s}$. The following formalization of these facts will also be useful for the proofs of (b) and (c). Let $v_{\vec{m},n} \in A_{|\vec{m}|+sn}$ be the unitary image under ρ of the braid described by:

$v_{\vec{m},n} =$

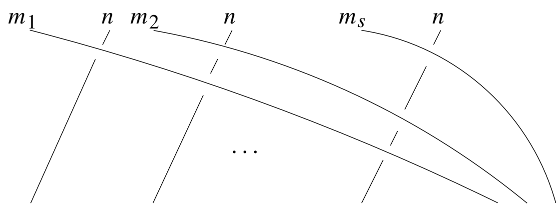


Figure 10

It is easy to see pictorially that for any element $a_1 \otimes \cdots \otimes a_s \in A_n^{\otimes s}$, the maps defined in the statement of (a) are given by

$$(a_1 \otimes \cdots \otimes a_n) \;\mapsto\; v_{\vec{m},n}(a_1 \otimes \cdots \otimes a_n \otimes \mathrm{id}_{|\vec{m}|})v^*_{\vec{m},n} \;\; \in A_{n+m_1} \otimes \cdots \otimes A_{n+m_s}.$$

That these maps extend to the von Neumann algebra inductive limit $\mathcal{N} = \lim\operatorname{ind} A_n^{\otimes s}$ follows from the fact that the following are commuting diagrams with respect to the canonical inclusions:

$$\begin{array}{ccccc} A_n^{\otimes s} & \hookrightarrow & A_n^{\otimes s} \otimes A_{|\vec{m}|} & \xrightarrow{\hat{v}_{\vec{m},n}} & A_{n+m_1} \otimes \cdots \otimes A_{n+m_s} \\ \downarrow & & \downarrow & & \downarrow \\ A_{n+1}^{\otimes s} & \hookrightarrow & A_{n+1}^{\otimes s} \otimes A_{|\vec{m}|} & \xrightarrow{\hat{v}_{\vec{m},n+1}} & A_{n+1+m_1} \otimes \cdots \otimes A_{n+1+m_s} \end{array} \tag{3-4}$$

and from the fact that the maps are norm and trace preserving. We denote the resulting endomorphism by $\mathrm{Shift}^{\mathcal{N}}_{\vec{m}}$.

(b) We shall extend the map $\mathrm{Shift}^{\mathcal{N}}_{\vec{m}}$ to $\mathcal{M}$ after embedding $\mathcal{N}$ in $\mathcal{M}$ via $\hat{u}$ (given by the inductive limit of conjugation of unitaries u_n or $u_{\vec{m},n}$ as in Figures 7 and 3). At the finite-dimensional level we define $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} : \lim\mathrm{ind}\, A_{sn} \to \lim\mathrm{ind}\, A_{|\vec{m}|+sn}$ by

$$\widehat{\omega_n} : \; A_{sn} \hookrightarrow A_{|\vec{m}|+sn} \to A_{|\vec{m}|+sn}, \tag{3-5}$$

where the first arrow stands for the standard inclusion $a \in A_{sn} \mapsto a \otimes 1 \in A_{|\vec{m}|+sn}$, and where the second arrow stands for conjugation by the unitary $\omega_n = \omega_n(s, \vec{m}) \in A_{|\vec{m}|+sn}$ defined by

$$\omega_n := u_{\vec{m},n} v_{\vec{m},n} (u_n^* \otimes \mathrm{id}_{|\vec{m}|}); \tag{3-6}$$

here $u_{\vec{m},n}$ and $v_{\vec{m},n}$ are given by Figures 6, 7, and 10. We give a diagrammatic representation for $s = 3$ in Figure 11, with $b \in A_{sn}$:

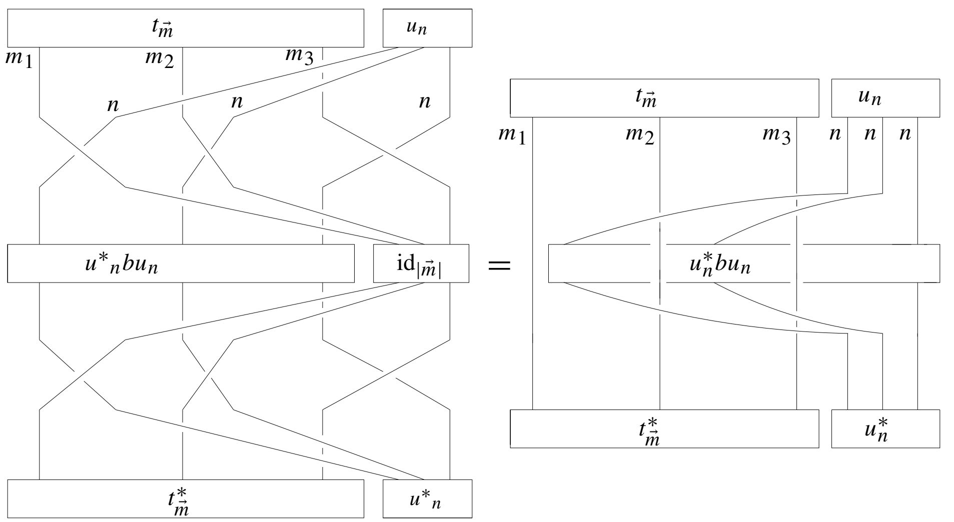


Figure 11. Pictorial representation of $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(b) \in A_{|\vec{m}|+sn}$, for $b \in A_{sn}$ (s=3).

We want to show that these maps extend to a well-defined map $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}$ on the inductive limit $\lim\mathrm{ind}\, A_{sn}$, i.e., we have to show that $\widehat{\omega}_{n+1}(\iota(b)) = \iota\big(\widehat{\omega}_n(b)\big)$, where we use the notation ι for the standard inclusions of $A_{sn} \to A_{s(n+1)}$ as well as for $A_{|\vec{m}|+sn} \to A_{|\vec{m}|+s(n+1)}$. To show this, we need the inductive property of the unitaries $u_n^{(s)}$ mentioned already at the braid level, seen in Figure 3, to write $u_{\vec{m},n+1}$ in terms of $u_{\vec{m},n}$ and of $\mathrm{i}d_s$. We then have for $b \in A_{sn}$ that $\widehat{\omega}_{n+1}(\iota(b)) = \iota\big(\widehat{\omega}_n(b)\big)$, as shown in Figure 12. Hence $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} = \lim\mathrm{ind}\,\widehat{\omega}_n$ is well defined.

We still need to show that $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}$ extends $\mathrm{Shift}^{\mathcal{N}}_{\vec{m}}$, i.e., that $(\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} \circ \hat{u}) = (\hat{u}_{\vec{m}} \circ \mathrm{Shift}^{\mathcal{N}}_{\vec{m}})$. From the definition, for $a = a_1 \otimes \cdots \otimes a_s \in A_n^{\otimes s}$,

$$\begin{aligned}(\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} \circ \lim\mathrm{ind}\, \hat{u}_n)(a) &= (\hat{\omega}_n \circ \iota \circ \hat{u}_n)(a) = (\hat{u}_{\vec{m},n} \circ \hat{v}_{\vec{m},n})(a \otimes \mathrm{id}_{|\vec{m}|}) \\ &= (\lim\mathrm{ind}\, \hat{u}_{\vec{m},n} \circ \mathrm{Shift}^{\mathcal{N}}_{\vec{m}})(a).\end{aligned}$$

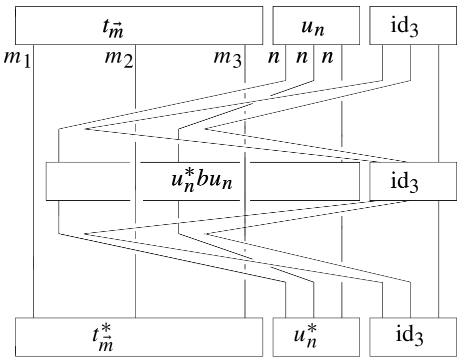
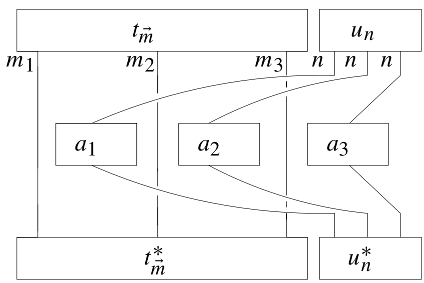


Figure 12. $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}$ is well-defined.

(c) This follows from the definition. Take $(a_1 \otimes \cdots \otimes a_s) \in A_n^{\otimes s}$. Using Figure 11, we obtain that $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(u_n(a_1 \otimes a_2 \otimes a_3)u_n^*)$ equals



and this in turn equals $1_{|\vec{m}|} \otimes u_n(a_1 \otimes a_2 \otimes a_3)u_n^*$. □

Proposition 3.6. *Let* $\mathrm{Shift}_{\vec{m}}$ *be as in Lemma 3.5.*

(a) $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(\mathcal{M}) \subset \mathcal{M}$ *is an inclusion of II_1 factors with index* $(\dim(X))^{2|\vec{m}|}$, *where* $|\vec{m}| = \sum m_i$ *and* $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(\mathcal{M})' \cap \mathcal{M}$ *has a subalgebra isomorphic to*

$$A_{m_1} \otimes \cdots \otimes A_{m_s}.$$

(b) $(\hat{u}_{\vec{m}} \circ \mathrm{Shift}_{\vec{m}}^{\mathcal{N}})(\mathcal{N}) \subset \mathcal{M}$ *is an inclusion of II_1 factors with index*

$$[\mathcal{M} : \mathcal{N}](\dim(X))^{2|\vec{m}|}$$

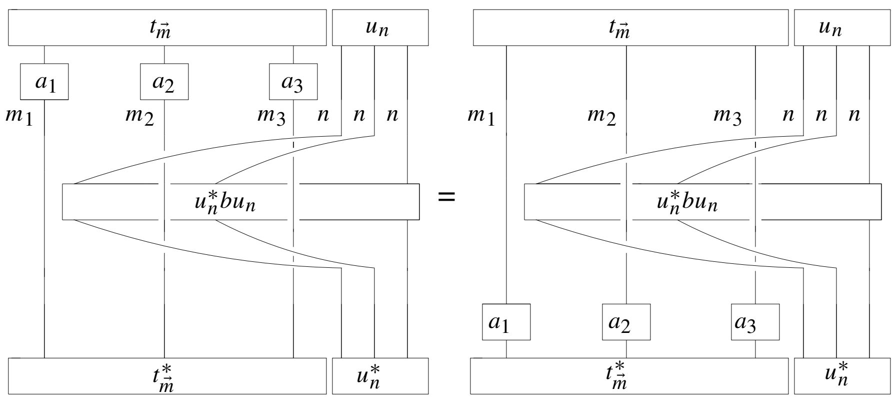
and relative commutant $(\hat{u}_{\vec{m}} \circ \mathrm{Shift}_{\vec{m}}^{\mathcal{N}})(\mathcal{N})' \cap \mathcal{M} \cong A_{|\vec{m}|}$.

