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CLASSIFICATION OF FIBER SURFACES OF GENUS 2 WITH AUTOMORPHISMS ACTING TRIVIALLY IN COHOMOLOGY

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Let S be a complex nonsingular projective surface of general type with a fibration of genus 2, and let $G \subset \operatorname{Aut} S$ be a nontrivial subgroup of automorphisms of S, inducing trivial actions on $H^2(S, \mathbb{Q})$. We give a classification for pairs (S, G) from the point of view of moduli. Consequently, we show that there exist surfaces S of general type (with p_g arbitrary large) with an involution acting trivially on $H^i(S, \mathbb{Z})$ for all i.

1. Introduction

Let S be a complex minimal nonsingular projective surface of general type, and let $G \subset \operatorname{Aut} S$ be a *nontrivial* subgroup of automorphisms of S inducing trivial actions on $H^2(S, \mathbb{Q})$. Peters [1979] proved that, if the canonical linear system $|K_S|$ is basepoint free, then either $K_S^2 = 8\chi(\mathbb{O}_S)$ or $K_S^2 = 9\chi(\mathbb{O}_S)$. Recently, we showed that $|G| \leq 4$ if $\chi(\mathbb{O}_S) > 188$ [Cai 2004]. When S has a fibration of genus 2, we have a numerical classification for pairs (S, G):

Theorem 1.1 [Cai 2006a; 2006b]. *Let S, G be as above. Assume that S has a relatively minimal fibration of genus 2 and $\chi(\mathbb{O}_S) \geq 5$. Then $|G| = 2$, and either*

(i) $K_S^2 = 4\chi(\mathbb{O}_S) - 4a$ $(a = 0,\ 1)$, *or*

(ii) $K_S^2 = 8\chi(\mathbb{O}_S) - 6b$ $(b = 0,\ 1,\ 2)$.

There are some examples in [Cai 2006a; 2006b] to show that such pairs (S, G) exist, besides the well known ones (products of two hyperelliptic curves). An interesting question is whether it could be possible to classify all possible pairs (S, G) in Theorem 1.1.

In this note we give a classification for pairs (S, G) in Theorem 1.1 from the more general point of view of moduli. Roughly speaking, our main result is this (see Theorems 2.5 and 4.7 for precise statements):

Theorem 1.2. *Let S, G be as in Theorem 1.1.*

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(i) *If S is as in Theorem 1.1*(*i*), *then S is birationally equivalent to a double cover of certain elliptic fiber bundle. The configuration of the ramification divisor of this covering is determined.*

(ii) *If S is as in Theorem 1.1*(*ii*) *with* $b = 0$, *then* $S \simeq (F \times \tilde{C})/\tilde{G}$, *where F and $\tilde{C}$ are curves of genus* $g(F) = 2$, $g(\tilde{C}) \geq 2$, *and $\tilde{G}$ is one of the following groups*: $\mathbb{Z}/m\mathbb{Z}$ ($m \leq 10$, $m \neq 7, 9$), $(\mathbb{Z}/2\mathbb{Z})^2$, $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, D_8 (*the dihedral group of order* 8); *a complete description for the action of $\tilde{G}$ on $F \times \tilde{C}$ is given.*

We note that, for K3 and Enriques surfaces S, $\operatorname{Aut} S$ acts faithfully on $H^2(S, \mathbb{Z})$ (see [Burns and Rapoport 1975; Ueno 1976]). As an interesting consequence of Theorem 2.5, we show that the analogous question for surfaces of general type has a negative answer:

Theorem 1.3 (Corollary 2.11). *Let $n \geq 3$ be an integer. There exist an infinite series of surfaces S_n of general type with* $K^2_{S_n} = 4n$, $p_g(S_n) = n$, $q(S_n) = 1$ *admitting an involution acting trivially on $H^i(S_n, \mathbb{Z})$ for all i.*

We work over the complex number field and use standard notation as exemplified by [Barth et al. 1984]. We also use freely the notation from [Cai 2006a; 2006b].

2. Surfaces whose canonical map being composite with a pencil

2.1. Let S be a complex nonsingular projective surface of general type with $p_g(S)$ at least 3 and let $f : S \to C$ be a relatively minimal fibration of genus 2. Consider a nontrivial subgroup $G \subset \operatorname{Aut} S$ of automorphisms of S inducing trivial actions on $H^2(S, \mathbb{Q})$. In this section, we assume that the canonical map Φ_S of S is composite with a pencil. By [Cai 2006a, Theorem 3.2], we have $|G| = 2$, the generator σ of G is a bielliptic involution of f (that is, $f \circ \sigma = f$, and for a general fiber F of f, $\sigma_{|F}$ is a bielliptic involution of F), and S has numerical invariants

(2.1.1) $K_S^2 = 4\chi(\mathcal{O}_S)$ and $q(S) = g(C) = 1$, or

(2.1.2) $K_S^2 = 4\chi(\mathcal{O}_S) - 4$, $q(S) = 1$ and $g(C) = 0$.

The hyperelliptic involutions of smooth fibers of f glue together to give a birational C-involution τ of S, which is everywhere defined by the uniqueness of the minimal model of f. We call τ the hyperelliptic involution of $f : S \to C$. Let $\lambda = \sigma \circ \tau$. Clearly λ is a bielliptic involution of f. We have a commutative diagram

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{\pi}"] \arrow[d, "\rho"'] & \tilde{S}/\tilde{\lambda} \arrow[r, "\alpha"] \arrow[d] & T \arrow[dl, "p"] \\
S \arrow[r, "f"] & C &
\end{tikzcd}$$

where ρ is the blowup of all isolated fixed points of λ, $\tilde{\lambda}$ is the induced involution on $\tilde{S}$, α is the blowdown of all -1-curves contained in fibers of $\tilde{S}/\tilde{\lambda} \to C$, and p is the induced relatively minimal elliptic fibration.

We can describe $p : T \to C$ explicitly:

Proposition 2.2. *Let E_2 be an elliptic curve, and set $E_4 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $E_3 = E_6 = \mathbb{C}/(\mathbb{Z} + \xi\mathbb{Z})$, for ξ a primitive third root of unit.*

(i) *If S is as in* (2.1.1), *then C is an elliptic curve, and*

$$(p : T \to C) \simeq (T_d := (C' \times E_d)/\mathbb{Z}_d \to C'/\mathbb{Z}_d)$$

for some $d \in \{2, 3, 4, 6\}$, where C' is an elliptic curve and $\mathbb{Z}_d$ acts on $C' \times E_d$ via a product action: $\mathbb{Z}_d$ acts on C' as a translation of order d such that $C'/\mathbb{Z}_d \simeq C$, and $\mathbb{Z}_d$ acts on E_d by (1) *$e \mapsto -e$ if $d = 2$;* (2) *$e \mapsto \xi e$ if $d = 3$;* (3) *$e \mapsto ie$ if $d = 4$;* (4) *$e \mapsto -\xi e$ if $d = 6$.*

Moreover, $K_T = p^*\eta$, where $\eta \in \operatorname{Pic}^0 C$, which determines the étale cover $C' \to C$.

(ii) *If S is as in* (2.1.2), *then $C = \mathbb{P}^1$, $T = C \times E$ and p is the projection to the first factor, where E is an elliptic curve.*

Proof. By [Cai 2006a, Proposition 4.12] and its proof, $p\colon T \to C$ is an elliptic fiber bundle with a section. By the proof of Theorem 3.2 of the same reference, we have $q(T) = g(C) = 1$ if S is as in (2.1.1), and $q(T) = 1$, $g(C) = 0$ if S is as in (2.1.2). Note that $p_g(T) = 0$. Now the proposition follows from the well-known result of Bagnera and de Franchis on the classification of bielliptic surfaces (see [Beauville 1983, VI, 20], for example). □

Proposition 2.3. *Let T_d be as in Proposition 2.2. Then $H_1(T_d, \mathbb{Z})_{\mathrm{tor}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, 0 if $d = 2, 3, 4, 6$, respectively.*

See [Iitaka 1971; Suwa 1969; Serrano 1991] for proofs.

