Pacific Journal of Mathematics

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Volume 232 No. 1 September 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 232, No. 1, 2007

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We prove that if $C \subset \mathbb{R}^N$ is of class C^2 and uniformly convex, the Cheeger set of C is unique. The Cheeger set of C is the set that minimizes, inside C, the ratio of perimeter over volume.

1. Introduction

For a nonempty open bounded subset Ω of $\mathbb{R}^N$, the *Cheeger constant* of Ω is the quantity

$$h_\Omega = \min_{K \subseteq \Omega} \frac{P(K)}{|K|}. \tag{1}$$

Here $|K|$ denotes the N-dimensional volume of K and $P(K)$ denotes the perimeter of K. The minimum in (1) is taken over all nonempty sets of finite perimeter contained in Ω. A *Cheeger set* of Ω is any set $G \subseteq \Omega$ which minimizes (1). If Ω minimizes (1), we say that it is Cheeger in itself. We observe that the minimum in (1) is attained at a subset G of Ω such that ∂G intersects $\partial\Omega$: otherwise we could diminish the quotient $P(G)/|G|$ by dilating G.

For any set K of finite perimeter in $\mathbb{R}^N$, define

$$\lambda_K := \frac{P(K)}{|K|}.$$

Thus $\lambda_G = h_G$ for any Cheeger set G of Ω. Moreover, G is a Cheeger set of Ω if and only if G minimizes

$$\min_{K \subseteq \Omega} P(K) - \lambda_G |K|. \tag{2}$$

We say that a set $\Omega \subset \mathbb{R}^N$ is *calibrable* if Ω minimizes the problem

$$\min_{K \subseteq \Omega} P(K) - \lambda_\Omega |K|. \tag{3}$$

MSC2000: primary 35J70; secondary 49J40, 52A20.
Keywords: convex bodies, variational problems.
Caselles was partially support by the Departament d'Universitats, Recerca i Societat de la Informació de la Generalitat de Catalunya and by PNPGC project, reference BFM2003-02125.

Any Cheeger set G of Ω is clearly calibrable. Thus, Ω is a Cheeger set of itself if and only if it is calibrable.

Finding the Cheeger sets of a given Ω is a difficult task. The task is simplified if Ω is a convex set and $N = 2$. In that case, the Cheeger set of Ω is unique and equals the set $\Omega^R \oplus B(0, R)$, where $\Omega^R := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > R\}$ is such that $|\Omega^R| = \pi R^2$ and $A \oplus B := \{a + b : a \in A,\ b \in B\}$, for $A, B \subset \mathbb{R}^2$ [Alter et al. 2005b; Kawohl and Lachand-Robert 2006]. In particular, in this case the Cheeger set is convex.

A convex set $\Omega \subseteq \mathbb{R}^2$ is Cheeger in itself if and only if $\operatorname{ess\,sup}_{x\in\partial\Omega}\kappa_\Omega(x) \le \lambda_\Omega$, where $\kappa_\Omega(x)$ denotes the curvature of $\partial\Omega$ at the point x. This has been proved in [Giusti 1978; Bellettini et al. 2002; Kawohl and Lachand-Robert 2006; Alter et al. 2005b; Kawohl and Novaga 2006], though it was stated in terms of calibrability in the second and fourth of these references. The proof in [Giusti 1978] had a complementary result: if Ω is Cheeger in itself then Ω is strictly calibrable, that is, for any set $K \subsetneq \Omega$, we have

$$0 = P(\Omega) - \lambda_\Omega|\Omega| < P(K) - \lambda_\Omega|K|.$$

(This implies that the gravity-less capillary problem with vertical contact angle at the boundary, given by

$$\begin{aligned} -\mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} &= \lambda_\Omega \quad \text{in } \Omega, \\ -\frac{Du}{\sqrt{1+|Du|^2}}\cdot \nu^\Omega &= 1 \qquad \text{in } \partial\Omega, \end{aligned} \tag{4}$$

has a solution. Indeed, the two problems are equivalent [Giusti 1978; Kawohl and Kutev 1995].)

Our purpose in this paper is to extend the preceding result to $\mathbb{R}^N$, that is, to prove the uniqueness and convexity of the Cheeger set contained in a convex set $\Omega \subset \mathbb{R}^N$. We have to assume, in addition, that Ω is uniformly convex and of class C^2. This regularity assumption is probably too strong, and its removal is the subject of current research [Alter and Caselles 2007]. The characterization of a convex set $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$ which is Cheeger in itself (also called calibrable) in terms of the mean curvature of its boundary was proved in [Alter et al. 2005a]. The precise result states that such a set Ω is Cheeger in itself if and only if $\kappa_\Omega(x) \le \lambda_\Omega$ for almost any $x \in \partial\Omega$, where $\kappa_\Omega(x)$ denotes the sum of the principal curvatures of the boundary of Ω, which is to say, $N-1$ times the mean curvature of $\partial\Omega$ at x. In [Alter et al. 2005a] it was also proved that for any convex set $\Omega \subset \mathbb{R}^N$ there exists a maximal Cheeger set contained in Ω which is convex. These results were extended to convex sets Ω satisfying a regularity condition and anisotropic norms in $\mathbb{R}^N$ (including the crystalline case) in [Caselles et al. 2005].