(c) $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}}(\mathcal{N}) \subset \mathcal{N}$ *is an inclusion of II_1 factors with index* $(\dim(X))^{2|\vec{m}|}$ *and relative commutant* $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}}(\mathcal{N})' \cap \mathcal{N} \cong A_{m_1} \otimes \cdots \otimes A_{m_s}$.

Proof. For (a), we first show that the maps $\hat{\omega}_n$ in (3-5) define periodic commuting squares with respect to n (which generate $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(\mathcal{M}) \subset \mathcal{M}$ by definition). For this, one simply uses the fact that these maps are compositions involving the maps $\hat{v}_{\vec{m},n}$,

$\hat{u}_{\vec{m},n}$ and $\hat{u}_n$ (see (3-6)). The desired diagrams are then built from compositions of the periodic commuting squares in diagrams (3-4), (3-2) and (3-3); see Lemma 3.2 and Lemma 3.4. Hence the desired diagrams are commuting squares. Periodicity is shown as in Lemma 3.2, and we can use the formula for the index, as done there. It follows from parts (b) and (d) of Lemma 3.1 that the ratio of the square lengths of the weight vectors for A_{sn} and $A_{sn+|\vec{m}|}$ is equal to $(\dim X)^{2|\vec{m}|}$.

The statement on the relative commutant follows from the definition of $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}$. Let us represent $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(b)$, for $b \in A_{sn}$ ($s = 3$ to make things simpler) as it appears in Figure 11. Then for $a \in (t_{\vec{m}} \otimes 1_{sn})(A_{m_1} \otimes \cdots \otimes A_{m_s} \otimes 1_{sn})(t^*_{\vec{m}} \otimes 1_{sn}) \in A_{|\vec{m}|+sn}$ we have $a\,\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(b) = \mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(b)a$, as follows from this figure representing the two sides:



Hence, $(t_{\vec{m}} \otimes 1_{sn})(A_{m_1} \otimes \cdots \otimes A_{m_s} \otimes 1_{sn})(t^*_{\vec{m}} \otimes 1_{sn}) \cong A_{m_1} \otimes \cdots \otimes A_{m_s}$ commutes with $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(b)$ for $b \in A_{sn}$, for every n, so that $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M}$ has a subalgebra isomorphic to $A_{m_1} \otimes \cdots \otimes A_{m_s}$. This proves the last statement of (a).

For (b), one observes that the generating square for $(\hat{u}_{\vec{m}} \circ \mathrm{Shift}^{\mathcal{N}}_{\vec{m}})(\mathcal{N}) \subset \mathcal{M}$ is obtained from the double-square given in (3-4) of proof of Lemma 3.5(a) (which defines the endomorphism $\mathrm{Shift}^{\mathcal{N}}_{\vec{m}}$), composed with the square given in (3-3) (which defines the inclusion $\hat{u}_{\vec{m}} : \mathcal{N} \to \mathcal{M}$). These squares are commuting squares (the one in (3-4) because it involves maps that are trace preserving, and the one in (3-3) was shown in Lemma 3.4). So their composition, which generates $(\hat{u}_{\vec{m}} \circ \mathrm{Shift}^{\mathcal{N}}_{\vec{m}})(\mathcal{N}) \subset \mathcal{M}$, gives also a commuting square. The indices for parts (b) and (c) can now be computed as before, using Lemma 3.1. It only remains to show the statement about the relative commutant.

Lemma 3.5(c) implies that $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(u_n A_n^{\otimes s} u_n^*) = 1_{|\vec{m}|} \otimes u_n A_n^{\otimes s} u_n^*$ for every n. So $A_{|\vec{m}|} \otimes 1_{sn}$ commutes with $\mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(u_n A_n^{\otimes s} u_n^*)$ for every n and $(\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} \circ \hat{u})(\mathcal{N})' \cap \mathcal{M}$ $(= (\hat{u}_{\vec{m}} \circ \mathrm{Shift}^{\mathcal{N}}_{\vec{m}})(\mathcal{N})' \cap \mathcal{M})$ has a subalgebra isomorphic to $A_{|\vec{m}|}$. Conversely, for the other inclusion, we apply a dimension upper bound result for relative commutants of inclusions generated by periodic commuting squares (see [Wenzl 1988], Theorem 1.6):

$$\dim\big((\mathrm{Shift}^{\mathcal{M}}_{\vec m}\circ\hat u)(\mathcal{N})'\cap\mathcal{M}\big)\le\dim\big((1_{|\vec m|}\otimes u_nA_n^{\otimes s}u_n^*)'_p\cap(A_{|\vec m|+sn})_p\big)$$
$$\le\dim(A_{|\vec m|+sn})_p,$$

for any projection $p\in 1_{|\vec m|}\otimes u_nA_n^{\otimes s}u_n^*$, and n large. If n is divisible by k and sufficiently large, then $X^{\otimes n}$ contains a subobject isomorphic to $\mathbb{1}$; let $p_{\mathbb{1}}\in A_n$ be the projection onto it. If $p=1_{|\vec m|}\otimes u_n(p_{\mathbb{1}}^{\otimes s})u_n^*\in A_{|\vec m|+ns}$, then we have $pA_{|\vec m|+ns}p\cong A_{|\vec m|}$. This shows (b).

For (c), it is even easier than in (a) to show that the generating Diagram (3-4) for $\mathrm{Shift}^{\mathcal{N}}_{\vec m}(\mathcal{N})\subset\mathcal{N}$ is a periodic commuting square; one can see that pictorially, as it was done in Lemmas 3.2 and 3.4, which is left to the reader. The statement about the relative commutant in (c) is proved in the same manner as in (b): By definition, $\mathrm{Shift}^{\mathcal{N}}_{\vec m}(a_1\otimes\cdots\otimes a_s)=(1_{m_1}\otimes a_1)\otimes\cdots\otimes(1_{m_s}\otimes a_s)$. Thus, $(A_{m_1}\otimes 1_n)\otimes\cdots\otimes(A_{m_s}\otimes 1_n)$ commutes with $\mathrm{Shift}^{\mathcal{N}}_{\vec m}(A_n^{\otimes s})$ for every n, and so $\mathrm{Shift}^{\mathcal{N}}_{\vec m}(\mathcal{N})'\cap\mathcal{N}$ has a subalgebra isomorphic to $A_{m_1}\otimes\cdots\otimes A_{m_s}$. For the other inclusion we apply again the upper bound result for the dimension of the relative commutant:

$$\dim\big(\mathrm{Shift}^{\mathcal{N}}_{\vec m}(\mathcal{N})'\cap\mathcal{N}\big)$$
$$\le\dim\big((1_{m_1}\otimes A_n)\otimes\cdots\otimes(1_{m_s}\otimes A_n)\big)'_p\cap(A_{n+m_1}\otimes\cdots\otimes A_{n+m_s})_p$$
$$\le\dim(A_{n+m_1}\otimes\cdots\otimes A_{n+m_s})_p,$$

for any projection $p\in(1_{m_1}\otimes A_n)\otimes\cdots\otimes(1_{m_s}\otimes A_n)$. One shows as in (b) that for $p=(1_{m_1}\otimes p_{\mathbb{1}})\otimes\cdots\otimes(1_{m_s}\otimes p_{\mathbb{1}})$ we have $(A_{n+m_1}\otimes\cdots\otimes A_{n+m_s})_p\cong A_{m_1}\otimes\cdots\otimes A_{m_s}$, from which one deduces (c). □

4. Bimodules and the principal graph

4A. ***Examples of bimodules.*** We are going to construct systems of bimodules in order to calculate the principal and the dual principal graph, as described in Proposition 1.8. This will be done using the endomorphisms Shift defined in the last section.

The $\mathcal{N}$–$\mathcal{N}$-bimodules. Let $\lambda_i\in\Lambda$ and let A_{m_i,λ_i} be the simple component of A_{m_i} corresponding to the simple object $X_{\lambda_i}\subset X^{\otimes m_i}$ with m_i being large multiples of k for $i=1,2,\dots,s$. We first fix minimal projections $p_{\lambda_i}\in A_{m_i,\lambda_i}$. Define $p_{\vec\lambda}=p_{\lambda_1}\otimes\cdots\otimes p_{\lambda_s}$, where $\vec\lambda=(\lambda_1,\dots,\lambda_s)$. The underlying Hilbert space will be given by

$$L^2(\mathcal{N},\mathrm{tr})p_{\vec\lambda}:=\{\zeta p_{\vec\lambda},\ \zeta\in L^2(\mathcal{N},\mathrm{tr})\}.$$

The $\mathcal{N}$–$\mathcal{N}$ bimodule structure is defined by

$$x.\xi.y=x\xi\,\mathrm{Shift}^{\mathcal{N}}_{\vec m}(y),\quad \text{for } x,y\in\mathcal{N},\ \xi\in L^2(\mathcal{N},\mathrm{tr})p_{\vec\lambda},$$

where we use the usual right and left multiplication in $\mathcal{N}$ on the right hand side. It follows from Proposition 3.6 that this indeed defines an $\mathcal{N}$–$\mathcal{N}$ bimodule structure on $L^2(\mathcal{N}, \mathrm{tr})p_{\vec{\lambda}}$.

Definition 4.1. The $\mathcal{N}$–$\mathcal{N}$ bimodules defined above will be denoted by $N_{\vec{\lambda},\vec{m}}$.