Notation 2.4. Let $p : T \to C$ be a fiber surface and $\Delta \subset T$ a bisection of T, that is, an irreducible curve with $\Delta P = 2$, where P is a fiber of p. We say that a point $t \in \Delta$ is a ramification point of $p_{|\Delta} : \Delta \to C$ if t is in the image of the set of ramification points of $p_{|\Delta} \circ \phi : \tilde{\Delta} \to \Delta \to C$ under ϕ, where $\phi : \tilde{\Delta} \to \Delta$ is the normalization of Δ.

For any point $t \in \Delta$, let $l(t; \Delta)$ be the number of times we must blow up $t \in T$ and its infinitely near points to get the strict transform of Δ being nonsingular at the inverse image of t.

For any two curves D, D' and $t \in D \cap D'$, we denote by $I(D, D'; t)$ the intersection number of D and D' at the point t.

Theorem 2.5. *Let $f: S \to C$, $p: T \to C$, $\tilde{\pi}$, and α be as in 2.1. Let $\pi: S' \to T$ be the Stein factorization of $\alpha \circ \tilde{\pi}$, and let (B, θ) be the singular double cover data corresponding to π. Then (B, θ) has the following properties*:

(i) $\theta = C_1 + p^*D$, *where C_1 is a section of p and D is a divisor on C of degree $n := p_g(S) \geq 3$*,

(ii) $B = \Delta + \sum_{i=1}^m p^*c_i$, *where $\Delta \in |2C_1 + p^*(2D - \sum_{i=1}^m c_i)|$ is a bisection of p and c_i $(i = 1, \dots, m)$ are different points of C*,

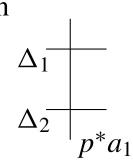
(iii) $\Delta \cap C_1$ *is contained in the set of ramification points of $p_{|\Delta}$. As a set, $\Delta \cap C_1 = \{t_1, \cdots, t_m\}$, where $t_i = p^*c_i \cap C_1$. For any i, $I(\Delta, C_1; t_i) = 2l(t_i; \Delta) + 1$. So $\sum_{i=1}^m l(t_i; \Delta) = n - m$.*

Conversely, let $p: T \to C$ be as in Proposition 2.2, and let (B, θ) be the singular double covers data satisfying conditions (*i*)–(*iii*) *above. Let $\pi: S' \to T$ be the double cover corresponding to (B, θ). Let S'' be the desingularization of S', and $f': S'' \to C$ the induced fibration. Let $f: S \to C$ be the relatively minimal fibration of f'. Denote by τ the hyperelliptic involution of f, and λ the involution corresponding to the double cover π. Let $\sigma = \tau \circ \lambda$. Then S is as in* (2.1.1) (*resp.* (2.1.2)) *with $p_g(S) = n$ if T is as in* (*i*) (*resp.* (*ii*)) *of Proposition 2.2 and σ acts trivially on $H^2(S, \mathbb{Q})$.*

Proof. We assume that T is as in Proposition 2.2(i). The proof of the other case is similar and is left to the reader. Since B has no essential singularities, by the formula for double covers, we have $h^0(K_T \otimes \theta) = n$. Note that $p: T \to C$ is a fiber bundle, and $(K_T \otimes \theta)P = 1$ for a fiber P of p. We have $K_T \otimes \theta \equiv C_1 + p^*D'$, where C_1 is a section of p and D' is an effective divisor on C. Clearly C_1 is the fixed part of $|C_1 + p^*D'|$. So $\deg D' = h^0(D') = h^0(C_1 + p^*D') = n$. Note that $K_T = p^*\eta$, where η is as in Proposition 2.2. So $\theta = C_1 + p^*D$, where $D = D' \otimes \eta$ is a divisor on C of degree n.

Since B is a reduced divisor, we may write $B = \Delta + \sum_{i=1}^m p^*c_i$, where Δ is a reduced horizontal divisor with respect to p, $m \geq 0$, and c_i $(i = 1, \dots, m)$ are different points of C.

2.6. We show that Δ is irreducible. Otherwise, $\Delta = \Delta_1 + \Delta_2$, where Δ_j are sections of p. Clearly $\Delta_1\Delta_2 = 0$. So $m > 0$. Then locally around p^*c_1 the branch locus B of π has the configuration



So $(p \circ \pi)^*c_1$ is a multiple fiber and S' has two rational double points on it, and hence f^*c_1 is a fiber of type (b_0). This contradicts [Cai 2006a, Lemma 4.7(ii)].

Lemma 2.7. *If $t \in \Delta \cap C_1$, then t is a ramification point of $p_{|\Delta}$, and*

$$I(\Delta, C_1; t)) = 2l(t; \Delta) + 1,$$

where $l(t; \Delta)$ is as in Notation 2.4.

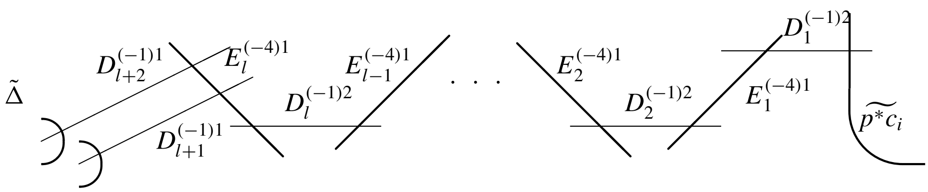
Proof. let $c = p(t)$ and $l = l(t; \Delta)$. First we assume that t is a smooth point of Δ. If t is not a ramification point of $p_{|\Delta}$, then $p^*c \cap \Delta$ consists of two different points, t and t'. We have $t + t' - 2t \equiv \Delta_{|p^*c} - 2C_{1|p^*c} \equiv p^*(2D - \sum_{i=1}^m c_i)_{|p^*c} \equiv 0$. This implies $t \equiv t'$ on p^*c, which is a contradiction since p^*c is not rational.

Now we may assume that t is a singular point of Δ. If $c \neq c_i$ for any i, then $\mathrm{mult}_t B = 2$. Let $\hat\rho : \hat T \to T$ be the blowing up at t, and E the exceptional curve. For any irreducible curve Z in T, we denote $\hat Z$ the strict transform of Z in $\hat T$. Set

$$\hat B = \hat\rho^* B - 2E, \quad \hat\theta = \hat\rho^*\theta - E = \hat C_1 + \hat\rho^* p^* D.$$

Let $\hat\pi : \hat S \to \hat T$ be the double cover corresponding to $(\hat B, \hat\theta)$. Clearly $\alpha \circ \tilde\pi$ (notation as in 2.1) factors through $\hat\pi$. Since C_1 and p^*c meet transversally only in one point t, we have $\widehat{C_1} \cap \widehat{p^*c} = \varnothing$. This implies $\hat\theta_{|\widehat{p^*c}}$ is trivial. So $\hat\pi^*\widehat{p^*c}$ has two disconnected components, and hence f^*c is of type (a_k). This contradicts [Cai 2006a, Lemma 4.6].

So we can assume $c = c_i$ for some i. If $t \in \Delta$ is not a ramification point of $p_{|\Delta}$, then $(p \circ \alpha)^*c$ has the following configuration:



where $\tilde\Delta$ and $\widetilde{p^*c_i}$ are the strict transforms of Δ and p^*c_i, thick lines mean branch locus of $\tilde\pi$, and superscript numbers without brackets are multiplicities and superscript numbers within brackets denote self-intersections. This implies f^*c_i is of type (b_{2l}), which is a contradiction by [Cai 2006a, Lemma 4.7(ii)].