In particular, we obtain that $\Omega \subset \mathbb{R}^N$ is the unique Cheeger set of itself, whenever Ω is a C^2, uniformly convex calibrable set. We point out that, by Theorems 1.1 and 4.2 in [Giusti 1978], this uniqueness result is equivalent to the existence of a solution $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ of the capillary problem (4).

In Section 2 we collect some definitions and recall results about the mean curvature operator in (4) and the subdifferential of the total variation. In Section 3 we state and prove our uniqueness result.

2. Preliminaries

2.1. ***BV functions.*** Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du on Ω turns out to be

$$\text{(5)} \quad \sup\left\{\int_\Omega u \operatorname{div} z \, dx : z \in C_0^\infty(\Omega; \mathbb{R}^N), \ \|z\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega}|z(x)| \le 1\right\},$$

(where for a vector $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. The map $u \mapsto |Du|(\Omega)$ is $L^1_{\text{loc}}(\Omega)$-lower semicontinuous. $BV(\Omega)$ is a Banach space when endowed with the norm $\int_\Omega |u| \, dx + |Du|(\Omega)$. We recall that $BV(\mathbb{R}^N) \subseteq L^{N/(N-1)}(\mathbb{R}^N)$.

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in $\mathbb{R}^N$ if (5) is finite when we substitute for u the characteristic function χ_E of E and $\Omega = \mathbb{R}^N$. The perimeter of E is defined as $P(E) := |D\chi_E|(\mathbb{R}^N)$. For more information on functions of bounded variation we refer to [Ambrosio et al. 2000].

Finally, we denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. We recall that when E is a finite-perimeter set with regular boundary (for instance, Lipschitz), its perimeter $P(E)$ also coincides with the more standard definition $\mathcal{H}^{N-1}(\partial E)$.

2.2. ***A generalized Green's formula.*** Let Ω be an open subset of $\mathbb{R}^N$. Following [Anzellotti 1983a], let

$$X_2(\Omega) := \{z \in L^\infty(\Omega; \mathbb{R}^N) : \operatorname{div} z \in L^2(\Omega)\}.$$

If $z \in X_2(\Omega)$ and $w \in L^2(\Omega) \cap BV(\Omega)$ we define the functional

$$(z \cdot Dw) : C_0^\infty(\Omega) \to \mathbb{R}$$

by the formula

$$\langle (z \cdot Dw), \varphi \rangle := -\int_\Omega w \, \varphi \operatorname{div} z \, dx - \int_\Omega w \, z \cdot \nabla\varphi \, dx.$$

Then $(z \cdot Dw)$ is a Radon measure in Ω,

$$\int_\Omega (z \cdot Dw) = \int_\Omega z \cdot \nabla w \, dx \quad \text{for } w \in L^2(\Omega) \cap W^{1,1}(\Omega).$$

Recall that the outer unit normal to a point $x \in \partial\Omega$ is denoted by $\nu^\Omega(x)$. We recall the following result proved in [Anzellotti 1983a].

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $z \in X_2(\Omega)$. Then there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ satisfying $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and such that for any $u \in BV(\Omega) \cap L^2(\Omega)$ we have*

$$\int_\Omega u \operatorname{div} z \, dx + \int_\Omega (z \cdot Du) = \int_{\partial\Omega} [z \cdot \nu^\Omega] u \, d\mathcal{H}^{N-1}.$$

Moreover, if $\varphi \in C^1(\overline{\Omega})$ then $[(\varphi z) \cdot \nu^\Omega] = \varphi[z \cdot \nu^\Omega]$.

This result is complemented with the following.

Theorem 2 [Anzellotti 1983b]. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a boundary of class C^1. Let $z \in C(\overline{\Omega}; \mathbb{R}^N)$ with $\operatorname{div} z \in L^2(\Omega)$. Then*

$$[z \cdot \nu^\Omega](x) = z(x) \cdot \nu^\Omega(x) \qquad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

2.3. ***Some auxiliary results.*** Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and let $\varphi \in L^1(\Omega)$. For all $\epsilon > 0$, we let $\Psi_\varphi^\varepsilon : L^2(\Omega) \to (-\infty, +\infty]$ be the functional defined by

$$\Psi_\varphi^\epsilon(u) := \begin{cases} \displaystyle\int_\Omega \sqrt{\epsilon^2 + |Du|^2} + \int_{\partial\Omega} |u - \varphi| & \text{if } u \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{6}$$

As it is proved in [Giusti 1976], if $f \in W^{1,\infty}(\Omega)$, then the minimum $u \in BV(\Omega)$ of the functional

$$\Psi_\varphi^\epsilon(u) + \int_\Omega |u(x) - f(x)|^2 \, dx \tag{7}$$

belongs to $u \in C^{2+\alpha}(\Omega)$, for every $\alpha < 1$. The minimum u of (7) is a solution of

$$\begin{cases} u - \dfrac{1}{\lambda} \operatorname{div} \dfrac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} = f(x) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where the boundary condition is taken in a generalized sense [Lichnewsky and Temam 1978], i.e.,

$$\left[\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \cdot \nu^\Omega \right] \in \operatorname{sign}(\varphi - u) \qquad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

Observe that (8) can be written as

$$u + \frac{1}{\lambda}\partial\Psi_{\varphi}^{\epsilon}(u) \ni f. \tag{9}$$

We are particularly interested in the case where $\varphi = 0$. As we shall show below (see also [Alter et al. 2005a]) in the case of interest to us we have $u > 0$ on $\partial\Omega$ and thus,

$$\left[\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \cdot \nu^{\Omega}\right] = -1 \qquad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

It follows that u is a solution of the first equation in (8) with vertical contact angle at the boundary.

