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For any $n \in \mathbb{N}$ we construct graph manifolds of genus $4n$ whose fundamental group is $3n$-generated.

1. introduction

A *Heegaard surface* of an orientable closed 3-manifold M is an embedded orientable surface S such that $\overline{M-S}$ consists of 2 handlebodies V_1 and V_2. This decomposition of M is called a *Heegaard splitting* and denoted by $M = V_1 \cup_S V_2$. We say that the splitting is of *genus* g if S is of genus g. It is not difficult to see that any orientable closed 3-manifold admits a Heegaard splitting. If M admits a Heegaard splitting of genus g but no Heegaard splitting of smaller genus then we say that M has *Heegaard genus* g and write $g(M) = g$.

Clearly any curve in a handlebody can be homotoped to its boundary. It follows that for any Heegaard splitting $M = V_1 \cup_S V_2$ every curve in M can be homotoped into V_1. Thus the map induced by the inclusion of V_1 into M maps a generating set of $\pi_1(V_1)$ to a generating set of $\pi_1(M)$. Since $\pi_1(V_1)$ is generated by g elements, $\pi_1(M)$ is also generated by g elements. Thus $g(M) \geq r(M)$, where $r(M)$ denotes the minimal number of generators of $\pi_1(M)$. Sometimes we will refer to $g(M)$ as the *geometric rank* and to $r(M)$ as the *algebraic rank* of M.

F. Waldhausen [1978] asked whether the converse inequality also holds: is $g(M) = r(M)$? A positive answer would have implied the Poincaré conjecture. First counterexamples however were found by M. Boileau and H. Zieschang [1984]. These examples were Seifert fibered manifolds with $g(M) = 3$ and $r(M) = 2$. The work of Y. Moriah and J. Schultens [1998] further shows that this class extends to higher-genus examples: Seifert manifolds with $g(M) = n+1$ and $r(M) = n$. In [Weidmann 2003] a class of graph manifolds was found for which $g(M) = 3$ and $r(M) = 2$. The original Boileau–Zieschang examples can be interpreted as a special case of these graph manifolds.

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We here show how the phenomenon observed in [Weidmann 2003] generalizes and how it can occur multiple times within a single graph manifold. This yields graph manifolds where the difference between the algebraic and the geometric rank is arbitrarily high.

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2. Formulation of the main results

Let M be a closed graph manifold. We will always assume that M comes equipped with its characteristic tori $\mathcal{T} = \mathcal{T}_M$ and a fixed Seifert fibration on every component of $\overline{M - \mathcal{T}}$. Recall that the Seifert fibrations are unique up to isotopy except for components homeomorphic to Q, the Seifert space with base orbifold the disk with two cone points of order 2. The space Q can also be fibered as the orientable circle bundle over the Möbius band. We will refer to the components of $\overline{M - \mathcal{T}}$ as the *Seifert pieces* of M. The Seifert pieces of M are up to isotopy precisely the maximal Seifert submanifolds of M. We will mostly work with *totally orientable* graph manifolds, that is, orientable graph manifolds whose Seifert pieces have orientable base orbifold. This makes the Seifert fibrations unique up to isotopy on all Seifert pieces.

Let N be a Seifert piece of M. Denote the fiber of N by f. Let $T_1, \ldots, T_n$ be the boundary components of N and let $\gamma_i \subset T_i$ be the curve corresponding to the fiber of that Seifert piece L_i which is reached by travelling from N transversely through T_i. Note that we possibly have $N = L_i$. The maximality of the Seifert piece N guarantees that for all i the intersection number of f with γ_i does not vanish.

We then define $\hat{N}$ to be the manifold $N(\gamma_1, \ldots, \gamma_n)$ obtained from N by performing a Dehn filling with slope γ_i at each boundary component T_i. It is clear that the Seifert fibration of N can be extended to a Seifert fibration of $\hat{N}$ as f has nontrivial intersection number with all γ_i.

In the following we will denote the base orbifold of a Seifert piece N by $\mathbb{O}(N)$. We will denote an orbifold by its topological type with a list of the orders of cone points, where ∞ stands for a boundary component. We will denote the disc by D, the sphere by S^2, the annulus by A, the orientable surface of genus g, for $g > 0$, by F_g and the projective plane by P^2.

Theorem 1. *Let M be a closed graph manifold consisting of two Seifert pieces N_1 and N_2 glued along T, where $\mathbb{O}(N_1) = F_g(r, \infty)$, $\mathbb{O}(N_2) = D(p, q)$ with $(p, q) = 1$ and* $\min(p, q) \leq 2g + 1$ *such that the intersection number of the fibers of N_1 and N_2 equals* 1.

Then $\pi_1(M)$ *is generated by* $2g+1$ *elements. Furthermore if* M *admits a Heegaard splitting of genus* $2g+1$ *then one of the following holds*:

(1) N_2 *is the exterior of a* s*-bridge knot with* $s \leq 2g+1$ *and the fiber of* N_1 *is identified with the meridian of* N_2, *that is*, $\hat{N}_2 = S^3$.

(2) $\hat{N}_1$ *admits a horizontal Heegaard splitting of genus* $2g$.

(Conversely, M has a Heegaard splitting of genus $2g+1$ if (1) or (2) is satisfied. This splitting is vertical in N_1 and pseudohorizontal in N_2 in case (1) and pseudohorizontal in N_1 and vertical in N_2 in case (2). This follows from the proof of Theorem 1.)

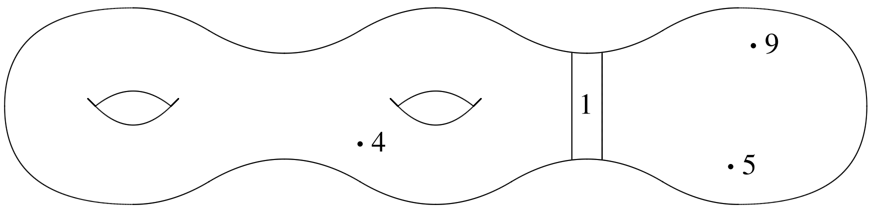

Figure 1. A graph manifold with 5-generated fundamental group.

We will further see that all manifolds of this type admit a Heegaard splitting of genus $2g+2$. Furthermore, most of these manifolds do not admit a Heegaard splitting of genus $2g+1$ as for any given pair of such manifolds N_1 and N_2 there are at most three gluing maps that yield a graph manifold of genus $2g+1$. It is also possible to show that $\pi_1(M)$ cannot be generated by less than $2g+1$ elements, but the argument is complicated.

A careful analysis of these examples shows that the phenomenon is of a local nature, it can therefore be reproduced multiple times within a graph manifold with a more complex underlying graph. Hence:

Theorem 2. *For any* $n \in \mathbb{N}$ *there exists a graph manifold* M_n *with* $3n$*-generated fundamental group that has Heegaard genus at least* $4n$.

This paper is organized as follows. In Section 3 we review the structure theorem for Heegaard splittings of totally orientable graph manifolds as proven in [Schultens 2004]. Then we study in more detail how Heegaard surfaces can intersect the Seifert pieces that are the building blocks of our examples. In Sections 5 and 6 we give the proofs of Theorems 1 and 2. We conclude by describing a class orientable Seifert manifolds with $2n$-generated fundamental group which we believe to be of Heegaard genus $3n$. However, these manifolds are not totally orientable.

3. Heegaard splittings of totally orientable graph manifolds

A graph manifold M is *totally orientable* if M is orientable and every Seifert piece N of M fibers over an orientable base space. In [Schultens 2004] it is shown that the Heegaard splittings of totally orientable graph manifolds have a structure that can be completely described. To do so, one considers a decomposition of M into edge manifolds and vertex manifolds. The edge manifolds are the submanifolds of the form $T \times I$, where T is one of the characteristic tori, $\mathcal{T}$, of M. The vertex manifolds are the components of the complement of the edge manifolds. Note that each vertex manifold is homeomorphic to a component of $M - \mathcal{T}$.

Heegaard splittings themselves are rather unwieldy. Instead we work with the surfaces arising in what is called a "strongly irreducible untelescoping" of a Heegaard splitting. We use the terms pseudohorizontal, horizontal, pseudovertical and vertical to describe the possible structure for the restriction of such a surface to the vertex manifolds. The restriction of such a surface to the edge manifolds takes three possible forms. It too plays a nontrivial role in the structure of the Heegaard splitting of a graph manifold.

A two-sided surface F in a 3-manifold M is said to be *weakly reducible* if there are disjoint essential curves a, b in F that bound disks D_a, D_b whose interior is disjoint from F and such that near their boundary D_a, D_b lie on opposite sides of F. A two-sided surface F in a 3-manifold M is said to be *strongly irreducible* if it is not weakly reducible.

Heegaard splittings correspond to handle decompositions. Given a 3-manifold M and a decomposition $M = V \cup_S W$ into two handlebodies, one handlebody, say V, provides the 0-handles and 1-handles and the other, W, provides the 2-handles and 3-handles. Without loss of generality, there is only one 0-handle and one 3-handle. Corresponding to $M = V \cup_S W$ we then have a handle decomposition in which all 1-handles are attached before any of the 2-handles. An *untelescoping* of a Heegaard splitting is a rearrangement of the order in which the 1-handles and 2-handles are attached. In the handle decomposition obtained we first attach the 0-handle, then some 1-handles, then some 2-handles, then some 1-handles, then some 2-handles, etc and finally, the 3-handle. We specify an untelescoping by a collection of surfaces S_1, F_1, S_2, F_2, $\dots$, F_{n-1}, S_n. These surfaces are obtained as follows: S_1 is the boundary of the submanifold of M obtained by attaching the 0-handle and the first batch of 1-handles. F_1 is the boundary of the submanifold of M obtained by attaching the 0-handle, the first batch of 1-handles and the first batch of 2-handles. S_2 is the boundary of the submanifold of M obtained by attaching the 0-handle, the first batch of 1-handles, the first batch of 2-handles and the second batch of 1-handles. F_2 is the boundary of the submanifold of M obtained by attaching the 0-handle, the first batch of 1-handles, the first batch of 2-handles, the second batch of 1-handles and the second batch of 2-handles, and so on. An untelescoping $S_1, F_1, S_2, F_2, \dots, S_n$ is said to be strongly irreducible if each S_i is a strongly irreducible surface in M and each F_i is an incompressible surface in M. Note that a Heegaard splitting can be considered a trivial untelescoping S. If it is strongly irreducible, then it is its own strongly irreducible untelescoping.

For the discussion here it will be useful to note that each of the S_i and each of the F_i is separating, and that each pair S_i, F_i cobound a submanifold homeomorphic to $S_i \times I$ with 2-handles attached to $S_i \times \{1\}$. In particular, $\chi(S_i) < \chi(F_i)$. Similarly for F_i and S_{i+1}. The following theorem summarizes the discussion in [Scharlemann and Thompson 1994], [Scharlemann 2002] and [Scharlemann and Schultens 1999, Lemma 2].

Theorem 3. *Let M be a 3-manifold and $M = V \cup_S W$ a Heegaard splitting. Then $M = V \cup_S W$ has a strongly irreducible untelescoping $S_1, F_1, S_2, F_2, \dots, S_n$. Furthermore,*

$$-\chi(S) = \sum_{i=1}^{n} \big(\chi(F_{i-1}) - \chi(S_i)\big).$$

A surface in a Seifert fibered space is *horizontal* if it is everywhere transverse to the fibration. It is *pseudohorizontal* if it is horizontal away from a fiber e and intersects a regular neighborhood $N(e)$ of e in an annulus that is a bicollar of e. (In [Moriah and Schultens 1998] the Heegaard splittings of a Seifert fibered space with pseudohorizontal splitting surface are called *horizontal Heegaard splittings*.)