Now $t \in \Delta$ is a ramification point of $p_{|\Delta}$. Let $H = (\alpha \circ \tilde\pi)^* C_1$. By [Cai 2006a, 4.8, 4.11 and 4.12], we have $(f \circ \rho)_{|H} : H \to C$ is étale. So the strict transform $\tilde C_1$ of C_1 in $\tilde S/\tilde\lambda$ does not meet the branch locus of $\tilde\pi$. This implies $I(\Delta, C_1; t) = 2l + 1$ by a standard calculation; see, for instance, [Hartshorne 1977, Chapter V, Propositions 3.2 and 3.6]. □

By the proof of Lemma 2.7, the image of $\Delta \cap C_1$ under p is contained in the set $\{c_1, \ldots, c_m\}$. Now suppose there is a point $c_i \in \{c_1, \ldots, c_m\} \setminus p(\Delta \cap C_1)$. If $p^*c_i \cap \Delta$ consists of two points, then $p_{|\Delta}$ is étale at c_i and we get a contradiction as in 2.6. Hence $p^*c_i \cap \Delta$ is a single point. By the choice of c_i, $p^*c_i \cap \Delta \notin C_1$.

So $p^*c_i \cap C_1 \neq p^*c_i \cap \Delta$, and hence $p^*c_i \cap C_1$ must be a smooth point of B. This implies the strict transform $\tilde{C}_1$ of C_1 does meet the branch locus of $\tilde{\pi}$. This is impossible since $H \to C$ is étale.

Now we prove the converse of the theorem. Let T be as in (i) of Proposition 2.2, and let (B, θ) be the singular double cover data satisfying conditions (i)–(iii) in Theorem 2.5. Let $\pi : S' \to T$ be the double cover corresponding to (B, θ). Then S' has only canonical singularities. Let $\epsilon : S \to S'$ be the minimal desingularization. We have

$$K_S = (\pi \circ \epsilon)^*(p^*(\eta + D) + C_1).$$

So S has the following numerical invariants:

$$K_S^2 = 4n, \ p_g(S) = n, \ q(S) = 1.$$

Now $f := p \circ \pi \circ \epsilon : S \to C$ is a fibration of genus 2. Denote by τ the hyperelliptic involution of f, and by λ the involution of S corresponding to π. Then $\langle \lambda, \tau \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Take $\sigma = \tau \circ \lambda$. Now the result follows by the following lemma. □

Lemma 2.8. *The involution σ acts trivially on $H^i(S, \mathbb{Q})$ for all i.*

The idea of the proof of Lemma 2.8 is to analyze the action of σ around the singular fibers of f, and to apply the topological Lefschetz formula to σ. The proof is longer and is postponed until the next section; see also [Cai 2006a, 3.3] for a proof in the special case when the bisection $\Delta < B$ is smooth.

Remark 2.9. Let Δ be as in Theorem 2.5. If Δ is smooth, then $l(t_i; \Delta) = 0$ for all i and hence $m = n$. In this case, by the proof of Lemma 2.7, the points in $\Delta \cap C_1$ are necessarily ramification points of $p_{|\Delta}$. So the only condition for (S, σ) being as in 2.1 is that the n fibers p^*c_i contained in B pass through the n points of $\Delta \cap C_1$.

Corollary 2.10. (i) *The moduli space $\mathcal{M}$ of surfaces (S, σ) as in* (2.1.1) *with $p_g(S) = n$ has four irreducible connected components. Among them one has dimension $2n + 1$ and the others have dimensions $2n$.*

(ii) *The moduli space $\mathcal{M}'$ of surfaces (S, σ) as in* (2.1.2) *with $p_g(S) = n$ is irreducible and of dimension $2n - 1$.*

Proof. We prove (i); the proof of (ii) is similar and is left to the reader. By Theorem 2.5, $\mathcal{M}$ is a disjoint union $\mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4 \cup \mathcal{M}_6$, where $\mathcal{M}_d = \{S \in \mathcal{M} \mid T \simeq T_d\}$, for T_d is as in Proposition 2.2. Let $\mathcal{B}_z \in |2\theta|$ be a flat family of curves such that $\mathcal{B}_0$ is the branch locus B of $\pi : S' \to T$ and $\mathcal{B}_1$ is smooth. Let $\mathcal{S}_z$ be the flat family of surfaces corresponding to the double cover data $(\mathcal{B}_z, \theta)$. Since the branch locus $\mathcal{B}_1$ of $\mathcal{S}_1 \to T$ is ample, we have $\pi_1(\mathcal{S}_1) \simeq \pi_1(T)$ by [Cornalba 1981]. Since $\mathcal{B}_0 = B$ has no essential singularities, $S' = \mathcal{S}_0$ has only rational double points. By [Atiyah 1958], the minimal desingularization S of $\mathcal{S}_0$ is diffeomorphic to $\mathcal{S}_1$. Hence we have $\pi_1(S) \simeq \pi_1(T)$. By Proposition 2.3, the sets $\mathcal{M}_d$ are open. Given

T_d, for generic $[S] \in \mathcal{M}_d$, S is determined by (B, θ), where $\theta = C_1 + p^*D$, D is a divisor of degree n on C, $B = \Delta + \sum_{i=1}^n p^*c_i$, $\Delta \in |2C_1 + p^*D|$ (cf. Remark 2.9). Up to automorphisms of T_d, C_1 is uniquely determined. Given a smooth curve $\Delta \in |2C_1 + p^*D|$, the choice of θ is unique up to a torsion element of order 2 of $\mathrm{Pic}^0\, T_d$. Clearly Δ depends on $h^0(2C_1 + p^*D) - 1 = 2n - 1$ (by Riemann–Roch) parameters. Note that T_d depends on two parameters if $d = 2$, and on one if $d = 3,\ 4,\ 6$. So the dimension of $\mathcal{M}_d$ is $2n + 1$ if $d = 2$, and $2n$ if $d = 3,\ 4,\ 6$. □

Corollary 2.11. *Let* (S, σ) *be as in* (2.1.1). *If* $S \in \mathcal{M}_6$, *where* $\mathcal{M}_6$ *is as in the proof of Corollary 2.10, then the involution* σ *acts trivially on* $H^i(S, \mathbb{Z})$ *for all* i.

Proof. If $S \in \mathcal{M}_6$ we have $\pi_1(S) \simeq \pi_1(T)$ by the proof of Corollary 2.10, and hence $H_1(S, \mathbb{Z})_{\mathrm{tor}} = 0$ by Proposition 2.3. By the Poincaré duality for the torsion part of homology, we have $H^2(S, \mathbb{Z})_{\mathrm{tor}} = 0$. Hence $H^*(S, \mathbb{Z})$ is torsion-free, and the result follows from Lemma 2.8. □

3. Proof of Lemma 2.8

We keep the notation of Theorem 2.5. Since $q(S) = g(C)$, by Hodge theory, σ acts trivially on $H^1(S, \mathbb{Q})$. To check that the involution σ acts trivially on $H^2(S, \mathbb{Q})$, we analyze the action of σ around the singular fibers of f. Let t_j $(j = 1, \dots, u)$ be the ramification points of $p_{|\Delta}$. After suitable reindexing, we may assume that $\{t_1, \dots, t_m\} = \Delta \cap C_1$ as a set. Let t_{u+k} $(k = 1, \dots, v)$ be the singular points of $\Delta \setminus \{t_j | 1 \le j \le u\}$. Set $l_j = l(t_j; \Delta)$, where $l(t_j; \Delta)$ is as in Notation 2.4. We have $l_j \ge 0$ for $j = 1, \dots, u$, and $l_j \ge 1$ for $j = u + 1, \dots, u + v$. By the definition of l_j, we have

$$p_a(\Delta) = g(\tilde{\Delta}) + \sum_{j=1}^{u+v} l_j,$$

where $\phi : \tilde{\Delta} \to \Delta$ is the normalization of Δ. Applying the Hurwitz formula to $p_{|\Delta} \circ \phi$, we get

$$2g(\tilde{\Delta}) - 2 = u.$$

By the adjunction formula,

$$2p_a(\Delta) - 2 = \Big(2C_1 + p^*\Big(2D - \sum_{j=1}^{m} c_j\Big)\Big)^2 = 4(2n - m).$$

