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**EXISTENCE OF INFINITELY MANY EQUILIBRIUM
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PRESCRIBING THE SAME NONCONSTANT BOUNDARY
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EXISTENCE OF INFINITELY MANY EQUILIBRIUM CONFIGURATIONS OF A LIQUID CRYSTAL SYSTEM PRESCRIBING THE SAME NONCONSTANT BOUNDARY VALUE

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In 1986, Hardt, Kinderlehrer and Lin established the existence and partial regularity of minimizers of the liquid crystal energy with the Oseen–Frank density. Motivated by the earlier results of Bethuel–Brezis–Coron and Riviere on harmonic maps, we prove the existence of infinitely many equilibrium configurations of the liquid crystal energy prescribing the same nonconstant boundary data.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary $\partial\Omega$, and let $\gamma : \partial\Omega \rightarrow S^2$ be smooth boundary data. The equilibrium configuration of a liquid crystal is described by a unit vector field u on Ω . For any map $u \in H_\gamma^1(\Omega, S^2)$ with $\gamma : \partial\Omega \rightarrow S^2$, the integral

$$\Phi(\gamma) = \frac{1}{2} \int_{\Omega} [\text{tr}(\nabla u)^2 - (\text{div } u)^2] dx$$

depends only on γ [Hardt et al. 1986]. According to the Ericksen–Leslie theory [Giaquinta et al. 1998], the Oseen–Frank bulk energy of a configuration $u \in H^1(\Omega, S^2)$ can be reduced to

$$(1-1) \quad E(u, \Omega) = \int_{\Omega} W(u, \nabla u) dx,$$

where $W(u, \nabla u)$ is the Oseen–Frank density

$$(1-2) \quad W(u, \nabla u) = \alpha |\nabla u|^2 + (k_1 - \alpha)(\text{div } u)^2 + (k_2 - \alpha)(u \cdot \text{curl } u)^2 + (k_3 - \alpha) |u \times \text{curl } u|^2,$$

with constants $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, and $\alpha = \min\{k_1, k_2, k_3\}$.

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For the above density $W(u, \nabla u)$, we set $V(u, p) = W(u, p) - \alpha |p|^2$ with $p = \nabla u$. As in [Hardt et al. 1986], the equilibrium system associated to E is of the form

$$(1-3) \quad -\operatorname{div} (W_p(u, \nabla u) - u \otimes u V_p(u, \nabla u)) + Y(u, \nabla u) = 0 \quad \text{in } \Omega,$$

where

$$Y(u, \nabla u) = (I - u \otimes u) W_u(u, \nabla u) - \nabla u (u V_p(u, \nabla u)) - (W_p(u, \nabla u) \cdot \nabla u) u$$

satisfies $|Y(u, p)| \leq C |p|^2$ for all $p = (p_i^j)_{3 \times 3}$ with $p_i^j \in \mathbb{R}$.

A static equilibrium configuration u corresponds to an extremal of the functional (1-1) in $H^1(\Omega, S^2)$, that is, $u \in H^1(\Omega, S^2)$ is a weak solution of system (1-3).

In a special case $k_1 = k_2 = k_3$, the equilibrium system (1-3) is

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } \Omega,$$

which is the equation for harmonic maps. When $k_1 = k_2 = k_3 = 1$, Bethuel et al. [1990] first proved the existence of infinitely many harmonic maps for some special boundary values γ . Rivière [1995] proved the existence of infinite many harmonic maps for all nonconstant boundary values. See further generalizations to higher dimensions in [Isobe 1995] and [Pakzad 2001].

In general, (1-3) is not always elliptic for every choice of the constants k_i , and so the system (1-3) is much more complicated than the harmonic map equation. Hardt et al. [1986], in a fundamental paper, proved the existence and partial regularity of a minimizer u , which is a weak solution of system (1-3) that gives the energy E in $H_\gamma^1(\Omega, S^2)$ given boundary data $\gamma : \partial\Omega \rightarrow S^2$. One questions whether one can prove there exist infinitely many weak solutions of the liquid crystal system (1-3) prescribing the same boundary data. Here, we prove the existence of infinitely many equilibrium configurations prescribing the same nonconstant boundary data γ in:

Theorem 1.1. *Let $\gamma : \partial\Omega \rightarrow S^2$ be a nonconstant smooth map. Assume that the constants k_1, k_2 and k_3 in (1-2) satisfy $|k_1 - k_2| \leq \min\{k_1, k_2\} 4(1 - \ln 2)/\ln 2$. Then there exist infinitely many stable weak solutions of system (1-3) in $H^1(\Omega, S^2)$ with the same boundary value γ .*

The key to proving Theorem 1.1 is generalizing the idea of Rivière [1995] to the liquid crystal energy. Rivière's idea relies on constructing dipoles and the relaxed energy of the Dirichlet energy in [Brezis et al. 1986] and [Bethuel et al. 1990]. More precisely, Rivière inserts a dipole into nonconstant maps and finds a way to confine the energy to strictly less than 8π times the length of the dipole.

In this paper, we extend a key result of Rivière to the liquid crystal:

Theorem 1.2. *Assume that the constants k_1, k_2 and k_3 in (1–2) satisfy the condition $|k_1 - k_2| \leq \min\{k_1, k_2\}4(1 - \ln 2)/\ln 2$. Let $u \in H^1(\Omega, S^2)$ be a nonconstant and smooth map, and let x_0 be a point inside Ω such that $\nabla u(x_0) \neq 0$. For any $\rho > 0$ with $B_\rho(x_0) \subset \Omega$, there exist two points $P, N \in B_\rho(x_0)$ with middle point x_0 and $|P - N| = 2\sigma \leq 2\rho$ and a map v in $H^1(\Omega, S^2) \cap C^{0,1}(\Omega \setminus \{P, N\}, S^2)$ such that*

$$v = u \quad \text{in } \Omega \setminus B_\rho(x_0), \quad \deg(v, P) = -\deg(v, N) = +1,$$

and

$$\int_{\Omega} W(v(x), \nabla v(x)) \, dx < \int_{\Omega} W(u(x), \nabla u(x)) \, dx + 8\pi\Gamma(k_1, k_2, k_3) |P - N|.$$

For the proof of Theorem 1.2, thanks to Giaquinta et al. [1990], the dipoles and relaxed functional of the liquid crystal energy E was given: For any $u \in H^1(\Omega, S^2)$, the vector field $D(u)$ is defined by

$$D(u) = (u \cdot u_{x_2} \wedge u_{x_3}, u \cdot u_{x_3} \wedge u_{x_1}, u \cdot u_{x_1} \wedge u_{x_2}).$$

Given $u^0 \in H^1_\gamma(\Omega, S^2)$, we set

$$L(u, u^0) := \frac{1}{4\pi} \sup_{\substack{\xi: \Omega \rightarrow \mathbb{R}, \\ \|\nabla \xi\|_{L^\infty} \leq 1}} \int_{\Omega} [D(u) - D(u^0)] \cdot \nabla \xi \, dx$$

for maps $u \in H^1_\gamma(\Omega, S^2)$. The relaxed functional F_{u^0} of the liquid crystal energy E is given by

$$F_{u^0}(u) = \int_{\Omega} W(u(x), \nabla u(x)) \, dx + 8\pi\Gamma(k_1, k_2, k_3)L(u, u^0),$$

where

$$\Gamma(k_1, k_2, k_3) = \sqrt{kk_3} \int_0^1 \sqrt{1 + (k/k_3 - 1)s^2} \, ds \geq \alpha, \quad k = \min\{k_1, k_2\}.$$

More precisely, $F_{u^0}(u)$ is lower semicontinuous in the weak H^1 -topology and any minimizer of $F_{u^0}(u)$ in H^1_γ is also a weak solution of system (1–3) prescribing the boundary value γ . One key to proving Theorem 1.2 is obtaining new estimates on the irrotational and solenoidal dipole in [Giaquinta et al. 1990] for inserting a small dipole into nonconstant map. More precisely, for $k_2 \geq k_1$, one finds the irrotational map u (with $u \cdot \text{curl } u = 0$) on $\mathbb{R}^2 \times (0, l)$ of the form

$$u(x) = \left(g(r) \frac{x_1}{r}, g(r) \frac{x_2}{r}, \text{sign}(1 - r) \sqrt{1 - g^2(r)} \right), \quad r = \sqrt{|x_1|^2 + |x_2|^2}$$

such that $E(u) = 8\pi l \Gamma(k_1, k_2, k_3)$, where $g \in C([0, \infty); [0, 1])$, $g'(r) > 0$ on $(0, 1)$, and $g'(r) < 0$ on $(1, \infty)$. Similarly, for $k_2 \leq k_1$, one finds the solenoidal map (with

$\operatorname{div} u = 0$) on $\mathbb{R}^2 \times (0, l)$ of the form

$$u(x) = \left(g(r) \frac{x_2}{r}, -g(r) \frac{x_1}{r}, \operatorname{sign}(1-r) \sqrt{1-g^2(r)} \right)$$

such that $E(u) = 8\pi l \Gamma(k_1, k_2, k_3)$, where $g \in C([0, \infty); [0, 1])$ and $g'(r) > 0$ on $(0, 1)$, and $g'(r) < 0$ on $(1, \infty)$. In this paper, we derive new estimates on $g(r)$ and also prove that $g'(0)$ exists, is positive, and is bounded by a constant depending on k_1, k_2 and k_3 .

The second key step is to improve the method in [Rivière 1995] (also [Brezis and Coron 1983]) of inserting a small dipole into a nonconstant map. During the proof, it is important that $g'(0)$ is positive and bounded. Due to the differing constants k_1, k_2 , and k_3 , the liquid crystal is more complicated and involved than the harmonic maps.

Theorem 1.1 is a consequence of Theorem 1.2. Rivière [1995] also used Theorem 1.2 to construct weak harmonic maps having singularities almost everywhere in Ω . With Theorem 1.2, we conjecture that one can construct a weak solution of system (1–3) having singularities almost everywhere in Ω for different constants k_1, k_2 , and k_3 .

In the last part, we deal with the partial regularity of weak solutions of system (1–3). The partial regularity of weak solutions of elliptic systems and weakly harmonic maps has been of great interest (for example [Giaquinta 1983; Giaquinta et al. 1998]). For the liquid crystal, Hardt et al. [1986; 1988], in fundamental papers, proved the partial regularity of minimizers of the liquid crystal energy E . Here, we investigate the partial regularity of the weak solutions that minimize a modified relaxed functional of the liquid crystal energy E .

For a parameter $\lambda \in [0, 1]$, as in [Bethuel and Brézis 1991], we consider the modified λ -energy

$$(1-4) \quad E_\lambda(u) := E(u) + \lambda 8\pi \Gamma(k_1, k_2, k_3) L(u, u^0)$$

for a map $u \in H_\gamma^1(\Omega, S^2)$. It follows from [Bethuel et al. 1990] and [Giaquinta et al. 1990] that there exists a minimizer u_λ of E_λ in $H_\gamma^1(\Omega, S^2)$, and u_λ is a weak solution of (1–3). The author in [Hong 2004] proved the partial regularity of minimizers u_λ for $0 \leq \lambda < \lambda_0 = \alpha / \Gamma(k_1, k_2, k_3)$ with $\Gamma(k_1, k_2, k_3) \geq \alpha$. It was not clear then whether one can establish the partial regularity of minimizers u_λ of (1–4) for $\lambda \in [\lambda_0, 1]$. Now we make progress with:

Theorem 1.3. *For any parameter λ with $0 \leq \lambda < 1$, let u_λ be a minimizer of E_λ in $H_\gamma^1(\Omega, S^2)$. Then u_λ is smooth in a set $\Omega_0 \subset \bar{\Omega}$ and $\mathcal{H}^\beta(\bar{\Omega} \setminus \Omega_0) = 0$ for some positive $\beta < 1$, where \mathcal{H}^β is the Hausdorff measure.*

The paper is organized as follows. In Section 2, we derive some new estimates for the irrotational dipole and the solenoidal dipole. In Section 3, we prove Theorems 1.1 and 1.2 for $k_2 \geq k_1$. In Section 4, we prove those theorems for $k_1 > k_2$. In Section 5, we complete a proof of Theorem 1.3.