Let F be a surface in a 3-manifold M and α an arc with interior in $M \backslash F$ and endpoints on F. Let $C(\alpha)$ be a collar of α in M. The boundary of $C(\alpha)$ consists of an annulus A together with two disks D_1, D_2, which we may assume to lie in F. We call the process of replacing F by $(F \backslash (D_1 \cup D_2)) \cup A$ *performing ambient 1-surgery on F along α*.

A surface S in a Seifert fibered space is *vertical* if it consists of regular fibers. It is *pseudovertical* if there is a vertical surface $\mathcal{V}$ and a collection of arcs Γ with interior disjoint from $\mathcal{V}$ that projects to an embedded collection of arcs such that S is obtained from $\mathcal{V}$ by ambient 1-surgery along Γ.

The definition of a standard Heegaard splitting for a graph manifold is rather lengthy. Let M be a graph manifold. A strongly irreducible untelescoping S_1, F_1, S_2, $F_2, \dots, S_n$ of a Heegaard splitting $M = V \cup_S W$ is standard if it is as follows:

(1) Each F_i intersects each vertex manifold either in a horizontal or in a vertical surface (or $\varnothing$).

(2) Each F_i is either a torus entirely contained in an edge manifold or intersects an edge manifold in spanning annuli (or $\varnothing$);

(3) Each S_i intersects each vertex manifold in either a horizontal, pseudohorizontal, vertical or pseudovertical surface (or $\varnothing$).

(4) $\bigcup_i S_i$ intersects each edge manifold $M_e = (\text{torus}) \times [0, 1]$ in one of three possible ways: $\left(\bigcup_i S_i\right) \cap M_e$ consists of incompressible annuli; or $S_i \cap M_e$ can be obtained from a collection of incompressible annuli by ambient 1-surgery along an arc that is isotopic to an embedded arc in the boundary of the edge manifold; or there is a pair of simple closed curves $c, c' \subset$ (torus) such that $c \cap c'$ consists of a single point p and $S_i \cap M_e$ is the portion of the boundary of a collar of $c \times \{0\} \cup p \times [0, 1] \cup c' \times \{1\}$ that lies in the interior of M_e. Furthermore, each edge manifold must be met by at least one of the S_i.

Recall that for each i, F_i and S_i are separating. Thus if F_i or S_i intersects an edge manifold M_e in spanning annuli, then it must do so in an even number of spanning annuli. It is a nontrivial fact that if $S_1, F_1, S_2, F_2, \dots, S_n$ meets M_e in spanning annuli, then between any two components of $F_i \cap M_e$ there must be at least two components of either $S_i \cap M_e$ or $S_{i+1} \cap M_e$. (This follows, for instance, by the argument used in the proof of Lemma 12.)

The Heegaard splitting $M = V \cup_S W$ is *standard* if every strongly irreducible untelescoping $S_1, F_1, S_2, F_2, \dots, S_n$ of $M = V \cup_S W$, the union $\bigcup_i F_i \cup \bigcup_i S_i$ can be isotoped to be standard.

We recall the main theorem in [Schultens 2004], some of whose many consequences we will need.

Theorem 4. *Let $M = V \cup_S W$ be an irreducible Heegaard splitting of a totally orientable graph manifold. Then $M = V \cup_S W$ is standard.*

We assume that M is a totally orientable graph manifold, $M = V \cup_S W$ a Heegaard splitting and $S_1, F_1, \dots, F_{n-1}, S_n$ a strongly irreducible untelescoping of $M = V \cup_S W$ that is standard. Then:

Fact 1. For N a vertex or edge manifold of M,

$$\sum_i \left(\chi(F_{i-1} \cap N) - \chi(S_i \cap N)\right) \geq 0.$$

Fact 2. Suppose e is an edge that abuts v. And suppose N_e, N_v, respectively, are the edge and vertex manifolds corresponding to e, v, respectively. Further suppose that $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_v$ is vertical and pseudovertical and a component $\tilde{S}$ of $\left(\bigcup_i S_i\right) \cap N_e$ is as in c). Then any annuli in $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_e$ that are parallel into ∂N_v can be isotoped to lie entirely in N_v. After this isotopy, $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_v$ is still vertical and pseudovertical.

Fact 3. Suppose e is an edge that abuts v. And suppose N_e, N_v, respectively, are the edge and vertex manifolds corresponding to e, v, respectively. Further suppose that $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_v$ is horizontal or pseudohorizontal. Then $\left(\bigcup_i F_i\right) \cap N_e$ does not contain a torus.

Fact 4. Suppose N_v is a vertex manifold and that a component $\tilde{S}$ of $\left(\bigcup_i S_i\right) \cap N_v$ is pseudohorizontal. Then $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_v = \tilde{S}$.

In the proof of the main theorem in [Schultens 2004] the strongly irreducible generalized Heegaard splitting $S_1, F_1, S_2, F_2, \ldots, S_n$ is isotoped to be standard. The first step in doing so involves isotoping $\bigcup_i F_i$ so that it intersects the boundaries of the edge manifolds in essential curves. After that, $\bigcup_i F_i$ remains fixed. We may assume without loss of generality that the number of components of this intersection is minimal. We will always assume that this is the case. As an immediate consequence we obtain our final fact:

Fact 5. Suppose N_e is an edge manifold. Then each annulus cut out by the intersection $\left(\bigcup_i F_i\right) \cap \partial N_e$ is essential in the compression body in which it is contained.

Lemma 5. *Let M be a graph manifold, N a Seifert piece and $S_1, F_1, S_2, F_2, \ldots, S_n$ a strongly irreducible untelescoping of a Heegaard splitting such that for some i $S_i \cap N$ is pseudohorizontal. Let f be the fiber contained in $S_i \cap N$. Then $(S_i \cap N) - \eta(f)$, for $\eta(f)$ a small regular neighborhood of f, is disconnected.*

Proof. Consider $(S_i \cap N) - \eta(f)$ in $N - \eta(f)$. Since S_i is separating in M, $(S_i \cap N) - \eta(f)$ is separating in $N - \eta(f)$. On the other hand, $(S_i \cap N) - \eta(f)$ is horizontal. A connected horizontal surface is not separating. It follows that $(S_i \cap N) - \eta(f)$ must have an even number of components, in this case at least two. Hence it is disconnected. □

Suppose that M is a closed totally orientable graph manifold and that $S_1, F_1, S_2, F_2, \ldots, S_n$ is a strongly irreducible untelescoping of a Heegaard splitting $M = V \cup_S W$. Suppose further that $S_1, F_1, S_2, F_2, \ldots, S_n$ has been isotoped to be standard. This implies in particular that for any vertex manifold N, $\left(\bigcup_i F_i \cup \bigcup_i S_i\right)$ meets ∂N in parallel simple closed curves. Thus to any vertex manifold N of M we associate the manifold N_S, which is the manifold obtained from N by performing a Dehn fillings at each components B of ∂N that meets $\left(\bigcup_i F_i \cup \bigcup_i S_i\right)$ along a slope represented by the curves $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap B$. Here N_S is not canonical. It depends on a specific (not necessarily unique) positioning of an (not necessarily unique) untelescoping. But we merely introduce this notation to discuss consequences of the existence of certain setups. N_S is a Seifert manifold if N contains a horizontal or pseudohorizontal component of $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N$, as $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap \partial N$ then consist of curves that have nontrivial intersection number with the fibre of N.

Lemma 6. *Suppose that for some i, $S_i \cap N$ is pseudohorizontal. Then the Seifert manifold N_S has a Heegaard surface S' such that $S' \cap N = S_i \cap N$. The corresponding Heegaard splitting is a horizontal Heegaard splitting of N_S. If $S_i \cap N$ is planar then S' is homeomorphic to S^2.*

Proof. Recall Fact 4 above: it tells us that if $S_i \cap N$ is pseudohorizontal, then $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N$ consists of a single component which we denote by $\tilde{S}$.

We may extend $\tilde{S}$ to a Heegaard surface of N_S by gluing meridional discs of the glued in solid tori to the boundary components of $\tilde{S}$. The corresponding Heegaard splitting for N_S is horizontal. If $\tilde{S}$ is planar then all boundary components get capped off which results in S^2. The assertion follows. $\square$

4. Some lemmata

The lemmata in this section will enable us to compute the Heegaard genus of certain graph manifolds in the next section. We start by discussing the possible pseudohorizontal surfaces in the relevant Seifert manifolds. Some proofs rely on the theory of 2-dimensional orbifolds and their covering theory as discussed in [Scott 1983]. These lemmata will be used in our discussion of Heegaard splittings and their untelescopings. But many of these results are more general. We do not necessarily require S to be the splitting surface of a Heegaard splitting or to be a surface in an untelescoping. Lemma 14 concerns vertical and pseudovertical surfaces.

Lemma 7. *Let M be a graph manifold and N be a Seifert piece with $\mathbb{O}(N) = D(p,q)$ and $(p,q) = 1$. If $S \cap N$ is a planar surface that is pseudohorizontal, then*

(1) *N_S is homeomorphic to S^3 and*

(2) *$S \cap \partial N$ contains exactly $2p$ or $2q$ components.*

Note that N_S being homeomorphic to S^3 is equivalent to N being the exterior of an r-bridge knot with meridian μ parallel to $\partial N \cap S$, where $r = \min(p,q)$.

Proof. Possibly after exchanging p and q we can assume that S is horizontal in the space $\bar{N}$ obtained from N after removing a regular neighborhood of the exceptional fiber corresponding to the cone point of order q or by removing a neighborhood of a regular fiber. Clearly $\bar{N}$ is a Seifert space with $\mathbb{O}(\bar{N}) = A(p)$ or $\mathbb{O}(\bar{N}) = A(p,q)$. Let T_1 be the boundary component of $\bar{N}$ that bounds the drilled out solid torus and T_2 be the boundary of N. Let $\bar{S}$ be a component of $S \cap \bar{N}$. Clearly $\bar{S}$ is planar as it is a subsurface of a planar surface.

Since we assume that S is pseudohorizontal in N, it follows that $\bar{S} \cap T_1$ consists of a single loop α. Let γ be one component of $\bar{S} \cap T_2$ and let g be an element of $\pi_1(\bar{N})$ corresponding to γ. Recall that all other components of $\bar{S} \cap T_2$ are parallel to γ. Let n be the intersection number of γ with the fiber.

Since $\bar{S}$ is horizontal in $\bar{N}$, there exists a finite sheeted orbifold covering $\pi : \bar{S} \to O(\bar{N})$, in particular $\pi_*(\pi_1(\bar{S}))$ is of finite index in $\pi_1(O(\bar{N}))$. We distinguish the cases $O(\bar{N}) = A(p)$ and $O(\bar{N}) = A(p,q)$.

Case 1: $O(\bar{N}) = A(p)$. We have $\pi_1(A(p)) = \langle x, y | x^p \rangle$, where the generator y corresponds to the boundary curve corresponding to T_2. This implies in particular that $\pi_*(g)$ is conjugate to y^n.

Since $\bar{S}$ is planar this implies that $\pi_1(\bar{S})$ is generated by homotopy classes that correspond to the components of $\bar{S} \cap T_2$; that is, $\pi_*(\pi_1(\bar{S}))$ is generated by conjugates of the element y^n. Let $N(y^n)$ be the normal closure of y in $\pi_1(A(p))$. Clearly $\pi_1(A(p))/N(y^n) \cong \mathbb{Z}_n * \mathbb{Z}_p$ is infinite unless $n = 1$. Since $\pi_*(\pi_1(\bar{S})) \subset N(y^n)$, this implies that $n = 1$ as otherwise $\pi_*(\pi_1(\bar{S}))$ is contained in a subgroup of infinite index in $\pi_1(A(p))$ and is therefore of infinite index itself. Thus we can assume that $n = 1$ and that $\pi_*(\pi_1(\bar{S})) \subset N(y)$.

The orbifold covering space $\tilde{S}$ corresponding to $N(y)$ is an orbifold without cone points and is homeomorphic to the $(p+1)$-punctured sphere. Denote the corresponding covering map by $\tilde{\pi}$.