Combining these three equalities, we have

$$4m + u + 2\sum_{j=1}^{u+v} l_j = 8n. \tag{3.0.1}$$

Let $\varrho\colon T' \to T$ be the morphism composed of l_j times blow-ups of t_j and its infinitely near points $(j = 1, \dots, u + v)$. The exceptional divisor $\varrho^*(t_j)$ equals

$\sum_{l=1}^{l_j} E'_{jl}$, where E'_{jl} is the exceptional curve corresponding to the $(l-1)$-th near points of t_j. Then the strict transform Δ' of Δ is smooth, and for $j=1,\dots,u$, Δ' meets E'_{jl_j} in one point t'_j and is tangent to it there. Let $\varrho'\colon T''\to T'$ be the blow-up of t'_j ($j=1,\ \dots,\ u$) and $s_{jl} := E'_{j,l-1}\cap E'_{jl}$ ($j=1,\ \dots,\ m,\ l=1,\ \dots,\ l_j$) (for convenience, here we set $E'_{j0}=(p^*c_j)'$). Let $E''_{j,l_j+1}=\varrho'^*(t'_j)$ and $D''_{jl}=\varrho'^*(s_{jl})$ be the exceptional curves. Then E''_{j,l_j+1} and the strict transform Δ'' of Δ' meet transversely at point t''_j. Let

$$\mu\colon \tilde T\to T''$$

be the blow-up of t''_j ($j=1,\dots,u$). Let $\tilde E_{j,l_j+2}=\mu^*(t''_j)$ ($j=1,\dots,u$) be the exceptional curves. For any irreducible curve Y in T, we denote by Y', Y'' and $\tilde Y$ the strict transform of Y in T', T'' and $\tilde T$, respectively. Set

$$\begin{aligned}\tilde B &:= \mu^*\Bigl(\varrho'^*\Bigl(\varrho^*B-2\sum_{j=1}^{u+v}\sum_{l=1}^{l_j} lE'_{jl}\Bigr)-2\sum_{j=1}^{m}E''_{j,l_j+1}-2\sum_{j=1}^{m}\sum_{l=1}^{l_j}D''_{jl}\Bigr)-2\sum_{j=1}^{u}\tilde E_{j,l_j+2}\\ &=\tilde\Delta+\sum_{j=1}^{m}\widetilde{p^*c_j}+\sum_{j=1}^{m}\sum_{l=1}^{l_j}\tilde E'_{jl}+\sum_{j=m+1}^{u}\tilde E''_{j,l_j+1}\end{aligned}$$

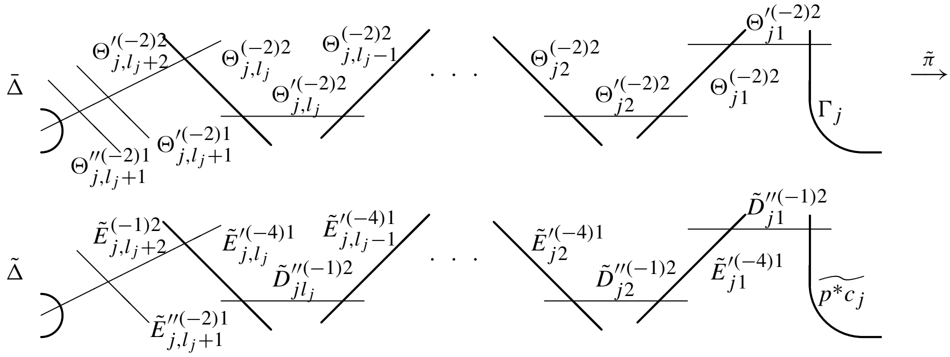
$$\begin{aligned}\tilde\theta &:= \mu^*\Bigl(\varrho'^*\Bigl(\varrho^*\theta-\sum_{j=1}^{u+v}\sum_{l=1}^{l_j} lE'_{jl}\Bigr)-\sum_{j=1}^{m}E''_{j,l_j+1}-\sum_{j=1}^{m}\sum_{l=1}^{l_j}D''_{jl}\Bigr)-\sum_{j=1}^{u}\tilde E_{j,l_j+2}\\ &=(\varrho\circ\varrho'\circ\mu)^*D+\tilde C_1-\sum_{j=m+1}^{u+v}\sum_{l=1}^{l_j} l\tilde E'_{jl}-\sum_{j=1}^{m}\sum_{l=1}^{l_j}\widetilde{D}''_{jl}\\ &\qquad -\sum_{j=m+1}^{u} l_j\tilde E''_{j,l_j+1}-\sum_{j=1}^{m}\tilde E_{j,l_j+2}-\sum_{j=m+1}^{u}(2l_j+1)\tilde E_{j,l_j+2}.\end{aligned}$$

We have $\tilde B$ is a smooth divisor on $\tilde T$, and $\tilde B\equiv 2\tilde\delta$. Let $\tilde\pi:\ \tilde S\to\tilde T$ be the morphism associated with the double cover data $(\tilde B,\tilde\delta)$. By the canonical resolution [Persson 1978], we have a commutative diagram

$$\begin{tikzcd} & \tilde S \arrow[r, "\tilde\pi"] \arrow[dl, "\varepsilon"'] \arrow[d, "\beta"] & \tilde T \arrow[d, "\gamma := \varrho\circ\varrho'\circ\mu"] \\ S \arrow[r, "\epsilon"] & S' \arrow[r, "\pi"] & T, \end{tikzcd}$$

where β is a desingularization of S', and ε is the contraction of -1-curves on $\tilde S$.

Clearly f has only $u+v$ singular fibers f^*c_j ($j=1,\ \dots,\ u+v$). For $j=1,\ \dots,\ m$, locally around a singular fiber, $\tilde\pi\colon (f\circ\varepsilon)^*c_j\to(p\circ\gamma)^*c_j$ has the following configurations:

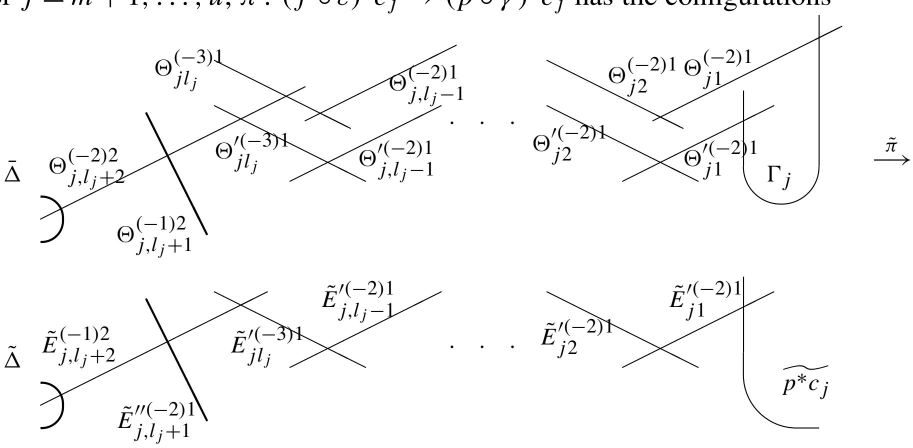


where $\bar{\Delta}$ is the inverse image of $\tilde{\Delta}$, thick lines mean ramification or branch locus of $\tilde{\pi}$, and superscript numbers without brackets are multiplicities and superscript numbers within brackets denote self-intersections. Hence

$$f^*c_j = \Theta'_{j,l_j+1} + \Theta''_{j,l_j+1} + 2\Theta'_{j,l_j+2} + 2\sum_{l=1}^{l_j} \Theta_{jl} + 2\sum_{l=1}^{l_j} \Theta'_{jl} + 2\Gamma_j$$

is as in (b_{2l_j+1}) of [Cai 2006a, 2.6]. Θ_{jl} $(l = 1, \ldots, l_j)$ are λ-fixed -2-curves and Γ_j is an λ-fixed elliptic curve.