The $\mathcal{M}$–$\mathcal{N}$-bimodules. Again let $\vec{m} := (m_1, \dots, m_s) \in \mathbb{N}^s$, with $m := m_1 + \cdots + m_s$. We fix a minimal projection $p_\mu \in (\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} \circ \hat{u})(\mathcal{N})' \cap \mathcal{M} \cong A_m$ (see Proposition 3.6) belonging to the simple component of A_m labeled $\mu \in \Lambda$. The underlying Hilbert space for all these bimodules will be given by

$$L^2(\mathcal{M}, \mathrm{tr})p_\mu := \{\zeta p_\mu \ / \ \zeta \in L^2(\mathcal{M}, \mathrm{tr})\}.$$

The $\mathcal{M}$–$\mathcal{N}$ bimodule structure is defined by

$$x.\xi.y = x\xi(\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} \circ \hat{u})(y), \quad \text{for } x \in \mathcal{M},\ y \in \mathcal{N},\ \xi \in L^2(\mathcal{M}, \mathrm{tr})p_\mu.$$

Definition 4.2. The $\mathcal{M}$–$\mathcal{N}$-bimodules defined above will be denoted by $H_{\mu,\vec{m}}$.

The $\mathcal{N}$–$\mathcal{M}$-bimodules. With notations as in the last definition, we define similarly $\mathcal{N}$–$\mathcal{M}$-bimodules based on Hilbert spaces $p_\mu L^2(\mathcal{M}, \mathrm{tr}) := \{p_\mu\zeta \ / \ \zeta \in L^2(\mathcal{M}, \mathrm{tr})\}$, and with the $\mathcal{N}$–$\mathcal{M}$ bimodule structure defined by

$$x.\xi.y = (\mathrm{Shift}^{\mathcal{M}}_{\vec{m}} \circ \hat{u})(x)\xi y, \quad \text{for } x \in \mathcal{N},\ y \in \mathcal{M},\ \xi \in p_\mu L^2(\mathcal{M}, \mathrm{tr}).$$

Definition 4.3. The $\mathcal{N}$–$\mathcal{M}$-bimodules defined above will be denoted by $K_{\mu,\vec{m}}$.

The $\mathcal{M}$–$\mathcal{M}$-bimodules. Similarly as for the $\mathcal{N}$–$\mathcal{N}$-bimodules, we fix minimal projections $p_{\lambda_i} \in A_{m_i,\lambda_i}$, with $\lambda_i \in \Lambda$, but now only requiring that $\sum m_i$ being divisible by k. The underlying Hilbert space for all these bimodules will be given by

$$p_{\vec{\lambda}}L^2(\mathcal{M}, \mathrm{tr}) := \{p_{\vec{\lambda}}\zeta / \ \zeta \in L^2(\mathcal{M}, \mathrm{tr})\}.$$

The $\mathcal{M}$–$\mathcal{M}$ bimodule structure is defined by

$$x.\xi.y = \mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(x)\xi y, \quad \text{for } x, y \in \mathcal{M},\ \xi \in p_{\vec{\lambda}}L^2(\mathcal{M}, \mathrm{tr}),$$

Definition 4.4. The $\mathcal{M}$–$\mathcal{M}$-bimodules defined above will be denoted by $M_{\vec{\lambda},\vec{m}}$.

Lemma 4.5. *Let the notation be as above.*

(a) *If we view both $N_{\vec{\lambda},\vec{m}}$ and $H_{\nu,\vec{m}}$ as left $\mathcal{N}$-modules, then*

$$\dim_{\mathcal{N}} N_{\vec{\lambda},\vec{m}} = d_{\vec{\lambda}}/(\dim X)^{|\vec{m}|} \quad \textit{and} \dim_{\mathcal{N}} H_{\nu,\vec{m}} = d_\nu[\mathcal{M} : \mathcal{N}]/(\dim X)^{|\vec{m}|}.$$

Moreover, we have $\mathrm{ind}(N_{\vec{\lambda},\vec{m}}) = d^2_{\vec{\lambda}} = \mathrm{ind}(M_{\vec{\lambda},\vec{m}})$, *where* $d_{\vec{\lambda}} = \prod d_{\lambda_i}$, *and* $\mathrm{ind}(H_{\nu,\vec{m}}) = d^2_\nu[\mathcal{M} : \mathcal{N}]$.

(b) *If* $|\vec{m}| = |\vec{k}|$, *then* $H_{\mu,\vec{m}} \cong H_{\mu,\vec{k}}$ *as $\mathcal{M}$–$\mathcal{N}$-bimodules, and* $K_{\mu,\vec{m}} \cong K_{\mu,\vec{k}}$ *as $\mathcal{N}$–$\mathcal{M}$-bimodules.*

(c) *If* $|\vec{m}| = |\vec{k}|$, *we have*

$$\mathrm{Hom}_{\mathcal{M}-\mathcal{M}}(M_{\vec{\lambda},\vec{m}}, M_{\vec{\mu},\vec{k}}) \subset \mathrm{Hom}_{\mathcal{N}-\mathcal{M}}(M_{\vec{\lambda},\vec{m}}, M_{\vec{\mu},\vec{k}}) \cong \mathrm{Hom}_{\mathscr{C}}(X_{\vec{\lambda}}, X_{\vec{\mu}}),$$

where $X_{\vec{\lambda}} = \otimes_{i=1}^s X_{\lambda_i}$ *and* $X_{\vec{\mu}} = \otimes_{i=1}^s X_{\mu_i}$.

Proof. It is well known (see [Jones 1983], for instance) that $\dim_{\mathcal{N}} L^2(\mathcal{N}, \mathrm{tr})p = \mathrm{tr}(p)$ and $\dim_{\mathcal{N}} L^2(\mathcal{M}, \mathrm{tr})q = \mathrm{tr}(q)[\mathcal{M} : \mathcal{N}]$ for any projections $p \in \mathcal{N}$, $q \in \mathcal{M}$. The dimension statements in (a) follow. For the index statements in (a), let ℓ and r denote left and right multiplication by $\mathcal{N}$ on $L^2(\mathcal{N}, \mathrm{tr})$ or suitable submodules of it. Observe that $\ell(\mathcal{N})'_{|L^2(\mathcal{N},\mathrm{tr})p}$ is equal to $r(p\mathcal{N}p)$ for any $p \in \mathrm{Shift}_{\vec{m}}(\mathcal{N})' \cap \mathcal{N}$. Recall that $\mathrm{Shift}_{\vec{m}}(\mathcal{N}) \subset \mathcal{N}$ has index $(\dim X)^{2|\vec{m}|}$. Moreover, $\mathrm{tr}(p_{\vec{\lambda}}) = d_{\vec{\lambda}}/(\dim X)^{|\vec{m}|}$ for a minimal idempotent $p_{\vec{\lambda}} \in \mathrm{Shift}_{\vec{m}}(\mathcal{N})' \cap \mathcal{N}$; see Proposition 3.6. Using the formula for local indices [Wenzl 1988, Theorem 1.5(iii)] and the index formula in Proposition 3.6(c), we obtain

$$\mathrm{ind}(N_{\vec{\lambda},\vec{m}}) = [p_{\vec{\lambda}}\mathcal{N}p_{\vec{\lambda}} : p_{\vec{\lambda}}\mathrm{Shift}_{\vec{m}}(\mathcal{N})] = \mathrm{tr}(p_{\vec{\lambda}})^2(\dim X)^{2|\vec{m}|} = (d_{\vec{\lambda}})^2.$$

The indices for $H_{\nu,\vec{m}}$ and $M_{\vec{\lambda},\vec{m}}$ are computed similarly. By Lemma 3.5, (c), we have $\mathrm{Shift}^{\mathcal{N}}_{\vec{m}} = \mathrm{Shift}^{\mathcal{N}}_{\vec{k}}$, from which (b) follows.

Let ${}_{\vec{m}}L^2(\mathcal{M}, \mathrm{tr})$ be the Hilbert space $L^2(\mathcal{M}, \mathrm{tr})$ with $\mathcal{M}$–$\mathcal{M}$ bimodule structure $x.\xi.y = \mathrm{Shift}^{\mathcal{M}}_{\vec{m}}(x)\xi y$ for $x, y \in \mathcal{M}$ and $\xi \in L^2(\mathcal{M}, \mathrm{tr})$. Define ${}_{\vec{k}}L^2(\mathcal{M}, \mathrm{tr})$ similarly. These bimodules are isomorphic as $\mathcal{N}$–$\mathcal{M}$ bimodules, again by Lemma 3.5(c). This, combined with Lemma 3.5(b), results in

$$\mathrm{Hom}_{\mathcal{M}-\mathcal{M}}({}_{\vec{m}}L^2(\mathcal{M}, \mathrm{tr}), {}_{\vec{k}}L^2(\mathcal{M}, \mathrm{tr})) \subset \mathrm{Hom}_{\mathcal{N}-\mathcal{M}}({}_{\vec{m}}L^2(\mathcal{M}, \mathrm{tr}), {}_{\vec{k}}L^2(\mathcal{M}, \mathrm{tr})) \cong$$

$$\cong \mathrm{End}_{\mathcal{N}-\mathcal{M}}({}_{\vec{m}}L^2(\mathcal{M}, \mathrm{tr})) \cong A_{|\vec{m}|} = \mathrm{End}_{\mathscr{C}}(X^{\otimes|\vec{m}|}), \qquad (*)$$

where the second isomorphism follows from Proposition 3.6(b), and (b). By construction, we have $M_{\vec{\lambda},\vec{m}} = p_{\vec{\lambda}}({}_{\vec{m}}L^2(\mathcal{M}, \mathrm{tr}))$ and $M_{\vec{\mu},\vec{k}} = p_{\vec{\mu}}({}_{\vec{k}}L^2(\mathcal{M}, \mathrm{tr}))$, where $p_{\vec{\lambda}} = p_{\lambda_1} \otimes \cdots \otimes p_{\lambda_s}$ and $p_{\vec{\mu}} = p_{\mu_1} \otimes \cdots \otimes p_{\mu_s}$. Hence we can interpret an element $f \in \mathrm{Hom}_{\mathcal{M}-\mathcal{M}}(M_{\vec{\lambda}}, M_{\vec{\mu}})$ as an element in $\mathrm{Hom}_{\mathcal{N}-\mathcal{M}}({}_{\vec{m}}L^2(\mathcal{M}, \mathrm{tr}), {}_{\vec{k}}L^2(\mathcal{M}, \mathrm{tr}))$ which satisfies $p_{\vec{\mu}} f p_{\vec{\lambda}} = f$. Using this together with $(*)$ proves claim (c). □

4B. *Principal graph.* Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s) \in (\Lambda')^s$, and let $L^{\nu}_{\vec{\lambda}}$ be the multiplicity of the object X_ν in $\otimes X_{\lambda_i}$. Observe that $L^{\nu}_{\vec{\lambda}}$ is also equal to the rank of the projection $\bigotimes p_{\lambda_i}$ in the simple component of $A_{|\vec{\lambda}|}$ labeled by ν.

