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Let M be a compact symplectic manifold, and $L \subset M$ be a closed Lagrangian submanifold which can be lifted to a Legendrian submanifold in the contactization of M. For any Legendrian deformation of L satisfying some given conditions, we get a new Lagrangian submanifold L'. We prove that the number of intersection $L \cap L'$ can be estimated from below by the sum of $\mathbb{Z}_2$-Betti numbers of L, provided they intersect transversally.

1. Introduction

V. I. Arnold formulated in [1965; 1978, Appendix 9] his famous conjectures on the number of fixed points of Hamiltonian diffeomorphisms of any compact symplectic manifold and the number of intersection points of any Lagrangian submanifold with its Hamiltonian deformations in a symplectic manifold. If M is a symplectic manifold, $L \subset M$ is a Lagrangian submanifold, and ψ_M is a Hamiltonian diffeomorphism, his conjectures can be written in topological terms as

$\#\mathrm{Fix}(\psi_M) \geq$ sum of Betti numbers of M, with all fixed points nondegenerate;
$\#\mathrm{Fix}(\psi_M) \geq$ cuplength of M, fixed points possibly degenerate;

$\#(L \cap \psi_M(L)) \geq$ sum of Betti numbers of L, with intersection points transverse;
$\#(L \cap \psi_M(L)) \geq$ cuplength of L, with intersection points possibly nontransverse.

Much effort has gone into proving these two conjectures. The pioneering works are [Conley and Zehnder 1983; Gromov 1985; Floer 1988a; 1988b; 1989a; 1989b]. Floer originally developed the seminal method, motivated by the variational method used by Conley and Zehnder and the elliptic PDE techniques introduced by Gromov, which is now called Floer homology theory, and solved many special cases of Arnold's conjectures. Fukaya and Ono [1999] and Liu and Tian [1998] independently proved the first conjecture for general compact symplectic manifolds in the nondegenerate case. The conjecture for general symplectic manifolds in the degenerate case is still open.

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For the second conjecture, Floer [1988a; 1989b] gave the proof under the additional assumption $\pi_2(M, L) = 0$. We write his result for the case that all intersections are transverse.

Theorem 1.1 (Floer). *Let L be a closed Lagrangian submanifold of a compact (or tame) symplectic manifold (M, ω) satisfying $\pi_2(M, L) = 0$, and ψ_M be a Hamiltonian diffeomorphism, then $\#(L \cap \psi_M(L)) \geq \dim H_*(L, \mathbb{Z}_2)$, if all intersections are transverse.*

In general, the condition $\pi_2(M, L) = 0$ can not be removed. For instance, let L be a circle in $\mathbb{R}^2$, then $\pi_2(\mathbb{R}^2, L) \neq 0$, however, there always exists a Hamiltonian diffeomorphism which can translate L arbitrarily far from its original position.

To prove his theorem, Floer introduced the so-called Floer homology group for Lagrangian pairs and showed that it is isomorphic to the homology of L under the condition above. The definition of Floer homology for Lagrangian pairs was generalized by Oh [1993a; 1993b; 1995] in the class of monotone Lagragian submanifolds with minimal Maslov number being at least 3. However, for general Lagrangian pairs, the Floer homology is hard to define due to the bubbling off phenomenon and some essentially topological obstructions [Fukaya et al. 2000], which is much different from the Hamiltonian fixed point case.

Therefore, if we want to throw away the additional assumption, we have to restrict the class of Hamiltonian diffeomorphisms. For the simplest case that ψ_M is C^0-small perturbation of the identity, the Lagrangian intersection problem is equivalent to the one for zero sections of cotangent bundles, which is proved by Hofer [1985] and Laudenbach and Sikorav [1985]. Yu. V. Chekanov [1996; 1998] also gave a version of Lagrangian intersection theorem which used the notion of symplectic energy introduced by Hofer [1990] (for $(\mathbb{R}^{2n}, \omega_0)$) and Lalonde and McDuff [1995] (for general symplectic manifolds). Following their notations, we denote by $\mathcal{H}(M)$ the space of compactly supported smooth functions on $[0, 1] \times M$. Any $H \in \mathcal{H}(M)$ defines a time dependent Hamiltonian flow ϕ_H^t on M. All such time-1 maps $\{\phi_H^1, H \in \mathcal{H}(M)\}$ form a group, denoted by $\mathrm{Ham}(M)$. Now we define a norm on $\mathcal{H}(M)$:

$$\|H\| = \int_0^1 \Big(\max_{x \in M} H(t, x) - \min_{x \in M} H(t, x)\Big)\, dt,$$

and we can define the energy of a $\psi \in \mathrm{Ham}(M)$ by

$$E(\psi) = \inf_H \big\{\|H\| \mid \psi = \phi_H^1,\ H \in \mathcal{H}(M)\big\}.$$

For a compact symplectic manifold (M, ω), there always exists an almost complex structure J compatible with ω, so (M, ω, J) is a compact almost complex manifold, and we denote by $\mathcal{J}$ the set of all such J. Let $\sigma_S(M, J)$ and $\sigma_D(M, L, J)$ denote the minimal area of a J-holomorphic sphere in M and of a J-holomorphic disc in M with boundary in L, respectively. If there is no such J-holomorphic curve, these values will be infinity. Otherwise, minimums are obtained by the Gromov compactness theorem [1985], and they are always positive. We write $\sigma(M, L, J) = \min(\sigma_S(M, J), \sigma_D(M, L, J))$, and $\sigma(M, L) = \sup_{J \in \mathcal{J}} \sigma(M, L, J)$.

Theorem 1.2 [Chekanov 1998]. *If $E(\psi) < \sigma(M, L)$, then*

$$\#(L \cap \psi(L)) \geq \dim H_*(L, \mathbb{Z}_2),$$

provided all intersections are transverse.

Remark. For the nontransverse case, under similar assumptions, C.-G. Liu [2005] also found an estimate for Lagrangian intersections by cup-length of L.

In this paper, we give an analogous Lagrangian intersection theorem, but the Hamiltonian deformation ψ will be replaced by a "Legendrian deformation" $\tilde{\psi}$ (which will be explained in the sequel). In fact, K. Ono has shown such a result, still under the assumption $\pi_2(M, L) = 0$, as we now discuss.

Suppose the symplectic structure ω is in an integral cohomology class and there exists a principal circle bundle $\pi : N \to M$ with a connection so that the curvature form is ω. This means for a connection form α, one has $d\alpha = \pi^*\omega$. We see that the horizontal distribution $\xi = \operatorname{Ker}\alpha$ is a cooriented contact structure on N. We say L satisfies the Bohr–Sommerfeld condition if $\alpha|_L$ is flat, or, in other words, it can be lifted to a Legendrian submanifold Λ in N.

Theorem 1.3 [Ono 1996]. *Let a contact isotopy $\{\tilde{\psi}_t \mid 0 \leq t \leq 1\}$ be given on N. If L is a Lagrangian submanifold of M that can be lifted to a Legendrian submanifold Λ in N, and $\pi_2(M, L) = 0$, then*

$$\#(L \cap \pi \circ \tilde{\psi}_1(\Lambda)) \geq \dim H_*(L, \mathbb{Z}_2),$$

provided L and $\pi \circ \tilde{\psi}_1(\Lambda)$ intersect transversally.

Remark. Since a Hamiltonian isotopy of M can be lifted to a contact isotopy of N, Ono's theorem is a generalization of the theorem of Floer already stated.