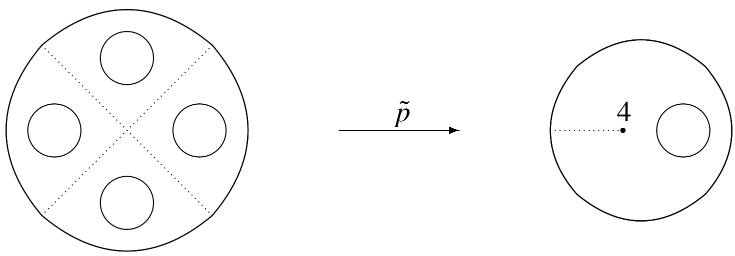

Figure 2. The 4-sheeted covering of $A(4)$ by a 5-punctured sphere.

Since $\pi_*(\pi_1(\bar{S})) \subset N(y)$, it follows that there exists a covering $\pi' : \bar{S} \to \tilde{S}$ such that $\pi = \tilde{\pi} \circ \pi'$.

Claim. *π' is a homeomorphism.*

As for both $\bar{S}$ and $\tilde{S}$, all but one boundary component map onto a curve corresponding to the element y it follows that π' is a homeomorphism when restricted to any of these boundary components. In particular π' extends to a covering $\pi'_\# : \bar{S}_\# \to \tilde{S}_\#$, where $\bar{S}_\#$ and $\tilde{S}_\#$ are the spaces obtained from $\bar{S}$ and $\tilde{S}$ by gluing discs to these boundary components. Since $\bar{S}_\#$ and $\tilde{S}_\#$ are discs, the map thus obtained is a homeomorphism. Thus the original π' was a homeomorphism, which proves the claim.

The second assertion is now immediate, because $S \cap \bar{N}$ is obtained from two copies of $\bar{S}$ by identifying two boundary components. All resulting boundary components lie in T_2. The first assertion follows from Lemma 6.

Case 2: $O(\bar{N}) = A(p,q)$. We have $\pi_1(A(p,q)) = \langle x, y, z \mid y^p, z^q \rangle$, where the generator x corresponds to the boundary curve corresponding to T_2. We see as in the first case that $\pi_*(O(\bar{N}))$ lies in the kernel of the map $\phi : \pi_1(A(p,q)) \to \pi_1(A(p,q))/N(x^n)$. As $\pi_1(A(p,q))/N(y^n) \cong \mathbb{Z}_n * \mathbb{Z}_p * \mathbb{Z}_q$ is infinite for all $n \in \mathbb{N}$, this implies that $\pi_*(O(\bar{N}))$ is of infinite index in $\pi_1(A(p,q))$, which contradicts our assumption. □

Lemma 8. *Let M be a graph manifold and let N be a Seifert piece with $\mathbb{O}(N) = F_g(p,\infty)$ or $\mathbb{O}(N) = F_g(p,\infty,\infty)$. Suppose that $S \cap N$ is pseudohorizontal and $\chi(S \cap N) > -8g$ or $\chi(S \cap N) > -8g-4$, respectively.*

(1) *$S \cap T$ has two components for every component T of ∂N.*

(2) *$S \cap N$ extends to the splitting surface of a horizontal Heegaard surface of genus $2g$ of N_S.*

Proof. We only deal with the case $\mathbb{O}(N) = F_g(p,\infty)$ the other case is analogous.

Suppose that $S \cap N$ is pseudohorizontal with respect to the exceptional fiber or a regular fiber and let $\bar{N}$ be the space obtained by drilling out the neighborhood of this fiber. Let $\bar{S}$ be a component of $\bar{N} \cap S$. Recall that $S \cap N$ is obtained from two copies of $\bar{S}$ by identifying them along a boundary component. In particular we have that $\chi(S \cap N) = 2\chi(\bar{S})$.

Now $\bar{S}$ is a finite sheeted covering of $\mathbb{O}(\bar{N})$, where $\mathbb{O}(\bar{N}) = F_g(\infty,\infty)$ or $\mathbb{O}(\bar{N}) = F_g(p,\infty,\infty)$ depending on what kind of fiber was drilled out. Suppose that the covering is n-sheeted. In the case $\mathbb{O}(\bar{N}) = F_g(p,\infty,\infty)$, we must have $n \geq p$; otherwise the covering space must be a orbifold with singularities. Thus we have

$$\chi(S \cap N) = 2\chi(\bar{S}) = 2n\chi(\mathbb{O}(\bar{N})).$$

Since $\chi(\mathbb{O}(\bar{N})) = -2g$ or $\chi(\mathbb{O}(\bar{N})) = -2g-1+1/p$, it follows immediately from the hypothesis on the Euler characteristic that $n = 1$. Thus $\mathbb{O}(\bar{N}) = F_g(\infty,\infty)$: the exceptional fiber was drilled out. Assertion (1) is now immediate and (2) follows from the proof of Lemma 6. □

It will be important that many Seifert manifolds do not admit a pseudohorizontal surface of small genus indiscriminately of what graph manifold they belong to.

Lemma 9. *Let N be a Seifert manifold with $\mathbb{O}(N) = F_g(p,\infty)$ such that the exceptional fiber has invariant (α,β) with $1 \leq \beta < \alpha$.*

(1) *If $\alpha = 2$, there exist two slopes γ on ∂N such that $N(\gamma)$ admits a horizontal Heegaard splitting of genus $2g$.*

(2) *If $\alpha \neq 2$ and $\beta \in \{1, \alpha-1\}$, there exists one slope γ on ∂N such that $N(\gamma)$ admits a horizontal Heegaard splitting of genus $2g$.*

(3) *In all other cases $N(\gamma)$ has no Heegaard splitting of genus $2g$ if $\gamma \neq f$.*

Proof. If γ is the fiber then $N(\gamma)$ is not a Seifert manifold. In particular $N(\gamma)$ admits no horizontal Heegaard splitting as those are only defined for Seifert manifolds. If the intersection number m of γ with the fiber is greater than 1 then $M(\gamma)$ is a Seifert space with base orbifold $F_g(p,m)$ which has no Heegaard splitting of genus $2g$ by [Boileau and Zieschang 1984, Proposition 1.4(i)]. Suppose now that $m=1$. Let $e\in\mathbb{Z}$ be the Euler class of the Seifert space. By [Boileau and Zieschang 1984, Proposition 1.4(iii)] it follows that $N(\gamma)$ admits no Heegaard splitting of genus $2g$ unless $\beta-e\alpha=\pm 1$. It is clear that there exists two values for e such that the equation holds if $\beta=1$ and $\alpha=2$, that there exists one solution if $\beta\in\{1,\alpha-1\}$ and none otherwise. The corresponding Heegaard splittings are constructed in [Boileau and Zieschang 1984, Section 1.10]. This proves the assertion. □

Lemma 10. *Let N be a Seifert manifold with $\mathbb{O}(N)=D(p,q)$ and $(p,q)=1$. Then N contains no compact planar horizontal surface.*

Proof. Suppose that S is a compact planar horizontal surface in N. Then there exists a finite sheeted orbifold covering $p:S\to D(p,q)$. Since all components of ∂S are parallel on ∂N, there exists a number $n\in\mathbb{N}$ such the restriction of p to any component of ∂S is a n-sheeted covering. This implies that we can extend p to a orbifold covering $p:S^2\to S^2(p,q,n)$ by gluing a disc to any component of ∂S and a disc with a cone point of order n to $D(p,q)$. If $n=1$ this yields a contradiction as $S^2(p,q,1)=S^2(p,q)$ is a bad orbifold which admits no covering by a manifold. If $n\neq 1$, then $S^2(p,q,n)$ must be a spherical orbifold with universal cover the sphere. Moreover, N_S is a Seifert manifold with $\mathbb{O}(N_S)=S(p,q,n)$. As such it is irreducible. This yields a contradiction, as $S\subset N$ extends to a horizontal, hence incompressible, sphere in N_S. □

Lemma 11. *Let M be a graph manifold and let N be a Seifert piece with $\mathbb{O}(N)=F_g(p,\infty)$ or $\mathbb{O}(N)=F_g(p,\infty,\infty)$. If $S\cap N$ is horizontal, then $\chi(S\cap N)\le -4g+1$ or $\chi(S\cap N)\le -4g-p+1$, respectively.*

Proof. Suppose that S is a horizontal incompressible surface in N that covers regular points of $F_g(p,\infty)$ k times. Here $k\ge p\ge 2$. By the Riemann–Hurwitz formula, $\chi(S)=k\big(-2g+\frac{1}{p}\big)\le p\big(-2g+\frac{1}{p}\big)=-2pg+1\le -4g+1$ or $\chi(S)=k\big(-2g-1+\frac{1}{p}\big)\le p\big(-2g-1+\frac{1}{p}\big)=-2pg-p+1\le -4g-p+1$, respectively. □

Lemma 12. *Let M be a graph manifold and let N be a Seifert piece with $\mathbb{O}(N)=F_g(p,\infty)$ or $\mathbb{O}(N)=F_g(p,\infty,\infty)$. Let $M=V\cup_S W$ be a Heegaard splitting and $S_1,F_1,\dots,F_{n-1},S_n$ an untelescoping. If $S_1,F_1,\dots,F_{n-1},S_n$ meets N in such a way that $F_i\cap N$ and $S_i\cap N$ are horizontal for each i, then*

$$\sum_i\big(\chi(F_{i-1}\cap N)-\chi(S_i\cap N)\big)\ge\begin{cases}8g-2 & \textit{if } \mathbb{O}(N)=F_g(p,\infty),\\ 8g+2p-2 & \textit{if } \mathbb{O}(N)=F_g(p,\infty,\infty).\end{cases}$$

Proof. The surfaces $S_1 \cap N$, $F_1 \cap N, \dots, F_{n-1} \cap N$, $S_n \cap N$ are disjoint and horizontal, hence they must be parallel. Let B be one of the components of ∂N. Consider the collection of torus knots $S_1 \cap B$, $F_1 \cap B, \dots, F_{n-1} \cap B$, $S_n \cap B$. Let γ be a torus knot on B that intersects each of the components in this collection of torus knots exactly once.

Now note that the untelescoping of the Heegaard splitting induces a Morse function of M and hence on B. If we assume that γ has as few critical points as possible, then near a maximum, γ meets two adjacent components of F_i or S_i for some i. If it meets two adjacent components of F_i, then the annulus cut out of B by these two components of F_i is inessential. But this contradicts Fact 5. Hence it meets two adjacent components of S_i. The same is true near a minimum of γ. Finally, if $\bigcup_i F_i$ meets N, then there are distinct adjacent components, as one such pair must lie above $\left(\bigcup_i F_i\right) \cap B$ and another below $\left(\bigcup_i F_i\right) \cap B$. Hence there are at least two more components of $\left(\bigcup_i S_i\right) \cap N$ than of $\left(\bigcup_i F_i\right) \cap N$. The lemma then follows from Lemma 11. □

Lemma 13. *Let N be a Seifert manifold with $\mathbb{O}(N) = D(p,q)$ with $(p,q) = 1$ and S be a properly embedded surface.*

(1) *If $S \cap N$ is horizontal, then there is an $l \geq 1$ such that $|S \cap N| = l$, $\chi(S \cap N) = lpq\left(-1 + \frac{1}{p} + \frac{1}{q}\right)$ and* $\text{genus}(S \cap N) \geq 1$.

(2) *If $S \cap N$ is pseudohorizontal, then $\chi(S \cap N) \leq -2\min(p,q) + 2$. Furthermore, either $S \cap N$ is as in Lemma 7, or* $\text{genus}(S \cap N) \geq 2$.