In the following we will fix a vector $\vec{m} = (m_i)$ where all its coordinates are divisible by k, and with m_i large enough that all simple objects of $\mathscr{C}'$ will appear in $X^{\otimes m_i}$ for $i = 1, \ldots, s$. We shall hence omit $\vec{m}$ in the indices of the bimodules and will just write $N_{\vec{\lambda}}$ and K_ν for $N_{\vec{\lambda},\vec{m}}$ and $K_{\nu,\vec{m}}$, respectively.

Theorem 4.6. *With the notation as above*:

(a) *The bimodules* $N_{\vec{\lambda}}$ *and* H_ν *defined above are irreducible.*

(b) *The principal graph for* $\mathcal{N} \subset \mathcal{M}$ *is the connected component of the fusion graph from* $(\mathscr{C}')^s$ *to* $\mathscr{C}'$ *which contains the trivial object of* $\mathscr{C}$. *Recall that the even vertices of the fusion graph are labeled by* s*-tuples of elements of* Λ', *the odd vertices are labeled by the elements of* Λ', *and the vertex labeled by* $\vec{\lambda} = (\lambda_1, \dots, \lambda_s)$ *is connected with the vertex labeled by* ν *by* $L_{\vec{\lambda}}^{\nu}$ *edges.*

(c) *The subfactor* $\mathcal{N} \subset \mathcal{M}$ *has finite depth.*

Proof. Statement (a) follows from Proposition 3.6. For (b), it suffices to calculate the principal graph for the isomorphic inclusion $\mathcal{N} \subset \mathcal{M}$ given by $\hat{u}_{\vec{m}}$ (see Lemma 3.4). In this case the $\mathcal{M}$–$\mathcal{N}$ bimodule structure of $L^2(\mathcal{M}, \mathrm{tr})$ is given by $x.\xi.y = x\xi\hat{u}_{\vec{m}}(y)$. It follows from the definitions that $L^2(\mathcal{M}, \mathrm{tr}) \otimes_{\mathcal{N}} N_{\vec{\lambda}} \cong L^2(\mathcal{M}, \mathrm{tr})\, p_{\vec{\lambda}} \cong \oplus L_{\vec{\lambda}}^{\nu} H_\nu$; the decomposition of $L^2(\mathcal{M}, \mathrm{tr})\, p_{\vec{\lambda}}$ into irreducible $\mathcal{M}$–$\mathcal{N}$ bimodules follows from Proposition 3.6(b) and the remarks at the beginning of this subsection. Hence our system of bimodules $(N_{\vec{\lambda}})_{\vec{\lambda} \in (\Lambda')^s}$ and $(H_\nu)_{\nu \in \Lambda'}$ is closed under induction. To prove closedness under restriction, observe that the multiplicity of the $\mathcal{N}$–$\mathcal{N}$ bimodule $N_{\vec{\lambda}}$ in the $\mathcal{M}$–$\mathcal{N}$-bimodule H_ν, viewed as an $\mathcal{N}$–$\mathcal{N}$-bimodule, is equal to $L_{\vec{\lambda}}^{\nu}$, by Frobenius reciprocity. To show that $H_\nu \cong \bigoplus_{\vec{\lambda}} L_{\vec{\lambda}}^{\nu} N_{\vec{\lambda}}$ as an $\mathcal{N}$–$\mathcal{N}$-bimodule, it suffices to prove that both sides have the same dimension, i.e., by Lemma 4.5(a), that

$$[\mathcal{M} : \mathcal{N}] d_\nu = \sum_{\vec{\lambda}} L_{\vec{\lambda}}^{\nu} d_{\vec{\lambda}}. \tag{4-1}$$

For this observe that the dimension vectors for $A_n^{\otimes s}$ and A_{ns}, with n a multiple of k, are given by $\vec{t}_{ns} = (d_{\vec{\lambda}}/(\dim X)^{ns})_{\vec{\lambda}}$ and $\vec{v}_{ns} = (d_\nu/(\dim X)^{ns})_\nu$, with $\vec{\lambda} \in (\Lambda')^s$ and $\nu \in \Lambda'$. Observe that the subfactor $\mathcal{N} \subset \mathcal{M}$ is generated by the periodic sequence $(A_n^{\otimes s} \subset A_{ns})$, with the inclusion matrix for $A_n^{\otimes s} \subset A_{ns}$ given by $G = (L_{\vec{\lambda}}^{\nu})$ with $\vec{\lambda}$ and ν as above, provided $k|n$. Hence it follows from [Wenzl 1988], Theorem 1.5(ii), that $G\vec{v}_{ns} = [\mathcal{M} : \mathcal{N}]\vec{t}_{ns}$. This implies (4-1). Finally, if we choose $\vec{\lambda} = (\mathbb{1}, \mathbb{1}, \dots, \mathbb{1})$ (s times), where $\mathbb{1}$ stands for the trivial object of $\mathscr{C}$, $\mathrm{ind}(N_{\vec{\lambda}}) = 1$ and hence $N_{\vec{\lambda}}$ is weakly reduced from the trivial $\mathcal{N}$–$\mathcal{N}$ bimodule by Proposition 1.8(c). This shows (b), by Proposition 1.8(a). Statement (c) is a consequence of (b). □

Remark 4.7. The fusion graph from $(\mathscr{C}')^s$ to $\mathscr{C}'$ may not be connected. An easy example is obtained for $\mathscr{C}$ being the representation category of a finite abelian group G, where it decomposes into $|G|$ connected components.

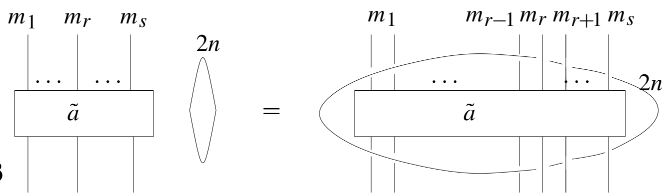
5. Dual principal graph

5A. ***Ring lemma.*** The precise structure of $\mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M}$ is still open after Proposition 3.6. To say more about this, we need the following lemma. Similar techniques have appeared before in topological quantum field theory, and within subfactors in work of Ocneanu and others; see [Evans and Kawahigashi 1998; Müger

2003], for example. Dual principal graphs in a similar setting (corresponding to the case $s = 2$) have also been calculated in [Izumi 2000] by somewhat different techniques.

Lemma 5.1. *If $a \in \mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M}$, take $\tilde{a} := t_{\vec{m}}^* a t_{\vec{m}}$ with $t_{\vec{m}} \in A_{|\vec{m}|}$ as in Figure 6. Then, for $r = 2, \ldots, s$, we have*

Figure 13

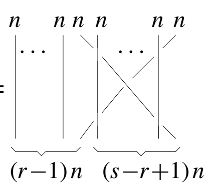


(*the picture is the translation of an algebraic expression of the form* $\iota_X^r(\tilde{a} \otimes 1_{sn})\iota_X^{r\,*} = \iota_X^r x_r(\tilde{a} \otimes 1_{sn}) x_r^* \iota_X^{r\,*}$ *for certain morphisms* x_r *and* ι_X^r *defined below*).

Proof. By Proposition 3.6(b) and our definition of the inductive limit, we have $\mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M} \cap A_{|\vec{m}|+ns} \subset A_{|\vec{m}|} \otimes 1_{ns}$. Take $t_{\vec{m}} \in A_{|\vec{m}|}$ as in Figure 6 in Lemma 3.4. If $a \in \mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M}$ then set

$$\tilde{a} \otimes 1_{sn} := (t_{\vec{m}}^* \otimes u_n^*) a (t_{\vec{m}} \otimes u_n) = t_{\vec{m}}^* a t_{\vec{m}} \otimes 1_{sn} \in A_{|\vec{m}|} \otimes 1_{sn},$$

and note that $\tilde{a} \otimes 1_{sn} \in \big((t_{\vec{m}}^* \otimes u_n^*)\mathrm{Shift}_{\vec{m}}(\mathcal{M})(t_{\vec{m}} \otimes u_n)\big)' \cap \mathcal{M}$. In particular, take the element $x_r := (t_{\vec{m}}^* \otimes u_n^*)\mathrm{Shift}_{\vec{m}}(u_n T_r u_n^*)(t_{\vec{m}} \otimes u_n)$, for $r = 2, \ldots, s$, where $T_r \in A_{sn}$ is obtained from the braiding morphisms and can be represented by

$T_r =$ 

We use Figure 11 in the proof of Lemma 3.5 to see that x_r is as in Figure 14.

Also note that x_r is a unitary, so that $(\tilde{a} \otimes 1_{sn}) x_r = x_r (\tilde{a} \otimes 1_{sn})$ implies $(\tilde{a} \otimes 1_{sn}) = x_r(\tilde{a} \otimes 1_{sn}) x_r^*$:

To obtain the relations in our statement in Figure 15, we proceed by closing strands in Figure 15 with cups and caps to form the loops (the caps and cups correspond to dual morphisms as described in Section 2B). This is done as follows: Let rh and lh be the left and right hand sides of Figure 15. Then we also obtain $rh \otimes 1_{(\bar{X})^{sn}} = lh \otimes 1_{(\bar{X})^{sn}}$. We now multiply both sides with $1_{X^{\otimes |\vec{m}|}} \otimes i_{X^{\otimes sn}}$ from the right (below) and by its conjugate from the left (above). The morphism $i_{X^{\otimes sn}}$ and its conjugate correspond to the pictures in Figure 16, which are obtained from the properties of the duality morphisms; see Section 2B. It is easy to check that we

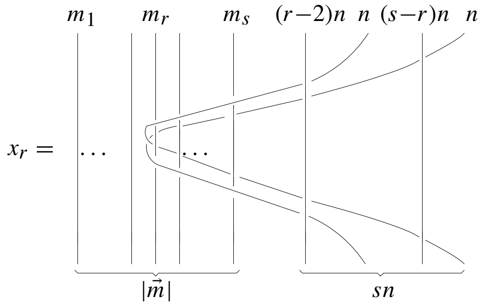


Figure 14. $x_r := (t_{\vec{m}}^* \otimes u_n^*)\mathrm{Shift}_{\vec{m}}(u_n T_r u_n^*)(t_{\vec{m}} \otimes u_n)$.