Eliashberg, Hofer, and Salamon [Eliashberg et al. 1995] independently obtained a result similar to Ono's. They overcame some difficulties due to the noncompactness of the symplectization manifold, and their arguments involve some complicated conditions for avoiding bubbling off.

In the present paper, we discard Ono's assumption that $\pi_2(M, L) = 0$, while adding a restrictive condition on the class of Legendrian deformation $\tilde{\psi}$. Let $\tilde{L}$ be the image of Λ under the principal S^1-action on N. We denote by (SN, ω_ξ) the symplectization of the contact manifold (N, ξ) with cooriented contact structure ξ, where the symplectic structure ω_ξ is induced from the standard 1-form of cotangent bundle T^*N. Then $\tilde{L}$ is a compact Lagrangian submanifold in SN. There is a natural projection $p : SN \to N$, and each section corresponds to a splitting $SN = N \times \mathbb{R}_+ = N \times (e^{-\infty}, +\infty]$. The contactomorphism $\tilde{\psi}$ can be lifted to a $\mathbb{R}_+$-equivariant Hamiltonian symplectomorphism Ψ on SN. We denote $\mathscr{L} = p^{-1}(\Lambda)$, which is also a Lagrangian submanifold in SN. Then we can see that there is a one-to-one correspondence between $\tilde{L} \cap \Psi(\mathscr{L})$ and $L \cap \pi \circ \tilde{\psi}_1(\Lambda)$. However, the symplectization SN is not compact. So, the ordinary method of Floer Lagrangian intersection needs to be modified.

Following [Ono 1996], we can replace the symplectization (SN, ω_ξ) manifold by another symplectic manifold (Q, Ω), which may be considered as a symplectic filling in the negative end, so Q coincides with SN in the part $N \times [e^{-C}, +\infty] \supset \tilde{L}$, where $C > 0$ is a sufficiently large number. We note that Q is a 2-plane bundle over M and is diffeomorphic to the associated complex line bundle $N \times_{S^1} \mathbb{C}$. We define the compatible almost complex structure by J' on Q in the following way. Since Q is the associated complex line bundle, the connection α on N gives the decomposition of $TQ = \mathrm{Ver}(Q) \oplus \mathrm{Hor}(Q)$. We have a ω-compatible almost complex structure J on M, and then we lift J to an almost complex structure on $\mathrm{Hor}(Q)$. Also we define the almost complex structure on each fiber by choosing the standard complex structure J_0 on complex plane $\mathbb{C}$. Then we let $J' = J \oplus J_0$, so J' is uniquely determined by the choice of J on M and a connection on N. Furthermore, Ono [1996, Section 6] showed that if we choose a generic J on M in the sense of the construction of Floer homology for (M, L), then J' is also a regular or generic almost complex structure on Q. If we write $\Pi : Q \to M$ for the natural projection, then it is a (J', J)-holomorphic map. Therefore, a map $u = \Pi \circ \tilde{u} : \Sigma \to M$ is J-holomorphic if and only if $\tilde{u} : \Sigma \to M$ is J'-holomorphic. We can see that, for $r > 1$, the image of the positive end $N \times \{r\} \subset SN$ in Q is *J'-convex*. Hence we can choose the Ω-compatible almost complex structure so that it coincides with J' outside of a compact set. For simplicity, we denote using J this almost complex structure on Q, if we can do so without danger of confusion.

Ono also proved that there is an a priori C^0-bound for connecting orbits in Q (note that all J-holomorphic curves we are concerned with are contained in a compact subset $K \subset Q$, while K depends on the choice of the contact isotopy $\{\psi_t\}$), and the bubbling off argument can go through as in the case of compact symplectic manifold. So the minimal area of J-holomorphic spheres and J-holomorphic discs bounding Lagrangian submanifolds $\tilde{L}$ and $\mathscr{L}$ can be achieved. We denote it by

$$\begin{aligned}\sigma(Q, \tilde{L}, \mathscr{L}, J)\\ &= \min\bigl(\sigma_S(Q, J)|_K, \sigma_D(Q, \tilde{L}, J)|_K, \sigma_D(Q, \mathscr{L}, J)|_K, \sigma_D(Q, \mathscr{L}, \tilde{L}, J)|_K\bigr)\end{aligned}$$

and we set

$$\sigma(Q, \tilde{L}, \mathscr{L}) = \sup_{J_M \in \mathcal{J}} \sigma(Q, \tilde{L}, \mathscr{L}, J_M \oplus J_0).$$

We will show that we can find a compactly supported Hamiltonian diffeomorphism $\Psi' \in \mathrm{Ham}(Q)$ such that for a compact set K, the two images of Ψ and Ψ' coincide. For detailed explanation, we refer to [Ono 1996], or to Section 2. We denote a contactomorphism by ψ, so our main result is:

Theorem 1.4. *Let M be a compact symplectic manifold and N be the principal S^1-bundle $\pi : N \to M$ defined above. Given a contact isotopy $\{\psi_t \mid 0 \le t \le 1\}$ on N, suppose L is a closed Lagrangian submanifold of M which can be lifted to a Legendrian submanifold Λ in N, and $E(\Psi') < \sigma(Q, \tilde{L}, \mathscr{L})$, then*

$$\#(L \cap \pi \circ \psi_1(\Lambda)) \ge \dim H_*(L, \mathbb{Z}_2),$$

provided L and $\pi \circ \psi_1(\Lambda)$ intersect transversally.

2. Preliminaries on symplectic and contact geometry

We say that a given $(2n+1)$-dimensional manifold N is a contact manifold if there exists a contact structure ξ which is a completely nonintegrable tangent hyperplane distribution. It is obvious that ξ can locally be defined as the kernel of a 1-form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$. If the contact structure is coorientable, then α can be globally defined. We only consider the cooriented contact structure in this paper. The contact manifold is denoted by (N, ξ), and α is called a contact form. A diffeomorphism ψ of N is called a contactomorphism if it preserves the cooriented contact structure ξ. We call $\{\psi_t, 0 \le t \le 1\}$ a contact isotopy if $\psi_0 = \mathrm{id}$ and every ψ_t is a contactomorphism, and $X_t = d\psi_t/dt$ is the contact vector field on N.

For any symplectic manifold (M, ω) there exists an almost complex structure J on M. We say the almost complex is *compatible* with the symplectic manifold if $\omega(J\,\cdot\,, J\,\cdot\,) = \omega(\,\cdot\,,\,\cdot\,)$ and $\omega(\,\cdot\,, J\,\cdot\,) > 0$, which can give the Riemannian metric on M.

Let N be an oriented codimension 1 submanifold in an almost complex manifold (Q, J), and ξ_x be the maximal J-invariant subspace of the tangent space $T_x N$, then ξ_x has codimension 1. N is said to be *J-convex* if for any defining 1-form α for ξ, we have $d\alpha(v, Jv) > 0$ for all nonzero $v \in \xi_x$. This implies ξ is a contact structure on N. It is a fact that if N is J-convex then no J-holomorphic curve in Q can touch (or be tangent to) N from the inside (from the negative side) [Gromov 1985; McDuff 1991].