Proof. (1) Clearly $S \cap N$ is a finite sheeted cover of $D(p,q)$. The degree of this covering must be a positive multiple of pq, say lpq. It is clear that $S \cap N$ has l components. The second assertion follows from the Riemann–Hurwitz formula as $\chi(D(p,q)) = -1 + \frac{1}{q} + \frac{1}{q}$. The last assertion holds as by Lemma 10, S is nonplanar, so $\text{genus}(S \cap N) \geq 1$.

(2) Suppose first that $S \cap N$ is pseudohorizontal with respect to the fiber e. Let $N' = N - \eta(e)$ and S' be a component of $S \cap N'$. Recall that S' is horizontal by the definition of a pseudohorizonal surface.

If e is a regular fiber then S' must cover $A(p,q)$ at least pq times, that is, we have $\chi(S') \leq pq\left(-2 + \frac{1}{p} + \frac{1}{q}\right) = -2pq + p + q$ and therefore $\chi(S) = 2\chi(S') \leq -4pq + 2p + 2q \leq -2\min(p,q) + 2$. The remaining assertion follows from the proof of Lemma 7 which implies that S' cannot be planar.

Thus we can assume that e is an exceptional fiber. Suppose that e is the exceptional fiber of index q and let $N' = N - \eta(e)$. Suppose that H' is a horizontal incompressible surface in N' that covers regular points k times. Clearly $k \geq p$. Then $\chi(H') = k\left(-1 + \frac{1}{p}\right) \leq p\left(-1 + \frac{1}{p}\right) = -p + 1$. Thus if $S \cap N$ is pseudohorizontal with respect to e, then

$$\chi(S \cap N) \leq 2\chi(H') \leq -2p + 2 \leq -2\min(p, q) + 2.$$

An analogous argument establishes this inequality in the case that e is the exceptional fiber of index p; the last comment follows immediately from Lemma 7. □

Lemma 14. *Let M be a graph manifold and let N be a vertex manifold. Let $M = V \cup_S W$ be a Heegaard splitting and $S_1, F_1, \ldots, F_{n-1}, S_n$ an untelescoping. Suppose that $F_i \cap N$ is vertical for each i and $S_i \cap N$ is vertical or pseudovertical for each i. Then*

$$\sum_i \big(\chi(F_{i-1} \cap N) - \chi(S_i \cap N)\big) \geq -2\chi(H) + 2s + \sum_i \big(|F_{i-1} \cap \partial N| - |S_i \cap \partial N|\big),$$

where H is the underlying surface of $\mathbb{O}(N)$ and s the number of exceptional fiberes.

Moreover, if If $\mathbb{O}(N) = F_g(p, \infty)$ and $\big(\bigcup_i S_i\big) \cap \partial N \neq \varnothing$, then

$$\sum_i \big(\chi(F_{i-1} \cap N) - \chi(S_i \cap N)\big) \geq 4g + 2 + \sum_i \big(|F_{i-1} \cap \partial N| - |S_i \cap \partial N|\big).$$

If $\mathbb{O}(N) = F_g(p, \infty, \infty)$, denote the components of ∂N by ∂N_1 and ∂N_2. If $\big(\bigcup_i S_i\big) \cap \partial N_j \neq \varnothing$ for $j = 1, 2$, then

$$\sum_i \big(\chi(F_{i-1} \cap N) - \chi(S_i \cap N)\big) \geq 4g + 4 + \sum_i \big(|F_{i-1} \cap \partial N| - |S_i \cap \partial N|\big).$$

Proof. We denote $\mathbb{O}(N)$ by F so long as we need not distinguish between the cases. Since $F_i \cap N$ is vertical, $F_i \cap N$ consists of saturated annuli and tori. Since $S_i \cap N$ is vertical or pseudovertical, $S_i \cap N$ is obtained from saturated annuli $A^i_1, \ldots, A^i_{n_i}$ and tori $T^i_1, \ldots, T^i_{k_i}$ (some of them parallel to components of $F_{i-1} \cap N$) by performing ambient 1-surgery along arcs $\beta^i_1, \ldots, \beta^i_{m_i}$ that project to disjoint imbedded arcs $b^i_1, \ldots, b^i_{m_i}$ disjoint from the projection of $A^i_1, \ldots, A^i_{n_i}$ and $T^i_1, \ldots, T^i_{k_i}$ except at their endpoints.

For the purposes of the computation in this lemma, we may amalgamate

$$\big(\textstyle\bigcup_i F_i \cup \bigcup_i S_i\big) \cap N.$$

Though it may not be possible to amalgamate $\bigcup_i F_i \cup \bigcup_i S_i$ without destroying its simultaneous structure on all vertex and edge manifolds, it is possible to perform an amalgamation without destroying the structure in a given vertex manifold. Said differently, a partial amalgamation in a given vertex manifold extends to a partial amalgamation in the graph manifold (though nothing can be said, for instance, about the structure of the resulting non strongly irreducible untelescoping of $M = V \cup_S W$ in edge manifolds adjacent to the given vertex manifold). Here the result of such an amalagamation with respect to N is a surface $\tilde{S}$ such that $\tilde{S} \cap N$

is pseudovertical. (For details on amalgamation involving vertical and pseudovertical surfaces see [Schultens 1993, Proposition 2.10], though note the difference in terminology.)

Since $\tilde{S} \cap N$ is pseudovertical, it is obtained from saturated annuli $A_1, \dots, A_{\tilde{n}}$ and tori $T_1, \dots, T_{\tilde{k}}$ by performing ambient 1-surgery along arcs $\beta_1, \dots, \beta_{\tilde{m}}$ that project to disjoint imbedded arcs $b_1, \dots, b_{\tilde{m}}$. These arcs are disjoint from the projections $a_1, \dots, a_{\tilde{n}}$ of $A_1, \dots, A_{\tilde{n}}$ and $t_1, \dots, t_{\tilde{k}}$ of $T_1, \dots, T_{\tilde{k}}$ except at their endpoints. Here each b_j corresponds either to b_l^i or to an arc dual to b_l^i for some l, i, and conversely. Furthermore,

$$-\chi(\tilde{S} \cap N) = 2\tilde{m} = 2\sum_i m_i = \sum_i \big(\chi(F_{i-1} \cap N) - \chi(S_i \cap N)\big)$$

and

$$|\tilde{S} \cap \partial N| = 2\tilde{n} = \sum_i \big(|S_i \cap \partial N| - |F_{i-1} \cap \partial N|\big).$$

Recall that $\tilde{S}$ cuts a submanifold of M that contains N into two compression bodies. Thus the (not necessarily connected) submanifolds into which $\tilde{S} \cap N$ cuts N can be analyzed from two perspectives: On the one hand, they result from cutting compression bodies along incompressible annuli. Recall that incompressible annuli are either essential or boundary parallel. Cutting a compression body along a boundary parallel annulus merely cuts off a solid torus. Cutting a compression body along an essential annulus yields either one or two compression bodies.

On the other hand, the submanifolds into which $\tilde{S} \cap N$ cuts N contain Seifert fibered submanifolds of N; specifically, the Seifert fibered submanifolds of N that project to the appropriate components of the complement of the graph $\Gamma = \big(\bigcup_j a_j\big) \cup \big(\bigcup_i t_i\big) \cup \big(\bigcup_l b_l\big) \cup \partial F$ in F. This is impossible unless the Seifert fibered spaces in question are fibered over a disk with at most one cone point (i.e., solid tori) or fibered over an annulus with no cone point. Each such solid torus or $(annulus) \times S^1$ must meet $\tilde{S}$. Furthermore, exactly one of the boundary components of any such $(annulus) \times S^1$ must lie in ∂N.

We denote the set of vertices of Γ by $V\Gamma$ and the set of edges by $E\Gamma$. We may assume that each vertex of Γ is either of valence two or of valence three. Each vertex on a circular component (corresponding either to a boundary component without attached b_i or to some t_i without attached b_i) is of valence two and each endpoint of an arc a_j and each endpoint of an arc b_l is a vertex of valence three. Then $\#V\Gamma = 2\tilde{n} + 2\tilde{m} + k$ and $\#E\Gamma = 3\tilde{n} + 3\tilde{m} + k$, where k is the number of circular components of Γ.

Denote the underlying surface of F by H. Now Γ induces a decomposition of H into 0-cells, 1-cells, 2-cells and annuli. Denote the union of the 2-cells and annuli by $D\Gamma$. Each such annulus must be cobounded by a component of ∂H. Let l be the number of annuli.

This implies that

$$\chi(H) = \#V\Gamma - (\#E\Gamma) + (\#D\Gamma - l).$$

Combining these insights we obtain

$$\begin{aligned}\sum_i \big(\chi(F_{i-1}\cap N) - \chi(S_i\cap N)\big) + \sum_i \big(|S_i\cap\partial N| - |F_{i-1}\cap\partial N|\big) \\ &= 2\tilde{m} + 2\tilde{n} \\ &= -4\tilde{n} - 4\tilde{m} + 6\tilde{n} + 6\tilde{m} - 2(\#D\Gamma - l) + 2(\#D\Gamma - l) \\ &= -2\chi(H) + 2(\#D\Gamma - l).\end{aligned}$$

Thus $\sum_i \big(\chi(F_{i-1}\cap N) - \chi(S_i\cap N)\big)$ is at least

$$-2\chi(H) + 2(\#D\Gamma - l) + \sum_i \big(|F_{i-1}\cap\partial N| - |S_i\cap\partial N|\big),$$

that is,

$$\sum_i \big(\chi(F_{i-1}\cap N) - \chi(S_i\cap N)\big) \geq -2\chi(H) + 2s + \sum_i \big(|F_{i-1}\cap\partial N| - |S_i\cap\partial N|\big),$$

because every cone point must lie in a disk component. Now note that $\tilde{S}$ induces a bicoloring on the components of the complement of Γ in F according to which side of $\tilde{S}$ the Seifert fibered space that projects to that component lies. Thus $\#D\Gamma \geq 2$.

In the cases $F = F_g(p,\infty)$ or $F = F_g(p,\infty,\infty)$, $\#D\Gamma - l \geq 1$ because there must be a disk containing the cone point. Furthermore, if $l > 0$, then the result of cutting H along Γ yields annuli cobounded by boundary components of ∂H. This is impossible if $F = F_g(p,\infty)$ and $\big(\bigcup_i S_i\big)\cap\partial N \neq \varnothing$ or if $F = F_g(p,\infty,\infty)$ and $\big(\bigcup_i S_i\big)\cap\partial N_j \neq \varnothing$, for $j = 1, 2$, where N_1 and N_2 are the boundary components of N. Thus the additional formulas hold. □

Lemma 15. *Let M be a graph manifold and N a Seifert fibered submanifold with $\mathcal{O}(N) = D(p,q)$. Let $M = V \cup_S W$ be a Heegaard splitting and $S_1, F_1, \ldots, F_{n-1}, S_n$ an untelescoping. If $F_i \cap N$ is vertical for each i and $S_i \cap N$ is vertical or pseudovertical for each i, then*

$$\sum_i \big(\chi(F_{i-1}\cap N) - \chi(S_i\cap N)\big) \geq 2 + \sum_i \big(|F_{i-1}\cap\partial N| - |S_i\cap\partial N|\big).$$

Proof. This follows immediately from Lemma 14. □

5. The proof of Theorem 1

In order to give the proof of Theorem 1 we will first show that the fundamental groups can in fact be generated by $2g+1$ elements and then that only the manifolds listed admit a Heegaard splitting of genus $2g+1$.

Lemma 16. *The manifolds described in Theorem 1 have* $2g+1$*-generated fundamental groups.*

Proof. We first recall the presentations of the fundamental groups of N_1 and N_2: $\pi_1(N_1) = \langle a_1, b_1, \dots, a_g, b_g, s, t, f_1 \mid R \rangle$, with

$$R = \{[a_1, f_1], \dots, [a_g, f_1], [b_1, f_1], \dots, [b_g, f_1], [s, f_1], [t, f_1], \\ s^r = f_1^\beta, [a_1, b_1] \dots [a_g, b_g] s t = f_1^e\}$$

and $\pi_1(N_2) = \langle x, y, f_2 \mid [x, f_2], [y, f_2], x^p = f_2^{\beta_1}, y^q = f_2^{\beta_2} \rangle$.