2. Improving estimates for irrotational dipole and solenoidal dipole

Proposition 2.1 [Giaquinta et al. 1990]. *There exists a C^∞ function $\tilde{u}(x)$ from \mathbb{R}^2 into $S^2 \subset \mathbb{R}^3$ such that*

- (i) $\tilde{u} = q$ at infinity, where q is the south pole of $S^2 \subset \mathbb{R}^3$;
- (ii) \tilde{u} , seen as a map from S^2 into S^2 , has degree 1;
- (iii) if $\Omega := \mathbb{R}^2 \times (0, l)$ and $u_0 : \Omega \rightarrow S^2$ is defined as

$$u_0(x_1, x_2, x_3) = \tilde{u}(x_1, x_2),$$

we have

$$(2-1) \quad E(u_0, \mathbb{R}^2 \times (0, l)) = 8\pi l \Gamma(k_1, k_2, k_3).$$

Now we will improve Proposition 2.1 so we can apply it to prove Theorem 1.2. Consider the dipole

$$T_0 = G_q + L \times [[S^2]], \quad \text{where } L = [[(0, 0, x_3) : 0 < x_3 \leq l]],$$

that is, $P = (0, 0, 0)$ and $N = (0, 0, l)$ with $l > 0$, where G_q is the current by the graph of the constant function q .

From [Giaquinta et al. 1990], we have

$$E(T_0, \mathbb{R}^2 \times (0, l)) = 2l \int_{S^2} \sqrt{k^2 n_3^2 + k k_3 (1 - n_3^2)} dH^2(n),$$

where $n = (n_1, n_2, n_3) \in S^2$. Then

$$\begin{aligned} E(u_0, \mathbb{R}^2 \times (0, l)) &= E(T_0, \mathbb{R}^2 \times (0, l)) = 8\pi k l \int_0^1 \sqrt{k_3/k + (1 - k_3/k)z^2} dz \\ &= 8\pi k l \int_0^1 \frac{\sqrt{1 - \beta y^2}}{\sqrt{1 - y^2}} y dy, \end{aligned}$$

where $\beta = 1 - k_3/k$.

The irrotational dipole. Assume $k_2 \geq k_1 = k$. We consider the all maps $u = (u_1, u_2, u_3) : \mathbb{R}^3 \rightarrow S^2$ of form

$$u_1(x) = g(r) \frac{x_1}{r}, \quad u_2 = g(r) \frac{x_2}{r}, \quad u_3(x) = \begin{cases} \sqrt{1 - g^2(r)}, & \text{for } 0 \leq r \leq 1, \\ -\sqrt{1 - g^2(r)}, & \text{for } r > 1, \end{cases}$$

where $g : [0, +\infty) \rightarrow [0, 1]$ is continuous and satisfies $g(0) = 0$, $g(1) = 1$, $g(r) \rightarrow 0$ as $r \rightarrow +\infty$, $g'(r) > 0$ on $(0, 1)$, and $g'(r) < 0$ on $(1, +\infty)$.

By a standard calculation, we have

$$\begin{aligned} (\operatorname{div} u)^2 &= (g')^2 + \frac{g^2}{r^2} + 2gg' \frac{1}{r}, \\ |u \times \operatorname{curl} u|^2 &= |\operatorname{curl} u|^2 = |\nabla u_3|^2 = \frac{g^2(g')^2}{1-g^2}, \end{aligned}$$

and

$$u \cdot \operatorname{curl} u = 0, \quad |\nabla u_1|^2 + |\nabla u_2|^2 = (g')^2 + \frac{g^2}{r^2}.$$

The Oseen–Frank density becomes

$$(2-2) \quad W(u, \nabla u) = k_1 \left[\frac{g^2}{r^2} + g'^2 \frac{(1-\beta)g^2}{1-g^2} \right] + 2(k_1 - \alpha)gg' \frac{1}{r},$$

where $\beta = 1 - k_3/k$. Then

$$E(u, \mathbb{R}^2 \times (0, l)) = 2\pi kl \int_0^\infty \left[\frac{g^2}{r^2} + g'^2 \frac{1-\beta g^2}{1-g^2} \right] r \, dr.$$

So

$$E(T_0) = 4\pi kl \int_0^\infty \frac{g}{r} g' \frac{\sqrt{1-\beta g^2}}{\sqrt{1-g^2}} r \, dr \leq 2\pi kl \int_0^\infty \left[\frac{g^2}{r^2} + g'^2 \frac{1-\beta g^2}{1-g^2} \right] r \, dr = E(u),$$

with equality if and only if

$$(2-3) \quad g' = \begin{cases} -\frac{g}{r} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}}, & \text{for } r \geq 1, \\ \frac{g}{r} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}}, & \text{for } 0 \leq r \leq 1. \end{cases}$$

First we consider the case $0 \leq r \leq 1$. We will prove that there is a solution of the equation

$$(2-4) \quad g' = \frac{g}{r} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}}$$

with $g(0) = 0$ and $g(1) = 1$. Moreover, $g'(0)$ exists and is positive and bounded.

Case I: $k \geq k_3$.

Since $0 \leq \beta \leq 1$, we have $1 - g^2 \leq 1 - \beta g^2 \leq 1$. Then

$$\frac{g}{r} \geq \frac{g}{r} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}} \geq \frac{g}{r} \sqrt{1-g^2}.$$

We consider two auxiliary equations:

$$\begin{aligned} g_1' &= \frac{g_1}{r}, \\ g_2' &= \frac{g_2}{r} \sqrt{1-g_2^2}. \end{aligned}$$

It is easy to see that $g_1(r) = r$ solves the first with $g_1(0) = 0$ and $g_1(1) = 1$ and $g_2(r) = 2r/1 + r^2$ solves the second with $g_2(0) = 0$ and $g_2(1) = 1$.

By the comparison theorem, there is a solution g of (2-4) such that

$$(2-5) \quad r \leq g \leq \frac{2r}{1+r^2}, \quad g(0) = 0, \quad g(1) = 1.$$

Using Equation (2-3), we have

$$\begin{aligned} g''(r) &= \frac{g'}{r} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}} - \frac{g}{r^2} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}} - \frac{g}{r} \frac{gg'}{\sqrt{1-g^2}\sqrt{1-\beta g^2}} + \frac{g}{r} \frac{\sqrt{1-g^2}\beta gg'}{(1-\beta g^2)^{3/2}} \\ &= \frac{g}{r^2} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}} \left(\frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}} - 1 \right) - \frac{g^2 g'}{r} \frac{1-\beta}{(1-g^2)^{1/2}(1-\beta g^2)^{2/3}} \leq 0. \end{aligned}$$

Therefore $g'(0) = \lim_{r \rightarrow 0} g'(r)$ exists and is finite because $g_1'(0) = 1$ and $g_2'(0) = 2$. More precisely, we know

$$1 \leq g'(0) \leq 2.$$

Case II: $k_3 \geq k$.

Since $\beta \leq 0$, we have $1 - \beta \geq 1 - \beta g^2 \geq 1$. Then

$$\frac{1}{\sqrt{1-\beta}} \frac{g}{r} \sqrt{1-g^2} \leq \frac{g}{r} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}} \leq \frac{g}{r} \sqrt{1-g^2}.$$

Then we consider two auxiliary equations:

$$\begin{aligned} g_1' &= \frac{1}{\sqrt{1-\beta}} \frac{g_1}{r} \sqrt{1-g_1^2}, \\ g_2' &= \frac{g_2}{r} \sqrt{1-g_2^2}. \end{aligned}$$

Note that $g_1(r) = 2r^c/1 + r^{2c}$ with $c = 1/\sqrt{1-\beta}$ solves the first with $g_1(0) = 0$ and $g_1(1) = 1$ and $g_2(r) = 2r/1 + r^2$ solves the second with $g_2(0) = 0$ and $g_2(1) = 1$.

By the comparison theorem, there is a solution g of the equation with $g(0) = 0$ and $g(1) = 1$ such that

$$\frac{2r}{1+r^2} \leq g \leq \frac{2r^c}{1+r^{2c}}.$$

Then

$$\frac{1}{\sqrt{1-\beta g^2}} \geq \frac{1}{1-\beta g^2} = 1 + \frac{\beta g^2}{1-\beta g^2} \geq 1 - |\beta| g^2 \geq 1 - 4|\beta| \frac{r^{2c}}{(1+r^{2c})^2}.$$

Note that as $r \rightarrow 0$, $g_1'(r) \rightarrow +\infty$. So we need to consider the auxiliary equation

$$(2-6) \quad g_3' = \left(1 - 4|\beta| \frac{r^{2c}}{(1+r^{2c})^2}\right) \frac{g_3}{r} \sqrt{1-g_3^2}$$

with $g_3(0) = 0$ and $g_3(1) = 1$. The solution to Equation (2-6) is

$$g_3(r) = 2r \exp\left(\frac{|\beta|}{c} \frac{1-r^{2c}}{1+r^{2c}}\right) \bigg/ \left(1 + r^2 \exp\left(\frac{2|\beta|}{c} \frac{1-r^{2c}}{1+r^{2c}}\right)\right).$$

By the comparison theory, we have

$$(2-7) \quad g_2(r) \leq g(r) \leq g_3(r) = 2r \exp\left(\frac{|\beta|}{c} \frac{1-r^{2c}}{1+r^{2c}}\right) \bigg/ \left(1 + r^2 \exp\left(\frac{2|\beta|}{c} \frac{1-r^{2c}}{1+r^{2c}}\right)\right)$$

It is easy to see that $g'(0)$ exists and is finite because $g_2'(0) = 2$ and $g_3'(0) = 2e^{|\beta|/c}$. More precisely, we have

$$2 \leq g'(0) \leq 2e^{|\beta|/c}.$$

In both Cases I and II, there is a solution with $g(0) = 0$, $g(1) = 1$ and with $g'(0)$ positive and finite.

If $r \geq 1$, take $h(r) = g(r^{-1})$, where $g(r)$ solves Equation (2-4) with $g(0) = 0$ and $g(1) = 1$. Using Equation (2-4), we have

$$h'(r) = g'(r^{-1}) (-r^{-2}) = \frac{g(r^{-1})}{r^{-1}} \frac{\sqrt{1-g^2}}{\sqrt{1-\beta g^2}} (-r^{-2}) = -\frac{h}{r} \frac{\sqrt{1-h^2}}{\sqrt{1-\beta h^2}}.$$

Then

$$\tilde{g}(r) = \begin{cases} g(r), & \text{for } 0 \leq r \leq 1, \\ h(r), & \text{for } r > 1, \end{cases}$$

is the required solution of Equation (2-3) with $\tilde{g}(r) = \tilde{g}(r^{-1})$.