Symplectization. We denote by $SN = S_\xi(N)$ the $\mathbb{R}_+$-subbundle of the cotangent bundle T^*N whose fibers at $q \in N$ are all nonzero linear forms in $T^*_q N$, which is compatible with the contact hyperplane $\xi_q \subset T_q N$. There is a canonical 1-form pdq on T^*N, and if we let $\alpha_\xi = pdq|_{SN}$, then $\omega_\xi = d\alpha_\xi$ is a symplectic structure on SN. Thus, we call $(SN,\ \omega_\xi)$ the symplectization of the contact manifold $(N,\ \xi)$.

We see that a contact form $\alpha : N \to SN$ is a section of this $\mathbb{R}_+$-bundle $p : SN \to N$. Hence we have a splitting $SN = N \times \mathbb{R}_+$.

An n-dimensional submanifold $\Lambda \subset (N, \xi)$ is called Legendrian if it is tangent to the distribution ξ, that is to say, Λ is Legendrian if and only if $\alpha|_\Lambda = 0$. The preimage $\mathscr{L} = p^{-1}(\Lambda)$ is an $\mathbb{R}_+$-invariant Lagrangian cone in (SN, ω_ξ). Conversely, any Lagrangian cone in the symplectization projects onto a Legendrian submanifold in (N, ξ).

SN carries a canonical conformal symplectic $\mathbb{R}_+$-action. Every contactomorphism φ uniquely lifts to a $\mathbb{R}_+$-equivariant symplectomorphism $\tilde{\varphi}$ of SN, which is also a Hamiltonian diffeomorphism of SN. Conversely, each $\mathbb{R}_+$-equivariant symplectomorphism of SN projects to a contactomorphism of (N, ξ). A function F on SN is called a contact Hamiltonian if it is homogeneous of degree 1. that is, if $F(cx) = cF(x)$ for all $c \in \mathbb{R}_+$, $x \in SN$.

The Hamiltonian flow generated by a contact Hamiltonian function is $\mathbb{R}_+$-equivariant; it defines a contact isotopy on (N, ξ). Therefore, any contact isotopy $\{\varphi_t\}$ is generated in this sense by a uniquely defined time-dependant contact Hamiltonian $F_t : SN \to \mathbb{R}$. There is a one-to-one correspondence between a contact vector field X_t and a function on N: $f_t = \alpha(X_t)$, also called a contact Hamiltonian function.

Contactization. If a symplectic manifold (M, ω) is exact ($\omega = d\alpha$), it can be contactized. The contactization $C(M, \omega)$ is the manifold $N = M \times S^1$ (or $M \times \mathbb{R}$) endowed with the contact form $dz - \alpha$. Here we denote by z the projection to the second factor and still denote by α its pull-back under the projection $N \to M$.

However, the contactization can sometimes be defined even when ω is not exact. Suppose that the form ω represents an integral cohomology class $[\omega] \in H^2(M)$. The contactization $C(M, \omega)$ of (M, ω) can be constructed as follows. Let

$$\pi : N \to M$$

be a principal S^1-bundle with the Euler class equal to $[\omega]$. This bundle admits a connection whose curvature form just is ω. This connection can be viewed as a S^1-invariant 1-form α on N. The nondegeneracy of ω implies that α is a contact form and, therefore, $\xi = \{\alpha = 0\}$ is a contact structure on N. The contact manifold (N, ξ) is, by definition, the contactization $C(M, \omega)$ of the symplectic manifold (M, ω). A change of the connection α leads to a contactomorphic manifold.

We note that a Hamiltonian vector field on (M, ω) can be lifted to a contact vector field on N. In fact, a Hamiltonian function H on M and its Hamiltonian vector field X_H satisfy $dH = \iota(X_H)\omega$. We know there exists a one-to-one correspondence between contact vector fields and functions on N, so we obtain a contact vector field $\tilde{X}_H$ on N by $\alpha(\tilde{X}_H) = \pi^* H$. Also, we have $\pi_* \tilde{X}_H = X_H$. Thus, any Hamiltonian isotopy on M is lifted to a contact isotopy on N.

If $L \subset M$ is a Lagrangian submanifold then the connection α over it is flat. The pull-back $\pi^{-1}(L) \subset N$ under the projection, which is also the image of the S^1-action of a Legendrian lift Λ, denoted by $\tilde{L}$, is a Lagrangian submanifold in SN and is foliated by Legendrian leaves obtained by integrating the flat connection over L. If the holonomy defined by the connection α is integrable over L then the Lagrangian submanifold $\tilde{L}$ is foliated by closed Legendrian submanifolds in N. In this case, the connection over L is trivial. If this condition is satisfied then L is called exact (Bohr–Sommerfeld condition), and the Lagrangian submanifold $\tilde{L}$ is foliated by closed Legendrian lifts of L.

A Legendrian submanifold $\Lambda \subset (N, \xi)$ has a neighborhood U contactomorphic to the 1-jet space $J^1(\Lambda)$. Then $\tilde{L} \cap U$ can be identified under the contactomorphism with the so-called "0-wall": $W = \Lambda \times \mathbb{R} \subset J^1(\Lambda)$, which is just the set of 1-jets of function with 0 differential.

Modifying (SN, ω_ξ). Now, given a contact isotopy $\{\psi_t \mid 0 \le t \le 1\}$ of (N, ξ), it can be lifted to a Hamiltonian isotopy $\{\Psi_t \mid 0 \le t \le 1\}$ of SN. Then, from the definition and properties listed above, we have a one-to-one correspondence between $L \cap \pi \circ \psi_1(\Lambda)$ and $\tilde{L} \cap \Psi_1(p^{-1}(\Lambda))$. They also coincide with $\tilde{L} \cap \psi_1(\Lambda)$, and all intersections are transversal. Therefore, it is natural to define Floer homology for such a pair of Lagrangian submanifolds $\tilde{L}$ and $\mathscr{L} = p^{-1}(\Lambda)$. However, as we all know, symplectization SN is not compact, so the ordinary method can not be applied directly. We adopt Ono's argument [1996] to overcome this difficulty.

We see that N is compact, thus there exists large $C > 0$ such that the trace of N under the isotopy $\{\Psi_t | 0 \le t \le 1\}$ is contained in a compact set $N \times [e^{-C}, e^C]$, and $N \times [e^{-C}, e^C]$ is disjoint from $\Psi_t(SN \setminus [e^{-D}, e^D])$, $t \in [0, 1]$, for some number $D > C$. The domain we are concerned with is $N \times [e^{-D}, +\infty)$. The isotopy $\{\Psi_t\}$ is generated by a Hamiltonian $H : [0, 1] \times SN \to \mathbb{R}$. We can find another function H', so that H' equals H on $N \times [e^{-C}, e^C]$ and equals zero outside of $N \times [e^{-D}, e^D]$. Then we get a new Hamiltonian isotopy $\{\Psi'_t | 0 \le t \le 1\}$ with compact support.

Since the boundary of the bundle $N \times [e^{-D-\epsilon}, +\infty)$ is of contact type, by symplectic filling techniques the symplectization $(SN = N \times \mathbb{R}_+, \omega_\xi)$ can be replaced by a new symplectic manifold (Q, Ω) which is diffeomorphic to the associated complex line bundle $N \times_{S^1} \mathbb{C} \to M$. In fact, Ono showed there exists a symplectic embedding $\mathscr{F}$ from $N \times (e^{-D-\epsilon}, +\infty)$ into (Q, Ω) ($\mathscr{F}$ is a symplectomorphism between $N \times (e^{-D-\epsilon}, +\infty)$ and $N \times_{S^1} \mathbb{C} - \{0\text{-section}\}$, see [Ono 1996, Appendix] for details). Therefore, we study the Lagrangian intersection problem for $Q, \mathscr{F}\tilde{L}, \mathscr{F}(\mathscr{L} \cap N \times (e^{-D-\epsilon}, +\infty))$ under Hamiltonian isotopy Φ_t generated by a Hamiltonian defined on Q, which equals $H' \circ \mathscr{F}^{-1}$ on $N \times_{S^1} \mathbb{C} - \{0\text{-section}\}$, and equals zero on the 0-section. For simplicity, we still denote them by $\tilde{L}, \mathscr{L}, H$.