The manifold M is obtained from N_1 and N_2 by identifying their boundaries, so it follows from van Kampen's theorem that

$$\pi_1(M) = \pi_1(N_1) *_C \pi_1(N_2) \text{ with } C \cong \mathbb{Z}^2.$$

Note that $f_1 = xyf_2^l$ for some $l \in \mathbb{Z}$ as we assume that the intersection number between f_1 and f_2 is 1. We will first establish a claim that is implicit in [Rost and Zieschang 1987].

Claim. *There exist* $n = \min(p, q)$ *conjugates of* f_1*in* $\pi_1(N_2)$ *that generate a subgroup that maps surjectively onto the orbifold group* $\pi_1(D(p, q))$.

Proof. It suffices to show that n conjugates of xy generated the quotient group $\pi_1(D(p,q)) = \langle x, y | x^p, y^q \rangle$. We can assume that $n = q < p$. The assertion then follows as we can choose the conjugates to be xy, $yx = x^{-1}(xy)x, \dots, x^{-n+2}yx^{n-1} = x^{-(n-1)}(xy)x^{n-1}$ which implies that their product (in the same order) is $xy^n x^{n-1} = xx^{n-1} = x^n$. As n and p are coprime it follows that $\langle x^n \rangle = \langle x \rangle$ which implies that x lies in the subgroup generated by the n conjugates, it follows that also $y = x^{-1}xy$ lies in the subgroup, which proves the claim. □

In fact we need something stronger:

Claim. *We can choose elements* $h_1, \dots, h_{n-1} \in \pi_1(N_2)$ *such that*

$$U = \langle f_1, h_1 f_1 h_1^{-1}, \dots, h_{n-1} f_1 h_{n-1}^{-1} \rangle$$

maps surjectively onto the base group and that additionally $h_i \in U$ *for* $1 \le i \le n-1$.

Proof. Choose k_i such that $\langle f_1, k_1 f_1 k_1^{-1}, \dots, k_{n-1} f_1 k_{n-1}^{-1} \rangle$ maps surjectively. For any k_i choose $h_i \in U$ and $z_i \in \mathbb{Z}$ such that $k_i = h_i f_2^{z_i}$. Clearly such h_i and z_i exist as we assume that U maps surjectively one $\pi_1(D(p, q))$ and as the kernel is generated by f_2. Since f_1 and f_2 commute, we have $k_i f_1 k_i^{-1} = h_i f_2^{z_i} f_1 f_2^{-z_i} h_i^{-1} = h_i f_1 h_i^{-1}$. This implies that $U = \langle f_1, h_1 f_1 h_1^{-1}, \dots, h_{n-1} f_1 h_{n-1}^{-1} \rangle$, and the claim follows. □

Note that U is a subgroup of finite index in $\pi_1(N_2)$ and that we can choose the elements h_i such that $\pi_1(N_2) = U$ if and only if N_2 is the exterior of a torus knot with meridian f_1. It is however always true that $\pi_1(N_2) = \langle U, C \rangle$ as $f_2 \in C$.

Note further that the subgroup $\langle s, f_1\rangle$ of $\pi_1(N_1)$ is generated by a single element g_0 which corresponds to the core of the solid torus corresponding to the exceptional fiber of N_1. It follows that $g_0^k = f_1$ for some $k \in \mathbb{Z}$. In order to prove the lemma we describe elements $g_1, \dots, g_{2g} \in \pi_1(M)$ such that $\pi_1(M) = \langle g_0, \dots, g_{2g}\rangle$.

Recall that by assumption $n \le 2g + 1$. Put $h_i = 1$ for $n \le i \le 2g$, and define

$$g_i := \begin{cases} h_i a_i & \text{for } 1 \le i \le g, \\ h_i b_{i-g} & \text{for } g+1 \le i \le 2g. \end{cases}$$

Claim. $U \subset \langle g_0, \dots, g_{2g}\rangle$.

Proof. It suffices to show that f_1 and the elements $h_i f_1 h_i^{-1}$ lie in $\langle g_0, \dots, g_{2g}\rangle$ for $1 \le i \le 2g$. Clearly $f_1 \in \langle g_0, \dots, g_{2g}\rangle$ as $f_1 = g_0^k$. Furthermore $h_i f_1 h_i^{-1} \in \langle g_0, \dots, g_{2g}\rangle$ for $1 \le i \le g$ as $g_i g_0^k g_i^{-1} = h_i a_i f_1 a_i^{-1} h_i^{-1} = h_i f_1 h_i^{-1}$. The same argument shows that $g_i g_0^k g_i^{-1} = h_i f_1 h_i^{-1}$ for $g+1 \le i \le 2g$, proving the claim. □

Since $h_i \in U$ for $1 \le i \le 2g$, this implies that $h_i \in \langle g_0, \dots, g_{2g}\rangle$ for $1 \le i \le 2g$ and therefore $h_i^{-1} g_i \in \langle g_0, \dots, g_{2g}\rangle$ for $1 \le i \le 2g$. Since $h_i^{-1} g_i = a_i$ for $1 \le i \le g$ and $h_i^{-1} g_i = b_{i-g}$ for $g+1 \le i \le 2g$, it follows that all a_i and b_i lie in $\langle g_0, \dots, g_{2g}\rangle$. Furthermore both f_1 and s are powers of g_0 and lie in $\langle g_0, \dots, g_{2g}\rangle$. The last generator t can be written as a product in the remaining generators by the last relation. Thus all generators of $\pi_1(N_1)$ lie in $\langle g_0, \dots, g_{2g}\rangle$ which shows that $\pi_1(N_1) \subset \langle g_0, \dots, g_{2g}\rangle$. Thus $C \subset \langle g_0, \dots, g_{2g}\rangle$ and therefore $\pi_1(N_2) = \langle U, C\rangle \subset \langle g_0, \dots, g_{2g}\rangle$. This shows that $\pi_1(M) = \langle g_0, \dots, g_{2g}\rangle$, proving Lemma 16. □

Lemma 17. *Let M be a manifold as described in Theorem 1 and let $M = V \cup_S W$ be a Heegaard splitting. Then one of the following holds*:

(1) *$S \cap N_1$ is vertical, $S \cap N_2$ is planar and pseudohorizontal with respect to the exceptional fiber e of index p and $q \le 2g + 1$.*

(2) *$S \cap N_1$ is as in Lemma 8, $S \cap N_2$ consists of a single annulus and* genus $S = 2g + 1$.

(3) genus $S \ge 2g + 2$.

Proof. Let M be a manifold as described in Theorem 1 and let $M = V \cup_S W$ be a Heegaard splitting. Furthermore, let $S_1, F_1, \dots, F_{n-1}, S_n$ be a strongly irreducible untelescoping of $M = V \cup_S W$ that is standard.

Case 1: $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_1$ and $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_2$ are vertical or pseudovertical. If $\left(\bigcup_i F_i \cup \bigcup_i S_i\right)$ meets the edge manifold N_e between N_1 and N_2 in annuli including spanning annuli, then M must be a Seifert fibered space. But this contradicts our assumption that the fibers of N_1 and N_2 have intersection number 1.

If $\bigcup_i F_i$ meets the edge manifold N_e in a torus, then we may assume that $\bigcup_i S_i$ is disjoint from N_e. (Annuli that are boundary parallel in N_e can be isotoped into the vertex manifolds.) Then Lemma 14 tells us that

$$\sum_i \big(\chi(F_{i-1}\cap N_1)-\chi(S_i\cap N_1)\big)\geq 4g+2+\sum_i \big(|F_{i-1}\cap\partial N_1|-|S_i\cap\partial N_1|\big)\geq 4g$$

and Lemma 15 tells us that

$$\sum_i \big(\chi(F_{i-1}\cap N_2)-\chi(S_i\cap N_2)\big)\geq 2+\sum_i \big(|F_{i-1}\cap\partial N_2|-|S_i\cap\partial N_2|\big)=2;$$

hence by Theorem 3, $2\,\text{genus}\, S-2=-\chi(S)\geq 4g+2$; thus $\text{genus}\, S\geq 2g+2$.

Otherwise $\big(\bigcup_i F_i\big)\cup\big(\bigcup_i S_i\big)$ meets the edge manifold between N_1 and N_2 in boundary parallel annuli and one component of Euler characteristic -2 contained in $\big(\bigcup_i S_i\big)\cap N_e$. Any boundary parallel annuli in $\big(\bigcup_i F_i\,\cup\,\bigcup_i S_i\big)\cap N_e$ can be isotoped into N_1 or N_2. It then follows from Lemmas 14 and 15 that

$$\begin{aligned}
&\sum_i \big(\chi(F_{i-1})-\chi(S_i)\big)\\
&\quad=\sum_i\big(\big(\chi(F_{i-1}\cap N_1)-\chi(S_i\cap N_1)\big)\\
&\qquad\qquad+\sum_i\big(\chi(F_{i-1}\cap N_2)-\chi(S_i\cap N_2)\big)+\sum_i\big(\chi(F_{i-1}\cap N_e)-\chi(S_i\cap N_e)\big)\\
&\quad\geq(4g+2-2)+(2-2)+2=4g+2.
\end{aligned}$$

Hence, by Theorem 3, $2\,\text{genus}\, S-2=-\chi(S)\geq 4g+2$, whence $\text{genus}\, S\geq 2g+2$.

Case 2: $\big(\bigcup_i F_i\cup\bigcup_i S_i\big)\cap N_1$ *is horizontal.* Recall Fact 1 following Theorem 4. It tells us that $\sum_i\big(\chi(F_{i-1}\cap N)-\chi(S_i\cap N)\big)\geq 0$ for any vertex or edge manifold N. It follows that

$$\sum_i \big(\chi(F_{i-1})-\chi(S_i)\big)\geq\sum_i \big(\chi(F_{i-1}\cap N_1)-\chi(S_i\cap N_1)\big).$$

By Lemma 12, $\sum_i \big(\chi(F_{i-1}\cap N_1)-\chi(S_i\cap N_1)\big)\geq 8g-2$, so

$$\sum_i \big(\chi(F_{i-1})-\chi(S_i)\big)\geq 8g-2.$$

Hence, by Theorem 3, we have $2\,\text{genus}\, S-2=-\chi(S)\geq 8g-2$, that is,

$$\text{genus}\, S\geq 4g\geq 2g+2.$$

Case 3: A component of $\big(\bigcup_i S_i\big)\cap N_1$ *is pseudohorizontal.* Denote the pseudohorizontal component of $\big(\bigcup_i S_i\big)\cap N_1$ by $\tilde{S}$. By Lemma 8, either $\tilde{S}$ is as in that lemma and $\big(\bigcup_i S_i\big)\cap N_2$ consists of a single annulus, or $\text{genus}\, S\geq 2g+2$. This puts us in situation (2) or (3), respectively.

Case 4: $\big(\bigcup_i F_i\,\cup\,\bigcup_i S_i\big)\cap N_2$ *is horizontal.* If $\big(\bigcup_i F_i\,\cup\,\bigcup_i S_i\big)\cap N_1$ is horizontal, then the result follows by Case 2. If a component of $\big(\bigcup_i S_i\big)\cap N_1$ is pseudohorizontal, then the result follows by Case 3. Thus we may assume that $\big(\bigcup_i F_i \cup \bigcup_i S_i\big) \cap N_1$ is vertical or pseudovertical.

We may assume that any boundary parallel annuli in the edge manifold N_e that are parallel into N_1 have been isotoped into N_1. (This does not change the Euler characteristics of the surfaces nor the fact that $\big(\bigcup_i F_i \cup \bigcup_i S_i\big) \cap N_1$ is vertical or pseudovertical.)