As $\epsilon \to 0^+$, the solution of (8) converges to the solution of

$$\begin{cases} u + \frac{1}{\lambda}\partial\Psi_{\varphi}(u) = f(x) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{10}$$

where $\Psi : L^2(\Omega) \to (-\infty, +\infty]$ is given by

$$\Psi_{\varphi}(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} |Du| + \int_{\partial\Omega} |u - \varphi| & \text{if } u \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{11}$$

In this case $\partial\Psi_{\varphi}$ represents the operator $-\operatorname{div} \dfrac{Du}{|Du|}$ with the boundary condition $u = \varphi$ in $\partial\Omega$, as shown by:

Lemma 2.1 [Andreu et al. 2001]. *The following assertions are equivalent*:

(a) $v \in \partial\Psi_{\varphi}(u)$.

(b) $u \in L^2(\Omega) \cap BV(\Omega)$, $v \in L^2(\Omega)$, *and there exists* $z \in X_2(\Omega)$ *with* $\|z\|_{\infty} \le 1$, *such that* $v = -\operatorname{div} z$ *in* $\mathcal{D}'(\Omega)$, $z \cdot Du = |Du|$, *and*

$$[z \cdot \nu^{\Omega}] \in \operatorname{sign}(\varphi - u) \qquad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

Notice that the solution $u \in L^2(\Omega)$ of (10) minimizes the problem

$$\min_{u \in BV(\Omega)} \int_{\Omega} |Du| + \int_{\partial\Omega} |u(x) - \varphi(x)|\, d\mathcal{H}^{N-1}(x) + \frac{\lambda}{2}\int_{\Omega} |u(x) - f(x)|^2\, dx, \tag{12}$$

and the two problems are equivalent.

3. The uniqueness theorem

We now state our main result.

Theorem 3. *Let C be a convex body in $\mathbb{R}^N$. Assume that C is uniformly convex, with boundary of class C^2. Then the Cheeger set of C is convex and unique.*

We do not believe that the regularity and the uniform convexity of C is essential for this result (see [Alter and Caselles 2007]).

Theorem 4 [Alter et al. 2005a, Theorems 6 and 8, Proposition 4]. *Let C be a convex body in $\mathbb{R}^N$ with boundary of class $C^{1,1}$. For any $\lambda, \varepsilon > 0$, there is a unique solution u_ε of the equation*

$$\begin{cases} u_\varepsilon - \dfrac{1}{\lambda}\operatorname{div}\dfrac{Du_\varepsilon}{\sqrt{\varepsilon^2+|Du_\varepsilon|^2}} = 1 & \text{in } C, \\ u_\varepsilon = 0 & \text{on } \partial C \end{cases} \tag{13}$$

such that $0 \le u_\varepsilon \le 1$. Moreover, there exist λ_0 and ε_0, depending only on ∂C, such that if $\lambda \ge \lambda_0$ and $\varepsilon \le \varepsilon_0$, then u_ε is a concave function such that $u_\varepsilon \ge \alpha > 0$ on ∂C for some $\alpha > 0$. Hence, u_ε satisfies

$$\left[\frac{Du^\epsilon}{\sqrt{\epsilon^2+|Du^\epsilon|^2}} \cdot \nu^C\right] = \operatorname{sign}(0-u^\epsilon) = -1 \quad \text{on } \partial C. \tag{14}$$

As $\varepsilon \to 0$, the functions u_ε converge to the concave function u minimizing the problem

$$\min_{u\in BV(C)} \int_C |Du| + \int_{\partial C} |u(x)|\, d\mathcal{H}^{N-1}(x) + \frac{\lambda}{2}\int_C |u(x)-1|^2\, dx; \tag{15}$$

equivalently, if u is extended with zero out of C, the extension minimizes

$$\int_{\mathbb{R}^N} |Du| + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u-\chi_C|^2\, dx.$$

The function u satisfies $0 \le u < 1$. The superlevel set $\{u \ge t\}$, for $t \in (0,1]$, is contained in C and minimizes the problem

$$\min_{F\subset C} P(F) - \lambda(1-t)|F|. \tag{16}$$

It was proved in [Alter et al. 2005a] (see also [Caselles et al. 2005]) that the set $C^* = \{u = \max_C u\}$ is the maximal Cheeger set contained in C, that is, the maximal set that solves (1). Moreover, one has $u = 1 - h_C/\lambda > 0$ in C^* and $h_C = \lambda_{C^*}$.