The solenoidal dipole. Assume $k_2 \geq k_1 = k$. Consider all maps $u = (u_1, u_2, u_3) : \Omega \rightarrow S^2$ of the form

$$u_1(x) = g(r) \frac{x_2}{r}, \quad u_2 = -g(r) \frac{x_1}{r}, \quad u_3(x) = \begin{cases} \sqrt{1-g^2(r)}, & \text{for } 0 \leq r \leq 1 \\ -\sqrt{1-g^2(r)}, & \text{for } r > 1, \end{cases}$$

where $g : [0, +\infty) \rightarrow [0, 1]$ is continuous and satisfies $g(0) = 0, g(1) = 1, g(r) \rightarrow 0$ as $r \rightarrow +\infty, g'(r) > 0$ on $(0, 1),$ and $g'(r) < 0$ on $(1, +\infty).$ Then we have

$$E(u, \mathbb{R}^2 \times (0, l)) = 2\pi kl \int_0^\infty \left[\frac{g^2}{r^2}(1 - \beta g^2) + \frac{(g')^2}{1 - g^2} \right] r dr$$

such that $E(T_0, \mathbb{R}^2 \times (0, l)) \leq E(u),$ with equality if and only if

$$(2-8) \quad \frac{dg}{dr} = \text{sign}(1 - r) \frac{g}{r} \sqrt{1 - g^2} \sqrt{1 - \beta g^2},$$

This equation has a solution $g(r)$ such that $g : [0, +\infty) \rightarrow [0, 1]$ is continuous and satisfies $g(0) = 0, g(1) = 1, g(r) \rightarrow 0$ as $r \rightarrow +\infty, g'(r) > 0$ on $(0, 1),$ and $g'(r) < 0$ on $(1, +\infty).$ Moreover, $g'(0)$ exists and is positive and bounded.

3. Proof of Theorems 1.1 and 1.2 for $k_2 \geq k_1$

3.1. The construction of u^δ for $k_2 \geq k_1.$ We assume Theorem 1.2 that $\nabla u(x_0) \neq 0.$ Without losing generality, we also assume $x_0 = 0.$ Note that

$$W(Qu, Q\nabla u Q^T) = W(u, \nabla u), \quad \text{for all } Q \in O(3).$$

After a rotation Q on both $x, u \in \mathbb{R}^3,$ we can choose an orthonormal basis $\{I, J, K\}$ of \mathbb{R}^3 for both $x, u \in \mathbb{R}^3$ as in [Brezis and Coron 1983] such that $u(0, 0, 0) = K.$ $u_{x_1}(0, 0, 0) \cdot u_{x_2}(0, 0, 0) = 0$ and $u_{x_1}(0, 0, 0) \neq 0.$

Without loss of generality, we may choose

$$K(x_3) = u(0, 0, x_3), \quad I(x_3) = \frac{u_{x_1}(0, 0, x_3)}{|u_{x_1}(0, 0, x_3)|}.$$

to form a basis $\{I(x_3), J(x_3), K(x_3)\}$ of \mathbb{R}^3 depending on $x_3.$ We write

$$u = \hat{u}_1 I(x_3) + \hat{u}_2 J(x_3) + \hat{u}_3 K(x_3)$$

with $\hat{u}_1(0, 0, x_3) = \hat{u}_2(0, 0, x_3) = 0$ and $\hat{u}_3(0, 0, x_3) = 1.$ More precisely, there are two numbers $a > 0$ and $b \geq 0$ —this is true after a rotation in $\mathbb{R}^3:$ after a rotation in the subspace \mathbb{R}^2 of $\mathbb{R}^3,$ the conclusion of Theorem 1.2 does not change—such that

$$u_{x_1}(0, 0, x_3) = (a + O(x_3))I(x_3), \quad u_{x_2}(0, 0, x_3) = O(x_3)I(x_3) + (b + O(x_3))J(x_3).$$

We use polar coordinates for $(x_1, x_2) \in \mathbb{R}^2,$ that is,

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta,$$

and consider the cylinder C^δ in \mathbb{R}^3 defined by

$$C^\delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq r \leq \delta + \delta^2, \quad -\delta - \delta^2 \leq x_3 \leq \delta + \delta^2\}.$$

As in [Rivière 1995], we construct a map

$$u^\delta = \hat{u}_1^\delta I(x_3) + \hat{u}_2^\delta J(x_3) + \hat{u}_3^\delta K(x_3)$$

such that

- (i) $u^\delta = u$ outside C^δ .
- (ii) Inside C^δ , for each $x_3 \in [-\delta + \delta^2, \delta - \delta^2]$ (that is, the subcylinder of C^δ), we construct u^δ in three different cases:
 - (a) If $r > 2\delta^2$, we set $u^\delta(x) = u(x)$;
 - (b) If $r < \delta^2$, we set

$$u^\delta(x_1, x_2, x_3) = g(r/\lambda) \frac{x_1}{r} I(x_3) + g(r/\lambda) \frac{x_2}{r} J(x_3) + \text{sign}(1-r) \sqrt{1-g^2(r/\lambda)} K(x_3).$$

- (c) If $\delta^2 \leq r \leq 2\delta^2$, we set

$$u^\delta(x) = (A_1 r + B_1) I(x_3) + (A_2 r + B_2) J(x_3) + \sqrt{1 - (A_1 r + B_1)^2 - (A_2 r + B_2)^2} K(x_3),$$

where A_1, A_2, B_1, B_2 , depending only on θ, δ and x_3 , are determined by

$$(3-1) \quad \begin{aligned} 2\delta^2 A_1 + B_1 &= \hat{u}_1(2\delta^2 \cos \theta, 2\delta^2 \sin \theta, x_3), \\ 2\delta^2 A_2 + B_2 &= \hat{u}_2(2\delta^2 \cos \theta, 2\delta^2 \sin \theta, x_3), \\ \delta^2 A_1 + B_1 &= g(\delta^2/\lambda) \cos \theta = g(\lambda/\delta^2) \cos \theta, \\ \delta^2 A_2 + B_2 &= g(\delta^2/\lambda) \sin \theta = g(\lambda/\delta^2) \sin \theta, \end{aligned}$$

and $g(r)$ is the solution of Equation (2-3) with $g(0) = 0, g'(0) > 0, g(1) = 1, g(r) = g(1/r)$, and $\lambda = c\delta^4$, where c will be determined later.

- (iii) Inside C^δ , for each

$$x_3 \in [-\delta, -\delta + \delta^2] \cup [\delta, \delta - \delta^2],$$

we let $P = (0, 0, \delta)$ and $N = (0, 0, -\delta)$ in a small cylinder c_P^δ (or c_N^δ). The cylinder is centered at P (or N) with radius $2\delta^2$, length $2\delta^2$, and its axis along the x_3 -axis. If we denote by Π^+ (or Π_-) the radial projection centered at P (or N) onto the boundary of c_P^δ (or c_N^δ), the transformed map u^δ is the composition of Π^+ (or Π_-) and the value of u^δ on this boundary.

3.2. The estimate of the energy of u^δ for $k_2 \geq k_1$. Case 1. The estimate of the energy of u^δ on the domain of $x_3 \in [-\delta + \delta^2, \delta - \delta^2]$ and $\delta^2 \leq r \leq 2\delta^2$.

Notice that $\hat{u}_i(0, 0, x_3) = 0$ for $i = 1, 2$ and

$$\begin{aligned}\frac{\partial \hat{u}_1}{\partial x_1}(0, 0, x_3) &= a + O(x_3), & \frac{\partial \hat{u}_2}{\partial x_1}(0, 0, x_3) &= O(x_3), \\ \frac{\partial \hat{u}_1}{\partial x_2}(0, 0, x_3) &= O(x_3), & \frac{\partial \hat{u}_2}{\partial x_2}(0, 0, x_3) &= b + O(x_3), \\ g(c\delta^2) &= c\delta^2 g'(0) + O(\delta^4)\end{aligned}$$

Then we have

$$\begin{aligned}2\delta^2 A_1 + B_1 &= 2a\delta^2 \cos \theta + O(\delta^3), \\ 2\delta^2 A_2 + B_2 &= 2b\delta^2 \sin \theta + O(\delta^3), \\ \delta^2 A_1 + B_1 &= g'(0)c\delta^2 \cos \theta + O(\delta^4), \\ \delta^2 A_2 + B_2 &= g'(0)c\delta^2 \sin \theta + O(\delta^4).\end{aligned}$$

Solving these equations, we have

$$(3-2) \quad \begin{aligned}A_1 &= (2a - cg'(0)) \cos \theta + O(\delta), \\ A_2 &= (2b - cg'(0)) \sin \theta + O(\delta), \\ B_1 &= 2\delta^2(g'(0)c - a) \cos \theta + O(\delta^3), \\ B_2 &= 2\delta^2(g'(0)c - b) \sin \theta + O(\delta^3).\end{aligned}$$

In a way similar to (3-2), it follows from (3-1) that

$$(3-3) \quad \begin{aligned}\frac{\partial A_1}{\partial \theta} &= -(2a - cg'(0)) \sin \theta + O(\delta), \\ \frac{\partial A_2}{\partial \theta} &= (2b - cg'(0)) \cos \theta + O(\delta), \\ \frac{\partial B_1}{\partial \theta} &= -2\delta^2(cg'(0) - a) \sin \theta + O(\delta^3), \\ \frac{\partial B_2}{\partial \theta} &= 2\delta^2(cg'(0) - b) \cos \theta + O(\delta^3).\end{aligned}$$

In polar coordinates, we know

$$\frac{\partial \theta}{\partial x_1} = -\frac{\sin \theta}{r}, \quad \frac{\partial \theta}{\partial x_2} = \frac{\cos \theta}{r}, \quad \frac{\partial r}{\partial x_1} = \cos \theta, \quad \frac{\partial r}{\partial x_2} = \sin \theta.$$

Using $\hat{u}_1(0, 0, x_3) = \hat{u}_2(0, 0, x_3) = 0$ in (3-1), we obtain, for $\delta^2 \leq r \leq 2\delta^2$,

$$(3-4) \quad \frac{\partial A_1}{\partial x_3} = O(\delta), \quad \frac{\partial A_2}{\partial x_3} = O(\delta), \quad \frac{\partial B_1}{\partial x_3} = O(\delta^3), \quad \frac{\partial B_2}{\partial x_3} = O(\delta^3).$$

and, by the chain rule,

$$\begin{aligned} \frac{\partial \hat{u}_1^\delta}{\partial x_2} - \frac{\partial \hat{u}_2^\delta}{\partial x_1} &= \frac{\partial (A_1 r + B_1)}{\partial x_2} - \frac{\partial (A_2 r + B_2)}{\partial x_1} \\ &= \frac{\partial (A_1 r + B_1)}{\partial \theta} \frac{\partial \theta}{\partial x_2} + \frac{\partial (A_1 r + B_1)}{\partial r} \frac{\partial r}{\partial x_2} - \frac{\partial (A_2 r + B_2)}{\partial \theta} \frac{\partial \theta}{\partial x_1} - \frac{\partial (A_2 r + B_2)}{\partial r} \frac{\partial r}{\partial x_1} \\ &= \left(\frac{\partial A_1}{\partial \theta} \frac{\partial \theta}{\partial x_2} - \frac{\partial A_2}{\partial \theta} \frac{\partial \theta}{\partial x_1} \right) r + \left(A_1 \frac{\partial r}{\partial x_2} - A_2 \frac{\partial r}{\partial x_1} \right) + \left(\frac{\partial B_1}{\partial \theta} \frac{\partial \theta}{\partial x_2} - \frac{\partial B_2}{\partial \theta} \frac{\partial \theta}{\partial x_1} \right). \end{aligned}$$

It follows from (3–3), (3–4) that

$$\begin{aligned} \left(\frac{\partial A_1}{\partial \theta} \frac{\partial \theta}{\partial x_2} - \frac{\partial A_2}{\partial \theta} \frac{\partial \theta}{\partial x_1} \right) r &= -(2a - cg'(0)) \sin \theta \cos \theta \\ &\quad - (2b - cg'(0)) r \cos \theta \left(-\frac{\sin \theta}{r} \right) + O(\delta) \\ &= 2(b - a) \sin \theta \cos \theta + O(\delta), \end{aligned}$$

$$\begin{aligned} A_1 \frac{\partial r}{\partial y} - A_2 \frac{\partial r}{\partial x_1} &= (2a - cg'(0)) \cos \theta \sin \theta - (2b - cg'(0)) \cos \theta \sin \theta + O(\delta) \\ &= 2(a - b) \sin \theta \cos \theta + O(\delta), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial B_1}{\partial \theta} \frac{\partial \theta}{\partial x_2} - \frac{\partial B_2}{\partial \theta} \frac{\partial \theta}{\partial x_1} &= (-2\delta^2 (cg'(0) - a) \sin \theta + O(\delta^3)) \left(\frac{\cos \theta}{r} \right) \\ &\quad - (2\delta^2 (cg'(0) - b) \cos \theta + O(\delta^3)) \left(-\frac{\sin \theta}{r} \right) \\ &= 2\delta^2 (a - b) \sin \theta \cos \theta \frac{1}{r} + O(\delta^3) \frac{1}{r}. \end{aligned}$$