For $j = m+1, \ldots, u$, $\tilde{\pi} : (f \circ \varepsilon)^* c_j \to (p \circ \gamma)^* c_j$ has the configurations

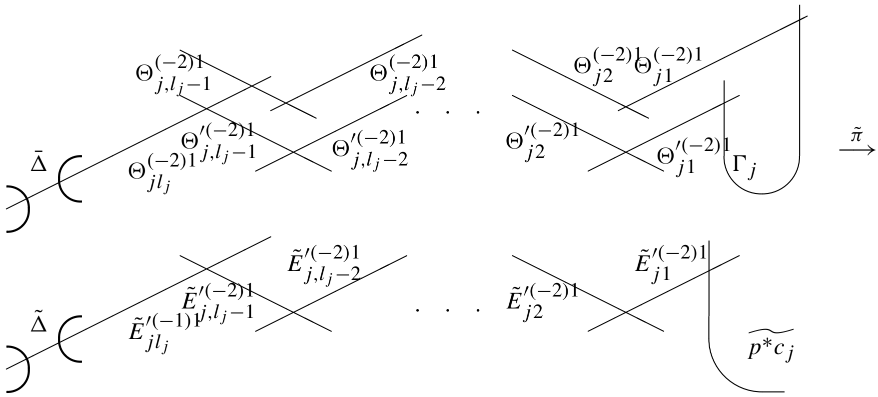


Since $\tilde{\delta}_{|\widetilde{p^*c_j}} = \tilde{C}_{1|\widetilde{p^*c_j}} - \tilde{E}'_{j1|\widetilde{p^*c_j}}$ $(= \tilde{C}_{1|\widetilde{p^*c_j}} - \tilde{E}_{j,l_j+2|\widetilde{p^*c_j}}$ when $l_j = 0)$ is nontrivial, the inverse image Γ_j of $\widetilde{p^*c_j}$ is connected. Hence

$$f^*c_j = \sum_{l=1}^{l_j} \Theta_{jl} + \sum_{l=1}^{l_j} \Theta'_{jl} + \Gamma_j$$

(here we also denote by Θ_{jl_j} and Θ'_{jl_j} the image of Θ_{jl_j} and Θ'_{jl_j} in S) is as in (v) of [Cai 2006a, Lemma 4.9]. The chain of -2-curves in f^*c_j is of type A_{2l_j} and $\Theta_{jl_j} \cap \Theta'_{jl_j}$ is a nonisolated λ-fixed point. (When $l_j = 0$ f^*c_j is an irreducible curve with exactly one node p_j, which is a nonisolated λ-fixed point. The normalization of f^*a_j is an elliptic curve.)

For $j = u+1, \dots, u+v$, $\tilde{\pi} : (f \circ \varepsilon)^* c_j \to (p \circ \gamma)^* c_j$ has the configurations



Since $\tilde{\delta}_{|\widetilde{p^*c_j}} = \tilde{C}_{1|\widetilde{p^*c_j}} - \tilde{E}'_{j1|\widetilde{p^*c_j}}$ is nontrivial, the inverse image Γ_j of $\widetilde{p^*c_j}$ is connected. Hence

$$f^*c_j = \sum_{l=1}^{l_j} \Theta_{jl} + \sum_{l=1}^{l_j-1} \Theta'_{jl} + \Gamma_j$$

is as in (v) of [Cai 2006a, Lemma 4.9]. The chain of -2-curves in f^*c_j is of type A_{2l_j-1}.

For $j = 1, \dots, m$, $\lambda|_{\Theta'_{j,l_j+2}}$ is an involution with fixed points

$$q_j = \Theta_{j,l_j} \cap \Theta'_{j,l_j+2}, \quad q'_j = \bar{\Delta} \cap \Theta_{j,l_j+2}$$

(the former equals $\Gamma_j \cap \Theta'_{j,l_j+2}$ when $l_j = 0$). See the picture above. Since $\bar{\Delta}$ is τ-invariant, q'_j is τ-fixed. From

$$\varepsilon^* K_S = (\gamma \circ \tilde{\pi})^*(p^*(\eta + D) + C_1), \tag{3.0.2}$$

we see that

$$(l_j+1)(\Theta'_{j,l_j+1} + \Theta''_{j,l_j+1}) + (2l_j+1)\Theta'_{j,l_j+2} + \sum_{l=1}^{l_j} 2l\Theta_{jl} + \sum_{l=1}^{l_j} (2l-1)\Theta'_{jl}$$

is contained in the fixed part of $|K_S|$. By [Cai 2006a, 2.9], f^*c_j is not of type V in the sense of Horikawa. So by [Cai 2006a, 2.8], q_j, q'_j are isolated τ-fixed points and there are three nonisolated τ-fixed points r_{1j}, r_{2j}, r_{3j} on Γ_j. So Θ'_{j,l_j+2} is σ-fixed (otherwise, $\langle \lambda, \tau \rangle \hookrightarrow \operatorname{Aut} \Theta'_{j,l_j+2}$, which is a contradiction since $\langle \lambda, \tau \rangle$ is not cyclic) and r_{1j}, r_{2j}, r_{3j} are σ-fixed points. Similarly we see easily that Θ'_{jl} ($l = 1, \dots, l_j$) are σ-fixed. Hence

$$e((f^*c_j)^\sigma) = 2(l_j+1) + 3 = 2l_j + 5 \quad \text{for } j = 1, \dots, m.$$

For $j = m+1, \dots, u+v$, since f^*c_j is reduced, by [Cai 2006a, 2.4], σ has no fixed curves on f^*c_j. Since each component of f^*c_j is σ-invariant, each node point of f^*c_j is σ-fixed. We show that they are isolated σ-fixed points. If there is a σ-fixed point $x \in f^*c_j$ which is not isolated, then there is a σ-fixed curve D (necessarily being horizontal with respective to f) passing through x. Since D is contained in the fixed part of $|K_S|$, $Df^*c_j = 2$. This implies there are three σ-invariant curves meeting in x with distinct tangent directions, and hence the induced linear action of σ on the tangent space at x must be $\varsigma\mathrm{id}$ for some $\varsigma \in \mathbb{C}$, a contradiction. (When $m+1 \le j \le u$ and $l_j = 0$, then p_j is a nonisolated τ-fixed point by [Cai 2001, Lemma 2.4], both τ and λ exchange the local branches at p_j. So σ fixes the local branches at p_j, implying that p_j is an isolated fixed point of σ.) Hence

$$e((f^*c_j)^\sigma) = \begin{cases} 2l_j + 1, & j = m+1, \dots, u; \\ 2l_j, & j = u+1, \dots, u+v. \end{cases}$$