If we want to consider what happens inside C^*, and in particular whether there are other Cheeger sets, we have to analyze the level sets of u_ε before passing to the limit as $\epsilon \to 0^+$. To do this, we introduce the following rescaling of u_ε:

$$v_\varepsilon = \frac{u_\varepsilon - m_\varepsilon}{\varepsilon} \le 0,$$

where $m_\varepsilon = \max_C u_\varepsilon \to 1 - h_C/\lambda$ as $\varepsilon \to 0$. The function v_ε is a generalized solution of the equation:

$$\begin{cases} \varepsilon v_\varepsilon - \dfrac{1}{\lambda}\operatorname{div} \dfrac{Dv_\varepsilon}{\sqrt{1+|Dv_\varepsilon|^2}} = 1 - m_\varepsilon & \text{in } C, \\ v_\varepsilon = -\dfrac{m_\varepsilon}{\varepsilon} & \text{on } \partial C. \end{cases} \tag{17}$$

We define the vector field

$$z_\varepsilon = Du_\varepsilon/\sqrt{\varepsilon^2 + |Du_\varepsilon|^2} = Dv_\varepsilon/\sqrt{1+|Dv_\varepsilon|^2};$$

it lies in $L^\infty(C)$, has uniformly bounded divergence, and satisfies $|z_\varepsilon| \le 1$ a.e. in C and, by (14), $[z_\varepsilon \cdot \nu_C] = -1$ on ∂C.

We now study the limit of v_ε and z_ε as $\varepsilon \to 0$. By the concavity of v_ε, for each $\varepsilon > 0$ small enough and each $s \in (0, |C|)$, there exists a (convex) superlevel set C_s^ε of v_ε such that $|C_s^\varepsilon| = s$. Moreover, $\{v_\varepsilon = 0\}$ is a null set: otherwise, since v_ε is concave, it would be a convex set of positive measure, hence with nonempty interior. We would then have $v_\varepsilon = \operatorname{div} z_\varepsilon = 0$, hence $1 - m_\varepsilon = 0$ in the interior of $\{v_\varepsilon = 0\}$. This is a contradiction with Theorem 4 for $\varepsilon > 0$ small enough.

Hence we may take $C_0^\varepsilon := \{v_\varepsilon = 0\}$ and $C_{|C|}^\varepsilon := C$. The boundaries $\partial C_s^\varepsilon \cap C$ define a foliation in C, in the sense that for all $x \in C$, there exists a unique value of $s \in [0, |C|]$ such that $x \in \partial C_s^\varepsilon$.

A sequence of uniformly bounded convex sets is compact both for the L^1 and Hausdorff topologies. Hence, up to a subsequence, we may assume that the C_s^ε converge to convex sets C_s, each of volume s, first for any $s \in \mathbb{Q} \cap (0, |C|)$ and then by continuity for any s. Possibly extracting a further subsequence, we may assume that there exists $s_* \in [0, |C|]$ such that v_ε goes to a concave function v in C_s for any $s < s_*$, and to $-\infty$ outside $C_* := C_{s_*}$. We may also assume that $z_\varepsilon \rightharpoonup z$ weakly* in $L^\infty(C)$, for some vector field z satisfying $|z| \le 1$ a.e. in C. From (13) we have in the limit

$$-\operatorname{div} z = \lambda(1 - u) \qquad \text{in } \mathcal{D}'(C). \tag{18}$$

Moreover, $-\operatorname{div} z \in \partial\Psi_0(u)$ by the results recalled in Section 2. We then see from (18) that

$$-\operatorname{div} z = h_C \qquad \text{in } C^*, \tag{19}$$

while $-\operatorname{div} z > h_C$ a.e. on $C \setminus C^*$.

Set $s^* := |C^*|$, so $C^* = C_{s^*}$. By Theorem 4, for $s \ge s^*$, the set C_s is a minimizer of the variational problem

$$\min_{E \subseteq C} P(E) - \mu_s |E|, \tag{20}$$

for some $\mu_s \geq h_C$ (μ_s is equal to the constant value of $-\mathrm{div}\, z = \lambda(1-u)$ on $\partial C_s \cap C$; see (16)). Notice that μ_s is bounded from above by $P(C)/(|C|-s)$: indeed,

$$-\int_{C\setminus C_s^\varepsilon} \mathrm{div}\, z_\varepsilon(x)\, dx = \mathcal{H}^{N-1}(\partial C \setminus \partial C_s^\varepsilon) - \int_{\partial C_s^\varepsilon \cap C} \frac{|Du_\varepsilon|}{\sqrt{1+|Du_\varepsilon|^2}} \;\leq\; P(C)$$

for $\varepsilon > 0$, since the inner normal to C_s^ε at $x \in \partial C_s^\varepsilon \cap C$ is $Du_\varepsilon(x)/|Du_\varepsilon(x)|$. On the other hand,

$$-\int_{C\setminus C_s^\varepsilon} \mathrm{div}\, z_\varepsilon(x)\, dx = \int_{C\setminus C_s^\varepsilon} \lambda(1-u_\varepsilon(x))\, dx \geq \mu_s^\varepsilon(|C|-s),$$

where μ_s^ε is the constant value of $\lambda(1-u_\varepsilon)$ on the level set $\partial C_s^\varepsilon \cap C$, and goes to μ_s as $\varepsilon \to 0$. A more careful analysis would show, in fact, that

$$\mu_s \leq \frac{P(C)-P(C_s)}{|C|-s}.$$

For $s > s^*$, we have $\mu_s > h_C$ and the set C_s is the unique minimizer of the variational problem (20). As a consequence (see [Alter et al. 2005a; Caselles et al. 2005]) for any $s > s^*$ the set C_s is also the unique minimizer of $P(E)$ among all $E \subseteq C$ of volume s.