These imply that, for $\delta^2 \leq r \leq 2\delta^2$,

$$\frac{\partial \hat{u}_1^\delta}{\partial x_2} - \frac{\partial \hat{u}_2^\delta}{\partial x_1} = 2\delta^2 (a - b) \sin \theta \cos \theta \frac{1}{r} + O(\delta).$$

Since $|\hat{u}_3^\delta|^2 = 1 - |\hat{u}_1^\delta|^2 - |\hat{u}_2^\delta|^2$, we have

$$\hat{u}_3^\delta \frac{\partial \hat{u}_3^\delta}{\partial x_1} = -\hat{u}_1^\delta \frac{\partial \hat{u}_1^\delta}{\partial x_1} - \hat{u}_2^\delta \frac{\partial \hat{u}_2^\delta}{\partial x_1}, \quad \hat{u}_3^\delta \frac{\partial \hat{u}_3^\delta}{\partial x_2} = -\hat{u}_1^\delta \frac{\partial \hat{u}_1^\delta}{\partial x_2} - \hat{u}_2^\delta \frac{\partial \hat{u}_2^\delta}{\partial x_2}$$

and $\hat{u}_3^\delta = 1 + O(\delta^2)$. Then for $\delta^2 \leq r \leq 2\delta^2$, we have

$$(3-5) \quad \frac{\partial \hat{u}_3^\delta}{\partial x_1} = O(\delta^2), \quad \frac{\partial \hat{u}_3^\delta}{\partial x_2} = O(\delta^2).$$

We consider a new map $\hat{u}^\delta = \hat{u}_1^\delta I + \hat{u}_2^\delta J + \hat{u}_3^\delta K$, and thus obtain

$$(3-6) \quad \begin{aligned} |\hat{u}^\delta \cdot \operatorname{curl} \hat{u}^\delta|^2 &= 4\delta^4 (a-b)^2 \sin^2 \theta \cos^2 \theta \frac{1}{r^2} + O(\delta), \\ |\hat{u}^\delta \times \operatorname{curl} \hat{u}^\delta|^2 &= O(\delta). \end{aligned}$$

Moreover, we have

$$(3-7) \quad \begin{aligned} \frac{\partial \hat{u}_1^\delta}{\partial x_1} &= \frac{\partial(A_1 r + B_1)}{\partial x_1} = \left(\frac{\partial A_1}{\partial \theta} r + \frac{\partial B_1}{\partial \theta} \right) \frac{\partial \theta}{\partial x_1} + A_1 \frac{\partial r}{\partial x_1} + O(\delta) \\ &= (2a - cg'(0)) + 2\delta^2 (cg'(0) - a) \frac{\sin^2 \theta}{r} + O(\delta). \end{aligned}$$

Similarly,

$$(3-8) \quad \frac{\partial \hat{u}_2^\delta}{\partial x_2} = \frac{\partial(A_2 r + B_2)}{\partial x_2} = (2b - cg'(0)) + 2\delta^2 (cg'(0) - b) \frac{\cos^2 \theta}{r} + O(\delta).$$

It follows from (3-4), (3-7) and (3-8) that

$$\begin{aligned} (\operatorname{div} \hat{u}^\delta)^2 &= 4 \left(a + b - cg'(0) + \frac{\delta^2 cg'(0)}{r} - \delta^2 \frac{(a \sin^2 \theta + b \cos^2 \theta)}{r} \right)^2 + O(\delta) \\ &= 4(a+b)^2 + 4c^2(g'(0))^2 + \frac{4\delta^4 c^2 (g'(0))^2}{r^2} + \frac{4\delta^4 (a \sin^2 \theta + b \cos^2 \theta)^2}{r^2} \\ &\quad - 8(a+b)cg'(0) + 8\delta^2 (a+b) \frac{cg'(0)}{r} - 8\delta^2 (a+b) \frac{a \sin^2 \theta + b \cos^2 \theta}{r} \\ &\quad - \frac{8\delta^2 c^2 (g'(0))^2}{r} + 8\delta^2 cg'(0) \frac{a \sin^2 \theta + b \cos^2 \theta}{r} - 8\delta^4 cg'(0) \frac{a \sin^2 \theta + b \cos^2 \theta}{r^2}. \end{aligned}$$

Combining this estimate with (3-6) yields

$$\begin{aligned} \int_{\delta^2 \leq r \leq 2\delta^2} (\operatorname{div} \hat{u}^\delta)^2 + |\operatorname{curl} \hat{u}^\delta|^2 dx_1 dx_2 &= \int_0^{2\pi} \int_{\delta^2}^{2\delta^2} (\operatorname{div} \hat{u}^\delta)^2 + |\operatorname{curl} \hat{u}^\delta|^2 r dr d\theta \\ &= 12\pi \delta^4 ((a+b)^2 + c^2(g'(0))^2 - 2(a+b)cg'(0)) + 16\pi \delta^4 (a+b)cg'(0) \\ &\quad - 8\pi \delta^4 (a+b)^2 - 16\pi \delta^4 c^2 (g'(0))^2 + 8\pi \delta^2 cg'(0)(a+b) \\ &\quad + 4\delta^4 \ln 2 \int_0^{2\pi} [c^2(g'(0))^2 - 2cg'(0)(a \sin^2 \theta + b \cos^2 \theta)] d\theta \\ &\quad + 4\delta^4 \ln 2 \int_0^{2\pi} [a^2 \sin^4 \theta + b^2 \cos^4 \theta + (a^2 + b^2) \sin^2 \theta \cos^2 \theta] d\theta \\ &= 4\pi \delta^4 ((a+b)^2 - g'(0)^2 c^2 + (a^2 + b^2 + 2(g'(0)c)^2 - 2ag'(0)c - 2bg'(0)c) \ln 2). \end{aligned}$$

It follows from [Brezis and Coron 1983] that

$$\begin{aligned} \int_{\delta^2 \leq r \leq 2\delta^2} |\nabla \hat{u}^\delta|^2 dx_1 dx_2 &= \int_{\delta^2 \leq r \leq 2\delta^2} [|\hat{u}_{x_1}^\delta|^2 + |\hat{u}_{x_2}^\delta|^2 + O(\delta)] dx_1 dx_2 \\ &= 4\pi \delta^4 (a^2 + b^2 - \frac{g'(0)^2 c^2}{2} + (a^2 + b^2 + 2(g'(0)c)^2 - 2ag'(0)c - 2bg'(0)c) \ln 2) \\ &\quad + O(\delta^5). \end{aligned}$$

It follows from (3–3), (3–4) and (3–5) that

$$\begin{aligned} (3-9) \quad &\int_{\delta^2 \leq r \leq 2\delta^2} W(\hat{u}^\delta, \nabla \hat{u}^\delta) dx_1 dx_2 \\ &= \alpha \int_{\delta^2 \leq r \leq 2\delta^2} |\nabla \hat{u}^\delta|^2 dx_1 dx_2 + \int_{\delta^2 \leq r \leq 2\delta^2} (k_1 - \alpha)(\operatorname{div} u)^\delta dx_1 dx_2 \\ &\quad + \int_{\delta^2 \leq r \leq 2\delta^2} [(k_2 - \alpha)|\hat{u}^\delta \cdot \operatorname{curl} \hat{u}^\delta|^2 + (k_3 - \alpha)|\hat{u}^\delta \times \operatorname{curl} \hat{u}^\delta|^2] dx_1 dx_2 \\ &= 4\alpha\pi \delta^4 (a^2 + b^2 - \frac{g'(0)^2 c^2}{2} + (a^2 + b^2 + 2(g'(0)c)^2 - 2ag'(0)c - 2bg'(0)c) \ln 2) \\ &\quad + (k_1 - \alpha)4\pi \delta^4 \left((a+b)^2 - g'(0)^2 c^2 + (a^2 + b^2 + 2(g'(0)c)^2) \ln 2 \right) \\ &\quad - 8\alpha\pi \delta^4 (ag'(0)c + bg'(0)c) \ln 2 + \pi(k_2 - k_1)\delta^4 (a-b)^2 \ln 2 + O(\delta^5). \end{aligned}$$

On other hand, we see

$$\frac{\partial u^\delta}{\partial x_1} = \frac{\partial \hat{u}^\delta}{\partial x_1} + O(\delta), \quad \frac{\partial u^\delta}{\partial x_2} = \frac{\partial \hat{u}^\delta}{\partial x_2} + O(\delta)$$

and

$$\frac{\partial u^\delta}{\partial x_3} = \frac{\partial \hat{u}_1^\delta}{\partial x_3} I(x_3) + \frac{\partial \hat{u}_2^\delta}{\partial x_3} J(x_3) + \frac{\partial \hat{u}_3^\delta}{\partial x_3} K(x_3) + \hat{u}_1^\delta \frac{dI(x_3)}{dx_3} + \hat{u}_2^\delta \frac{dJ(x_3)}{dx_3} + \hat{u}_3^\delta \frac{dK(x_3)}{dx_3}.$$

For $\delta^2 \leq r \leq 2\delta^2$, we have

$$\frac{\partial \hat{u}_1^\delta}{\partial x_3} = O(\delta^2), \quad \frac{\partial \hat{u}_2^\delta}{\partial x_3} = O(\delta^2), \quad \frac{\partial \hat{u}_3^\delta}{\partial x_3} = O(\delta^4),$$

and, moreover,

$$\hat{u}^\delta(x_1, x_2, x_3) = \hat{u}^\delta(0, 0, x_3) + O(\delta^2) = K(x_3) + O(\delta^2).$$

It follows from $u(0, 0, x_3) = K(x_3)$ that for $\delta^2 \leq r \leq 2\delta^2$, we have

$$\frac{\partial u^\delta}{\partial x_3}(x_1, x_2, x_3) = \frac{\partial u}{\partial x_3}(0, 0, x_3) + O(\delta^2).$$

Using $|u| = 1$ and $u_3(0, 0, 0) = (0, 0, 1)$, we have $\partial u_3 / \partial x_3(0, 0, 0) = 0$. Thus

$$\frac{\partial u_3^\delta}{\partial x_3}(x_1, x_2, x_3) = \frac{\partial u_3}{\partial x_3}(0, 0, x_3) + O(\delta^2) = O(\delta).$$

For $\delta^2 \leq r \leq 2\delta^2$, we have

$$(3-10) \quad (\operatorname{div} u^\delta)^2 = (\operatorname{div} \hat{u}^\delta)^2 + O(\delta), \quad (u^\delta \cdot \operatorname{curl} u^\delta)^2 = (\hat{u}^\delta \cdot \operatorname{curl} \hat{u}^\delta)^2 + O(\delta)$$

and

$$(3-11) \quad (u^\delta \times \operatorname{curl} u^\delta)^2 = (\hat{u}^\delta \times \operatorname{curl} \hat{u}^\delta)^2 + \left| \frac{\partial u_1^\delta}{\partial x_3} \right|^2 + \left| \frac{\partial u_2^\delta}{\partial x_3} \right|^2 + O(\delta) \\ = (\hat{u}^\delta \times \operatorname{curl} \hat{u}^\delta)^2 + d^2 + f^2 + O(\delta),$$

where we have set

$$d = \frac{\partial u_1}{\partial x_3}(0, 0, 0), \quad f = \frac{\partial u_2}{\partial x_3}(0, 0, 0).$$

It follows from (3-9), (3-10)-(3-11) that

$$\int_{-\delta+\delta^2}^{\delta-\delta^2} dx_3 \int_{\delta^2 \leq r \leq 2\delta^2} dx_1 dx_2 W(u^\delta, \nabla u^\delta) \\ = \int_{-\delta+\delta^2}^{\delta-\delta^2} dx_3 \int_{\delta^2 \leq r \leq 2\delta^2} dx_1 dx_2 W(\hat{u}^\delta, \nabla \hat{u}^\delta) + 6\pi k_3 \delta^5 (d^2 + f^2) + O(\delta^6) \\ = 8\alpha\pi \delta^5 \left(a^2 + b^2 - \frac{g'(0)^2 c^2}{2} + (a^2 + b^2 + 2(g'(0)c)^2 - 2ag'(0)c - bg'(0)c) \ln 2 \right) \\ + 8\pi \delta^5 (k_1 - \alpha) \left((a+b)^2 - g'(0)^2 c^2 + (a^2 + b^2 + 2(g'(0)c)^2) \ln 2 \right) \\ - 16\pi \delta^5 (k_1 - \alpha) (ag'(0)c + bg'(0)c) \ln 2 + 2\pi (k_2 - k_1) \delta^5 (a-b)^2 \ln 2 \\ + 6\pi k_3 \delta^5 (d^2 + f^2) + O(\delta^6).$$

Next, we estimate $\int_{-\delta+\delta^2}^{\delta-\delta^2} dx_3 \int_{0 \leq r \leq \delta^2} dx_1 dx_2 W(u^\delta, \nabla u^\delta)$.