The positive end of Q is J-convex; that is, for a given $E > 1$, the product $N \times \{E\} \subset Q$ is a J-convex codimension-1 submanifold. So no J-holomorphic curves can touch it, and, especially, there exists a C^0 bound for every J-holomorphic disc $u : D^2 \to Q$ with boundary in Lagrangian submanifolds $\tilde{L}$ and $\Phi_t(\mathscr{L})$ (compare also [Ono 1996]). For the general case, we consider $u : \Pi = \mathbb{R} \times [0, 1] \to Q$ with $u(\tau, 0) \subset \mathscr{L}$ and $u(\tau, 1) \subset \tilde{L}$, $\tau \in \mathbb{R}$, which is regarded as the connecting orbit between $x_-(t) = \lim_{\tau \to -\infty} u(\tau, t)$ and $x_+(t) = \lim_{\tau \to +\infty} u(\tau, t)$, solving the perturbed Cauchy–Riemann equation

$$\frac{\partial u}{\partial \tau} = -J \frac{\partial u}{\partial t} + \nabla H(t, u(\tau, t)).$$

In this situation, Gromov [1985] showed how to define an almost complex structure $\tilde{J}_H$ on the product $\tilde{Q} = \Pi \times Q$, such that the $\tilde{J}_H$-holomorphic sections of $\tilde{Q}$ are precisely the graph $\tilde{u}$ of solutions of the equation above. We can see that $\tilde{Q}$ is $\tilde{J}_H$-convex, so there is a C^0-bound for $\tilde{J}_H$-holomorphic curves in $\tilde{Q}$. The same then thing happens to the connecting orbits in Q.

3. Variation and functional

From the discussion above, we know that we have got a symplectic manifold (Q, Ω), and two Lagrangian submanifolds $\tilde{L}$ and $\mathscr{L}$. Then we will establish a homology theory for the pair $(\tilde{L}, \mathscr{L})$ in Q, and study critical points of the symplectic action functional defined on (some covering of) the space of paths in Q, starting from $\mathscr{L}$ with ends on $\tilde{L}$.

Let $H \in \mathscr{H}(Q)$ satisfy $\|H\| < \sigma(Q, \tilde{L}, \mathscr{L}, J)$, and $\Psi^t_{(s)}$, $s \in [0, 1]$, be the time-t flow generated by Hamiltonian sH (note that $\Psi^1_{(s)}$ is the lift of the contactomorphism $\psi^1_{(s)}$). And set

$$\mathscr{L}_s = \Psi^1_{(s)}(\mathscr{L}), \quad \Lambda_s = \psi^1_{(s)}(\Lambda) \subset N.$$

We suppose that $\tilde{L}$ intersects $\mathscr{L}_1$ transversally.

Let Σ be the connected component of constant paths in the path space

$$\{\gamma \in C^\infty([0, 1], Q) | \gamma(0) \in \mathscr{L},\ \gamma(1) \in \tilde{L}\}.$$

We define the closed 1-form α on Σ by

$$\langle \alpha(\gamma), v \rangle = \int_0^1 \Omega(\dot{\gamma}(t), v(t))\, dt, \quad v(t) \in TQ|_{\gamma(t)}, \text{ for } t \in [0, 1].$$

We also write the function $\theta : \Sigma \to \mathbb{R}$ as

$$\theta(\gamma) = -\int_0^1 H(t, \gamma(t))\, dt.$$

Note that the zeroes of $\alpha_s = \alpha + s d\theta$ are just time-1 trajectories generated by the flow $\Psi^t_{(s)}$ that start from $\mathscr{L}$ and end on $\tilde{L}$. If γ is the zero of α_s, then the ends of all $\gamma(1)$ are just the intersection points of $\tilde{L}$ with $\mathscr{L}_s$, which are one-to-one correspondent to the zeroes of α_s. The purpose of this paper is to estimate from below the number of zeroes of α_1 .

Since H_t is compactly supported on Q, let $b_+ = \int_0^1 \max_{x\in Q} H(t,x)\,dt$, and $b_- = \int_0^1 \min_{x\in Q} H(t,x)\,dt$. Then $\|H\| = b_+ - b_-$, $-b_+ \le \theta(\gamma) \le -b_-$, for all $\gamma \in \Sigma$. We introduce the Riemannian structure on Σ through the metric

$$(v_1, v_2) = \int_0^1 \Omega(v_1(t), Jv_2(t))\,dt.$$

Since

$$\begin{aligned}(\operatorname{grad}_\alpha(\gamma), v) &= \langle \alpha(\gamma), v\rangle \\ &= \int_0^1 \Omega(\dot\gamma(t), v(t))\,dt = \int_0^1 \Omega(J\dot\gamma(t), Jv(t))\,dt = (J\dot\gamma, v),\end{aligned}$$

so the gradient of the closed 1-form α is given by $J\dot\gamma$, similarly, the gradient of the closed 1-form α_s is $\operatorname{grad}_{\alpha_s} = J\dot\gamma - s\nabla H$.

Now, we consider the minimal covering $\pi : \tilde\Sigma \to \Sigma$ such that the form $\pi^*\alpha$ is exact (meaning that there is a functional F on $\tilde\Sigma$ such that $\pi^*\alpha = dF$), and its structure group Γ is free abelian. Denote $F_s = F + s(\theta\circ\pi)$, so $dF_s = \pi^*\alpha_s$. The gradient ∇F_s of the functional F_s, with respect to the lift of the Riemannian structure on Σ, is a Γ-invariant vector field on $\tilde\Sigma$, and $\pi_*\nabla F_s = \operatorname{grad}_{\alpha_s}$. Then we consider the moduli space of thus gradient flows connecting a pair of critical points (x_-, x_+) of F_s

$$\begin{aligned}&M_s(x_-, x_+)\\ &\quad = \left\{ u : \mathbb{R} \to \tilde\Sigma \;\middle|\; \frac{du(\tau)}{d\tau} = -\nabla F_s(u(\tau)), u \text{ is not constant}, \lim_{\tau\to\pm\infty} u(\tau) = x_\pm \right\}.\end{aligned}$$

Denote by $\mathcal{M}_s = \bigcup_{x_\pm} M_s(x_-, x_+)$ the collection, and the nonparameterized space by $\hat{M}_s(x_-, x_+) = M_s(x_-, x_+)/\mathbb{R}$, and the natural quotient map $q : M_s \to \hat{M}_s$. Choosing a regular Ω-compatible almost complex structure J on Q [Ono 1996], we can assume that there is a dense set $T \subset [0, 1]$ such that for all $s \in T$, $M_s(x_-, x_+)$ are finite dimensional smooth manifolds, consequently, $\tilde{L}$ intersects $\mathscr{L}_s$ transversally.[1]

[1] Recall that the J used here is just the $J' = J \oplus J_0$ given in the introduction; by choosing a generic ω-compatible almost complex structure J on M we can obtain the regular or generic Ω-compatible structure J' on Q. The arguments in [Ono 1996] for J'-holomorphic maps can apply to our H-perturbed J'-holomorphic map by similar statements as those in [Floer et al. 1995]. We can overcome the similar problem which appears in the continuation argument of Section 6.