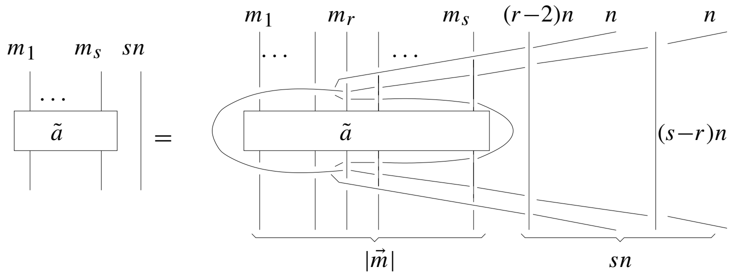


Figure 15. $(\tilde{a} \otimes 1_{sn}) = x_r(\tilde{a} \otimes 1_{sn})x_r^*$.

Figure 16. $\iota_{X^{\otimes(sn)}}^*$ and $\iota_{X^{\otimes(sn)}}$.

obtain $(s-2)n$ unlinked circles on the right hand side, which correspond to the scalar $(\dim X)^{(s-2)n}$. Canceling this with the same number of circles on the left hand side, we obtain the picture as claimed in the statement. □

Corollary 5.2. *The equality in Lemma 5.1 still holds if the rings on both sides are labeled by an irreducible object in $\mathscr{C}'$.*

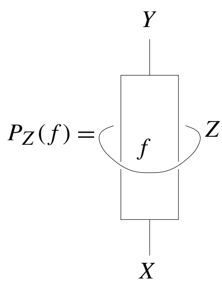
Proof. Assume that $k|n$. Then the proof of Lemma 5.1 works as well if we multiply T_r by $1_{(r-1)n} \otimes p_{\mathbb{1}} \otimes 1_{(s-r+1)n} \otimes p_\mu$ where $p_{\mathbb{1}}$ and p_μ are projections onto irreducible objects appearing in $X^{\otimes n}$ isomorphic to $\mathbb{1}$ and to X_μ, respectively. Going through the proof of Lemma 5.1, we obtain the statement of the corollary at the end. □

5B. ***Notations and preliminaries.*** For any braided semisimple tensor category $\mathscr{C}$ we can define a scalar $s_{\lambda\mu} = \mathrm{Tr}(c_{\mu,\lambda}c_{\lambda,\mu})$, where $c_{\lambda,\mu}$ is the braiding morphism for $X_\lambda \otimes X_\mu$. The S-matrix is then given by $(s_{\lambda\mu})$, where the rows and columns are labeled by the simple objects of $\mathscr{C}$.

Let now $\mathscr{D}$ be a full subcategory of $\mathscr{C}$. We define $\mathscr{T}_\mathscr{D}$ to be the set of simple objects X_λ in $\mathscr{D}$ for which $s_{\lambda\mu} = \dim(X_\lambda)\dim(X_\mu)$ for all simple objects X_μ in $\mathscr{C}'$. We will primarily be interested only in the cases $\mathscr{D} = \mathscr{C}$ and $\mathscr{D} = \mathscr{C}'$. We usually assume $\mathscr{D}$ to be fixed, in which case we may just write $\mathscr{T}$ for $\mathscr{T}_\mathscr{D}$.

Let $X = \bigoplus_\lambda m_\lambda X_\lambda$, $Y = \bigoplus n_\lambda X_\lambda$ be objects in $\mathscr{C}$, and let $f : X \to Y$ be a morphism. Then f can be written as $f = \bigoplus f_\lambda$, where $f_\lambda : m_\lambda X_\lambda \to n_\lambda X_\lambda$. For given $f : X \to Y$, we define the morphism $f_\mathscr{T} : X_\mathscr{T} \to Y_\mathscr{T}$, where $f_\mathscr{T} = \bigoplus_{X_\lambda \in \mathscr{T}} f_\lambda$, and $X_\mathscr{T}$, $Y_\mathscr{T}$ are defined accordingly. Also, we define $p_\mathscr{T}(X) \in \mathrm{End}(X)$ to be the projection from X onto $X_\mathscr{T}$.

For a fixed object Z in $\mathscr{C}$ and a morphism $f : X \to Y$ we define the morphism $P_Z(f) : X \to Y$ by



Of course this picture corresponds to an algebraic expression involving rigidity and braiding morphisms. One can also check that for $Z = Z_1 \otimes Z_2$, the operation P_Z is also given by a picture involving two parallel rings labeled by Z_1 and Z_2. If X_λ, X_μ are simple objects in $\mathscr{C}$, it follows from the definitions that $P_{X_\mu}(1_{X_\lambda}) = (s_{\lambda\mu}/d_\lambda)1_{X_\lambda}$. For a formal linear combination $\Omega = \sum_\mu \omega_\mu X_\mu$, with X_μ simple objects in $\mathscr{C}$, the morphism $P_\Omega(f)$ can also be expressed as the sum $\sum_\mu \omega_\mu P_{X_\mu}(f)$. The following lemma is well-known and follows from the definitions:

Lemma 5.3. *With notations above*,

$$P_{X_\mu}(f) = \sum_\lambda \frac{s_{\lambda\mu}}{d_\lambda} f_\lambda \quad \text{and} \quad P_\Omega(f) = \sum_{\lambda,\mu} \omega_\mu \frac{s_{\lambda\mu}}{d_\lambda} f_\lambda.$$

We now state a straightforward generalization of the results in [Bruguières 2000, Lemma 1.3].

Proposition 5.4. *Fix the category $\mathscr{D}$ and let $\mathscr{T} = \mathscr{T}_\mathscr{D}$. There exists a linear combination $\Omega = \sum_{\mu\in\Lambda'} \omega_\mu X_\mu$ such that $P_\Omega(f) = f_\mathscr{T}$ for any morphism f in $\mathscr{D}$. Moreover, $\sum_\mu \omega_\mu d_\mu = 1$.*

Proof. We adapt the arguments in the proofs of [Bruguières 2000, Lemmas 1.2 and 1.3] to our setting. By Lemma 5.3, we have to find scalars ω_μ, $\mu \in \Lambda'$ such that $\sum_{\mu\in\Lambda'} \omega_\mu (s_{\lambda\mu}/d_\lambda)$ is equal to 1 or 0 depending on whether $X_\lambda \in \mathcal{T}$ or not. Observe that the second statement will also follow from this as $s_{\lambda\mu} = d_\lambda d_\mu$ for $X_\lambda \in \mathcal{T}$.

To do so, pick an object $X = \bigoplus_{\lambda\in\Lambda(\mathcal{D})} m_\lambda X_\lambda$ in $\mathcal{D}$ with $m_\lambda \neq 0$ for all $\lambda \in \Lambda(\mathcal{D})$. Let z_λ denote the corresponding minimal idempotent in the center of $\mathrm{End}(X)$. Then $P_{X_\mu}(z_\lambda) = \frac{s_{\lambda\mu}}{d_\lambda} z_\lambda$. It also follows immediately by drawing pictures that $P_{Z_1\otimes Z_2}(f) = P_{Z_1}(P_{Z_2}(f))$ for any $f \in \mathrm{End}(X)$ (see also the proof of [Bruguières 2000], Lemma 1.2). Hence we obtain a representation of the fusion algebra of $\mathcal{C}'$ on V, the $\mathbb{C}$-span of the idempotents z_λ, $\lambda \in \Lambda(\mathcal{D})$, with each P_{X_μ} acting via a diagonal matrix with respect to the basis of z_λ's. It follows from Lemma 5.3 that P_{X_μ} acts via the same scalar on the central idempotent z_λ as on $z_{\mathbb{1}}$, for all simple objects X_μ in $\mathcal{C}'$, if and only if $\lambda \in \mathcal{T}$. Hence the projection onto $\mathrm{span}\{z_\lambda, X_\lambda \in \mathcal{T}\}$ is in the image of the fusion algebra, which is spanned by the P_{X_μ}'s. So we can find scalars ω_μ such that this projection is written as $\sum_{\mu\in\Lambda'} \omega_\mu P_{X_\mu}$. The claim follows from this. □

5C.. Let $f : \bigotimes_{i=1}^s X_{\lambda_i} \to \bigotimes_{i=1}^s X_{\mu_i}$ be a morphism. We define, for $r = 1, \dots, s$, the morphism $\hat{f}_r : \bigotimes_{i=r+1}^s X_{\lambda_i} \otimes \overline{X}_{\mu_i} \to \bigotimes_{i=1}^r \overline{X}_{\lambda_i} \otimes X_{\mu_i}$ using rigidity and braiding morphisms for suitable objects as indicated in Figure 17; if $r = s$, the source of $\hat{f}_s$ is defined to be $\mathbb{1}$. For instance, we have

$$\hat{f}_1 = \alpha \circ (1_{\bar{\lambda}_1} \otimes f \otimes 1_{\bar{\mu}_2} \otimes \cdots \otimes 1_{\bar{\mu}_s}) \circ \beta,$$

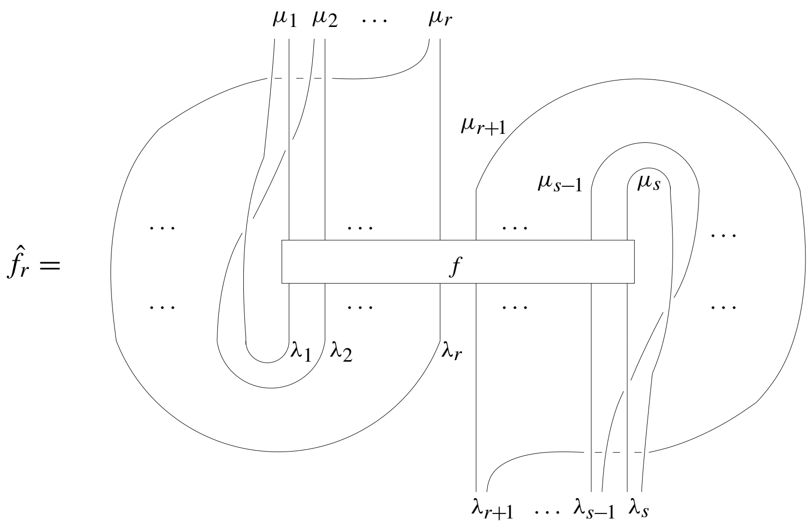


Figure 17. $\hat{f}_r : \bigotimes_{i=r+1}^s X_{\lambda_i} \otimes \overline{X}_{\mu_i} \to \bigotimes_{i=1}^r \overline{X}_{\lambda_i} \otimes X_{\mu_i}$.

for suitable morphisms α and β. We set $\hat{f} = \hat{f}_s$.