Let u_0 be the map defined by $u_0 = \tilde{u}(x_1, x_2)$ in the Section 2. By (2-1) and (2-2), we know

$$(3-12) \quad \int_{r \leq \delta^2} W(\hat{u}^\delta, \nabla \hat{u}^\delta) dx_1 dx_2 = \int_{r \leq 1/c\delta^2} W(u_0, \nabla u_0) dx_1 dx_2 \\ = 4\pi k_1 \int_0^{1/c\delta^2} \frac{g^2(r)}{r} dr + 4\pi (k_1 - \alpha) \int_0^{1/c\delta^2} gg' dr \\ = 4\pi k_1 \int_0^\infty \frac{g^2(r)}{r} dr - 4\pi k_1 \int_{1/c\delta^2}^\infty \frac{g^2}{r^2} r dr + 2\pi (k_1 - \alpha) \delta^4 c^2 (g'(0))^2 + O(\delta^5) \\ = 8\pi \Gamma(k_1, k_2, k_3) - 2\pi \alpha \delta^4 c^2 [g'(0)]^2 + O(\delta^5),$$

because $g(r) = g'(0)r + O(r^2)$ and $g(r) = g(1/r)$.

On other hand, we see

$$(3-13) \quad \frac{\partial u^\delta}{\partial x_1} = \frac{\partial \hat{u}^\delta}{\partial x_1} + O(\delta), \quad \frac{\partial u^\delta}{\partial x_2} = \frac{\partial \hat{u}^\delta}{\partial x_2} + O(\delta)$$

and

$$(3-14) \quad \frac{\partial u^\delta}{\partial x_3} = \hat{u}_1^\delta \frac{dI(x_3)}{dx_3} + \hat{u}_2^\delta \frac{dJ(x_3)}{dx_3} + \hat{u}_3^\delta \frac{dK(x_3)}{dx_3},$$

which implies

$$\begin{aligned} \frac{\partial u_1^\delta}{\partial x_3} &= \hat{u}_2^\delta \frac{dJ(x_3)}{dx_3} \cdot I + \hat{u}_3^\delta \frac{dK(x_3)}{dx_3} \cdot I + O(\delta), \\ \frac{\partial u_2^\delta}{\partial x_3} &= \hat{u}_1^\delta \frac{dI(x_3)}{dx_3} \cdot J + \hat{u}_3^\delta \frac{dK(x_3)}{dx_3} \cdot J + O(\delta). \end{aligned}$$

From the results in the Section 2, we have

$$(3-15) \quad |\nabla \hat{u}^\delta| \leq C \frac{g(r/\lambda)}{r} \leq C \frac{\delta^4}{\delta^8 + r^2}.$$

We estimate the term

$$\begin{aligned} u_1^\delta \frac{\partial u_3^\delta}{\partial x_2} \frac{\partial \hat{u}_2^\delta}{\partial x_3} &= (\hat{u}_1^\delta + O(\delta)) \left(\frac{\partial \hat{u}_3^\delta}{\partial x_2} + O(\delta) \right) \left(\hat{u}_1^\delta \frac{dI(x_3)}{dx_3} \cdot J + \hat{u}_3^\delta \frac{dK(x_3)}{dx_3} \cdot J + O(\delta) \right) \\ &= \left(-\frac{g^2(\rho) g'(\rho) \cos \theta \sin \theta}{\lambda \sqrt{1-g(\rho)^2}} \right) \left(g(\rho) \sin \theta \frac{dJ(x_3)}{dx_3} \cdot I + \sqrt{1-g(\rho)^2} \frac{dK(x_3)}{dx_3} \cdot I \right) \\ &\quad + O(\delta) \left| \frac{\partial \hat{u}_3^\delta}{\partial x_2} \right| + O(\delta), \end{aligned}$$

where $\rho = r/\lambda$. Integrating the above identity and using (3-15), we have

$$\int_{r \leq \delta^2} u_1^\delta \frac{\partial u_3^\delta}{\partial x_2} \frac{\partial u_2^\delta}{\partial x_3} dx_1 dx_2 = \int_0^{\delta^2} \int_0^{2\pi} u_1^\delta \frac{\partial u_3^\delta}{\partial x_2} \frac{\partial u_2^\delta}{\partial x_3} d\theta r dr = O(\delta^5 \ln(1/\delta)).$$

Similarly, we obtain

$$\int_{r \leq \delta^2} u_2^\delta \frac{\partial u_3^\delta}{\partial x_1} \frac{\partial u_1^\delta}{\partial x_3} dx_1 dx_2 = O(\delta^5 \ln \delta).$$

By a similar argument, we also have

$$\int_{r \leq \delta^2} \frac{\partial u_3^\delta}{\partial x_1} \frac{\partial u_1^\delta}{\partial x_3} dx_1 dx_2 = O(\delta^5 \ln \delta), \quad \int_{r \leq \delta^2} \frac{\partial u_3^\delta}{\partial x_2} \frac{\partial u_2^\delta}{\partial x_3} dx_1 dx_2 = O(\delta^5 \ln \delta).$$

From (3-14), it is easy to see that

$$\int_{r \leq \delta^3} \left| \frac{\partial u^\delta}{\partial x_3}(x_1, x_2, x_3) \right|^2 dx_1 dx_2 = O(\delta^6).$$

By combining above estimates with the identity $|\operatorname{curl} u^\delta|^2 = |u \cdot \operatorname{curl} u^\delta|^2 + |u \times \operatorname{curl} u^\delta|^2$, we obtain

$$\begin{aligned} (3-16) \quad & \int_{-\delta+\delta^2}^{\delta-\delta^2} dx_3 \int_{r \leq \delta^3} dx_1 dx_2 W(u^\delta, \nabla u^\delta) \\ & = \int_{-\delta+\delta^2}^{\delta-\delta^2} dx_3 \int_{r \leq \delta^3} dx_1 dx_2 W(\hat{u}^\delta, \nabla \hat{u}^\delta) + O(\delta^6 \ln \delta). \end{aligned}$$

For $\delta^3 \leq r \leq \delta^2$, it follows from (2-5) and (2-7) that

$$(3-17) \quad g(r/\lambda) \leq C \frac{r\lambda}{\lambda^2 + r^2} = O(\delta)$$

for some constant C . Using (3-17), it follows from (3-13)-(3-14) that

$$\begin{aligned} & \int_{-\delta+\delta^2}^{\delta-\delta^2} dx_3 \int_{\delta^3 \leq r \leq \delta^2} dx_1 dx_2 W(u^\delta, \nabla u^\delta) \\ & = \int_{-\delta+\delta^2}^{\delta-\delta^2} dx_3 \int_{\delta^3 \leq r \leq \delta^2} dx_1 dx_2 \left(W(\hat{u}^\delta, \nabla \hat{u}^\delta) + k_3 \left[\left| \frac{\partial u_1^\delta}{\partial x_3} \right|^2 + \left| \frac{\partial u_2^\delta}{\partial x_3} \right|^2 \right] \right) + O(\delta^6 \ln \delta). \end{aligned}$$

Combining this with (3-12) and (3-16) yields

$$\begin{aligned} \int_{-\delta+\delta^2}^{\delta-\delta^2} dx_3 \int_{r \leq \delta^2} dx_1 dx_2 W(u^\delta, \nabla u^\delta) & = 16\pi(\delta-\delta^2)\Gamma(k_1, k_2, k_3) - 4\pi\alpha\delta^5 c^2 [g'(0)]^2 \\ & \quad + 2\pi k_3 \delta^5 (d^2 + f^2) + O(\delta^6 \ln \delta). \end{aligned}$$

Case 2. Estimate for $E(u^\delta)$ in c_P^δ and c_N^δ .

Let G_P be the little cone inside c_P^δ with vertex $P = (0, 0, \delta)$ given by

$$G_P = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1)^2 + (x_2)^2 \leq (\delta - x_3)^2, \quad \delta - \delta^2 \leq x_3 \leq \delta \right\}.$$

Its end is the disk

$$D_{\delta^2} = \left\{ (x'_1, x'_2, x'_3) \in \mathbb{R}^3 \mid r'^2 = x'^2 + y'^2 \leq \delta^4, \quad x'_3 = \delta - \delta^2 \right\}.$$

Let $x = (x_1, x_2, x_3)$ be a point in G_P and let $x' = (x'_1, x'_2, x'_3)$ be its projection $x' = \Pi^+(x)$ on the disk D_{δ^2} as

$$x' = \Pi^+(x) = \left(\frac{\delta^2 x_1}{\delta - x_3}, \frac{\delta^2 x_2}{\delta - x_3}, \delta - \delta^2 \right).$$

Now u^δ is constant on the rays passing by P , that is,

$$u^\delta(x_1, x_2, x_3) = u^\delta(x'_1, x'_2, \delta - \delta^2).$$

Using the chain rule, it follows from the previous two equations that for a point $(x_1, x_2, x_3) \in G_P$,

$$\begin{aligned} \frac{\partial u^\delta}{\partial x_1}(x_1, x_2, x_3) &= \frac{\delta^2}{\delta - x_3} \frac{\partial u^\delta}{\partial x'_1}(x'_1, x'_2, \delta - \delta^2) \\ (3-18) \quad \frac{\partial u^\delta}{\partial x_2}(x_1, x_2, x_3) &= \frac{\delta^2}{\delta - x_3} \frac{\partial u^\delta}{\partial x'_2}(x'_1, x'_2, \delta - \delta^2) \\ \frac{\partial u^\delta}{\partial x_3}(x_1, x_2, x_3) &= \frac{x'_1}{\delta - x_3} \frac{\partial u^\delta}{\partial x'_1}(x'_1, x'_2, \delta - \delta^2) + \frac{x'_2}{\delta - x_3} \frac{\partial u^\delta}{\partial x'_2}(x'_1, x'_2, \delta - \delta^2). \end{aligned}$$