We define the length of a gradient trajectory $u \in M_s(x_-, x_+)$ by $l_s(u) = F_s(x_-) - F_s(x_+)$. If $\hat{u} \in \hat{M}_s$, then we define its length naturally by $l_s(\hat{u}) = l_s(u)$, where $\hat{u} = q \circ u$. Denote $\Pi = \mathbb{R} \times [0, 1]$, then the map $\bar{u} : \Pi \to Q$, defined by $\bar{u}(\tau, t) = \pi(u(\tau))(t)$, satisfies the following perturbed Cauchy–Riemann equation

$$\frac{\partial \bar{u}(\tau, t)}{\partial \tau} = -J(\bar{u}(\tau, t)) \frac{\partial \bar{u}(\tau, t)}{\partial t} + s \nabla H(t, \bar{u}(\tau, t)),$$

with limits

$$\lim_{\tau \to \pm\infty} \bar{u}(\tau, t) = \pi(x_\pm) = \bar{x}_\pm(t).$$

It is easy to see that $l_0(u) = \int_{-\infty}^{+\infty} u^* dF = \int_\Pi \bar{u}^* \Omega$.

If $u \in M_0$, then $\bar{u}$ is a J-holomorphic map from Π to Q. From Oh's removing of boundary singularities theorem [1992], $\bar{u}$ can be extended to a J-holomorphic curve $\bar{u}' : (D^2, \partial^+ D^2, \partial^- D^2) \to (Q, \tilde{L}, \mathcal{L})$, where $D^2 = \bar{\Pi}$ is the two-point compactification of Π. Since $l_0(u) = \int_\Pi \bar{u}^* \Omega = \int_{D^2} (\bar{u}')^* \Omega$, we know that $l_0(u) \geq \sigma_D(Q, \tilde{L}, \mathcal{L}, J)$.

4. Defining and computing homology for C_ε^0

We denote by Y_s the set of critical points of F_s, and by C_s the vector space spanned by Y_s over $\mathbb{Z}_2$.

Since Y_s is Γ-invariant, C_s has a structure of free K-module with rank$= \#(\tilde{L} \cap \mathcal{L}_s)$, $s \in T$, where $K = \mathbb{Z}_2[\Gamma]$. Our aim in this section is to establish some homology for the complex C_ε, where ε is small enough. We write the following definition similar as the one given by Chekanov [1998].

Definition 4.1. Fix $\delta > 0$, satisfying $\Delta := \|H\| + \delta < \sigma(Q, \tilde{L}, \mathcal{L}, J)$. A gradient trajectory $u \in M_s$ is said to be short if $l_s(u) \leq \Delta$, and be very short if $l_s(u) \leq \delta$.

Now we denote the area by $A(u) = \int_\Pi \bar{u}^* \Omega$, and $h(u) = s \int_{-\infty}^{+\infty} u^* d(\theta \circ \pi)$, then still write $l(u) = l_s(u) = A(u) + h(u)$, we have

Lemma 4.2. *If u is very short, in the sense that $l(u) \leq \delta$, then the area $A(u) \leq \Delta$.*

Proof. Since $\theta = -\int_0^1 H(t, \gamma(t))\, dt \in [-b_+, -b_-]$, then

$$h(u) = s \int_{-\infty}^{+\infty} u^* d(\theta \circ \pi) = s\theta(\pi(u(\tau)))\big|_{-\infty}^{+\infty} \geq s(b_- - b_+),$$

so

$$A(u) = l(u) - h(u) \leq \delta - (b_- - b_+) = \|H\| + \delta = \Delta. \qquad \square$$

The next lemma was essentially proved by Chekanov [1998, Lemma 6]; we adapt it here for our setting, with some modifications.

Lemma 4.3. . *For a small neighborhood U of $\tilde{L}$ in Q, there exists a $\varepsilon_0 > 0$, such that for any positive $\varepsilon < \varepsilon_0$, every short gradient trajectory $u \in M_\varepsilon$ is very short, and for every short u we have $\bar{u}(\Pi) \subset U$.*

Proof. We work it by contradiction. For the first claim, we suppose there is a sequence $u_n \in M_{s_n}$ and a positive number c with $\delta \leq c \leq \Delta$ so that when $s_n \to 0$ then $l_{s_n}(u_n) \to c$. By Gromov's compactness theorem, there are some subsequence of $\bar{u}_n = \pi(u_n)$ convergent to $\bar{u}_\infty$ which is a collection of J-holomorphic spheres and J-holomorphic discs bounding $\tilde{L}$ and/or $\mathscr{L}$. Then the total symplectic area of this limit collection is just $l_0(u_\infty) = c$ which by the assumptions of Theorem 1.4 is larger than $\sigma(Q, \tilde{L}, \mathscr{L})$, but $c \leq \Delta < \sigma(Q, \tilde{L}, \mathscr{L})$, so the claim holds. For the second claim, the argument is similar. Note that if the image $\bar{u}_\infty(\Pi)$ of the limit collection is not contained in U, then at least one of the J-curve is not contained in U which is nonconstant and its area will be larger than $\sigma(Q, \tilde{L}, \mathscr{L}, J) > \Delta$, this contradicts the (very) shortness condition. □

Then, we denote by $M'_\varepsilon \subset M_\varepsilon$ the set of all short gradient trajectories (nonparameterized short gradient trajectories), and likewise for $\hat{M}'_\varepsilon \subset \hat{M}_\varepsilon$. And we can define the $\mathbb{Z}_2$-linear map $\partial : C_\varepsilon \to C_\varepsilon$ by

$$\partial(x) = \sum_{y \in Y_\varepsilon} \#\{\text{isolated points of } \hat{M}'_\varepsilon(x, y)\} y \quad \text{for all } x \in Y_\varepsilon.$$

Let $\varepsilon \in T$ be sufficiently small and satisfy the conditions of Lemma 4.3. Choose an element $x_0 \in Y_\varepsilon$, then we can define a subclass $Y^0_\varepsilon \subset Y_\varepsilon$ by

$$Y^0_\varepsilon = \{x \in Y_\varepsilon \mid |F_\varepsilon(x) - F_\varepsilon(x_0)| \leq \delta\}.$$

Then we see that the projection π bijectively maps Y^0_ε onto the set of zeroes of the form α_ε, and we get the bijection $Y^0_\varepsilon \times \Gamma \to Y_\varepsilon$ given by $(y, a) \mapsto a(y)$, which induces the isomorphism $C^0_\varepsilon \otimes K \to C_\varepsilon$, where $C^0_\varepsilon \subset C_\varepsilon$ is spanned over $\mathbb{Z}_2$ by Y^0_ε. Now, for sufficiently small $\varepsilon \in T$, we can establish the homology for $(C_\varepsilon, \partial)$

Lemma 4.4. (1) *The map ∂ is K-linear, well defined, and $\partial(C^0_\varepsilon) \subset C^0_\varepsilon$.*

(2) *If $\varepsilon \in T$ is sufficiently small, then $\partial^2 = 0$.*

(3) *The homology $H(C^0_\varepsilon, \partial) \cong H_*(\Lambda, \mathbb{Z}_2)$.*

Proof. Step 1. Since the gradient flow is Γ-invariant, ∂ is naturally K-linear. We know that the bubbling off can not occur. Indeed, since ε is sufficiently small, then $u \in M_\varepsilon$ is very short, $l_\varepsilon(u) \leq \delta$, by Lemma 4.2, the area $A(u) \leq \Delta < \sigma(Q, \tilde{L}, \mathscr{L}, J)$, and from the assumption in our theorem, the area of any J-holomorphic sphere or J-holomorphic disc bounding $\tilde{L}$ and $\mathscr{L}$ is larger than $\sigma(Q, \tilde{L}, \mathscr{L}, J)$. Thus, $\hat{M}'_\varepsilon(x, y)$ is compact and the number of its isolated points is finite.