Corollary 5.5. *Let* $f \in \mathrm{Hom}_{\mathcal{M}-\mathcal{M}}(M_{\vec{\lambda}}, M_{\vec{\mu}})$ (*with the notation as explained at the beginning of Section 4B*), *viewed as an element in* $\mathrm{Hom}_{\mathcal{C}}(X_{\vec{\lambda}}, X_{\vec{\mu}})$ (see Lemma 4.5(c)), *and let* P_Ω *be as in Proposition 5.4. Then* $\hat{f}_r = P_\Omega(\hat{f}_r) = (\hat{f}_r)_{\mathcal{T}}$.

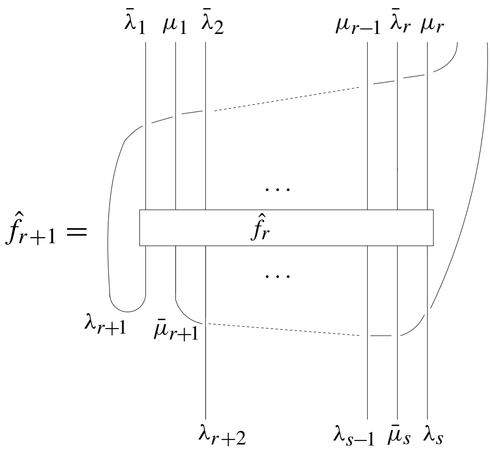
Proof. Fix r, and put a ring around f as it was done for $\tilde{a}$ in Lemma 5.1. By Corollary 5.2 the equality there also holds if we label the ring by $\Omega = \sum \omega_\mu X_\mu$, with the ω_μ as in Proposition 5.4. Observe that the ring on the left hand side becomes the scalar $\sum_\mu \omega_\mu d_\mu = 1$, by Proposition 5.4. Now multiply both sides with suitable morphisms which change f to $\hat{f}_r$, such that all strands ending up go under the ring, and all strands ending at the bottom go above the ring. Then the right-hand side is equal to $P_\Omega(\hat{f}_r)$, which is equal to the left-hand side, $\hat{f}_r$. But by Proposition 5.4 $P_\Omega(\hat{f}_r) = (\hat{f}_r)_{\mathcal{T}}$. □

Lemma 5.6. *If* $f \in \mathrm{Hom}(M_{\vec{\lambda}}, M_{\vec{\mu}})$ *then* $\hat{f} = (\otimes_{i=1}^{s} p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i}))\hat{f}$.

Proof. We will prove by induction on r that $\hat{f}_r = (\otimes_{i=1}^{r} p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i}))\hat{f}_r$. For $r = 1$, we have

$$\hat{f}_1 = P_\Omega(\hat{f}_1) = (\hat{f}_1)_{\mathcal{T}},$$

by Corollary 5.5. This proves the claim for $r = 1$, as the target of the morphism $\hat{f}_1$ is $\bar{X}_{\lambda_1} \otimes X_{\mu_1}$. For the induction step we use the inductive formula for $\hat{f}_{r+1}$, as given in the figure:



We obtain from this and the induction assumption that

$$\hat{f}_{r+1} = [(\otimes_{i=1}^{r} p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i})) \otimes 1_{\overline{X}_{\lambda_{r+1}} \otimes X_{\mu_{r+1}}}]\hat{f}_{r+1}.$$

By Corollary 5.5 (as for the case $r = 1$) we also obtain

$$\hat{f}_{r+1} = P_\Omega(\hat{f}_{r+1}) = p_{\mathcal{T}}(\otimes_{i=1}^{r+1} \overline{X}_{\lambda_i} \otimes X_{\mu_i})\hat{f}_{r+1}.$$

If X_λ is an object in $\mathcal{T}$, then so is $\bar{X}_\lambda$. It follows that the tensor product of simple objects $X_\lambda \otimes X_\mu$ is in $\mathcal{T}$ for $X_\lambda \in \mathcal{T}$ only if also X_μ is in $\mathcal{T}$. One deduces from this and the last two formulas that

$$\begin{aligned}\hat{f}_{r+1} &= p_{\mathcal{T}}(\textstyle\bigotimes_{i=1}^{r+1} \overline{X}_{\lambda_i} \otimes X_{\mu_i})\big((\bigotimes_{i=1}^{r} p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i})) \otimes 1_{\overline{X}_{\lambda_{r+1}} \otimes X_{\mu_{r+1}}}\big)\hat{f}_{r+1} \\ &= \textstyle\bigotimes_{i=1}^{r+1} p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i})\hat{f}_{r+1}.\end{aligned}$$

This proves the claim by induction on r. □

5D.. It can be shown under fairly weak conditions that the category $\mathcal{T}$ is equivalent to the representation category of a finite group G, see the papers [Bruguières 2000] and [Müger 2000]. In the following, we shall require in addition that $\mathcal{T}$ is equivalent to the representation category of a finite *abelian* group G, for any choice of $\mathcal{D}$. In this case, every simple object in the subcategory $\mathcal{T}$ is invertible. Moreover, we can and will label the simple objects of $\mathcal{T}$ by the elements of G in such a way that $X_g \otimes X_h \cong X_{gh}$ for any $g, h \in G$. Then we get a G-action on the index set Λ defined by $X_{g.\lambda} = X_g \otimes X_\lambda$. We shall also need the subgroup G_1^s of G^s consisting of all s-tuples $(g_1, g_2, \dots, g_s)$ which satisfy $g_1 g_2 \cdots g_s = 1$. The just defined G-action extends to an action of G_1^s on Λ^s in the obvious way.

Proposition 5.7. *Under the above assumptions we have*

(a) $\mathrm{Hom}(M_{\vec{\lambda}}, M_{\vec{\mu}}) \neq 0$ *only if there exists a* $g \in G_1^s$ *such that* $\vec{\mu} = g.\vec{\lambda}$.

(b) $\dim \mathrm{End}(M_{\vec{\lambda}}) \leq |Stab_{G_1^s}\vec{\lambda}|$.

Proof. We use notations as in Lemma 5.6. By our assumptions, we have $p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i}) = 0$ unless we can find an element $g_i \in G$ such that $X_{g_i} \subset \overline{X}_{\lambda_i} \otimes X_{\mu_i}$. This implies $g_i.\lambda_i = \mu_i$, and hence $\vec{\mu} = g.\vec{\lambda}$ for some $g \in G^s$. Moreover, we have a nonzero morphism from $\mathbb{1}$ to $\otimes X_{g_i}$ if and only if $\prod g_i = 1$. This shows that $g \in G_1^s$, by Lemma 5.6.

By the discussion in the previous paragraph, the dimension of

$$\mathrm{Hom}(\mathbb{1}, \otimes_i p_G(\overline{X}_{\lambda_i} \otimes X_{\lambda_i}))$$

is equal to the cardinality of all s-tuples $g = (g_i)$ of elements of G for which $g.\vec{\lambda} = \vec{\lambda}$ and whose product $\prod g_i$ is equal to 1. These are exactly the elements of $\mathrm{Stab}_{G_1^s}\vec{\lambda}$. The claim now follows from the fact that the map $f \mapsto \hat{f}$ is injective; indeed, it is easy to construct a left-inverse by multiplying $\hat{f}$ by a suitable combination of $\cap$'s and $\cup$'s to get back f. □

Theorem 5.8. *If the S-matrix for the category $\mathscr{C}'$ is invertible, the dual principal graph for the inclusion $\mathcal{N} \subset \mathcal{M}$ coincides with its principal graph. In particular, each $\mathcal{M}$–$\mathcal{M}$ bimodule $M_{\vec{\lambda}}$, with $\vec{\lambda} = (\lambda_i)$ such that each λ_i labels a simple object in $\mathscr{C}'$ is irreducible.*

Proof. We will use the results of Lemma 5.6 and of Proposition 5.7 for the category $\mathscr{C}'$ (recall that its simple objects appear in tensor powers of X whose exponents are divisible by k). If the S-matrix is invertible, the group G corresponding to the category $\mathscr{T}_{\mathscr{C}'}$ is the trivial group. Hence there are no nonzero morphisms between $M_{\vec{\lambda}}$ and $M_{\vec{\mu}}$ for $\vec{\lambda} \neq \vec{\mu}$, and each $\mathcal{M}$–$\mathcal{M}$-bimodule $M_{\vec{\lambda}}$ is irreducible by Proposition 5.7. It follows from the definitions (see before Theorem 4.6) that the multiplicity of a simple $\mathcal{N}$–$\mathcal{M}$ bimodule K_ν in the simple $\mathcal{M}$–$\mathcal{M}$ bimodule $M_{\vec{\lambda}}$ is equal to $L_{\vec{\lambda}}^{\nu}$.

Observe that $\mathrm{ind}(K_\nu) = d_\nu^2[\mathcal{M} : \mathcal{N}]$ and $\mathrm{ind}(M_{\vec{\lambda}}) = \prod_i d_{\lambda_i}^2$. It follows that

$$\sum_{\nu\in\Lambda'} d_\nu^2[\mathcal{M} : \mathcal{N}] = \Big(\sum_{\nu\in\Lambda'} d_\nu^2\Big)^s = \sum_{\vec{\lambda}\in(\Lambda')^s} \prod_i d_{\lambda_i}^2.$$

Hence $\sum_{\nu\in\Lambda'} \mathrm{ind}(K_\nu) = \sum_{\vec{\lambda}\in(\Lambda')^s} \mathrm{ind}(M_{\vec{\lambda}})$. Hence any simple $\mathcal{N}$–$\mathcal{M}$-bimodule in a higher relative commutant has as a weak reduction an element in $(K_\nu)_{\nu\in\Lambda'}$, by Theorem 4.6. As our original inclusion $\mathcal{N} \subset \mathcal{M}$ is of finite depth by Theorem 4.6(c), it follows from Lemma 1.10(a) that there can not be any additional $\mathcal{M}$–$\mathcal{M}$-bimodules in the higher relative commutants. □

5E. ***Noninvertible S-matrix.*** We shall make the following assumptions: We assume that the category $\mathscr{T}$ for our chosen category $\mathscr{D} = \mathscr{C}$ is equivalent to the representation category of a finite abelian group G, and, moreover, that $|G| = k$, with k as defined in Section 3A. This also implies that $|G_1^s| = k^{s-1}$. For $\lambda \in \Lambda$ we also define $|\lambda|$ to be the residue class mod k such that $|\lambda| \equiv n \bmod k$ whenever $X_\lambda \subset X^{\otimes n}$.