Using the third identity of (3-18), we have

$$\begin{aligned} &\int_{G_P} \left| \frac{\partial u^\delta}{\partial x_3}(x_1, x_2, x_3) \right|^2 dx_1 dx_2 dx_3 \\ &= \int_{\delta - \delta^2}^{\delta} dx_3 \int_{r^2 \leq (\delta - x_3)^2} dx_1 dx_2 \frac{1}{(\delta - x_3)^2} \left(x'_1 \frac{\partial u^\delta}{\partial x'_1} + x'_2 \frac{\partial u^\delta}{\partial x'_2} \right)^2 (x'_1, x'_2, \delta - \delta^2) \\ &= \int_{r' \leq \delta^2} \frac{1}{\delta^2} \left(x'_1 \frac{\partial u^\delta}{\partial x'_1} + x'_2 \frac{\partial u^\delta}{\partial x'_2} \right)^2 (x'_1, x'_2, \delta - \delta^2) dx'_1 dx'_2. \end{aligned}$$

On other hand, from the results in the Section 2, we obtain

$$\begin{aligned} &|\nabla_{x'_1} u^\delta|^2(x'_1, x'_2, \delta - \delta^2) + |\nabla_{x'_2} u^\delta|^2(x'_1, x'_2, \delta - \delta^2) \\ &\leq C \frac{g(r'/c\delta^2)}{r'^2} \leq C \frac{\lambda^2}{(\lambda^2 + r'^2)^2} \leq C \frac{\delta^8}{(\delta^8 + r'^2)^2}, \end{aligned}$$

where $\lambda = c\delta^4$. Combining the previous two equations, we obtain

$$\int_{G_P} \left| \frac{\partial u^\delta}{\partial x_3}(x_1, x_2, x_3) \right|^2 dx_1 dx_2 dx_3 \leq C \int_0^{\delta^2} \frac{\delta^6 r'^2}{(\delta^8 + r'^3)^2} dr' = O(\delta^6 \ln(\delta)).$$

A simple calculation yields

$$\begin{aligned} (3-19) \quad &\left(u_2^\delta \frac{\partial u_3^\delta}{\partial x'_1} \left(x'_1 \frac{\partial u_1^\delta}{\partial x'_1} + x'_2 \frac{\partial u_1^\delta}{\partial x'_2} \right) \right) (x'_1, x'_2, \delta - \delta^2) \\ &= \left(g^2(r'/\lambda) + g(r'/\lambda) \left(\frac{1}{r'} g(r'/\lambda) \right)_{r'} \right) \left(\pm \sqrt{1 - g^2(r'/\lambda)} \right)_{r'} \sin \theta \cos \theta, \end{aligned}$$

where we use polar coordinates $x'_1 = r' \cos \theta$ and $x'_2 = r' \sin \theta$.

By (3–18) and (3–19), we calculate

$$\begin{aligned} & \int_{\delta-\delta^2}^{\delta} dx_3 \int_{r \leq \delta-x_3} dx_1 dx_2 u_2^\delta(x_1, x_2, x_3) \frac{\partial u_3^\delta}{\partial x_1}(x_1, x_2, x_3) \frac{\partial u_1^\delta}{\partial x_3}(x_1, x_2, x_3) \\ &= \frac{1}{\delta^2} \int_{\delta-\delta^2}^{\delta} dx_3 \int_{r' \leq \delta^2} dx_1 dx_2 \left(u_2^\delta \frac{\partial u_3^\delta}{\partial x_1'}(x_1', x_2', \delta - \delta^2) + x_2' \frac{\partial u_1^\delta}{\partial x_2'}(x_1', x_2', \delta - \delta^2) \right) = 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \int_{\delta-\delta^2}^{\delta} dx_3 \int_{r \leq \delta-x_3} dx_1 dx_2 \frac{\partial u_3^\delta}{\partial x_1}(x_1, x_2, x_3) \frac{\partial u_1^\delta}{\partial x_3}(x_1, x_2, x_3) = 0, \\ & \int_{\delta-\delta^2}^{\delta} dx_3 \int_{r \leq \delta-x_3} dx_1 dx_2 u_1^\delta(x_1, x_2, x_3) \frac{\partial u_3^\delta}{\partial x_2}(x_1, x_2, x_3) \frac{\partial u_2^\delta}{\partial x_3}(x_1, x_2, x_3) = 0, \\ & \int_{\delta-\delta^2}^{\delta} dx_3 \int_{r \leq \delta-x_3} dx_1 dx_2 \frac{\partial u_3^\delta}{\partial x_2}(x_1, x_2, x_3) \frac{\partial u_2^\delta}{\partial x_3}(x_1, x_2, x_3) = 0. \end{aligned}$$

Combining these estimates with (3–12) yields

$$\begin{aligned} & \int_{G_P} W(u^\delta, \nabla u^\delta)(x_1, x_2, x_3) dx_1 dx_2 dx_3 \\ &= \int_{\delta-\delta^2}^{\delta} dx_3' \int_{r' \leq \delta^2} dx_1' dx_2' W(u^\delta, \nabla u^\delta)(x_1', x_2', \delta - \delta^2) + O(\delta^6 \ln \delta) \\ &= 8\pi \Gamma(k_1, k_2, k_3) \delta^2 + O(\delta^6 \ln \delta). \end{aligned}$$

Since u is regular and $|\nabla u^\delta|$ is bounded by a constant, we obtain

$$\int_{c_P^\delta \setminus G_P} W(u^\delta, \nabla u^\delta) dx_1 dx_2 dx_3 = O(\delta^6).$$

Therefore, it follows from the previous two equations that

$$\int_{c_P^\delta} W(u^\delta, \nabla u^\delta) dx_1 dx_2 dx_3 = 8\pi \delta^2 \Gamma(k_1, k_2, k_3) + O(\delta^6 \ln \delta).$$

Similarly, we get $\int_{c_N^\delta} W(u^\delta, \nabla u^\delta) dx_1 dx_2 dx_3 = 8\pi \delta^2 \Gamma(k_1, k_2, k_3) + O(\delta^6 \ln \delta)$.

Proof of Theorem 1.2 for $k_2 \geq k_1$. Since u is smooth, we have

$$\begin{aligned} & \int_{-\delta-\delta^2}^{\delta+\delta^2} dx_3 \int_{r \leq 2\delta^2} dx_1 dx_2 W(u, \nabla u)(x_1, x_2, x_3) \\ &= \int_{-\delta-\delta^2}^{\delta+\delta^2} dx_3 \int_{r \leq 2\delta^2} dx_1 dx_2 [\alpha(a^2 + b^2) + (k_1 - \alpha)(a + b)^2 + k_3(d^2 + f^2)] + O(\delta^6) \\ &= 8\pi \delta^5 [\alpha(a^2 + b^2) + (k_1 - \alpha)(a + b)^2 + k_3(d^2 + f^2)] + O(\delta^6). \end{aligned}$$

Also

$$\int_{c_N^\delta} W(u, \nabla u) dx_1 dx_2 dx_3 = O(\delta^6), \quad \int_{c_p^\delta} W(u, \nabla u) dx_1 dx_2 dx_3 = O(\delta^6).$$

Finally, we have

$$\begin{aligned} E(u^\delta, \Omega) - E(u, \Omega) - 16\pi \Gamma(k_1, k_2, k_3)\delta \\ = -8k_1\pi\delta^5 \left(g'(0)^2 c^2 - (a^2 + b^2 + 2(g'(0)c)^2 - 2ag'(0)c - bg'(0)c) \ln 2 \right) \\ + 2\pi\delta^5 (k_2 - k_1)(a - b)^2 \ln 2 + O(\delta^6 \ln \delta). \end{aligned}$$

When $k_2 - k_1 \leq k_1 4(1 - \ln 2)/\ln 2$, we choose $g'(0)c = \max\{a, b\}$ to obtain

$$g'(0)^2 c^2 - (a^2 + b^2 + 2(g'(0)c)^2 - (2a + b)g'(0)c) \ln 2 - \frac{k_2 - k_1}{4k_1} (a - b)^2 \ln 2 > 0.$$

Choosing δ sufficiently small, Theorem 1.2 is proved. \square

Remark 3.1. Let $u_0 \in H_\gamma^1(\Omega, S^2)$ for smooth boundary data γ with $\deg(\gamma) \neq 0$. For a map $u \in H_\gamma^1(\Omega, S^2)$, in similar fashion to arguments of [Giaquinta et al. 1998, Chapter 4], there is a one-dimensional rectifiable current L_{u, u_0} with $-\partial L_{u, u_0} = P(u) - P(u_0)$ that minimizes the mass among all one-dimensional rectifiable currents L with $-\partial L = P(u) - P(u_0)$, where $P(u)$ is the zero-dimensional current in Ω determined by u (see [Giaquinta et al. 1998, Chapter 4]). Moreover, we have

$$M(L_{u, u_0}) = L(u, u_0).$$

For $u \in H_\gamma^1(\Omega, S^2)$, consider

$$[[u]] := \{ T = G_{u_T} + L_{u, u_0} \times [[S^2]] \mid T - [[G_{u_0}]] \in \text{Cart}^{2,1}(\Omega \times \mathbb{R}^3), \quad u_T = u \}.$$

Define

$$\mathcal{E}([[u]]) = \int_\Omega W(u(x), \nabla u(x)) dx + 8\pi \Gamma(k_1, k_2, k_2) M(L_{u, u_0}) = F_{u_0}(u, \Omega).$$

Then the semicontinuity \mathcal{E} implies that $F_{u_0}(u, \Omega)$ is also lower semicontinuous with respect to the weak convergence in $H_{u_0}^1(\Omega, S^2)$ (see [Giaquinta et al. 1989; 1998]).

Proof of Theorem 1.1 for $k_2 \geq k_1$. If there are infinitely many distinct minimizers for E in $H_\gamma^1(\Omega, S^2)$, the proof of Theorem 1.1 is completed. Now we assume that there are only a finite number of minimizers w_1, \dots, w_m for E in $H_\gamma^1(\Omega, S^2)$.

By the partial regularity of [Hardt et al. 1986], with the fact that γ is not a constant, there is a new subdomain Ω_1 of Ω such that w_1 is smooth in Ω_1 , and

there is some $x_0 \in \Omega_1$ with $\nabla w_1(x_0) \neq 0$. For sufficiently small ρ , it follows from taking $v_1 = w^\delta$ in Theorem 1.2 that

$$E(v_1) < F_{v_1}(w_1) = E(w_1) + 8\pi\Gamma(k_1, k_2, k_3)L(v_1, w_1).$$

Let u_1 be a minimizer of F_{v_1} in $H_\gamma^1(\Omega, S^2)$, and let u_1 be a weak solution of Equation (1–3) with boundary value γ .

For δ sufficiently small, we shall prove that u_1 is different from all minimizers w_i of E in $H_\gamma^1(\Omega, S^2)$. We have two cases.

- (i) $L(w_k, w_1) = 0$ for some k . It is easy to see $L(v_1, w_1) = L(v_1, w_k)$. Noticing $E(w_1) = E(w_k)$, it follows the minimality of u_1 that

$$F_{v_1}(u_1) \leq E(v_1) < E(w_1) + 8\pi\Gamma(k_1, k_2, k_3)L(v_1, w_1) = F_{v_1}(w_k).$$

This implies that $u_1 \neq w_k$.

- (ii) $L(w_k, w_1) > 0$. We know

$$L(w_k, v_1) + L(v_1, w_1) \geq L(w_k, w_1).$$

This implies

$$\begin{aligned} F_{v_1}(w_k) &= E(w_k) + 8\pi\Gamma(k_1, k_2, k_3)L(w_k, v_1) \\ &\geq E(w_1) + 8\pi\Gamma(k_1, k_2, k_3)(L(w_k, w_1) - 2\rho). \end{aligned}$$

Choose $\rho > 0$ sufficiently small so that

$$0 < 2\rho < \frac{L(w_k, w_1)}{2}.$$

This gives $L(w_k, w_1) - 2\rho > 2\rho > L(v_1, w_1)$. Therefore

$$F_{v_1}(w_k) > F_{v_1}(w_1) \geq F_{v_1}(u_1).$$

So u_1 is different from all minimizers w_k of E .