Step 2. Suppose $\varepsilon \in T$ satisfy the conditions in Lemma 4.3. If $\|H\| = 0$, $\Delta = \delta$, then $H \equiv const.$ and $\psi_H \equiv \mathrm{id}$, it is a trivial case. If $\|H\| > 0$, we can always choose a fixed $\delta < \frac{1}{2}\|H\| < \frac{1}{2}\Delta$. Consider a pair of isolated trajectories $u_1 \in \hat{M}'_\varepsilon(x, y)$, $u_2 \in \hat{M}'_\varepsilon(y, z)$. Then there exists a unique 1-dimensional connected component $\mathscr{C} \subset \hat{M}_\varepsilon(x, z)$ such that (u_1, u_2) is one of the two ends of compactification of $\mathscr{C}$ [Floer 1988a]. Since the length is additive under gluing, we have $l_\varepsilon(u) = l_\varepsilon(u_1) + l_\varepsilon(u_2) < 2\delta < \Delta$, for all $u \in \mathscr{C}$. By Lemma 4.3, u is also very short. From Lemma 4.2, we know the bubbling off doesn't occur, too. Then the other end of $\mathscr{C}$ can be compactified by a pair of isolated trajectories $u'_1 \in \hat{M}_\varepsilon(x, y)$, $u'_2 \in \hat{M}_\varepsilon(y, z)$. Also $l_\varepsilon(u'_1) + l_\varepsilon(u'_2) = l_\varepsilon(u_1) + l_\varepsilon(u_2) < \Delta$, thus u'_1, u'_2 are short trajectories. So we know that, for $x, z \in Y_\varepsilon$, the number of isolated trajectories $(u_1, u_2) \in \hat{M}'_\varepsilon(x, y) \times \hat{M}'_\varepsilon(y, z)$ is even, that is, $\partial^2(x) = 0$.

Step 3. From Lemma 4.3, we know that for any $u \in \hat{M}'_\varepsilon(x, y)$, $\bar{u}(\bar{\Pi}) \subset U$. That is to say, if ε is small enough, then $\Lambda_\varepsilon = \psi^1_{(\varepsilon)}(\Lambda)$ is always contained in a small neighborhood $U' = U \cap N$ of the Legendrian submanifold Λ in N. By Darboux's theorem, U' is contactomorphic to a neighborhood of the 0-section in the 1-jet space $J^1(\Lambda)$. This contactomorphism moves $\tilde{L} \cap U'$ onto the 0-wall W, which is defined as the space of 1-jets of functions with 0 differential. Thus a Legendrian submanifold Λ', which is C^1-close to Λ and transverse to W, corresponds to a Morse function $\beta : \Lambda \to \mathbb{R}$ so that the intersection points of $\tilde{L}$ and Λ' are in one-to-one correspondence with the critical points of the function β. We can explicitly choose a metric on Λ and a generic almost complex structure J (recall the footnote in Section 3) on the symplectization SN in such a way that the gradient trajectories in $\mathcal{M}_\varepsilon$ would be in one-to-one correspondence with the gradient trajectories of the function β connecting the corresponding critical points of this function. Thus we can identify our complex C^0_ε with the Morse chain complex for the function β (here we may also reduce the problem to Lagrangian intersections in M by applying continuation argument and projecting the manifold to M, the method of equating Floer and Morse complex is standard, we refer the reader to [Schwarz 1993]), so we have an isomorphism $H(C^0_\varepsilon, \partial) \cong H_*(\Lambda, \mathbb{Z}_2)$. □

5. Homology algebra

Under the condition $\pi_2(Q, \cdot) = 0$ or the monotonicity assumption [Floer 1988a; Oh 1993a; 1993b; 1995], the Floer homology $HF_*(C_s, \partial)$ of the complex C_s, $s \in T \subset [0, 1]$, can be defined. Then we can use the classical continuation method (see [Floer 1989b; McDuff 1990] or the papers by Oh just cited) to prove the isomorphism between $HF_*(C_\varepsilon, \partial)$ and $HF_*(C_1, \partial)$, that means to construct chain homotopy $\Phi'\Phi \sim \mathrm{id}_\varepsilon$ (and $\Phi\Phi' \sim \mathrm{id}_1$), where $\Phi : (C_\varepsilon, \partial) \to (C_1, \partial)$, $\Phi' : (C_1, \partial) \to (C_\varepsilon, \partial)$ are chain homomorphisms defined similarly as the definition of ∂, except for considering the moduli space of continuation trajectories. That is to say, in order to prove $\Phi'\Phi \sim \mathrm{id}_\varepsilon$, we should show there exists a chain homomorphism $\boldsymbol{h} : (C_\varepsilon, \partial) \to (C_\varepsilon, \partial)$, so that

$$\Phi'\Phi - \mathrm{id} = \boldsymbol{h}\partial - \partial\boldsymbol{h}.$$

However, in general case, we can not define appropriately any homology for C_s unless s is small enough. Then we may only prove a weaker "homotopy", which was called λ-*homotopy* by Chekanov. In fact, for the aim of estimating from below the number of critical points of the functional F_s, this λ-homotopy is enough.

We shall use the following homology algebraic result, introduced and proved by Chekanov [1998].

Let Γ be a free abelian group equipped with a monomorphism $\lambda : \Gamma \to \mathbb{R}$, which we call a *weight function*. Set

$$\Gamma^+ = \{a \in \Gamma \mid \lambda(a) > 0\}, \qquad \Gamma^- = \{a \in \Gamma \mid \lambda(a) < 0\}.$$

Let k be a commutative ring. Consider the group ring $K = k[\Gamma]$. For a k-module M, we have the natural decomposition $M \otimes K = M^+ \oplus M^0 \oplus M^-$. where $M^+ = \Gamma^+(M)$, $M^0 = M$, $M^- = \Gamma^-(M)$. Consider the projections

$$p^+ : M \otimes K \to M^+ \oplus M^0, \qquad p^- : M \otimes K \to M^0 \oplus M^-.$$

Assume that (M, ∂) is a differential k-module, then ∂ naturally extends to a K-linear differential on $M \otimes K$.

Definition 5.1. We say two linear maps $\phi_0, \phi_1 : M \otimes K \to M \otimes K$ are λ-homotopic if there exists a K-linear map $\boldsymbol{h} : M \otimes K \to M \otimes K$ such that

$$p^+(\phi_0 - \phi_1 + \boldsymbol{h}\partial + \partial\boldsymbol{h})p^- = 0.$$

Lemma 5.2 [Chekanov 1998]. *Let λ be a weight function on a free abelian group Γ. Assume (M, ∂) to be a differential k-module and N to be a K-module, where $K = k[\Gamma]$. If the maps $\Phi^+ : M \otimes K \to N$ and $\Phi^- : N \to M \otimes K$ are K-linear and $\Phi^-\Phi^+$ is λ-homotopic to the identity, then* $\mathrm{rank}_K N \geq \mathrm{rank}_k H(M, \partial)$.