Theorem 5.9. *Let the conditions be as just stated.*

(a) $\mathrm{End}_{\mathcal{M}-\mathcal{M}}(M_{\vec{\lambda}})$ *has dimension* $|\mathrm{Stab}_{G_1^s}\vec{\lambda}|$ *for any* $\vec{\lambda} \in \Lambda_0^s := \{\vec{\lambda} \in \Lambda^s : k | \sum |\lambda_i|\}$.

(b) *The even vertices of the dual principal graph of the inclusion* $\mathcal{N} \subset \mathcal{M}$ *are labeled by the equivalence classes of irreducible components of the bimodules* $M_{\vec{\lambda}}$, *with* $\vec{\lambda} \in \Lambda_0^s$.

Proof. Let $M_{\vec{\lambda}} = \bigoplus_i m_i Q_{\vec{\lambda},i}$ be the decomposition of the $\mathcal{M}$–$\mathcal{M}$ bimodule $M_{\vec{\lambda}}$ into irreducible $\mathcal{M}$–$\mathcal{M}$-bimodules, the m_i being multiplicities. Then it follows from Lemma 1.10(b), and Proposition 5.7 that

$$\sum_i \mathrm{ind}(Q_{\vec{\lambda},i}) \geq \frac{\mathrm{ind}(M_{\vec{\lambda}})}{\dim(\mathrm{End}(M_{\vec{\lambda}}))} \geq \frac{\mathrm{ind}(M_{\vec{\lambda}})}{|\mathrm{Stab}_{G_1^s}\vec{\lambda}|}.$$

Now let $(Q_j)_j = \bigcup_{\vec{\lambda}} (Q_{\vec{\lambda},i})_i$ be the collection of nonisomorphic representatives of irreducible $\mathcal{M}$–$\mathcal{M}$ submodules of any module $M_{\vec{\lambda}}$ with $\vec{\lambda} \in \Lambda_0^s$. Then

$$\sum_j \mathrm{ind}(Q_j) \geq \sum_{G_1^s-\mathrm{orbits}\in\Lambda_0^s} \frac{\mathrm{ind}(M_{\vec{\lambda}})}{|\mathrm{Stab}_{G_1^s}\vec{\lambda}|} = \frac{1}{k^{s-1}} \sum_{\vec{\lambda}\in\Lambda_0^s} \mathrm{ind}(M_{\vec{\lambda}}).$$

Using Lemma 4.5(a) and Lemma 3.1(d) one sees that this equals

$$= \frac{1}{k^{s-1}}\left(\frac{1}{k}\sum_{\vec{\lambda}\in\Lambda^s} d_{\vec{\lambda}}^2\right) = \left(\sum_{\lambda\in\Lambda'} d_{\lambda}^2\right)^s.$$

But the last sum is equal to $\sum_{\nu\in\Lambda'}\operatorname{ind}(K_\nu)$, as was already shown in the proof of Theorem 5.8. Hence the inequalities above must be equalities, and our set of bimodules $(Q_j)_j$ must already exhaust all possible $\mathcal{M}$–$\mathcal{M}$-bimodules in the higher relative commutant, by Lemma 1.10. □

Remark 5.10. If the stabilizer $\mathrm{Stab}_{G_1^s}\vec{\lambda}$ is trivial, which usually is the case for most labels, the bimodule $M_{\vec{\lambda}}$ is irreducible, and its decomposition into $\mathcal{N}$–$\mathcal{M}$-bimodules is again determined by the fusion coefficients $L_{\vec{\lambda}}^{\nu}$. Unfortunately, our theorem does not say anything about what $\mathrm{End}(M_{\vec{\lambda}})$ looks like if $|\mathrm{Stab}_{G_1^s}\vec{\lambda}| \geq 4$. For example, if the stabilizer has four elements, $\mathrm{End}(M_{\vec{\lambda}})$ could be isomorphic to $\mathbb{C}^4$ or to the 2×2 matrices. Neither does it say how the submodules of $\mathcal{M}_{\vec{\lambda}}$ decompose into irreducible $\mathcal{N}$–$\mathcal{M}$ modules in these cases.

6. Examples

6A. ***Examples of C*-tensor categories.*** (1) The easiest example for our set-up is the representation category $Rep(G)$ of finite-dimensional unitary representations of a finite group. In order to avoid degenerate trivial cases, we take for X in our construction an object such that some tensor power of it contains the whole group ring $\mathbb{C}G$ as a subobject. For example, for G a finite cyclic group, we could take the direct sum of the trivial and of a faithful one-dimensional representation. For these examples, the braiding structure is just given by the permutation of tensor factors, which commutes with the group action. This makes the S-matrix a rank 1 matrix, meaning it is noninvertible unless G is trivial. However, at least in principle, the dual principal graph can be computed from a general result about fixed point algebras of a group K and its subgroup H. In our setting, $K = G^s$ and $H \cong G$, which is embedded by $g \in G \mapsto (g, g, \dots, g)$ (s times). See [Kosaki et al. 1997] for details.

In the special case when the subgroup K is normal, we obtain principal and dual principal graphs of the factor group H/K. This is the case in our setting if G is abelian.

(2) Let ρ be a II_1 factor representation of the infinite braid group $\mathcal{B}_\infty$ such that the Jones index for the inclusion of factors $\rho(\mathcal{B}_{2,\infty})'' \subset \rho(\mathcal{B}_\infty)''$ is finite. Let us define $A_n = \rho(\mathcal{B}_{n+1,\infty})' \cap \rho(B_\infty)''$ (recall that finite index implies that the relative commutant is finite-dimensional). We moreover assume that there exists, for some $k \in \mathbb{N}$, a projection $p \in A_k$ such that $p\rho(\mathcal{B}_\infty)''p = p\rho(B_{k+1,\infty})''$. It is possible to

define from this a C^*-tensor category, with the objects being the projections in A_n. Most of this has already been done in [Wenzl 1993], Section 2, without mentioning categories. We shall not do this here. We just remark that the constructions of this paper will work in this setting without explicitly exhibiting the category; this has already been done in [Erlijman 2001]. In particular, this can be applied to the Jones subfactors as well as to the Hecke algebra and BCD type subfactors.

(3) Let $U_q\mathfrak{g}$ be the Drinfeld–Jimbo deformation of the universal enveloping algebra $U\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$. It is well-known that the category of its finite-dimensional representations has a braiding structure. It can not be unitarized except for $q = 1$. If q is a root of unity $\neq 1$, one can define a special class of representations called tilting modules which again forms a braided tensor category. It can be shown that the category of tilting modules has a semisimple quotient with only finitely many simple modules up to equivalence; this is often referred to as a fusion category (see [Andersen 1992],[Andersen and Paradowski 1995]). Moreover, for q being certain roots of unity (usually of the form $q = e^{\pm 2\pi i/l}$ for suitable integers l (see [Wenzl 1998] for precise values), this quotient can be unitarized. This yields a large and important class of C^* tensor categories. Using the one-sided subfactor construction, one obtains the Jones subfactors for X being the $U_q sl_2$-analog of the 2-dimensional representation of sl_2. Similarly, Hecke algebra subfactors and BCD type subfactors can be obtained from fusion categories of quantum groups of classical Lie types.

These C^*-fusion categories can also be obtained by a completely different construction using the category of positive energy representation of a loop group. The difficulty in this construction comes from the fact that one can not use the usual tensor product for representations; instead one has to define a new, so-called fusion tensor product (see [Wassermann 1998]).

(4) Let $N \subset M$ be an inclusion of II_1 factors with finite index and finite depth. Then the category of N–N bimodules obtained as direct sums of summands of the bimodules

$$M^{\otimes n} = M \otimes_N M \otimes_N \cdots \otimes_N M$$

(n times), $n \in \mathbb{N}$ defines a C^*-tensor category which may or may not be braided. One can similarly also define the C^*-tensor category of M-M bimodules generated by $M^{\otimes n}$.

If these categories are not braided, one can apply a general construction, called the categorical quantum double construction to construct from our category of bimodules a larger braided C^* tensor category. It was shown that this category is equivalent to the category of $\mathcal{M}$–$\mathcal{M}$-bimodules for the asymptotic inclusion $\mathcal{N} \subset \mathcal{M}$ derived from $N \subset M$; see [Müger 2003]. If the original category already was braided, the asymptotic inclusion coincides with the 2-sided inclusion constructed in this paper.

(5) Our constructions of bimodules in this paper are based on certain endomorphisms of II_1 factors. The approach to categories via endomorphisms has been used for a long time for type III factors in the framework of algebraic quantum field theory (see [Longo and Roberts 1997; Fredenhagen et al. 1989; Xu 2000], for example). Here subtleties involving coupling constants do not matter, and objects are given directly by morphisms.

6B. *Examples for our construction.* (1) We first list examples of C^*-tensor categories with invertible S-matrix.

(a) The S-matrix for the full fusion tensor categories as constructed in [Andersen 1992],[Andersen and Paradowski 1995] is invertible under the conditions for unitarizability, as stated in [Wenzl 1998]. Hence if we can find an object X such that *all* irreducible representation of the fusion category appear in some tensor power of X, we have $\mathscr{C}' = \mathscr{C}$ and the dual principal graph is equal to the principal graph. Such representations can be found in all cases, but usually can not be chosen to be irreducible. For instance, for Lie type A (the case of Jones subfactors and Hecke algebra subfactors), one can choose $X = \mathbb{1} \oplus V$, where V is the analog of the vector representation.

(b) Similarly, the S-matrix for the quantum double of a C^* tensor category is always invertible (see [Müger 2003], for instance). Hence, as soon as we have found an object X for which all irreducible representations of the double category appear in some tensor power of X, the dual principal graph of our s-sided inclusion with respect to X is equal to the principal graph.

(2) It turns out that our construction not only depends on the category $\mathscr{C}$, but also on the choice of the object X. Even though in the case of the fusion tensor categories the S-matrix for $\mathscr{C}$ is invertible, the S matrix for the category $\mathscr{C}'$ may not be invertible. For instance, for type A if one takes $X = V$, the S-matrix for $\mathscr{C}'$ is invertible only if the degree of the root of unity is coprime to k. If this is not the case, however, our results for noninvertible S-matrices apply. This will be shown in more detail in the following subsection at an example.