We construct by induction a sequence u_j of distinct weak solutions of (1–3) in $H_\gamma^1(\Omega, S^2)$ which are also different from the minimizer w_i . Choose ρ_{j+1} such that

$$0 < 2\rho_{j+1} < \min \{L(w_k, w_1)/2, \text{ with } L(w_k, w_1) > 0\}$$

and

$$(3-20) \quad 0 < 2\rho_{j+1} < \min \left\{ \frac{E(u_i) - E(w_1)}{8\pi\Gamma(k_1, k_2, k_3)}, \quad i = 1, \dots, j \right\}.$$

By taking $\rho = \rho_{j+1}$ and $u = w_1$ in Theorem 1.2, there exists a v_{j+1} and $\delta_{j+1} \leq \rho_{j+1}$ such that

$$E(v_{j+1}) < E(w_1) + 16\pi\Gamma(k_1, k_2, k_3)\delta_{j+1}.$$

Let u_{j+1} be a minimizer of $F_{v_{j+1}}$ in $H_\gamma^1(\Omega, S^2)$. The same argument as above assures that u_{j+1} is different from all w_i . Next we prove that $u_{j+1} \neq u_i$ for all $i \leq j$. From the above estimates, we know

$$(3-21) \quad F_{v_{j+1}}(u_{j+1}) \leq F_{v_{j+1}}(v_{j+1}) = E(v_{j+1}) < E(w_1) + 16\pi\Gamma(k_1, k_2, k_3)\rho_{j+1}.$$

From (3-20) we have

$$16\pi\rho_{j+1}\Gamma(k_1, k_2, k_3) < E(u_i) - E(w_1).$$

Combining this with (3-21) yields

$$E(v_{j+1}) \leq F_{v_{j+1}}(u_{j+1}) < E(u_i),$$

which implies $u_{j+1} \neq u_i$ for $i = 2, \dots, j$. Letting $j \rightarrow \infty$, we see that there exist infinitely many solutions $\{u_j\}_{j=1}^\infty$ of (1-3) in $H_\gamma^1(\Omega, S^2)$. This proves Theorem 1.1 for $k_2 \geq k_1$. \square

4. Proof of Theorems 1.1 and 1.2 for $k_1 > k_2$

As in Section 3.1, for a sufficiently small δ and x_3 in $[-\delta + \delta^2, \delta - \delta^2]$, we may choose

$$K(x_3) = u(0, 0, x_3), \quad I(x_3) = \frac{u_{x_1}(0, 0, x_3)}{|u_{x_1}(0, 0, x_3)|}.$$

to form a basis $\{I(x_3), J(x_3), K(x_3)\}$ of \mathbb{R}^3 depending on x_3 . We write

$$u = \hat{u}_1 I(x_3) + \hat{u}_2 J(x_3) + \hat{u}_3 K(x_3)$$

with $\hat{u}_1(0, 0, x_3) = \hat{u}_2(0, 0, x_3) = 0$, $\hat{u}_3(0, 0, x_3) = 1$. There are two numbers $a > 0$ and $b \leq 0$ (for a suitable rotation of \mathbb{R}^3) such that

$$u_{x_1}(0, 0, x_3) = (a + O(x_3))J(x_3), \quad u_{x_2}(0, 0, x_3) = (b + O(x_3))I(x_3) + O(x_3)J(x_3).$$

We consider the cylinder C^δ in \mathbb{R}^3 defined by

$$C^\delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq r \leq \delta + \delta^2, \quad -\delta - \delta^2 \leq x_3 \leq \delta + \delta^2\}.$$

As in Section 3.1, we construct a map $u^\delta = \hat{u}_1^\delta I(x_3) + \hat{u}_2^\delta J(x_3) + \hat{u}_3^\delta K(x_3)$ as follows:

- (i) $u^\delta = u$ outside C^δ .
- (ii) Inside C^δ , for each $x_3 \in [-\delta + \delta^2, \delta - \delta^2]$ (that is, the subcylinder of C^δ), we construct u^δ in three different cases:
 - (a) If $r > 2\delta^2$, we set $u^\delta(x) = u(x)$.

(b) If $r < \delta^2$, we set, with $\rho = r/\lambda$,

$$u^\delta(x_1, x_2, x_3) = g(\rho) \frac{x_2}{r} I(x_3) - g(\rho) \frac{x_1}{r} J(x_3) + \text{sign}(1-r) \sqrt{1-g^2(\rho)} K(x_3).$$

(c) If $\delta^2 \leq r \leq 2\delta^2$, we set

$$u^\delta(x) = (A_1 r + B_1) I(x_3) + (A_2 r + B_2) J(x_3) + \sqrt{1 - (A_1 r + B_1)^2 - (A_2 r + B_2)^2} K(x_3),$$

where A_1, A_2, B_1, B_2 depend only on θ, δ , and x_3 and are determined by

$$\begin{aligned} 2\delta^2 A_1 + B_1 &= \hat{u}_1(2\delta^2 \cos \theta, 2\delta^2 \sin \theta, x_3), \\ 2\delta^2 A_2 + B_2 &= \hat{u}_2(2\delta^2 \cos \theta, 2\delta^2 \sin \theta, x_3), \\ \delta^2 A_1 + B_1 &= g(\delta^2/\lambda) \sin \theta = g(\lambda/\delta^2) \sin \theta, \\ \delta^2 A_2 + B_2 &= -g(\delta^2/\lambda) \cos \theta = -g(\lambda/\delta^2) \cos \theta, \end{aligned} \tag{4-1}$$

and $g(r)$ is the solution of Equation (2-8) with $g(0) = 0, g'(0) > 0, g(1) = 1, g(r) = g(1/r)$ and $\lambda = c\delta^4$, where c will be determined later.

(iii) Inside C^δ , for each

$$x_3 \in [-\delta, -\delta + \delta^2] \cup [\delta, \delta - \delta^2],$$

we let $P = (0, 0, \delta)$ and $N = (0, 0, -\delta)$, in a small cylinder c_P^δ (or c_N^δ). The cylinder is centered at P (or N) with radius $2\delta^2$, length $2\delta^2$, and its axis along the x_3 -axis. If we denote by Π^+ (or Π_-) the radial projection centered at P (or N) onto the boundary of c_P^δ (or c_N^δ), the transformed map u^δ is the composition of Π^+ (or Π_-) and the value of u^δ on this boundary.

The proof of Theorem 1.2 for $k_2 \leq k_1$ is very similar to the one for $k_1 \leq k_2$ on page 195. We only need to make a few modifications.

For $\delta^2 \leq r \leq 2\delta^2$ and for each

$$x_3 \in [-\delta + \delta^2, \delta - \delta^2],$$

we solve Equation (4-1) for A_1, A_2, B_1 and B_2 to obtain

$$\begin{aligned} A_1 &= (2a - cg'(0)) \sin \theta + O(\delta), \\ A_2 &= (2b + cg'(0)) \cos \theta + O(\delta), \\ B_1 &= 2\delta^2(g'(0)c - a) \sin \theta + O(\delta^3), \\ B_2 &= -2\delta^2(g'(0)c + b) \cos \theta + O(\delta^3). \end{aligned} \tag{4-2}$$

and

$$(4-3) \quad \begin{aligned} \frac{\partial A_1}{\partial \theta} &= (2a - cg'(0)) \cos \theta + O(\delta), \\ \frac{\partial A_2}{\partial \theta} &= -(2b + cg'(0)) \sin \theta + O(\delta), \\ \frac{\partial B_1}{\partial \theta} &= 2\delta^2(cg'(0) - a) \cos \theta + O(\delta^3), \\ \frac{\partial B_2}{\partial \theta} &= 2\delta^2(cg'(0) + b) \sin \theta + O(\delta^3). \end{aligned}$$

Using (4-2) and (4-3), we have

$$\begin{aligned} \frac{\partial \hat{u}_1^\delta}{\partial x_1} &= \left(\frac{\partial A_1}{\partial \theta} r + \frac{\partial B_1}{\partial \theta} \right) \frac{\partial \theta}{\partial x_1} + A_1 \frac{\partial r}{\partial x_1} + O(\delta) \\ &= -2\delta^2(cg'(0) - a) \frac{\cos \theta \sin \theta}{r} + O(\delta), \\ \frac{\partial \hat{u}_2^\delta}{\partial x_2} &= \frac{\partial(A_2 r + B_2)}{\partial x_2} = 2\delta^2(cg'(0) + b) \frac{\sin \theta \cos \theta}{r} + O(\delta), \\ \frac{\partial \hat{u}_1^\delta}{\partial x_2} &= \frac{\partial(A_1 r + B_1)}{\partial x_2} = (2a - cg'(0)) + 2\delta^2(cg'(0) - a) \frac{\cos^2 \theta}{r} + O(\delta), \\ \frac{\partial \hat{u}_2^\delta}{\partial x_1} &= \frac{\partial(A_2 r + B_2)}{\partial x_1} = (2b + cg'(0)) - 2\delta^2(cg'(0) + b) \frac{\sin^2 \theta}{r} + O(\delta). \end{aligned}$$

Consider a new map $\hat{u}^\delta = \hat{u}_1^\delta I + \hat{u}_2^\delta J + \hat{u}_3^\delta K$. Then we have

$$\begin{aligned} |\operatorname{div} \hat{u}^\delta|^2 &= 4\delta^4(a+b)^2 \sin^2 \theta \cos^2 \theta \frac{1}{r^2} + O(\delta), \\ |\hat{u}^\delta \times \operatorname{curl} \hat{u}^\delta|^2 &= O(\delta), \\ (\operatorname{curl} \hat{u}^\delta)^2 &= \left[2(a-b) - 2cg'(0) + \frac{2\delta^2 cg'(0)}{r} - 2\delta^2 \frac{(a \cos^2 \theta - b \sin^2 \theta)}{r} \right]^2 + O(\delta). \end{aligned}$$

Using this, we have

$$\begin{aligned} &\int_{\delta^2 \leq r \leq 2\delta^2} (\operatorname{div} \hat{u}^\delta)^2 + |\operatorname{curl} \hat{u}^\delta|^2 dx_1 dx_2 \\ &= 4\pi \delta^4 \left((a-b)^2 - g'(0)^2 c^2 + (a^2 + b^2 + 2(g'(0)c)^2 - (2a-2b)g'(0)c) \ln 2 \right). \end{aligned}$$

We know that

$$\begin{aligned}
 |\hat{u}_{x_1}^\delta|^2 + |\hat{u}_{x_2}^\delta|^2 &= 4\delta^4(g'(0)c - a)^2 \frac{\cos^2 \theta}{r^2} + 4\delta^4(g'(0)c + b)^2 \frac{\sin^2 \theta}{r^2} + (2a - g'(0)c)^2 \\
 &\quad + (2b + g'(0)c)^2 + 4\delta^2(2a - g'(0)c)(g'(0)c - a) \frac{\cos^2 \theta}{r} \\
 &\quad - 4\delta^2(2b + g'(0)c)(g'(0)c + b) \frac{\sin^2 \theta}{r}.
 \end{aligned}$$

Using this, we have

$$\begin{aligned}
 \int_{\delta^2 \leq r \leq 2\delta^2} |\nabla \hat{u}^\delta|^2 dx_1 dx_2 &= \int_{\delta^2 \leq r \leq 2\delta^2} [|\hat{u}_{x_1}^\delta|^2 + |\hat{u}_{x_2}^\delta|^2 + O(\delta)] dx_1 dx_2 \\
 &= 4\pi\delta^4 \left(a^2 + b^2 - \frac{g'(0)^2 c^2}{2} + (a^2 + b^2 + 2(g'(0)c)^2 - 2(a - b)g'(0)c) \ln 2 \right) \\
 &\quad + O(\delta^5).
 \end{aligned}$$

From arguments similar to those in Section 3.2, we finally have

$$\begin{aligned}
 E(u^\delta, \Omega) - E(u, \Omega) - 16\pi\Gamma(k_1, k_2, k_3)\delta \\
 &= -8k_1\pi\delta^5 \left(g'(0)^2 c^2 - (a^2 + b^2 + 2(g'(0)c)^2 - 2(a - b)g'(0)c) \ln 2 \right) \\
 &\quad + 2\pi\delta^5 (k_2 - k_1)(a + b)^2 \ln 2 + O(\delta^6 \ln \delta).
 \end{aligned}$$

When $0 \leq k_1 - k_2 \leq k_2 4(1 - \ln 2)/\ln 2$, we choose $g'(0)c = \max\{a, -b\}$ to obtain

$$g'(0)^2 c^2 - (a^2 + b^2 + 2(g'(0)c)^2 - 2(a - b)g'(0)c) \ln 2 - \frac{k_1 - k_2}{4k_1} (a + b)^2 \ln 2 > 0.$$

Theorem 1.2 follows from choosing δ sufficiently small.