6. Proofs of the main results

Given a (s_-, s_+) continuation function $\rho : \mathbb{R} \to [0, 1]$ satisfying

$$\rho(\tau) = \begin{cases} s_- & \text{if } \tau < -r, \\ s_+ & \text{if } \tau > r, \end{cases}$$

where $r \in \mathbb{R}$, we can define the moduli space of continuation trajectories

$$M_\rho(x_-, x_+) = \left\{ u : \mathbb{R} \to \tilde{\Sigma} \;\middle|\; \frac{du(\tau)}{d\tau} = -\nabla F_{\rho(\tau)}(u(\tau)),\ \lim_{\tau \to \pm\infty} u(\tau) = x_\pm \right\},$$

where $x_\pm \in Y_{s_\pm}$. And we denote the collection by $\mathcal{M}_\rho = \bigcup_{x_-,x_+} M_\rho(x_-, x_+)$. The length of a continuation trajectory is defined by $l_\rho(u) = F_{s_-}(x_-) - F_{s_+}(x_+)$.

Choose a monotone $(\varepsilon, 1)$ continuation function ρ_+ and a monotone $(1, \varepsilon)$ continuation function ρ_-. For generic H, $M_{\rho_\pm}(x_-, x_+)$ are smooth manifolds. We will say a continuation trajectory $u \in M_{\rho_+}(x_-, x_+)$ (or $u \in M_{\rho_-}(x_-, x_+)$) is short if $l_{\rho_+}(u) \le \delta + (1-\varepsilon)b_+$ (resp. $l_{\rho_-}(u) \le \delta + (\varepsilon - 1)b_-$). The subspace of all short trajectories is denoted by $M'_{\rho_\pm}(x_-, x_+) \subset \mathcal{M}_{\rho_\pm}$.

In the best possible situation, that is, if no sequence of continuation trajectories reaches the negative end (the zero section of $Q \to M$),[2] we can simply construct the $\mathbb{Z}_2$-linear continuation map $\Phi^+ : C_\varepsilon \to C_1$, $\Phi^- : C_1 \to C_\varepsilon$ as

$$\Phi^+(x) = \sum_{y \in Y_1} \#\{\text{isolated points of } M'_{\rho_+}(x, y)\} y,$$

$$\Phi^-(y) = \sum_{z \in Y_\varepsilon} \#\{\text{isolated points of } M'_{\rho_-}(x, z)\} z,$$

where $x \in Y_\varepsilon$. The next lemma implies that this definition of $\Phi^\pm$ is sound.

Lemma 6.1. *If $u \in M_{\rho_+}(x_-, x_+)$, then $l_{\rho_+}(u) \ge (1-\varepsilon)b_-$. If $u \in M_{\rho_-}(x_-, x_+)$, then $l_{\rho_-}(u) \ge (\varepsilon - 1)b_+$. And the sum in the definition of $\Phi^\pm$ is finite.*

Proof. Recall $F_s = F + s\theta \circ \pi$, $s \in [0, 1]$. For a (s_-, s_+) continuation function $\rho : \mathbb{R} \to [0, 1]$, we have $F_{\rho(\tau)} = F + \rho(\tau)\theta \circ \pi$. So the length of a continuation trajectory is

$$\begin{aligned} l_\rho(u) &= F_{s_-}(x_-) - F_{s_+}(x_+) \\ &= -\int_{-\infty}^{+\infty} u^* dF_{\rho(\tau)} = -\int_{-\infty}^{+\infty} u^* dF - \int_{-\infty}^{+\infty} u^* d(\rho(\tau)\theta \circ \pi) \\ &= A(u) + h(u), \end{aligned}$$

where we set

$$\begin{aligned} A(u) &= -\int_{-\infty}^{+\infty} u^* dF = \int_\Pi \bar{u}^* \Omega = \int_{-\infty}^{+\infty} \left(\frac{du(\tau)}{d\tau}, \frac{du(\tau)}{d\tau} \right) d\tau \\ &= \int_{-\infty}^{+\infty} \|\nabla F_{\rho(\tau)}\|^2 d\tau \ge 0, \end{aligned}$$

$$h(u) = -\int_{-\infty}^{+\infty} u^* d(\rho(\tau)\theta \circ \pi) = -\int_{-\infty}^{+\infty} \frac{d\varrho(\tau)}{d\tau} \theta(\pi(u(\tau)))\, d\tau.$$

Recall that

$$\theta = -\int_0^1 H\, dt \in [-b_+, -b_-],$$

[2] The possibility that there is such a sequence was pointed out to the author by one of referees. In this case, we have to modify the continuation map.

if $u \in M_{\rho_+}(x_-, x_+)$,

$$l(u) \geq h(u) \geq (1-\varepsilon)b_-;$$

if $u \in M_{\rho_-}(x_-, x_+)$,

$$l(u) \geq h(u) \geq (\varepsilon - 1)b_+.$$

Thus, For a short trajectory $u \in M'_{\rho_\pm}(x_-, x_+)$,

$$A(u) = l(u) - h(u) \leq \delta + (1-\varepsilon)(b_+ - b_-) = \|H\| + \delta = \Delta.$$

Since $A(u) = \int_\Pi \bar{u}^*\Omega \leq \Delta < \sigma(Q, \tilde{L}, J)$, by Gromov's arguments, no bubbling can occur, then spaces $M'_{\rho_\pm}(x_-, x_+)$ are compact, and the finiteness of the set of isolated points of $M'_{\rho_\pm}(x, y)$ is verified. □

However, in the general case, the ideal assumption is not always satisfied. If a sequence of continuation trajectories converges to a curve reaching the negative end of $\mathscr{L}$, the boundary of moduli space of continuation trajectories will contain such curve. So we need modify the definition of $\Phi^\pm$ to get a well-defined continuation map. In fact, Ono [1996, p. 218] had dealt with a similar problem by considering the algebraic intersection number of the continuation trajectories with the zero section O_M of $Q \to M$. Under Ono's assumption $\pi_2(M, L) = 0$, bubbling off of holomorphic discs contained in the zero section of Q with boundary on L never occurs. In our case, since we have the bound for energy, such kind of bubbling off of discs is also avoided. Thus, we can define homomorphisms $\Phi_k^+ : C_\varepsilon \to C_1$, $\Phi_k^- : C_1 \to C_\varepsilon$ as

$$\Phi_k^+(x) = \sum_{y \in Y_1} \mathrm{Int}_{2,k}^+(x, y)y, \quad \Phi_k^-(y) = \sum_{z \in Y_\varepsilon} \mathrm{Int}_{2,k}^-(y, z)z,$$

where $\mathrm{Int}_{2,k}^+(x, y)$ ($\mathrm{Int}_{2,k}^-(y, z)$) is the mod-2 number of isolated points u in the continuational moduli spaces $M'_{\rho_+}(x, y)$ and $M'_{\rho_-}(y, z)$, respectively, having algebraic intersection number $u \cdot O_M = k/2$.

With the restriction of bound of energy, we can make similar discussions as in [Ono 1996] to get the finiteness of $\mathrm{Int}_{2,0}^\pm$. So $\Phi_0^\pm$ are just our favorite continuation maps. We will not list the detailed arguments here and refer the reader to [Ono 1996] for original discussion. In the following, we will still denote the continuation maps by $\Phi^\pm$ for simplicity.