Fact 3 tells us that $\big(\bigcup_i F_i\big) \cap N_e$ does not contain a torus. Hence $\bigcup_i S_i \cap \partial N_1 \neq \varnothing$. It also follows from Fact 5 that

$$\big|\big(\bigcup_i F_i\big) \cap \partial N_1\big| = \big|\big(\bigcup_i F_i\big) \cap \partial N_2\big|.$$

Since there are no annuli in $\big(\bigcup_i F_i \cup \bigcup_i S_i\big) \cap N_e$ that are parallel into N_1, we obtain $\sum\limits_i |S_i \cap \partial N_1| \leq \sum\limits_i |S_i \cap \partial N_2|$, and hence

$$\sum_i \big(|F_{i-1} \cap \partial N_1| - |S_i \cap \partial N_1|\big) \geq \sum_i \big(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|\big).$$

The components of $\big(\bigcup_i F_i \cup \bigcup_i S_i\big) \cap N_2$ are all parallel. If H is such a component, then

$$\chi(H) = 2 - 2\,\text{genus}\, H - |H \cap \partial N_2|.$$

Recall that by Lemma 10, genus $H \geq 1$. Thus

$$\begin{aligned}
&\sum_i \big(\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2)\big) \\
&\quad = (2\,\text{genus}\, H - 2) \sum_i \big(|S_i \cap N_2| - |F_{i-1} \cap N_2|\big) - \sum_i \big(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|\big) \\
&\quad \geq - \sum_i \big(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|\big).
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_i \big(\chi(F_{i-1}) - \chi(S_i)\big) \\
&\qquad = \sum_i \big(\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1)\big) + \sum_i \big(\chi(F_{i-1} \cap N_e) - \chi(S_i \cap N_e)\big) \\
&\qquad\qquad + \sum_i \big(\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2)\big).
\end{aligned}$$

Then Fact 1 tells us that $\sum_i \big(\chi(F_{i-1} \cap N_e) - \chi(S_i \cap N_e)\big) \geq 0$, so

$$\begin{aligned}
&\sum_i \big(\chi(F_{i-1}) - \chi(S_i)\big) \\
&\qquad \geq \sum_i \big(\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1) + \chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2)\big).
\end{aligned}$$

Thus, by Lemma 14,

$$\begin{aligned}\sum_i \big(\chi(F_{i-1}) - \chi(S_i)\big)&\\ &\geq 4g+2+\sum_i \big(|F_{i-1}\cap\partial N_1| - |S_i\cap\partial N_1|\big) - \sum_i \big(|F_{i-1}\cap\partial N_2| - |S_i\cap\partial N_2|\big)\\ &\geq 4g+2.\end{aligned}$$

By Theorem 3, therefore, we conclude that $2\,\text{genus}\,S - 2 = -\chi(S) \geq 4g+2$, whence $\text{genus}\,S \geq 2g+2$.

Case 5: A component of $\big(\bigcup_i S_i\big)\cap N_2$ *is pseudohorizontal.* Here, too, if

$$\big(\textstyle\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_1$$

is horizontal, the result follows by Case 2. If a component of $\big(\bigcup_i S_i\big)\cap N_1$ is pseudohorizontal, it follows by Case 3. Thus we assume that $\big(\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_1$ is vertical or pseudovertical.

By the same reasoning as in Case 4, we can assume that

$$\sum_i \big(|F_{i-1}\cap\partial N_1| - |S_i\cap\partial N_1|\big) \geq \sum_i \big(|F_{i-1}\cap\partial N_2| - |S_i\cap\partial N_2|\big).$$

Denote the pseudohorizontal component of $\big(\bigcup_i S_i\big)\cap N_2$ by $\tilde S$ and note that here $\big(\big(\bigcup_i F_i \cup \bigcup_i S_i\big) - \tilde S\big)\cap N_2 = \varnothing$. Thus

$$\chi(\tilde S) = 2 - 2\,\text{genus}\,\tilde S - |\tilde S\cap\partial N_2|.$$

By Lemma 13, either $\tilde S$ is as in Lemma 7 or $\text{genus}\,\tilde S \geq 2$. In the former case, we have $|\partial\tilde S| = 2q$ and $\chi(\tilde S) = 2-2q$. If, moreover, $\text{genus}\,S \leq 2g+1$, then $-4g \leq \chi(S) = \sum_i \big(\chi(S_i) - \chi(F_{i-1})\big) \leq \chi(\tilde S) = 2-2q$. Thus $q \leq 2g+1$.

In the second case ($\text{genus}\,\tilde S \geq 2$), we have

$$\sum_i \big(\chi(F_{i-1}\cap N_2) - \chi(S_i\cap N_2)\big) = -\chi(\tilde S) = 2\,\text{genus}\,\tilde S - 2 + |\tilde S\cap\partial N_2| \geq |\tilde S\cap\partial N_2|.$$

Arguing as in Case 4, we obtain

$$\begin{aligned}\sum_i \big(\chi(F_{i-1}) - \chi(S_i)\big)&\\ &\geq \sum_i \big(\chi(F_{i-1}\cap N_1) - \chi(S_i\cap N_1) + \chi(F_{i-1}\cap N_2) - \chi(S_i\cap N_2)\big)\\ &\geq 4g+2+\sum_i \big(|F_{i-1}\cap\partial N_1| - |S_i\cap\partial N_1|\big) - \sum_i \big(|F_{i-1}\cap\partial N_2| - |S_i\cap\partial N_2|\big)\\ &\geq 4g+2.\end{aligned}$$

Again, by Theorem 3, we have $2\,\text{genus}\,S - 2 = -\chi(S) \geq 4g+2$, whence $\text{genus}\,S \geq 2g+2$. □

Proof of Theorem 1. Consider the options allowed by Lemma 17. If option (1) occurs, then Lemma 7 implies that N_2 is a q-bridge knot complement and the fiber of N_1 is identified with the meridian of N_2. This puts us in case (1) of Theorem 1. If option (2) occurs in Lemma 17, then $\hat{N}_1$ admits a horizontal Heegaard splitting of genus $2g$ by Lemma 8 and we are in case (2) of Theorem 1. If option (3) occurs there is nothing to show. □

6. The proof of Theorem 2

In this section we construct for any $n \in \mathbb{N}$ such that $n \geq 3$ a graph manifold M_n such that $\pi_1(M_n)$ is $3n$-generated but that the Heegaard genus of M_n is $4n$. We denote the graph underlying M_n by Γ_n. Γ_n is a tree on $2n+1$ vertices $z, c_1, \ldots, c_n, d_1, \ldots, d_n$ and $2n$ edges $e_1, \ldots, e_n, f_1, \ldots, f_n$ such that c_i and d_i are the endpoints of e_i and that d_i and z are the endpoints of f_i.

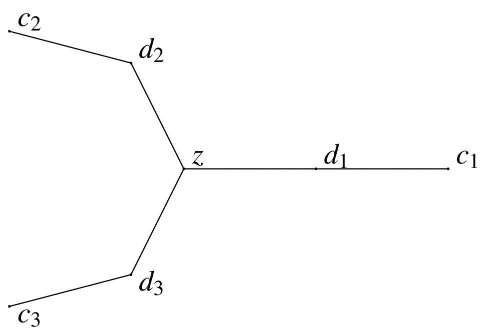

Figure 3. The tree Γ_3.

The closed graph manifold M_n is then constructed as follows, where we denote the Seifert piece corresponding to a vertex v by N_v.

(1) The intersection number between the fibers of the adjacent Seifert spaces is 1 at any torus of the JSJ decomposition.

(2) $\mathbb{O}(N_z)$ is a n-punctured sphere with one cone point of order $20n$ and $\hat{N}_z = S^3$.

(3) $\mathbb{O}(N_{d_i}) = T^2(\infty, \infty, 20n)$ and N_{d_i} admits no pseudohorizontal surface that has genus 2.

(4) $\mathbb{O}(N_{c_i})$ is of type $D(3, q)$ with $q \geq 20n$ and $(3, q) = 1$ but N_{c_i} is not homeomorphic to the exterior of a 2-bridge knot in S^3.

Remark 18. Note that (2) is equivalent to stating that N_z is the exterior of a Seifert fibered n component n-bridge link in S^3, in particular $\pi_1(N_z)$ is generated by the fibers of the N_{d_i}. The existence of the spaces N_{d_i} satisfying (3) is an immediate consequence of Lemma 9.

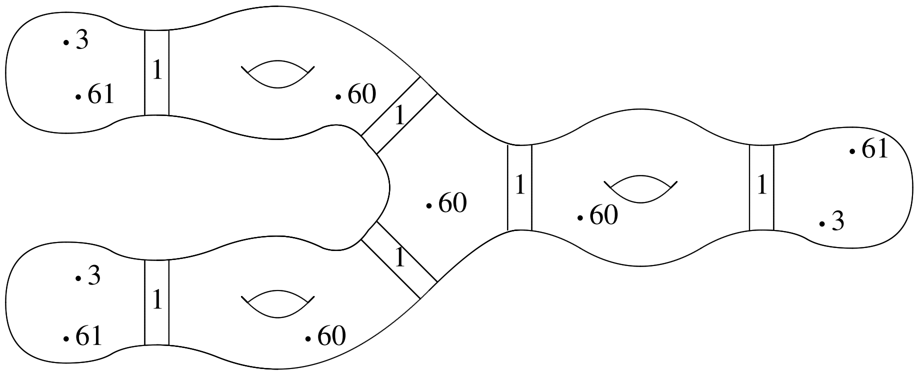

Figure 4. A graph manifold M with $g(M) = 12$ and $r(M) \leq 9$.

The first part of the proof of Theorem 2 is again a simple calculation:

Lemma 19. *$\pi_1(M_n)$ can be generated by $3n$ elements.*

Proof. The proof is almost identical to the proof of Lemma 16 and we frequently omit explicit calculations if they are identical. Recall that

$$\pi_1(N_{d_i}) = \langle a_i, b_i, s_i, t_{i1}, t_{i2}, f_i \mid R_i \rangle \text{ with}$$

$$R_i = \{[a_i, f_i], [b_i, f_i], [s_i, f_i], [t_{i1}, f_i], [t_{i2}, f_i], s_i^{5n} = f_i^{\beta_i}, [a_i b_i] t_{i1} t_{i2} s_i = f_i^{e_i}\}$$

where t_{i1} corresponds the the boundary component between N_{d_i} and N_z and t_{i2} corresponds to the boundary component between N_{d_i} and N_{c_i}.

Recall from the proof of Lemma 16 that there exist elements $h_{i1}, h_{i2} \in \pi_1(N_{c_i})$ such that $U_i = \langle f_i, h_{i1} f_i h_{i1}^{-1}, h_{i2} f_i h_{i2}^{-1} \rangle$ is a subgroup of finite index in $\pi_1(N_{c_i})$ that maps surjectively onto the fundamental group of $\mathbb{O}(N_{c_i})$ and that $h_{i1}, h_{i2} \in U_i$.

We will show that $\pi_1(M_n)$ is generated by the generators $g_1, \ldots, g_{3n}$ defined as follows:

(1) g_i is the generator of the cyclic group $\langle f_i, s_i \rangle$ for $1 \leq i \leq n$.

(2) $g_{n+i} = h_{i1} a_i$ for $1 \leq i \leq n$.

(3) $g_{2n+i} = h_{i2} b_i$ for $1 \leq i \leq n$.

Let $H = \langle g_1, \ldots, g_{3n} \rangle$. We show that $H = \pi_1(M_n)$.

Note first that $\pi_1(N_z) \subset H$ as $g_i \in H$ implies $f_i \in H$ for $1 \leq i \leq n$ and $\pi_1(N_z)$ is generated by the f_i. This implies that $t_{i1} \in H$ for $1 \leq i \leq n$.