6C. *Subfactors related to Jones subfactors.* We illustrate our examples in some detail for the fusion category $\mathscr{C}$ of $U_q sl_2$, with $q = e^{2\pi i/l}$. There also exist other, more elementary methods to construct these categories using the Temperley–Lieb algebras; see [Turaev 1994], for example. As mentioned before, this is also one of the cases where the subfactor constructions can be done on the level of braid representations, as it was carried out in the original paper [Erlijman 2001].

We give a brief description of this category. Up to isomorphism, we have exactly $l-1$ simple objects in $\mathcal{C}$, which are denoted by $[i]$, $1 \le i \le l-1$. The decomposition of tensor products is given by

$$[i] \otimes [j] = [|i-j|+1] \oplus [|i-j|+3] \oplus \cdots \oplus [m], \tag{6-1}$$

where m is the minimum of $i+j-1$ and $2l-1-i-j$. One sees easily that $[1]$ corresponds to the trivial object. It follows from the tensor product rules by induction on n that all simple objects in $[2]^{\otimes n}$ are labeled by even numbers if n is odd, and by odd numbers if n is even. Hence $k=2$ and the simple objects of $\mathcal{C}'$ are labeled by odd numbers. This explicitly describes the principal graph for $\mathcal{N} \subset \mathcal{M}$, constructed with $X=[2]$, by Theorem 4.6.

Observe that $[i] \otimes [l-1] = [l-i]$ for all $1 \le i < l$. Hence the objects $[1]$ and $[l-1]$ together with the operation $\otimes$ form a group G which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover, the S matrix is well-known to be of the form $S = (\sin(ij\pi/l))$, up to a scalar.

It is very easy to check that if l is even, then $\sin(i(l-1)\pi/l) = \sin(i\pi/l)$ for any odd $i = 1, 3, \ldots, l-1$. Hence the category $\mathcal{T}$ contains at least the objects $[1]$ and $[l-1]$. It contains no more simple objects as obviously $\sin(i\pi/l) = \sin(ij\pi/l)$ for $1 < j < l$ only if $j = l-1$. So the conditions at the beginning of Section 5B are satisfied with $|G| = 2 = k$. We have shown most of the following

Proposition 6.1. *Let $\mathcal{N} \subset \mathcal{M}$ be the subfactor constructed from the s-sided inclusion from the Jones subfactor at an l-th root of unity, with l even. Then we have*

(a) *The even vertices of the principal graph are labeled by all s-tuples of odd positive numbers less than l and the odd vertices are labeled by all odd positive numbers less than l. The number of edges between two vertices can be computed from the tensor product rule stated in (6-1).*

(b) *Each s-tuple of positive integers less than l whose sum is even and which contains the number $l/2$ at most once labels an even vertex of the dual principal graph; the number of edges emanating from such a vertex can be computed as in (a). The $\mathcal{M}$–$\mathcal{M}$ bimodules $M_{\vec{\lambda}}$ labeled by an s-tuple $\vec{\lambda}$ containing the number $l/2$ exactly $r > 1$ times satisfies* $\dim(\mathrm{End}(M_{\vec{\lambda}})) = 2^{r-1}$.

Proof. Part (a) follows from Theorem 4.6 and our explicit description of the simple objects of $\mathcal{C}'$. For part (b), we have already checked the conditions stated at the beginning of Section 5B. It remains to calculate $\mathrm{Stab}_{G_1^s}\vec{\lambda}$ for any $\vec{\lambda} \in \Lambda^s$. Recall that the action of the nontrivial element of G on our labeling set is given by $i \mapsto l-i$. Obviously, the only fixed point is $l/2$ for l odd. It is now not hard to show that $\vec{\lambda} \in \Lambda^s$ has a nontrivial stabilizer in G_1^s if and only if $r \ge 2$ of its components are equal to $l/2$, and that in this case the stabilizer has exactly 2^{r-1} elements. Statement (b) now follows from Theorem 5.9. □

Remark 6.2. If $s = 3$, part (b) of the last proposition completely determines the number of edges in the dual principal graph except for the decomposition of the bimodule $M_{\vec{\lambda}}$ with $\vec{\lambda} = (l/2, l/2, l/2)$, which could decompose into the direct sum of four nonisomorphic irreducible $\mathcal{M}$–$\mathcal{M}$ bimodules or into the direct sum of two isomorphic irreducible $\mathcal{M}$–$\mathcal{M}$ bimodules.

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References

[Andersen 1992] H. H. Andersen, "Tensor products of quantized tilting modules", *Comm. Math. Phys.* **149**:1 (1992), 149–159. MR 94b:17015 Zbl 0760.17004

[Andersen and Paradowski 1995] H. H. Andersen and J. Paradowski, "Fusion categories arising from semisimple Lie algebras", *Comm. Math. Phys.* **169**:3 (1995), 563–588. MR 96e:17026 Zbl 0827.17010

[Asaeda 2006] M. Asaeda, "Quantum multiple construction of subfactors", preprint, UC Riverside, 2006. math.OA/0602265

[Barrett and Westbury 1999] J. W. Barrett and B. W. Westbury, "Spherical categories", *Adv. Math.* **143**:2 (1999), 357–375. MR 2000c:18007 Zbl 0930.18004

[Bisch 1997] D. Bisch, "Bimodules, higher relative commutants and the fusion algebra associated to a subfactor", pp. 13–63 in *Operator algebras and their applications* (Waterloo, ON, 1994/1995), edited by P. A. Fillmore and J. A. Mingo, Fields Inst. Commun. **13**, Amer. Math. Soc., Providence, RI, 1997. MR 97i:46109 Zbl 0894.46046

[Bruguières 2000] A. Bruguières, "Catégories prémodulaires, modularisations et invariants des variétés de dimension 3", *Math. Ann.* **316**:2 (2000), 215–236. MR 2001d:18009 Zbl 0943.18004

[Erlijman 2001] J. Erlijman, "Multi-sided braid type subfactors", *Canad. J. Math.* **53**:3 (2001), 546–564. MR 2002f:46119 Zbl 1031.46072

[Erlijman 2003] J. Erlijman, "Multi-sided braid type subfactors, II", *Canad. Math. Bull.* **46**:1 (2003), 80–94. MR 2004a:46060 Zbl 1032.46074

[Evans and Kawahigashi 1998] D. E. Evans and Y. Kawahigashi, "Orbifold subfactors from Hecke algebras, II: Quantum doubles and braiding", *Comm. Math. Phys.* **196**:2 (1998), 331–361. MR 2000i:46058 Zbl 0910.46043

[Fredenhagen et al. 1989] K. Fredenhagen, K.-H. Rehren, and B. Schroer, "Superselection sectors with braid group statistics and exchange algebras, I: General theory", *Comm. Math. Phys.* **125**:2 (1989), 201–226. MR 91c:81047 Zbl 0682.46051

[Freyd 1964] P. Freyd, *Abelian categories: An introduction to the theory of functors*, Harper & Row, New York, 1964. MR 29 #3517 Zbl 0121.02103

[Izumi 2000] M. Izumi, "The structure of sectors associated with Longo-Rehren inclusions, I: General theory", *Comm. Math. Phys.* **213**:1 (2000), 127–179. MR 2002k:46160 Zbl 1032.46529

[Jones 1983] V. F. R. Jones, "Index for subfactors", *Invent. Math.* **72**:1 (1983), 1–25. MR 84d:46097 Zbl 0508.46040

[Kassel 1995] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**, Springer, New York, 1995. MR 96e:17041 Zbl 0808.17003

[Kosaki et al. 1997] H. Kosaki, A. Munemasa, and S. Yamagami, "On fusion algebras associated to finite group actions", *Pacific J. Math.* **177**:2 (1997), 269–290. MR 98i:46064 Zbl 0882.46030

[Longo and Roberts 1997] R. Longo and J. E. Roberts, "A theory of dimension", *K-Theory* **11**:2 (1997), 103–159. MR 98i:46065 Zbl 0874.18005

[Mac Lane 1998] S. Mac Lane, *Categories for the working mathematician*, Second ed., Graduate Texts in Mathematics **5**, Springer, New York, 1998. MR 2001j:18001 Zbl 0906.18001

[Müger 2000] M. Müger, "Galois theory for braided tensor categories and the modular closure", *Adv. Math.* **150**:2 (2000), 151–201. MR 2001a:18008 Zbl 0945.18006

[Müger 2003] M. Müger, "From subfactors to categories and topology, II: The quantum double of tensor categories and subfactors", *J. Pure Appl. Algebra* **180**:1-2 (2003), 159–219. MR 2004f:18014 Zbl 1033.18003

[Orellana and Wenzl 2002] R. C. Orellana and H. G. Wenzl, "q-centralizer algebras for spin groups", *J. Algebra* **253**:2 (2002), 237–275. MR 2003i:17022 Zbl 1085.17012

[Turaev 1994] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics **18**, de Gruyter, Berlin, 1994. MR 95k:57014 Zbl 0812.57003

[Wassermann 1998] A. Wassermann, "Operator algebras and conformal field theory, III: Fusion of positive energy representations of $\mathrm{LSU}(N)$ using bounded operators", *Invent. Math.* **133**:3 (1998), 467–538. MR 99j:81101 Zbl 0944.46059

[Wenzl 1988] H. Wenzl, "Hecke algebras of type A_n and subfactors", *Invent. Math.* **92**:2 (1988), 349–383. MR 90b:46118 Zbl 0663.46055

[Wenzl 1993] H. Wenzl, "Braids and invariants of 3-manifolds", *Invent. Math.* **114**:2 (1993), 235–275. MR 94i:57021 Zbl 0804.57007

[Wenzl 1998] H. Wenzl, "C^* tensor categories from quantum groups", *J. Amer. Math. Soc.* **11**:2 (1998), 261–282. MR 98k:46123 Zbl 0888.46043

[Xu 2000] F. Xu, "Jones–Wassermann subfactors for disconnected intervals", *Commun. Contemp. Math.* **2**:3 (2000), 307–347. MR 2001f:46094 Zbl 0966.46043

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JULIANA ERLIJMAN
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF REGINA
REGINA, SASKATCHEWAN S4S 0A2
CANADA
erlijman@math.uregina.ca
http://www.math.uregina.ca/~erlijman/

HANS WENZL
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, SAN DIEGO
9500 GILMAN DRIVE, DEPT 0112
LA JOLLA, CA 92093-0112
UNITED STATES
hwenzl@ucsd.edu
http://www.math.ucsd.edu/~wenzl/

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