Lemma 3.1. *We have $s_* > 0$ and the sets C_s are Cheeger sets in C for any $s \in [s_*, s^*]$.*

Proof. Let $s_* < s \leq |C|$. If $x \in \partial C_s^\epsilon \setminus \partial C$, then

$$0 - v_\varepsilon(x) \;\leq\; Dv_\varepsilon(x)\cdot(\bar{x}_\varepsilon - x)$$

where $v_\varepsilon(\bar{x}_\varepsilon) = \max_C v_\varepsilon$. Hence, $\lim_{\varepsilon\to 0} \inf_{\partial C_s^\varepsilon \setminus \partial C} |Dv_\varepsilon| = +\infty$. Since $[z_\varepsilon \cdot \nu^C] = -1$ on ∂C and $P(C_s^\varepsilon) \to P(C_s)$, we deduce

$$\begin{aligned}
-\int_{\partial C_s^\varepsilon} &[z_\varepsilon(x)\cdot \nu^{C_s^\varepsilon}(x)]\, d\mathcal{H}^{N-1}(x) \\
&= \int_{\partial C_s^\varepsilon \setminus \partial C} \frac{|Dv_\varepsilon(x)|}{\sqrt{1+|Dv_\varepsilon(x)|^2}}\, d\mathcal{H}^{N-1}(x) \;+\; \mathcal{H}^{N-1}(\partial C_s^\varepsilon \cap \partial C) \;\to\; P(C_s)
\end{aligned}$$

as $\varepsilon \to 0^+$. Hence,

$$\begin{aligned}
\int_{\partial C_s} \left[z\cdot \nu^{C_s}\right] d\mathcal{H}^{N-1} &= \int_{C_s} \mathrm{div}\, z = \lim_{\varepsilon\to 0} \int_{C_s^\varepsilon} \mathrm{div}\, z_\varepsilon \\
&= \lim_{\varepsilon \to 0} \int_{\partial C_s^\varepsilon} [z_\varepsilon \cdot \nu_{C_s^\varepsilon}]\, d\mathcal{H}^{N-1} = -\,P(C_s).
\end{aligned}$$

Since $|z| \le 1$ a.e. in C, we deduce that $[z \cdot \nu^{C_s}] = -1$ on ∂C_s for any $s > s_*$ (in particular, $|z| = 1$ a.e. in $C \setminus C_*$). Using this and (19), we have for all $s_* < s \le s^*$

$$\frac{P(C_s)}{|C_s|} = h_C. \tag{21}$$

This has two consequences. First, from the isoperimetric inequality, we obtain

$$h_C = \frac{P(C_s)}{|C_s|} \ge \frac{P(B_1)}{\big(|B_1|^{N-1} s\big)^{1/N}},$$

if $s \in (s_*, s^*]$, so that $s_* > 0$. Moreover, C_s is a Cheeger set for any $s \in (s_*, s^*]$, and by continuity C_* is also a Cheeger set. □

Since the sets C_s are convex minimizers of $P(E) - \mu_s |E|$ among all $E \subseteq C$, for $s \ge s_*$, their boundary is of class $C^{1,1}$ [Brézis and Kinderlehrer 1974; Stredulinsky and Ziemer 1997], with curvature at most μ_s, and equal to μ_s in the interior of C (note that $\mu_s = h_C$ for $s \in [s_*, s^*]$).

Remark 3.2. Either $s_* = s^*$, and so $C_* = C^*$, or $s_* < s^*$, and so $C^* = \bigcup_{s \in (s_*, s^*)} C_s$. In the latter case, the supremum of the sum κ_{C^*} of the principal curvatures on ∂C^* is equal to h_C. Indeed, if this were not the case, by considering $C' \subset \mathrm{int}(C^*)$ with curvature strictly below h_C, together with the smallest set C_s with $s > s_*$ containing C', we would get $\kappa_{C'}(x) \ge \kappa_{C_s}(x) = h_C$ at all $x \in \partial C' \cap \partial C_s$, a contradiction. In particular, $C = C_*$ if the supremum of κ_C on ∂C is strictly less than $P(C)/|C|$; this condition also implies $C = C^*$ by [Alter et al. 2005a].

From the strong convergence of Dv_ε to Dv (in $L^2(C_s)$ for any $s < s_*$), we deduce that $z = Dv/\sqrt{1 + |Dv|^2}$ in C_*. It follows that v satisfies the equation

$$-\operatorname{div} \frac{Dv}{\sqrt{1 + |Dv|^2}} = h_C \quad \text{in } C_*. \tag{22}$$

Integrating both sides of (22) in C_*, we deduce that

$$\left[\frac{Dv}{\sqrt{1 + |Dv|^2}} \cdot \nu^{C_*} \right] = -1 \quad \text{on } \partial C_*.$$

Lemma 3.3. *The set C_* is the* ***minimal*** *Cheeger set of C; that is, any Cheeger set of C must contain C_*.*