Let $H \subset S$ be the inverse image of C_1. Both $\tau_{|H}$ and $\lambda_{|H}$ are involution of H. (Clearly by (3.0.2), H is contained in the fixed part of $|K_S|$. So H is τ-invariant and $H_{|F}$ is a g_2^1 on F, where F is a general fiber of f. If $\tau_{|H}$ = identity, let $H \cap F = \{s, s'\}$, then $s + s' = H_{|F} \equiv 2s$, which implies $s' \equiv s$ on F, a contradiction.) So H is a σ-fixed curve. Clearly H is the only σ-fixed curve which is horizontal with respective to f. we show that $f_{|H} : H \to C$ is étale. In particular, this implies r_{1j}, r_{2j}, r_{3j} are isolated σ-fixed points. Suppose $x \in H$ is a ramification point of $f_{|H}$. Let $F' = f^*(f(x))$. Since $HF' = 2$, we have $H \cap F' = \{x\}$. Since H is λ-invariant, we have x is $\langle \tau, \lambda \rangle$ -fixed. Since $\langle \tau, \lambda \rangle$ is not cyclic, x is a singular point of F'. If $F' = f^*c_j$ for some j, $m+1 \le j \le u+v$, then x is one of the node points of f^*c_j, which is a contradiction since these points are isolated fixed points of σ. Now we suppose $F' = f^*c_j$ for some j, $1 \le j \le m$. Since Θ'_{j,l_j+2} is σ-fixed, Θ'_{j,l_j+1} is not σ-fixed. So there is a σ-fixed point o_j on $\Theta'_{j,l_j+1} \setminus \Theta'_{j,l_j+1} \cap \Theta'_{j,l_j+2}$. By [Cai 2001, Lemma 2.4], H passes through o_j, which is a contradiction. Since H is étale over C, $e(H) = 0$. Summing-up, we have

$$\begin{aligned} e(S^\sigma) &= \sum_{j=1}^{u+v} e((f^*c_j)^\sigma) + e(H) = \sum_{j=1}^{m} (2l_j + 5) + \sum_{j=m+1}^{u} (2l_j + 1) + \sum_{j=u+1}^{u+v} 2l_j \\ &= 2\sum_{j=1}^{u+v} l_j + 4m + u. \end{aligned}$$

By the Noether formula, $e(S) = 8n$. Applying the topological Lefschetz formula to σ [Atiyah and Singer 1968, p. 566], namely

$$e(S) + 8(q(S) - \dim H^0(S, \Omega_S^1)^\sigma) - 2(H^2(S, \mathbb{Q}) - \dim H^2(S, \mathbb{Q})^\sigma) = e(S^\sigma),$$

we get

$$2(\dim H^2(S,\mathbb{Q}) - \dim H^2(S,\mathbb{Q})^\sigma) = 8n - \Big(2\sum_{j=1}^{u+v} l_j + 4m + u\Big) = 0$$

by (3.0.1). Thus σ acts trivially on $H^2(S,\mathbb{Q})$, and Lemma 2.8 is proved. □

Remark 3.1. Here is a sketch of an alternative proof of Lemma 2.8 suggested by the referee if T is as in Proposition 2.2(i). In this case $q(S) = g(C)$, and we can use Theorem 3 of [Shioda 1999] to compute the rank of the Néron–Severi group $\mathrm{NS}(S)_\mathbb{Q} = \mathrm{NS}(S) \otimes \mathbb{Q}$ of S. Consequently, $\mathrm{NS}(S)_\mathbb{Q}$ is generated by H, F and all irreducible components of singular fibers of f. By the construction of S, we can check that H, F and each such component are σ-invariant. Hence σ acts trivially on $\mathrm{NS}(S)_\mathbb{Q}$. Let $\mathrm{T}(S)$ be the orthogonal complement of $\mathrm{NS}(S)_\mathbb{Q}$ in $H^2(S,\mathbb{Q})$. Note that $\mathrm{T}(S)$ is the smallest rational subspace of $H^2(S,\mathbb{Q})$ such that the complexification of $\mathrm{T}(S)$ contains $H^{2,0}(S)$. Since the involution σ acts trivially on $H^0(\omega_S)$, we have $\mathrm{T}(S)^\sigma = \mathrm{T}(S)$. Hence σ acts trivially on $H^2(S,\mathbb{Q})$.

4. Surfaces with $K_S^2 = 8\chi(\mathcal{O}_S)$

In this section, we describe explicitly families of pairs (S,σ), where S is a fiber surface of genus 2 with $K_S^2 = 8\chi(\mathcal{O}_S)$, and σ is an involution of S inducing trivial action on $H^2(S,\mathbb{Q})$.

Throughout the section, we denote by τ_D the hyperelliptic involution of a hyperelliptic curve D; for a point e in an elliptic curve E, we denote by t_e the translation by e.

Example 4.1. Let $(S,\sigma) = (F \times C,\ \tau_F \times \tau_C)$, where F and C are hyperelliptic curves with $g(F) = 2$ and $g(C) \geq 2$. This example is well known.

Example 4.2. Let F be a curve of genus 2 with a bielliptic involution λ_F. Let $\tilde{B} = \mathbb{P}^1$ and $\gamma_{\tilde{B}}$ an involution of $\tilde{B}$. Let $\pi : C \to B := \tilde{B}/\langle\gamma_{\tilde{B}}\rangle$ be a double cover with $g(C) \geq 2$, such that the branch points of $\tilde{B} \to \tilde{B}/\langle\gamma_{\tilde{B}}\rangle = B$ are contained in that of π. Let $\tilde{C}$ be the normalization of $C \times_B \tilde{B}$ and $\gamma_{\tilde{C}} \in \mathrm{Aut}\tilde{C}$ the lift of $\gamma_{\tilde{B}}$. (Note that $\tilde{C}$ is hyperelliptic since the involution corresponding to $\tilde{C} \to \tilde{B}$ is the hyperelliptic one.)

Let $(S,\sigma) = ((F \times \tilde{C})/\langle\lambda_F \times \gamma_{\tilde{C}}\rangle,\ \overline{\tau_F \times \tau_{\tilde{C}}})$, where $\overline{\tau_F \times \tau_{\tilde{C}}}$ is the involution of $(F \times \tilde{C})/\langle\lambda_F \times \gamma_{\tilde{C}}\rangle$ induced by $\tau_F \times \tau_{\tilde{C}}$.

Example 4.3. Let G be one of the groups $\mathbb{Z}_a$ ($a = 2, 3, 4, 5, 6, 8, 10$) or $\mathbb{Z}_b \oplus \mathbb{Z}_2$ ($b = 2, 6$). Let F be a curve of genus 2 on which G acts faithfully and $g(F/G) = 0$. Let $\tilde{B}$ be an elliptic curve and G a subgroup of translations of $\tilde{B}$. Let $C \to B := \tilde{B}/G$ be a double cover with $g(C) \geq 2$. Let $\tilde{C} = C \times_B \tilde{B}$. Then G induces a faithful action on $\tilde{C}$. Let $\lambda_{\tilde{C}}$ be the involution of $\tilde{C}$ corresponding to the double cover $\tilde{C} \to \tilde{B}$.

Let $(S, \sigma) = ((F \times \tilde{C})/G,\ \overline{\tau_F \times \lambda_{\tilde{C}}})$, where G acts on $F \times \tilde{C}$ via a product action.

Example 4.4. Let F be a curve of genus 2 with a bielliptic involution λ_F. Let $\tilde{B}$ be an elliptic curve, and $e \in \tilde{B}$ a nontrivial 2-torsion point. Let $\pi : C \to B := \tilde{B}/\langle t_e, -1_{\tilde{B}}\rangle$ be a double cover such that the branch locus of $\tilde{B} \to B$ is contained in that of π. Let $\tilde{C}$ be the normalization of $C \times_B \tilde{B}$, and $\widetilde{-1}_{\tilde{B}}, \tilde{t}_e \in \operatorname{Aut} \tilde{C}$ the lifts of $-1_{\tilde{B}}, t_e \in \operatorname{Aut} \tilde{B}$ respectively. Let $\lambda_{\tilde{C}}$ be the involution of $\tilde{C}$ corresponding to the double cover $\tilde{C} \to \tilde{B}$.

Let $(S, \sigma) = ((F \times \tilde{C})/\langle \tau_F \times \tilde{t}_e,\ \lambda_F \times (\widetilde{-1}_{\tilde{B}})\rangle,\ \overline{\tau_F \times \lambda_{\tilde{C}}})$.