The proof of Theorem 1.1 for the case $k_1 > k_2$ is the same as one for the case $k_2 \geq k_1$ in Section 3.2. We omit details here. \square

5. Partial regularity of the weak solutions

We will now complete a proof of Theorem 1.3. We recall that

$$W(u, p) = \alpha|p|^2 + (k_1 - \alpha)(\operatorname{tr} p)^2 + (k_2 - \alpha)(g \cdot u)^2 + (k_3 - \alpha)|g \times u|^2,$$

where $p = (p_i^j)_{3 \times 3}$ and g is the axial vector of $p - p^T$, that is, the vector defined in coordinates by

$$g_i = \varepsilon_{ijk} p_j^k$$

with ε_{ijk} being the components of the Levi-Civita tensor. For simplicity, we assume $\alpha = 1$.

There exists a positive constant $\Lambda > 0$ such that

$$|p|^2 \leq W(u, p) \leq \Lambda|p|^2.$$

Lemma 5.1. For any $\lambda \in [0, 1)$, let u_λ be a minimizer of E_λ in $H_\gamma^1(\Omega, S^2)$. Then u_λ is a quasiminimizer of the functional E in $H_\gamma^1(\Omega, S^2)$, that is, $E(u_\lambda, B) \leq QE(w; B)$ for any $w \in H_{u_\lambda}^1(B, S^2)$ and any subdomain $B \subset \Omega$ with $Q = (1 + \lambda)/(1 - \lambda)$.

Proof. Let R_γ^∞ be a set of all maps in $H_\gamma^1(\Omega, S^2)$ having a finite number of singular points, of which $\{P_i\}$ are of positive degree $+1$ and $\{N_i\}$ are of negative degree -1 inside Ω . Let u be a map in R_γ^∞ .

As in [Giaquinta et al. 1989; 1998], the function

$$\Gamma(n, \xi) := \inf \{W(n, G) \mid M_2(G) = \xi, \quad G^T n = 0\}$$

is given at every $n \in S^2$ and $\xi = t \wedge \epsilon(n) \in \wedge_3(\mathbb{R}^3 \times T_n S^2)$, where $|t| = 1$, $\epsilon(n)$ is the unit 2-vector associated to $T_n S^2$, and G^T is the transpose of the matrix G . A calculation (see [Giaquinta et al. 1989; 1998]) yields

$$\Gamma(n, \xi) = 2\sqrt{k^2(t, n)^2 + kk_3(1 - (t, n)^2)^2} + (k - \alpha)(t, n).$$

Thus

$$W(u(x), \nabla u(x)) \geq \Gamma(n, M_2(\nabla u(x))) = \Gamma\left(n, \frac{D(u(x))}{|D(u(x))|} \wedge \epsilon(n)\right) |M_2(\nabla u(x))|.$$

Integrating over B and using the co-area formula, we then have

$$\begin{aligned} E(u, B) &\geq \int_B \Gamma\left(n, \frac{D(u(x))}{|D(u(x))|} \wedge \epsilon(n)\right) |M_2(\nabla u(x))| dx \\ &\geq \int_{S^2} d\mathcal{H}^2(n) \int_{u^{-1}(n)} \Gamma\left(n, \frac{D(u(x))}{|D(u(x))|} \wedge \epsilon(n)\right) d\mathcal{H}^1. \end{aligned}$$

We know that $u^{-1}(n)$ is the union of curves of two kind of curves oriented by $D(u)/|D(u)|$:

- (i) closed curves $\Gamma_1^u \cup \Gamma_2^u \cup \dots \cup \Gamma_l^u$;
- (ii) curves joining $\partial B \cap \{P_i^u\}_{i=1}^k \cap \{N_i^u\}_{i=1}^k$, where $\{P_i^u, N_i^u\}_{i=1}^k$ are all singularities of u inside B .

For any positive singularity P_i^u , there is a curve $C_i(u)$ joining P_i^u to another point \tilde{N}_i^u , which is either a negative singularity of the map u or a point y_i on the boundary ∂B .

Since $\Gamma(n, \cdot \wedge \epsilon(n))$ is convex and one-homogeneous, Jensen's inequality implies

$$\int_{C_i} \Gamma\left(n, \frac{D(u(x))}{|D(u(x))|} \wedge \epsilon(n)\right) d\mathcal{H}^1 \geq \Gamma\left(n, \int_{C_i} \frac{D(u(x))}{|D(u(x))|} d\mathcal{H}^1 \wedge \epsilon(n)\right)$$

Note that for any vector t with $|t| = 1$, we have [Giaquinta et al. 1990]

$$\int_{S^2} \Gamma(n, t \wedge \epsilon(n)) d\mathcal{H}^2(n) = 8\pi \Gamma(k_1, k_2, k_3).$$

Let $w \in R_v^\infty$ and $w - u \in H_0^1(B, \mathbb{R}^3)$. Then $w^{-1}(n)$ is the union of curves of two kind of curves oriented by $D(w)/|D(w)|$:

(iii) closed curves $\Gamma_1^w \cup \Gamma_2^w \cup \dots \cup \Gamma_l^w$;

(iv) curves joining $\partial B \cap \{P_i^w\}_{i=1}^k \cap \{N_i^w\}_{i=1}^m$, where $\{P_i^w, N_i^w\}_{i=1}^m$ are all singularities of w inside B .

If \tilde{N}_i^u is a boundary point y_i with $u(y_i) = n$ joining a curve to a positive singularity P_i^u by a curve $C^i(u)$ inside the set $u^{-1}(n)$, there is a positive singularity P_i^w of w joining to y_i by a curve $C^i(w)$ inside the set $w^{-1}(n)$. As in [Giaquinta et al. 1998], we note that $D(u(x))$ is the tangent to the level line $u(x) = n$. For an oriented curve $C_i(u)$ joining P_i^u to \tilde{N}_i^u and a curve $C_i(w)$ joining P_i^w to \tilde{N}_i^w , we have

$$\int_{C_i(u)} \frac{D(u(x))}{|D(u(x))|} d\mathcal{H}^1 = -(P_i^u - \tilde{N}_i^u), \quad \int_{C_i(w)} \frac{D(w(x))}{|D(w(x))|} d\mathcal{H}^1 = -(P_i^w - \tilde{N}_i^w),$$

where \tilde{N}_i^u is either a negative singularity of u or a boundary y_i with $u(y) = n$, and \tilde{N}_i^w is either a negative singularity of w or a boundary y_i with $w(y) = n$.

Then

$$\begin{aligned} E(u, B) + E(w, B) &\geq \int_{S^2} d\mathcal{H}^2(n) \sum_{i=1}^k \Gamma\left(n, \int_{C_i(u)} \frac{D(u(x))}{|D(u(x))|} d\mathcal{H}^1 \wedge \epsilon(n)\right) \\ &\quad + \int_{S^2} d\mathcal{H}^2(n) \sum_{i=1}^m \Gamma\left(n, \int_{C_i(w)} \frac{D(w(x))}{|D(w(x))|} d\mathcal{H}^1 \wedge \epsilon(n)\right) \\ &= \sum_{i=1}^k |P_i^u - \tilde{N}_i^u| \int_{S^2} \Gamma\left(n, \frac{P_i^u - \tilde{N}_i^u}{|P_i^u - \tilde{N}_i^u|} \wedge \epsilon(n)\right) d\mathcal{H}^2(n) \\ &\quad + \sum_{i=1}^m |P_i^w - \tilde{N}_i^w| \int_{S^2} \Gamma\left(n, \frac{P_i^w - \tilde{N}_i^w}{|P_i^w - \tilde{N}_i^w|} \wedge \epsilon(n)\right) d\mathcal{H}^2(n) \\ &= 8\pi \Gamma(k_1, k_2, k_3) \left(\sum_{i=1}^k |P_i^u - \tilde{N}_i^u| + \sum_{i=1}^m |P_i^w - \tilde{N}_i^w| \right) \\ &\geq 8\pi \Gamma(k_1, k_2, k_3) L(u, w), \end{aligned}$$

where $L(u, w)$ is the minimal connection of w and u , that is, the minimal connection between $\{P_i^u\}_{i=1}^k \cup \{N_i^w\}_{i=1}^m$ and $\{N_i^u\}_{i=1}^k \cup \{P_i^w\}_{i=1}^m$.

By the density result of [Bethuel 1990], the last equation is true for all $w, u \in H_\gamma^1(B, S^2)$ with $w - u \in H_0^1(B, S^2)$.

Now, taking $u = u_\lambda$ for $0 \leq \lambda < 1$, let w be any map $H_\gamma^1(\Omega, S^2)$ with $u_\lambda - w \in H_0^1(B, S^2)$ with an arbitrary subdomain $B \subset \Omega$. By the minimality of u_λ , we have

$$E(u_\lambda, \Omega) + \lambda 8\pi \Gamma(k_1, k_2, k_3)L(u_\lambda, u_0) \leq E(w; \Omega) + \lambda 8\pi \Gamma(k_1, k_2, k_3)L(w, u_0),$$

Moreover, we know

$$L(w, u_0) - L(u_\lambda, u_0) \leq L(w, u_\lambda).$$

For $0 \leq \lambda < 1$, we have

$$E(u_\lambda; B) \leq \frac{1 + \lambda}{1 - \lambda} E(w; B)$$

for all $w \in H_{u_\lambda}^1(B, S^2)$. This proves our claim. \square

Using Lemma 5.1 with an extension lemma in [Hardt et al. 1988], we have:

Proposition 5.2 (Caccioppoli's inequality). *For any $0 \leq \lambda < 1$, let u_λ be a minimizer of E_λ in $H_\gamma^1(\Omega, S^2)$. Then for all $x_0 \in \Omega$ and $R < \text{dist}(x_0, \partial\Omega)$, we have*

$$(5-1) \quad \int_{B_{R/2}(x_0)} |\nabla u_\lambda|^2 \leq C R^{-2} \int_{B_R(x_0)} |u_\lambda - u_{\lambda, x_0, R}|^2 dx.$$

Next, we have:

Proposition 5.3 [Hong 2004]. *Let $u \in H^1(\Omega, S^2)$ be any weak solution of (1-3) and assume that u satisfies the Caccioppoli inequality (5-1). Then u is smooth in an open set $\Omega_0 \subset \Omega$ and $\mathcal{H}^\beta(\Omega \setminus \Omega_0) = 0$ for some positive $\beta < 1$.*

Proof of Theorem 1.3. It follows from Propositions 5.2 and 5.3. \square

Finally, it seems that there exists no monotonicity formula for the minimizers of F_{u_0} in H^1 . It is a challenging question whether one can establish the partial regularity of minimizers of F_{u_0} in $H^1(\Omega, S^2)$ for a given map $u_0 \in R_\gamma^\infty$.

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