Proof. Let $K \subseteq C^*$ be a Cheeger set in C. We have

$$h_C |K| = -\int_K \operatorname{div} z = -\int_{\partial K} [z \cdot \nu^K] \, d\mathcal{H}^{N-1} = P(K),$$

so $[z \cdot \nu^K] = -1$ a.e. on ∂K. Let ν^ϵ and ν be the vector fields of unit normals to the sets C_s^ϵ and C_s, $s \in [0, |C|]$, respectively. By the Hausdorff convergence of C_s^ϵ

to C_s as $\epsilon \to 0^+$ for any $s \in [0, |C|]$, we have $\nu^\epsilon \to \nu$ a.e. in C. On the other hand, $|z_\epsilon + \nu^\epsilon| \to 0$ locally uniformly in $C \setminus \bar{C}_*$: indeed, in C,

$$|z_\epsilon + \nu^\epsilon| = \left| \frac{Dv_\varepsilon}{\sqrt{1+|Dv_\varepsilon|^2}} - \frac{Dv_\varepsilon}{|Dv_\varepsilon|} \right| = \left| \frac{|Dv_\varepsilon|}{\sqrt{1+|Dv_\varepsilon|^2}} - 1 \right|.$$

Since $|Dv_\epsilon| \to \infty$ uniformly in any subset of C at positive distance from C_* (see the first lines of the proof of Lemma 3.1), this shows the uniform convergence of $|z_\epsilon + \nu^\epsilon|$ to 0 in such subsets.

These two facts imply that $z = -\nu$ a.e. on $C \setminus C_*$. By modifying z in a set of null measure, we may assume that $z = -\nu$ on $C \setminus C_*$. We recall that the sets C_s, $s \geq s_*$ are minimizers of variational problems of the form $\min_{K \subseteq C} P(K) - \mu|K|$, for some values of μ (with $\mu = h_C$ as long as $s \leq s^*$ and $\mu = \mu_s > h_C$ continuously increasing with $s > s^*$). Since these sets are convex, with boundary (locally) uniformly of class $C^{1,1}$, and the map $s \to C_s$ is continuous in the Hausdorff topology, we conclude that the normal $\nu(x)$ is a continuous function in $C \setminus \text{int}(C_*)$.

Since $|z| < 1$ inside C_* and $[z \cdot \nu^K] = -1$ a.e. on ∂K, by [Anzellotti 1983a, Theorem 1]) we have that the boundary of K must be outside the interior of C_*, hence either $K \supseteq C_*$ or $K \cap C_* = \varnothing$ (modulo a null set). Let us prove that the last situation is impossible. Indeed, assume that $K \cap C_* = \varnothing$ (modulo a null set). Since ∂K is of class C^1 out of a closed set of zero $\mathcal{H}^{N-1}$-measure (see [Gonzalez et al. 1983]) and z is continuous in $C \setminus \text{int}(C_*)$, by Theorem 2 we have

$$z(x) \cdot \nu^K(x) = -1 \qquad \mathcal{H}^{N-1}\text{-a.e. on } \partial K. \tag{23}$$

Now, since $K \cap C_* = \varnothing$ (modulo a null set), then there is some $s \geq s_*$ and some $x \in \partial C_s \cap \partial K$ such that $\nu^K(x) + \nu(x) = 0$. Fix $0 < \epsilon < 2$. By a slight perturbation, if necessary, we may assume that $x \in \partial C_s \cap \partial K$ with $s > s_*$, (23) holds at x and

$$|\nu^K(x) + \nu(x)| < \epsilon. \tag{24}$$

Since by (23) we have $\nu(x) = -z(x) = \nu^K(x)$ we obtain a contradiction with (24). We deduce that $K \supseteq C_*$. □

Therefore, in order to prove the uniqueness of the Cheeger set of C, it is enough to show that

$$C_* = C^*. \tag{25}$$

Recall that the boundary of both C_* and C^* is of class $C^{1,1}$, and the sum of its principal curvatures is less than or equal h_C, and constantly equal to h_C in the interior of C. We now show that if $C_* \neq C^*$ and under additional assumptions, the sum of the principal curvatures of the boundary of C^* (or of any C_s for $s \in (s_*, s^*]$) must be h_C out of C_*.

Lemma 3.4. *Assume that C has C^2 boundary. Let $s \in (s_*, s^*]$ and $x \in \partial C_s \setminus \partial C_*$. If the sum of the principal curvatures of ∂C_s at x is strictly below h_C, then the Gaussian curvature of ∂C at x is* 0.

Proof. Let $x \in \partial C_s \setminus \partial C_*$ and assume the sum of the principal curvatures of ∂C_s at x is strictly below h_C (assuming x is a Lebesgue point for the curvature on ∂C_s). Necessarily, this implies that $x \in \partial C$. Assume then that the Gauss curvature of ∂C at x is positive: by continuity, in a neighborhood of x, C is uniformly convex and the sum of the principal curvatures is less than h_C. We may assume that near x, ∂C is the graph of a nonnegative, C^2 and convex function $f : B \to \mathbb{R}$ where B is an $(N-1)$-dimensional ball centered at x. We may as well assume that ∂C_s is the graph of $f_s : B \to \mathbb{R}$, which is $C^{1,1}$ [Brézis and Kinderlehrer 1974; Stredulinsky and Ziemer 1997], and also nonnegative and convex. In B, we have $f_s \ge f \ge 0$, and

$$D^2 f \ \ge \ \alpha I \ \text{ and } \ \operatorname{div} \frac{Df}{\sqrt{1+|Df|^2}} = h$$

with $h \in C^0(\bar{B})$, $h < h_C$, $\alpha > 0$, while

$$\operatorname{div} \frac{Df_s}{\sqrt{1+|Df_s|^2}} = h\chi_{\{f=f_s\}} + h_C\chi_{\{f_s>f\}}$$

(where $\chi_{\{f=f_s\}}$ has positive density at x).