Example 4.5. Let $\tilde{B}$ be an elliptic curve, and $e \in \tilde{B}$ a nontrivial 4-torsion point. Let $G := \langle t_e, -1_{\tilde{B}}\rangle \simeq D_8$ (the dihedral group of order 8). Let F be a curve of genus 2 on which G acts faithfully. Let $\pi : C \to B := \tilde{B}/G$ be a double cover such that the branch locus of $\tilde{B} \to B$ is contained in that of π. Let $\tilde{C}$ be the normalization of $C \times_B \tilde{B}$. Then G induces a faithful action on $\tilde{C}$. Let $\lambda_{\tilde{C}}$ be the involution of $\tilde{C}$ corresponding to the double cover $\tilde{C} \to \tilde{B}$.

Let $(S, \sigma) = ((F \times \tilde{C})/G,\ \overline{\tau_F \times \lambda_{\tilde{C}}})$.

Remark 4.6. Let (S, σ) be as in one of Examples 4.1–4.5. Clearly S has a fibration of genus 2 with $K_S^2 = 8\chi(\mathbb{O}_S)$. Applying the topological and holomorphic Lefschetz formula to σ (see [Atiyah and Singer 1968, p. 566]) or by [Cai 2006b, 3.1], we can check easily that σ induces trivial actions on $H^2(S, \mathbb{Q})$.

Theorem 4.7. *Let S be a complex nonsingular projective surface of general type with $\chi(\mathbb{O}_S) \geq 5$, and $f : S \to C$ be a relatively minimal fibration of genus 2. Let $G \subset \operatorname{Aut} S$ be a nontrivial subgroup of automorphisms of S, inducing trivial actions on $H^2(S, \mathbb{Q})$. Assume that the canonical map ϕ_S of S is generically finite. Then $|G| = 2$, $g(C) \geq 2$, the generator σ of G induces a hyperelliptic involution or a bielliptic involution $\bar{\sigma}$ of C such that $\bar{\sigma} \circ f = f \circ \sigma$, and either*

(4.7.1) $K_S^2 = 8\chi(\mathbb{O}_S)$ *and* $g(C) \leq q(S) \leq g(C) + 2$,

(4.7.2) $K_S^2 = 8\chi(\mathbb{O}_S) - 6$ *and* $g(C) \leq q(S) \leq g(C) + 1$, *or*

(4.7.3) $K_S^2 = 8\chi(\mathbb{O}_S) - 12$ *and* $q(S) = g(C)$.

Moreover, if S is as in (4.7.1), *then (S, σ) belongs to one of Examples 4.1–4.5.*

Proof. The first part of this theorem follows from [Cai 2006b, Theorem 1.1]. Now we let $f : S \to C, \sigma$ be as in (4.7.1). Let τ be the hyperelliptic involution of

$f: S \to C$, and $\lambda = \sigma \circ \tau$. We have a commutative diagram

$$\begin{tikzcd} S \arrow[dr, "f"'] & \tilde{S} \arrow[l, "\rho"'] \arrow[r, "\tilde{\pi}"] \arrow[d] & \tilde{S}/\tilde{\lambda} \arrow[r, "\eta"] \arrow[d] & T \arrow[dl, "h"] \\ & C \arrow[r, "\pi"] & B := C/\bar{\sigma} & \end{tikzcd}$$

where ρ is the blowup of all isolated fixed points of λ, $\tilde{\lambda}$ the induced involution on $\tilde{S}$, and η is the blowdown of all -1-curves contained in fibers of $\tilde{S}/\tilde{\lambda} \to B$. Then $p_g(T) = 0$, and $h: T \to B$ is a relatively minimal fibration of genus 2. The configurations of reducible fibers of h is as in Table 1 (see [Cai 2006b, 2.9]), where $q_f = q(S) - g(C)$, and $4(b_0)$, etc (column 5) means h having 4 reducible fibers of type (b_0) and no other reducible fibers.

	q_f	$g(B)$	$q(T)$	K_T^2	configurations of reducible singular fibers of h
1	0	0	0	0	$4(b_0)$
2	0	1	1	0	a nontrivially analytic fiber bundle
3	1	0	1	-4	$2(b_0)$
4	2	0	2	-8	a trivial fiber bundle

Table 1

Since f is a fiber bundle by [Xiao 1985, p. 18], h has constant moduli. Let F be a general fiber of h. There exists a finite group G acting on F and on some smooth curve $\tilde{B}$ such that h is birationally isomorphic to $(F \times \tilde{B})/G \to \tilde{B}/G$.

If h is as in line 4 of Table 1, then clearly (S, σ) is as in Example 4.1.

Case 1: h is as in line 3 of Table 1. In this case $g(F/G) = q(T) - g(\tilde{B}/G) = 1$. So $|G| = 2$ by the Hurwitz formula. Since $p_g(T) = 0$, we have $\tilde{B} \simeq \mathbb{P}^1$. So T is birationally isomorphic to $(F \times \tilde{B})/\langle \lambda_F \times \gamma_{\tilde{B}} \rangle$, where λ_F is a bielliptic involution of F, and $\gamma_{\tilde{B}}$ is an involution of $\tilde{B}$. We have a commutative diagram

$$\begin{tikzcd} \tilde{C} \arrow[r, "\mu"] \arrow[dr] & C \times_B \tilde{B} \arrow[r] \arrow[d] & \tilde{B} \arrow[d] \\ & C \arrow[r, "\pi"] & B \end{tikzcd}$$

where π is as in the beginning of the proof and μ is the normalization. Let $\lambda_{\tilde{C}}$ be the involution of $\tilde{C}$ corresponding to the double cover $\tilde{C} \to \tilde{B}$, and $\gamma_{\tilde{C}} \in \operatorname{Aut} \tilde{C}$ is the lift of $\gamma_{\tilde{B}}$. Since the image of reducible fibers of h is contained in the set of branch points of π, the branch points of $\tilde{B} \to B$ are contained in that of π. This implies $\tilde{C} \to C \simeq \tilde{C}/\langle \gamma_{\tilde{C}} \rangle$ is étale. We have a commutative diagram

$$\begin{array}{ccc} F \times (C \times_B \tilde{B}) \simeq C \times_B (F \times \tilde{B}) & \longrightarrow & F \times \tilde{B} \\ \downarrow & & \downarrow \\ C \times_B (F \times \tilde{B})/\langle \lambda_F \times \gamma \rangle & \longrightarrow & (F \times \tilde{B})/\langle \lambda_F \times \gamma \rangle \end{array}$$

Hence $S = (F \times \tilde{C})/\langle \lambda_F \times \gamma_{\tilde{C}} \rangle$ and $\sigma = (\overline{\tau_F \times \mathrm{id}_{\tilde{C}}})(\overline{\mathrm{id}_F \times \tau_{\tilde{C}}}) = \overline{\tau_F \times \tau_{\tilde{C}}}$. So (S, σ) is as in Example 4.2.

Case 2: h is as in line 2 of Table 1. In this case, $T \simeq (F \times \tilde{B})/G$, where F, $\tilde{B}$ and G are as in Example 4.3. (Since G is an abelian subgroup of $\mathrm{Aut}F$, we have $|G| \le 4g(F)+4 = 12$ ($\le 4g(F)+2 = 10$ if G is cyclic). Moreover, when $\tau_F \notin G$, since $\langle \tau_F, G\rangle$ is also abelian, we have $|G| = \frac{1}{2}|\langle \tau_F, G\rangle| \le 2g(F)+2 = 6$. Finally $G \not\simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4$ by the Riemann's existence theorem (see, for instance, [Broughton 1991, Proposition 2.1 or Theorem 4.1]). By the same argument as in Case 1, we get (S, σ) is as in Example 4.3.