The same calculation as in the proof of Lemma 16 further shows that $U_i \subset H$ for $1 \leq i \leq n$. It follows that $a_i, b_i \in H$ for $1 \leq i \leq n$. Thus $\pi_1(N_{d_i}) \subset H$ as $\pi_1(N_{d_i})$ is generated by $a_i, b_i, s_i, f_i, t_{i_1}$ and s_i and f_i are powers of g_i.

It follows further that $\pi_1(N_{c_i}) \subset H$ as $\pi_1(N_{c_i})$ is generated by U_i and C_i, where $C_i = \pi_1(N_{c_i}) \cap \pi_1(N_{d_i})$. □

To conclude the proof of Theorem 2 it clearly suffices to establish the following:

Proposition 20. *The Heegaard genus of M_n is at least* $4n$.

In proving this we will tacitly use that small genus Heegaard splittings have very special untelescopings — a fact that deserves its own result since it has independent interest:

Lemma 21. *Let $M_n = V \cup_S W$ be a Heegaard splitting of M_n. Then either $g(S) \geq 4n$ or there is a strongly irreducible untelescoping $S_1, F_1, \dots, F_{k-1}, S_k$ of $M_n = V \cup_S W$ such that for any vertex manifold N no component of $S_i \cap N$ or $F_i \cap N$ is horizontal. In particular all F_i are vertical incompressible tori.*

Proof. Suppose that some component F of $S_i \cap N$ or $F_i \cap N$ is horizontal for some i and some vertex manifold N. Note first that no component of ∂F bounds a disk as any component is an essential curve in an incompressible torus. It follows that $\chi(F) \geq \chi(F_i)$ (or $\chi(F) \geq \chi(S_i)$), where F_i (or S_i) is the surface containing F.

Note first that $F \cap N$ is a covering of the base space $\mathbb{O}$ of N of degree at least $20n$. It is furthermore easy to see that we have $\chi(\mathbb{O}) \leq -\frac{1}{2}$ for any choice of N. If follows that $\chi(F \cap N) \leq -10n$ and therefore $\chi(F_i) \leq -10n$ (or $\chi(S_i) \leq -10n$). This however implies that the genus of F_i (or S_i) is greater than $5n$ which implies that the Heegaard surface S is of genus at least $5n$. This proves the assertion. □

Proof of Proposition 20. To see that M_n admits no Heegaard splitting of genus less than $4n$, proceed along the same lines as in the proof of Lemma 17. Let $M_n = V \cup_S W$ be a Heegaard splitting and let $S_1, F_1, \dots, F_{k-1}, S_k$ be a strongly irreducible untelescoping of $M_n = V \cup_S W$. We consider the various possible cases for $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{c_j}$ and $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{d_j}$.

Case 1: Fix j and suppose that $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{c_j}$ and $\left(\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{d_j}\right.$ are vertical or pseudovertical. In this case it is impossible for $\left(\left(\bigcup_i F_i \cup \bigcup_i S_i\right)\right)$ to meet the edge manifold N_{e_j} between N_{c_j} and N_{d_j} in spanning annuli. Moreover, any boundary parallel annuli in N_{e_j} can be isotoped into N_{c_j} and N_{d_j} and any boundary parallel annuli in N_{g_j} that are parallel into N_{d_j} can be isotoped into N_{d_j}. (This does not change the fact that $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{c_j}$ and $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{d_j}$ are vertical or pseudovertical and serves to facilitate our counting argument.) In conjunction with Fact 5, this tells us that

$$\sum_i \left(-|S_i \cap \partial N_{d_j}| + |F_{i-1} \cap \partial N_{d_j}|\right)$$
$$\geq \sum_i \left(-|S_i \cap \partial N_{c_j}| + |F_{i-1} \cap \partial N_{c_j}| - |S_i \cap \partial N_z^j| + |F_{i-1} \cap \partial N_z^j|\right),$$

where ∂N_z^j is the component of ∂N_z that meets the edge manifold N_{g_j} between N_z and N_{d_j}.

Now either $\bigcup_i F_i$ meets N_{e_j} in an essential torus, or $\bigcup_i S_i$ meets N_{e_j} in the only other possible configuration. In the first case, we obtain

$$\sum_i \big(-|S_i \cap \partial N_{c_j}| + |F_{i-1} \cap \partial N_{c_j}|\big) = 0$$

and

$$\sum_i \big(-|S_i \cap \partial N_{d_j}| + |F_{i-1} \cap \partial N_{d_j}|\big) \geq \sum_i \big(-|S_i \cap \partial N_z^j| + |F_{i-1} \cap \partial N_z^j|\big).$$

In the second case we obtain $\sum_i \big(-|S_i \cap \partial N_{c_j}| + |F_{i-1} \cap \partial N_{c_j}|\big) = -2$ and

$$\sum_i \big(-|S_i \cap \partial N_{d_j}| + |F_{i-1} \cap \partial N_{d_j}|\big) \geq -2 + \sum_i \big(-|S_i \cap \partial N_z^j| + |F_{i-1} \cap \partial N_z^j|\big).$$

We further distinguish the cases in which $\bigcup_i F_i$ meets or does not meet the edge manifold N_{f_j} in an essential torus.

Case 1.1: $\bigcup_i F_i$ meets N_{e_j} in an essential torus. By Lemmas 14 and 15, we have

$$\begin{aligned}
&\sum_i \big(\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j})\big) \\
&\quad \geq 4+2+\sum_i \big(|F_{i-1} \cap \partial N_{d_j}| - |S_i \cap \partial N_{d_j}|\big) + 2 + \sum_i \big(|F_{i-1} \cap \partial N_{c_j}| - |S_i \cap \partial N_{c_j}|\big) \\
&\quad \geq 4+2+\sum_i \big(|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|\big) + 2 \\
&\quad \geq 8+\sum_i \big(|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|\big).
\end{aligned}$$

Case 1.2: $\bigcup_i F_i$ meets neither N_{f_j} nor N_{e_j} in an essential torus. By Lemmas 14 and 15, we have

$$\begin{aligned}
&\sum_i \big(\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) \\
&\qquad\qquad + \chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})\big) \\
&\geq 4+4+\sum_i \big(|F_{i-1} \cap \partial N_{d_j}| - |S_i \cap \partial N_{d_j}|\big) + 2 + \sum_i \big(|F_{i-1} \cap \partial N_{c_j}| - |S_i \cap \partial N_{c_j}|\big) + 2 \\
&\geq 4+4-2+\sum_i \big(|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|\big) + 2 - 2 + 2 \\
&\geq 8+\sum_i \big(|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|\big).
\end{aligned}$$

Case 1.3: $\bigcup_i F_i$ meets N_{f_j} in an essential torus but does not meet N_{e_j} in an essential torus. Here Lemmas 14 and 15 yield only

$$\begin{aligned}
&\sum_i \big(\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) \\
&\qquad\qquad + \chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})\big) \\
&\geq 4+2+\sum_i \big(|F_{i-1} \cap \partial N_{d_j}| - |S_i \cap \partial N_{d_j}|\big) + 2 + \sum_i \big(|F_{i-1} \cap \partial N_{c_j}| - |S_i \cap \partial N_{c_j}|\big) + 2 \\
&\geq 4+2-2+2-2+2 \geq 6.
\end{aligned}$$

In this case $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_z$ must be vertical or pseudovertical. (See Fact 3 above.)

Note that in all cases we have

$$\sum_i \left(\chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) + \chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})\right) \geq 2.$$

Case 2: Fix j and suppose a component of $\bigcup_i S_i \cap N_{d_j}$ is pseudohorizontal. As we have seen, in this case

$$S' = \left(\bigcup_i S_i \cup \bigcup_i F_i\right) \cap N_{d_j}$$

is connected. In particular, $\bigcup_i F_i \cap N_{d_j} = \varnothing$. By construction, the genus of a pseudohorizontal surface is even. Recall our assumption that N_{d_j} admits no pseudohorizontal surface of genus 2. Thus the genus of S' is at least 4. Hence

$$\begin{aligned}\sum_i \left(\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j})\right) &= 0 - \chi(S' \cap N_{d_j}) \\ &\geq 6 + b = 6 - \sum_i \left(|F_{i-1} \cap \partial N_{d_j}| - |S_i \cap \partial N_{d_j}|\right),\end{aligned}$$

where b is the number of boundary components of S'. Since S' is a separating surface, it meets each boundary component of N_{d_j} at least twice. Consequently, $b \geq 4$. It thus follows from Fact 1 that

$$\begin{aligned}\sum_i \big(\chi(F_{i-1} \cap N_{f_j}) - \chi(S_i \cap N_{f_j}) + \chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j}) \\ + \chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j}) + \chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j})\big) \\ \geq 6 + 4 = 10.\end{aligned}$$

Case 3: Fix j and suppose a component of $\left(\bigcup_i S_i\right) \cap N_{c_j}$ is pseudohorizontal. It will suffice to consider the case in which $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{d_j}$ is vertical or pseudovertical. Denote the pseudohorizontal component of $\left(\bigcup_i S_i\right) \cap N_{c_j}$ by $\tilde{S}$ and note that here $\left(\left(\bigcup_i F_i \cup \bigcup_i S_i\right) - \tilde{S}\right) \cap N_{c_j} = \varnothing$. By assumption, N_{c_j} is not the exterior of a 2-bridge knot in $\mathbb{S}^3$, thus by Lemmas 7 and 13, genus $\tilde{S} \geq 2$. Hence,

$$\chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j})) = -\chi(\tilde{S} \cap N_{c_j}) \geq -\chi(\tilde{S}) \geq 2 + c,$$

where c is the number of boundary components of $\tilde{S}$.

Recall that when $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{d_j}$ is vertical or pseudovertical, we may isotope any annuli in $\left(\bigcup_i S_i\right) \cap N_{f_i}$ or $\left(\bigcup_i S_i\right) \cap N_{e_i}$ that are parallel into N_{d_j} into N_{d_j} without altering the fact that $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{d_j}$ is vertical or pseudovertical. Therefore we may assume that there are no annuli in $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{f_j}$ or $\left(\bigcup_i F_i \cup \bigcup_i S_i\right) \cap N_{e_j}$ that are parallel into N_{d_j}. Thus

$$\begin{aligned}\sum_i \big(|F_{i-1}\cap\partial N_{d_j}| - |S_i\cap\partial N_{d_j}|\big)\\ &\ge \sum_i \big(|F_{i-1}\cap\partial N_z^j| - |S_i\cap\partial N_z^j|\big) + \sum_i \big(|F_{i-1}\cap\partial N_{c_j}| - |S_i\cap\partial N_{c_j}|\big)\\ &= \sum_i \big(|F_{i-1}\cap\partial N_z^j| - |S_i\cap\partial N_z^j|\big) - c.\end{aligned}$$

Hence, by Lemma 14,

$$\begin{aligned}\sum_i \big(\chi(F_{i-1}\cap N_{d_j}) - \chi(S_i\cap N_{d_j}) + \chi(F_{i-1}\cap N_{c_j}) - \chi(S_i\cap N_{c_j})\big)\\ &\ge 4+4+\sum_i \big(|F_{i-1}\cap\partial N_{d_j}| - |S_i\cap\partial N_{d_j}|\big) + 2 + c\\ &\ge 10 + \sum_i \big(|F_{i-1}\cap\partial N_z^j| - |S_i\cap\partial N_z^j|\big).\end{aligned}$$

Putting these computations together we must consider the various options for $\big(\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_z$:

Case A: $\big(\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_z$ is vertical and pseudovertical. In this case the options for $\big(\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_{f_j}$ are severely limited. If $\big(\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_{d_j}$ is vertical and pseudovertical, then $\big(\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_{f_j}$ cannot consist of spanning annuli. So either $\bigcup_i F_i$ meets N_{f_j} in an essential torus, or $\bigcup_i S_i$ meets N_{f_j} in the only other possible configuration. If $\big(\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_{d_j}$ is pseudohorizontal, then N_{f_j} cannot meet a toral component of $\bigcup_i F_i$. So it must consist either of spanning annuli or the only other possible configuration.