We let $g = f_s - f \ge 0$. Introducing the Lagrangian $\Psi : \mathbb{R}^{N-1} \to [0, +\infty)$ given by $\Psi(p) = \sqrt{1+|p|^2}$, we obtain, for a.e. $y \in B$,

$$\begin{aligned}(h_C - h(y))&\chi_{\{g>0\}}(y)\\ &= \operatorname{div}\big(D\Psi(Df_s(y)) - D\Psi(Df(y))\big)\\ &= \operatorname{div}\left(\left(\int_0^1 D^2\Psi(Df(y) + t(Df_s(y) - Df(y)))\,dt\right) Dg(y)\right),\end{aligned}$$

so that, letting $A(y) := \int_0^1 D^2\Psi(Df(y) + tDg(y))\,dt$ (which is a positive definite matrix and Lipschitz continuous inside B), we see that g is the minimizer of the functional

$$w \mapsto \int_B \big(A(y)Dw(y)\cdot Dw(y) + (h_C - h(y))w(y)\big)\,dy$$

under the constraint $w \ge 0$ and with boundary condition $w = f_s - f$ on ∂B. Adapting the results in [Caffarelli and Rivière 1976] we get that $\{f = f_s\} = \{g = 0\}$ is the closure of a nonempty open set with boundary of zero $\mathcal{H}^{N-1}$-measure.

We therefore have found an open subset $D \subset \partial C \cap \partial C_s$, disjoint from ∂C_*, on which C is uniformly convex, with curvature less than h_C. Let φ be a smooth, nonnegative function with compact support in D. One easily shows that if $\varepsilon > 0$ is small enough, $\partial C_s - \varepsilon\varphi\nu^{C_s}$ is the boundary of a set C'_ϵ which is still convex, with

$P(C'_\epsilon)/|C'_\epsilon| > P(C_s)/|C_s| = h_C$ (just differentiate the map $\epsilon \to P(C'_\epsilon)/|C'_\epsilon|$), and the sum of its principal curvatures is less than h_C. This implies that for $\epsilon > 0$ small enough, the set $C' := C'_\epsilon$ is calibrable [Alter et al. 2005a], which in turn implies that $\min_{K \subset C'} P(K)/|K| = P(C')/|C'|$. But this contradicts $C_* \subset C'$, which is true for ε small enough. □

Proof of Theorem 3. Assume that C is C^2 and uniformly convex. Let us prove that its Cheeger set is unique. Assume by contradiction that $C^* \neq C_*$. From Lemma 3.4 we have that the sum of the principal curvatures of ∂C^* is h_C outside of C_*.

Let now $\bar{x} \in \partial C^* \cap \partial C_*$ be such that $\partial C^* \cap B_\rho(\bar{x}) \neq \partial C_* \cap B_\rho(\bar{x})$ for all $\rho > 0$ ($\partial C^* \cap \partial C_* \neq \varnothing$ since otherwise both C^* and C_* would be balls, which is impossible). Letting T be the tangent hyperplane to ∂C^* at $\bar{x}$, we can write ∂C^* and ∂C_* as the graph of two positive convex functions v^* and v_*, respectively, over $T \cap B_\rho(\bar{x})$ for $\rho > 0$ small enough. Identifying $T \cap B_\rho(\bar{x})$ with $B_\rho \subset \mathbb{R}^{N-1}$, we have that $v_*, v^* : B_\rho \to \mathbb{R}$ both solve the equation

$$-\operatorname{div} \frac{Dv}{\sqrt{1+|Dv|^2}} = f, \tag{26}$$

for some function $f \in L^\infty(B_\rho)$. Moreover, it holds $v_* \geq v^*$, $v_*(0) = v^*(0)$ and $v_*(y) > v^*(y)$ for some $y \in B_\rho$. Notice that $f = \lambda_C$ in the (open) set where $v_* > v^*$, in particular both functions are smooth in this set. Let D be an open ball such that $\overline{D} \subset B_\rho$, $v_* > v^*$ on D and $v_*(y) = v^*(y)$ for some $y \in \partial D$. Notice that, since both v^* and v_* belong to $C^\infty(D) \cap C^1(\overline{D})$, the fact that $v_*(y) = v^*(y)$ also implies that $Dv_*(y) = Dv^*(y)$. In D, both functions solve (26) with $f = \lambda_C$. Letting $w = v_* - v^*$, we obtain $w(y) = 0$ and $Dw(y) = 0$, while $w > 0$ inside D. Recalling the function $\Psi(p) = \sqrt{1+|p|^2}$, we have, for any $x \in D$,

$$\begin{aligned} 0 &= \operatorname{div}\big(D\Psi(Dv_*(x)) - D\Psi(Dv^*(x))\big) \\ &= \operatorname{div}\left(\left(\int_0^1 D^2\Psi(Dv^*(x) + t(Dv_*(x) - Dv^*(x)))\,dt\right) Dw(x)\right), \end{aligned}$$

so that w solves a linear, uniformly elliptic equation with smooth coefficients. Then Hopf's lemma [Gilbarg and Trudinger 1983] implies that $Dw(y) \cdot \nu_D(y) < 0$, a contradiction. Hence $C_* = C^*$. □

Remark 3.5. As a consequence of Theorem 3 and the results of [Giusti 1978], if C is of class C^2 and uniformly convex, Equation (22) has a solution on the whole of C, if and only if C is a Cheeger set of itself, i.e., if and only if the sum of the principal curvatures of ∂C is less than or equal to $P(C)/|C|$.