Case 3: h is as in line 1 of Table 1. Let $B' \to B$ be the double cover branched at four points, which are the image of four singular fibers of type (b_0) of h. Let $T'' \to T \times_B B'$ be the normalization, and $h' : T' \to B'$ the relatively minimal fibration induced by contracting -1-curves contained in the fibers of $T'' \to B'$. Since h has only 4 reducible fibers of type $(b_0(I_0))$, $(b_0(I_1))$ or $(b_0(II))$ (see [Cai 2006b, Table 1]) and no other reducible fibers, by the construction, each singular fiber (if any) of h' is irreducible and reduced. Since h' has constant moduli, this implies h' is a fiber bundle. By [Cai 2006b, Lemma 2.5], $q(T') = 1$. So $(h' : T' \to B') \simeq ((F \times \tilde{B})/G \to \tilde{B}/G)$, where F, $\tilde{B}$ and G are as in Example 4.3. This implies h has only 4 reducible fibers of type $(b_0(I_0))$ and no other singular fibers. Hence, for any $z \in \tilde{B}$, the order of the stabilizer G_z of z in G is at most 2, and if G_z is not trivial for some $z \in \tilde{B}$, then $|G|/|G_z| \le 4$ and the generator of G_z acts on F as a bielliptic involution. So $|G| = 4$ or 8. If $|G| = 4$, then $G \simeq \mathbb{Z}_2^2$ and T is birationally isomorphic to $(F \times \tilde{B})/\langle \tau_F \times t_e,\ \lambda_F \times (-1_{\tilde{B}})\rangle$, where $e \in \tilde{B}$ is a nontrivial 2-torsion point, and λ_F is the involution of F corresponding to the generator of G_z. If $|G| = 8$, then $G \simeq \mathbb{Z}_8$, Q_8 or D_8 by [Broughton 1991, Theorem 4.1]. Since $G \hookrightarrow \mathrm{Aut}\tilde{B}$, $G \simeq G_1 \rtimes G_2$ (a semidirect product), where G_1 is a group of translations and $G_2 \subset \mathrm{Aut}\tilde{B}$ is a subgroup preserving the group structure. Since $\tilde{B}/G = \mathbb{P}^1$, $G_2 \ne 0$, thus $G_2 \simeq \mathbb{Z}_m$ ($m = 2$ or 4). This implies $G \not\simeq \mathbb{Z}_8$ or Q_8. Hence $G \simeq \langle t_e,\ -1_{\tilde{B}}\rangle \simeq D_8$, where $e \in \tilde{B}$ is a nontrivial 4-torsion point. Now by the similar argument as in Case 1, we get (S, σ) is as in Examples 4.4 and 4.5. □

Remark 4.8. Let S be a surface isogenous to a product of curves of genus at least 2 (see [Catanese 2000; 2003] for properties of these surfaces), and $G \subset \mathrm{Aut}\, S$ be a *nontrivial* subgroup of automorphisms of S, inducing trivial actions on $H^2(S, \mathbb{Q})$.

It is interesting to classify pairs (S, G). Note that fiber surfaces of genus 2 with $K_S^2 = 8\chi(\mathbb{O}_S)$ are isogenous to products of curves. Theorem 4.7 gives a classification for such pairs under the condition that one curve of the products has genus 2.

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References

[Atiyah 1958] M. F. Atiyah, "On analytic surfaces with double points", *Proc. Roy. Soc. London. Ser. A* **247** (1958), 237–244. MR 20 #2472 Zbl 0135.21301

[Atiyah and Singer 1968] M. F. Atiyah and I. M. Singer, "The index of elliptic operators. III", *Ann. of Math.* (2) **87** (1968), 546–604. MR 38 #5245 Zbl 0164.24301

[Barth et al. 1984] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik **4**, Springer, Berlin, 1984. MR 86c:32026 Zbl 0718.14023

[Beauville 1983] A. Beauville, *Complex algebraic surfaces*, London Mathematical Society Lecture Note Series **68**, Cambridge Univ. Press, Cambridge, 1983. MR 85a:14024 Zbl 0512.14020

[Broughton 1991] S. A. Broughton, "Classifying finite group actions on surfaces of low genus", *J. Pure Appl. Algebra* **69**:3 (1991), 233–270. MR 92b:57021 Zbl 0722.57005

[Burns and Rapoport 1975] D. Burns, Jr. and M. Rapoport, "On the Torelli problem for kählerian $K-3$ surfaces", *Ann. Sci. École Norm. Sup.* (4) **8**:2 (1975), 235–273. MR 56 #5945 Zbl 0324.14008

[Cai 2001] J.-X. Cai, "On abelian automorphism groups of fibred surfaces of small genus", *Math. Proc. Cambridge Philos. Soc.* **130**:1 (2001), 161–174. MR 2002a:14049 Zbl 0974.14028

[Cai 2004] J.-X. Cai, "Automorphisms of a surface of general type acting trivially in cohomology", *Tohoku Math. J.* (2) **56**:3 (2004), 341–355. MR 2005f:14078 Zbl 1076.14052

[Cai 2006a] J.-X. Cai, "Automorphisms of fiber surfaces of genus 2, inducing the identity in cohomology", *Trans. Amer. Math. Soc.* **358**:3 (2006), 1187–1201. MR 2007b:14090 Zbl 1087.14029

[Cai 2006b] J.-X. Cai, "Automorphisms of fiber surfaces of genus 2, inducing the identity in cohomology. II", *Internat. J. Math.* **17**:2 (2006), 183–193. MR 2007b:14091 Zbl 05018011

[Catanese 2000] F. Catanese, "Fibred surfaces, varieties isogenous to a product and related moduli spaces", *Amer. J. Math.* **122**:1 (2000), 1–44. MR 2001i:14048 Zbl 0983.14013

[Catanese 2003] F. Catanese, "Moduli spaces of surfaces and real structures", *Ann. of Math.* (2) **158**:2 (2003), 577–592. MR 2005d:14053 Zbl 1042.14011

[Cornalba 1981] M. Cornalba, "Una osservazione sulla topologia dei rivestimenti ciclici di varietà algebriche", *Boll. Un. Math. Ital. A* (5) **18** (1981), 323–328. Zbl 0462.14007

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977. MR 57 #3116

[Iitaka 1971] S. Iitaka, "Deformations of compact complex surfaces, III", *J. Math. Soc. Japan* **23** (1971), 692–705. MR 44 #7598

[Persson 1978] U. Persson, "Double coverings and surfaces of general type", pp. 168–195 in *Algebraic geometry* (Tromsø, 1977), edited by L. D. Olson, Lecture Notes in Math. **687**, Springer, Berlin, 1978. MR 80h:14017 Zbl 0396.14003

[Peters 1979] C. A. M. Peters, "Holomorphic automorphisms of compact Kähler surfaces and their induced actions in cohomology", *Invent. Math.* **52**:2 (1979), 143–148. MR 80h:32050

[Serrano 1991] F. Serrano, "The Picard group of a quasi-bundle", *Manuscripta Math.* **73**:1 (1991), 63–82. MR 93b:14020

[Shioda 1999] T. Shioda, "Mordell-Weil lattices for higher genus fibration over a curve", pp. 359–373 in *New trends in algebraic geometry* (Warwick, 1996), edited by K. Hulek et al., London Math. Soc. Lecture Note Ser. **264**, Cambridge Univ. Press, Cambridge, 1999. MR 2000g:11053 Zbl 0947.14012

[Suwa 1969] T. Suwa, "On hyperelliptic surfaces", *J. Fac. Sci. Univ. Tokyo Sect. I* **16** (1969), 469–476. MR 42 #2035

[Ueno 1976] K. Ueno, "A remark on automorphisms of Enriques surfaces", *J. Fac. Sci. Univ. Tokyo Sect. I A Math.* **23**:1 (1976), 149–165. MR 53 #8071

[Xiao 1985] G. Xiao, *Surfaces fibrées en courbes de genre deux*, Lecture Notes in Mathematics **1137**, Springer, Berlin, 1985. MR 88a:14042

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