Then we can use the homology algebraic result given in Section 5 to prove Theorem 1.4, provided there exists a λ-homotopy between $\Phi^-\Phi^+$ and the identity.

Proof of Theorem 1.4. Take $k = \mathbb{Z}_2$, $K = \mathbb{Z}_2[\Gamma]$, $M = C_\varepsilon^0$, $M \otimes K = C_\varepsilon$, $N = C_1$, and let Γ be the structure group of the covering. The weight function $\lambda : \Gamma \to \mathbb{R}$ can be defined as $\lambda(a) = F(a(x)) - F(x)$. We also have decompositions $Y_\varepsilon = Y_\varepsilon^+ \cup Y_\varepsilon^0 \cup Y_\varepsilon^-$, $C_\varepsilon = C_\varepsilon^+ \oplus C_\varepsilon^0 \oplus C_\varepsilon^-$, where $Y_\varepsilon^\pm = \Gamma^\pm(Y_\varepsilon^0)$.

Assume that we have got a λ-homotopy $\boldsymbol{h} : C_\varepsilon \to C_\varepsilon$. By Lemmas 5.2 and 6.1 we have

$$\#(L \cap \pi \circ \psi_1(\Lambda)) = \#(\tilde{L} \cap \psi_1(\Lambda)) = \#(\tilde{L} \cap \Psi_1(\mathscr{L})) = \operatorname{rank}_K C_1$$
$$\geq \operatorname{rank}_k H(C_\varepsilon^0, \partial) = \dim H_*(\Lambda, \mathbb{Z}_2) = \dim H_*(L, \mathbb{Z}_2). \quad \square$$

In the rest of this section, we show a sketchy proof of the existence of λ-homotopy.

Lemma 6.2. *There exists a λ-homotopy $\boldsymbol{h} : C_\varepsilon \to C_\varepsilon$ between $\Phi^-\Phi^+$ and the identity.*

Proof. We follow the arguments of Chekanov and state his main thought. For constructing the homomorphism $\boldsymbol{h}$, we use a family of $(\varepsilon, \varepsilon)$ continuation functions $\mu_c,\ c \in [0, +\infty)$ satisfying these conditions:

(1) $\mu_0(\tau) \equiv \varepsilon$.

(2) $du_c(\tau)/d\tau \geq 0$ if $\tau < 0$ and $du_c(\tau)/d\tau \leq 0$ if $\tau > 0$.

(3) $c \mapsto \mu_c(0)$ is a monotone map from $[0, +\infty)$ onto $[\varepsilon, 1]$.

(4) when c is large enough $\mu_c(\tau) = \begin{cases} \rho_+(\tau + c), & \text{if } \tau \leq 0; \\ \rho_-(\tau - c), & \text{if } \tau \geq 0. \end{cases}$

We denote the moduli space by

$$M_\mu(x_-, x_+) = \{(c, u) \mid u \in M_{\mu_c}(x_-, x_+)\},\ \ x_\pm \in Y_\varepsilon.$$

For generic H, $M_\mu(x_-, x_+)$ are smooth manifolds.

As for the $(\varepsilon, 1)$ or $(1, \varepsilon)$ continuation trajectories earlier in this section, under the assumption that any sequence of μ_c-continuation trajectories reaches the negative end of $\mathscr{L}$,[3] we can define the $\mathbb{Z}_2$-linear map for C_ε^0 as

$$\boldsymbol{h}(x) = \sum_{y \in Y_\varepsilon^0} \#\{\text{isolated points of } M'_\mu(x, y)\} y,\ \ x \in Y_\varepsilon^0,$$

where $M'_\mu(x, y)$ is the subset of the moduli space $M_\mu(x, y)$ which contains only short μ_c-continuation trajectories, a μ_c-continuation trajectory $u \in M_{\mu_c}(x_-, x_+)$ is called short if its length $l(u) = l_{\mu_c}(u) \leq \delta$. Moreover, For any $u \in M_{\mu_c}(x_-, x_+)$, we have $l_{\mu_c}(u) = A(u) + h(u) \geq h(u) \geq (\mu_c(0) - \varepsilon)b_- + (\varepsilon - \mu_c(0))b_+ \geq b_- - b_+ = -\|H\|$, and if $l(u) \leq \delta$, then $A(u) = l(u) - h(u) \leq \delta + \|H\| \leq \Delta$.

The map $\boldsymbol{h}$ can be extended naturally to a K-linear map on C_ε. Since for $u \in M'_\mu(x, y)$, $l_{\mu_c}(u) \leq \delta,\ A(u) \leq \Delta$, the bubbling off does not occur, $M'_\mu(x, y)$ is compact and the sum is finite, thus the map $\boldsymbol{h}$ is well defined.

[3]Otherwise, we will again adopt Ono's argument to take into consideration the algebraic intersection number. The way to modify the definition of the map $\boldsymbol{h}$ is similar to the previously mentioned way to modify $\Phi^\pm$.

To prove $\boldsymbol{h}$ is a λ-homotopy, we have to verify

$$p^+(x + \Phi^-\Phi^+x + \boldsymbol{h}\partial x + \partial\boldsymbol{h}x) = 0,$$

for all $x \in Y_\varepsilon^0 \cup Y_\varepsilon^-$. This will follow from the standard gluing argument involving the ends of the 1-dimensional part $\aleph$ of $M_\mu(x, z)$, $z \in Y_\varepsilon^+ \cup Y_\varepsilon^0$. Since $x \in Y_\varepsilon^0 \cup Y_\varepsilon^-$, $z \in Y_\varepsilon^+ \cup Y_\varepsilon^0$, and ε is small enough, we know $l(u) \le \delta$ for $u \in \aleph$. Indeed, $l(u) = F_\varepsilon(x) - F_\varepsilon(z)$, and there exist x' and z' in Y_ε^0 and $a \in \Gamma^+ \cup \Gamma^0$, $b \in \Gamma^0 \cup \Gamma^-$ such that $z = a(z')$, $x = b(x')$, and $F_\varepsilon(x) - F_\varepsilon(x') = \lambda(b) \le 0$, $F_\varepsilon(z') - F_\varepsilon(z) = -\lambda(a) \le 0$, also we know that $F_\varepsilon(x') - F_\varepsilon(z') \le \delta$ since $x', z' \in Y_\varepsilon^0$, this implies $l(u) \le \delta$, so $A(u) \le \Delta$.[4] This gets rid of the bubbling off. Then the compactification of $\aleph$ shows that the left-hand side of the formula above equals

$$\sum_{z \in Y_\varepsilon^+ \cup Y_\varepsilon^0} \#\{S(x, z)\} z,$$

and the number $\#\{S(x, z)\}$ is even.[5] This ends the proof of the lemma and of Theorem 1.4. For more details, see [Chekanov 1998; Floer 1989b; McDuff 1990]. □

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[4]From this proof of the inequality $l(u) \le \delta$, the reader can see why we only check λ-homotopy.

[5]Beyond the standard gluing argument, one should verify that the other ends of the compactification, which are pairs of continuation trajectories, are still all short. This is not difficult to do. Also, for the modified continuation maps $\Phi_0^\pm$, we can verify that the continuation trajectories in the other ends have the same 0-algebraic intersection number with the zero section O_M, see [Ono 1996].

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