Remark 3.6. The results of this paper can be easily extended to the anisotropic setting (see [Caselles et al. 2005]) provided the anisotropy is smooth and uniformly elliptic.

References

[Alter and Caselles 2007] F. Alter and V. Caselles, "Uniqueness of the Cheeger set of a convex body", preprint, 2007, Available at benasque.ecm.ub.es/2007pde/talks_contr/072cheeger8.pdf.

[Alter et al. 2005a] F. Alter, V. Caselles, and A. Chambolle, "A characterization of convex calibrable sets in $\mathbb{R}^N$", *Math. Ann.* **332**:2 (2005), 329–366. MR 2006g:35091 Zbl 02172739

[Alter et al. 2005b] F. Alter, V. Caselles, and A. Chambolle, "Evolution of characteristic functions of convex sets in the plane by the minimizing total variation flow", *Interfaces Free Bound.* **7**:1 (2005), 29–53. MR 2006b:35154 Zbl 1084.49003

[Ambrosio et al. 2000] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, New York, 2000. MR 2003a:49002 Zbl 0957.49001

[Andreu et al. 2001] F. Andreu, C. Ballester, V. Caselles, and J. Mazón, "The Dirichlet problem for the total variation flow", *J. Funct. Anal.* **180**:2 (2001), 347–403. MR 2001k:35143 Zbl 0973.35109

[Anzellotti 1983a] G. Anzellotti, "Pairings between measures and bounded functions and compensated compactness", *Ann. Mat. Pura Appl.* (4) **135** (1983), 293–318 (1984). MR 85m:46042 Zbl 0572.46023

[Anzellotti 1983b] G. Anzellotti, "Traces of bounded vector fields and the divergence theorem", preprint, Dipartimento di Matematica, Università di Trento, 1983.

[Bellettini et al. 2002] G. Bellettini, V. Caselles, and M. Novaga, "The total variation flow in $\mathbb{R}^N$", *J. Differential Equations* **184**:2 (2002), 475–525. MR 2003g:35105 Zbl 1036.35099

[Brézis and Kinderlehrer 1974] H. Brézis and D. Kinderlehrer, "The smoothness of solutions to nonlinear variational inequalities", *Indiana Univ. Math. J.* **23** (1974), 831–844. MR 50 #13881 Zbl 0278.49011

[Caffarelli and Rivière 1976] L. A. Caffarelli and N. M. Rivière, "On the rectifiability of domains with finite perimeter", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **3**:2 (1976), 177–186. MR 53 #14288 Zbl 0362.49031

[Caselles et al. 2005] V. Caselles, A. Chambolle, S. Moll, and M. Novaga, "A characterization of convex calibrable sets in $\mathbb{R}^N$ with respect to anisotropic norms", preprint, 2005.

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Math. Wissenschaften **224**, 1983. Corrected reprint, 1998. MR 2001k:35004 Zbl 0562.35001

[Giusti 1976] E. Giusti, "Boundary value problems for non-parametric surfaces of prescribed mean curvature", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **3**:3 (1976), 501–548. MR 58 #2572 Zbl 0344.35036

[Giusti 1978] E. Giusti, "On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions", *Invent. Math.* **46**:2 (1978), 111–137. MR 58 #7337 Zbl 0381.35035

[Gonzalez et al. 1983] E. Gonzalez, U. Massari, and I. Tamanini, "On the regularity of boundaries of sets minimizing perimeter with a volume constraint", *Indiana Univ. Math. J.* **32**:1 (1983), 25–37. MR 84d:49043 Zbl 0486.49024

[Kawohl and Kutev 1995] B. Kawohl and N. Kutev, "Global behaviour of solutions to a parabolic mean curvature equation", *Differential Integral Equations* **8**:8 (1995), 1923–1946. MR 96f:35088 Zbl 0844.35050

[Kawohl and Lachand-Robert 2006] B. Kawohl and T. Lachand-Robert, "Characterization of Cheeger sets for convex subsets of the plane", *Pacific J. Math.* **225**:1 (2006), 103–118. MR 2007e:52002 Zbl 05170768

[Kawohl and Novaga 2006] B. Kawohl and M. Novaga, "The p-Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric", preprint, 2006, Available at http://cvgmt.sns.it/papers/kawnov06/KawNovFeb27.pdf.

[Lichnewsky and Temam 1978] A. Lichnewsky and R. Temam, "Pseudosolutions of the time-dependent minimal surface problem", *J. Differential Equations* **30**:3 (1978), 340–364. MR 80a:35059 Zbl 0368.49016

[Stredulinsky and Ziemer 1997] E. Stredulinsky and W. P. Ziemer, "Area minimizing sets subject to a volume constraint in a convex set", *J. Geom. Anal.* **7**:4 (1997), 653–677. MR 99k:49089 Zbl 0940.49025

Received April 6, 2006. Revised July 21, 2006.

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