Define

$$J_0 = \Big\{ j : \sum_i \big(|F_{i-1}\cap\partial N_z^j| - |S_i\cap\partial N_z^j|\big) = 0\Big\}.$$

Then $\sum_i \big(\chi(F_{i-1}\cap N_{f_j}) - \chi(S_i\cap N_{f_j})\big) = 0$ for $j\in J_0$.

Denote by J_1 the set of j not in J_0 such that $\big(\bigcup_i F_i\big)\cup\big(\bigcup_i S_i\big)\cap N_{d_j}$ are vertical or pseudovertical. Then

$$\sum_i \big(|F_{i-1}\cap\partial N_z^j| - |S_i\cap\partial N_z^j|\big) = -2$$

and

$$\sum_i \big(\chi(F_{i-1}\cap N_{f_j}) - \chi(S_i\cap N_{f_j})\big) = 2 \quad \text{for } j\in J_1.$$

Denote by J_2 the set of j such that $\big(\bigcup_i F_i \cup \bigcup_i S_i\big)\cap N_{d_j}$ is pseudohorizontal; it is easy to see that J_0, J_1, J_2 are disjoint and their union equals J. We have

$$\sum_i \big(|F_{i-1}\cap\partial N_{d_j}| - |S_i\cap\partial N_{d_j}|\big) \ge \sum_i \big(|F_{i-1}\cap\partial N_z^j| - |S_i\cap\partial N_z^j|\big) \quad \text{for } j\in J_2.$$

Further, by Lemma 14,

$$\sum_i \big(\chi(F_{i-1}\cap N_z) - \chi(S_i\cap N_z)\big) \ge -2(2-n)+2+\sum_i \big(|F_{i-1}\cap\partial N_z| - |S_i\cap\partial N_z|\big).$$

Thus $-\chi(S)$ equals

$$
\begin{aligned}
&\sum_i \big(\chi(F_{i-1}) - \chi(S_i)\big) \\
&\quad \ge \sum_i \big(\chi(F_{i-1}\cap N_z) - \chi(S_i\cap N_z)\big) \\
&\qquad + \sum_j \Big(\chi(F_{i-1}\cap N_{c_j}) - \chi(S_i\cap N_{c_j}) + \chi(F_{i-1}\cap N_{d_j}) - \chi(S_i\cap N_{d_j}) \\
&\qquad\qquad + \chi(F_{i-1}\cap N_{e_j}) - \chi(S_i\cap N_{e_j}) + \chi(F_{i-1}\cap N_{f_j}) - \chi(S_i\cap N_{f_j})\big)\Big) \\
&\quad = \sum_i \big(\chi(F_{i-1}\cap N_z) - \chi(S_i\cap N_z)\big) \\
&\qquad + \sum_j\sum_i \big(\chi(F_{i-1}\cap N_{c_j}) - \chi(S_i\cap N_{c_j}) + \chi(F_{i-1}\cap N_{d_j}) - \chi(S_i\cap N_{d_j}) \\
&\qquad\qquad + \chi(F_{i-1}\cap N_{e_j}) - \chi(S_i\cap N_{e_j}) + \chi(F_{i-1}\cap N_{f_j}) - \chi(S_i\cap N_{f_j})\big) \\
&\quad \ge -2(2-n-1) + \sum_i \big(|F_{i-1}\cap \partial N_z| - |S_i\cap \partial N_z|\big) \\
&\qquad + \sum_{j\in J_0}\sum_i \big(\chi(F_{i-1}\cap N_{c_j}) - \chi(S_i\cap N_{c_j}) + \chi(F_{i-1}\cap N_{d_j}) - \chi(S_i\cap N_{d_j}) \\
&\qquad\qquad + \chi(F_{i-1}\cap N_{e_j}) - \chi(S_i\cap N_{e_j}) + \chi(F_{i-1}\cap N_{f_j}) - \chi(S_i\cap N_{f_j})\big) \\
&\qquad + \sum_{j\in J_1}\sum_i \big(\chi(F_{i-1}\cap N_{c_j}) - \chi(S_i\cap N_{c_j}) + \chi(F_{i-1}\cap N_{d_j}) - \chi(S_i\cap N_{d_j}) \\
&\qquad\qquad + \chi(F_{i-1}\cap N_{e_j}) - \chi(S_i\cap N_{e_j}) + \chi(F_{i-1}\cap N_{f_j}) - \chi(S_i\cap N_{f_j})\big) \\
&\qquad + \sum_{j\in J_2}\sum_i \big(\chi(F_{i-1}\cap N_{c_j}) - \chi(S_i\cap N_{c_j}) + \chi(F_{i-1}\cap N_{d_j}) - \chi(S_i\cap N_{d_j}) \\
&\qquad\qquad + \chi(F_{i-1}\cap N_{e_j}) - \chi(S_i\cap N_{e_j}) + \chi(F_{i-1}\cap N_{f_j}) - \chi(S_i\cap N_{f_j})\big) \\
&\quad \ge -2(1-n) + \sum_i\sum_j \big(|F_{i-1}\cap \partial N_z^j| - |S_i\cap \partial N_z^j|\big) \\
&\qquad + \sum_{j\in J_0} 6 + \sum_{j\in J_1}(8-2+2) + \sum_{j\in J_2}\Big(6 - \sum_i \big(|F_{i-1}\cap \partial N_z^j| - |S_i\cap \partial N_z^j|\big)\Big) \\
&\quad = -2(1-n) + \sum_{j\in J_0}\sum_i \big(|F_{i-1}\cap \partial N_z^j| - |S_i\cap \partial N_z^j|\big) \\
&\qquad + \sum_{j\in J_1}\sum_i \big(|F_{i-1}\cap \partial N_z^j| - |S_i\cap \partial N_z^j|\big) + \sum_{j\in J_2}\sum_i \big(|F_{i-1}\cap \partial N_z^j| - |S_i\cap \partial N_z^j|\big) \\
&\qquad + \sum_{j\in J_0} 6 + \sum_{j\in J_1}(8-2+2) + \sum_{j\in J_2}\Big(6 - \sum_i \big(|F_{i-1}\cap \partial N_z^j| - |S_i\cap \partial N_z^j|\big)\Big) \\
&\quad = -2+2n+0+\sum_{j\in J_1}(-2) + \sum_{j\in J_2}\sum_i \big(|F_{i-1}\cap \partial N_z^j| - |S_i\cap \partial N_z^j|\big) \\
&\qquad + \sum_{j\in J_0} 6 + \sum_{j\in J_1}(8-2+2) + \sum_{j\in J_2} 6 - \sum_{j\in J_2}\sum_i \big(|F_{i-1}\cap \partial N_z^j| - |S_i\cap \partial N_z^j|\big) \\
&\quad = -2+2n+\sum_{j\in J_0} 6 + \sum_{j\in J_1} 8 + \sum_{j\in J_2} 6 \ge -2+2n+6n = 8n-2.
\end{aligned}
$$

This shows that genus $S \ge 4n$.

Case B: A component of $\big(\bigcup_i S_i\big) \cap N_z$ *is pseudohorizontal.* Denote this component by $\tilde{S}$ and note that $\big(\big(\bigcup_i F_i \cup \bigcup_i S_i\big) - \tilde{S}\big) \cap N_z = \varnothing$. Now $\chi(\tilde{S}) = 2 - 2\,\text{genus}\,\tilde{S} - |\partial\tilde{S}|$ and

$$\sum_i \big(|F_{i-1} \cap \partial N_z| - |S_i \cap \partial N_z|\big) = -|\partial\tilde{S}|.$$

Define J_0, J_1, J_2 as above and note that here $J_0 = \varnothing$. Then $-\chi(S)$ is given by

$$\begin{aligned}
&\sum_i \big(\chi(F_{i-1}) - \chi(S_i)\big)\\
&\quad \geq \sum_i \big(\chi(F_{i-1} \cap N_z) - \chi(S_i \cap N_z)\big)\\
&\qquad + \sum_i \sum_j \big(\chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) + \chi(F_{i-1} \cap N_{d_j})\\
&\qquad\qquad - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})\big)\\
&\quad = \sum_i \big(\chi(F_{i-1} \cap N_z) - \chi(S_i \cap N_z)\big)\\
&\qquad + \sum_i \sum_{j \in J_1} \big(\chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) + \chi(F_{i-1} \cap N_{d_j})\\
&\qquad\qquad - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})\big)\\
&\qquad + \sum_i \sum_{j \in J_2} \big(\chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) + \chi(F_{i-1} \cap N_{d_j})\\
&\qquad\qquad - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})\big)\\
&\quad = -2 + 2\,\text{genus}\,\tilde{S} + |\partial\tilde{S}| + \sum_{j \in J_1} \Big(8 + \sum_i \big(|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|\big)\Big) + \sum_{j \in J_2} 10\\
&\quad = -2 + 2\,\text{genus}\,\tilde{S} + \sum_j 8 \geq -2 + 8n.
\end{aligned}$$

Hence genus $S \geq 4n$. □

7. Some comments on nontotally orientable graph manifolds

In the proofs of Theorem 1 and Theorem 2 we make extensive use of the structure theorem for Heegaard splittings of totally orientable graph manifolds [Schultens 2004]. We believe however that similar statements are true for graph manifolds in general. This suggests a more straightforward generalization of the examples provided in [Weidmann 2003] which are not totally orientable.

Note that the verification that the manifolds constructed in [Weidmann 2003] are not of Heegaard genus 2 relies on the classification of 3-manifolds with nonempty characteristic submanifold that have a genus 2 Heegaard splitting as given by T. Kobayashi [1984].

Thus we conjecture that the manifolds M_n constructed below are of Heegaard genus $3n$, the same argument as above shows that they can be generated by $2n$ elements.

The graph underlying the manifold M_n is again Γ_n and the Seifert piece corresponding to the vertex v is again denoted by N_v. Moreover:

(1) The intersection number between the fibers of the adjacent Seifert spaces is 1 at any torus of the JSJ decomposition.

(2) $\mathbb{O}(N_z)$ is a n-punctured sphere with at most one cone point and $\hat{N}_z = S^3$.

(3) $\mathbb{O}(N_{d_i}) = P^2(\infty, \infty, 5n)$ and N_{d_i} admits no pseudohorizontal surface that has genus 2.

(4) $\mathbb{O}(N_{c_i})$ is of type $D(2, q)$ with odd q but N_{c_i} is not homeomorphic to the exterior of a 2-bridge knot in S^3.

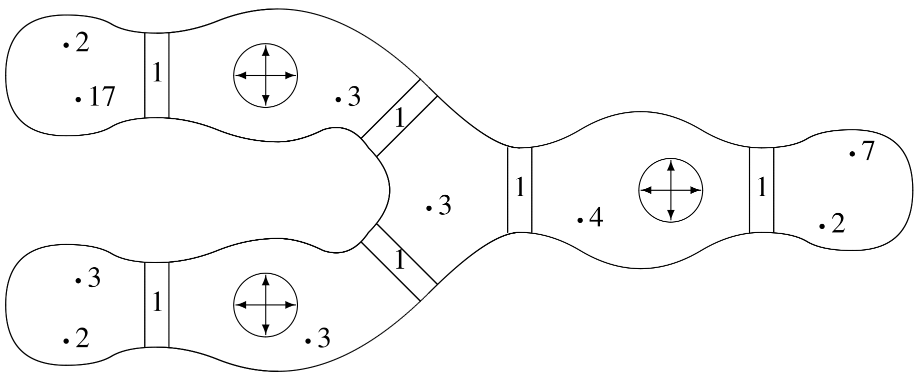

Figure 5. A graph manifold M with $g(M) = 9$ and $r(M) \leq 6$?

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