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CHUN-GEN LIU

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We study the multiplicity of the solutions of certain asymptotically linear Hamiltonian systems with a Lagrangian boundary condition.

1. Introduction and main results

We consider the solutions of the nonlinear Hamiltonian systems with Lagrangian boundary condition

$$\dot{x}(t) = JH'(t, x(t)), \qquad x(0) \in L, \quad x(1) \in L. \tag{1–1}$$

where $x(t) \in \mathbb{R}^{2n}$ and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

is the standard symplectic matrix with I_n the identity in $\mathbb{R}^n$, and $L \in \Lambda(n)$, where $\Lambda(n)$ is the set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$ with standard symplectic form $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. The Hamiltonian function $H \in C^2([0, 1] \times \mathbb{R}^{2n}, \mathbb{R})$ satisfies these conditions:

- (H_0): $H'(t, 0) \equiv 0$, $t \in [0, 1]$.
- (H_∞): There exist continuous symmetric matrix functions $B_1(t)$ and $B_2(t)$ with $i_L(B_1) = i_L(B_2)$, $\nu_L(B_2) = 0$ such that

  $$B_1(t) \le H''(t, x) \le B_2(t)$$

  for all (t, x) with $|x| \ge r$ for some large $r > 0$ and for all $t \in [0, 1]$.

For two symmetric matrices A and B, $A \ge B$ means that $A - B$ is a semipositive definite matrix, and $A > B$ similarly means that $A - B$ is a positive definite matrix.

For a Lagrangian subspace L of the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, [Liu 2007] defined the Maslov-type index pair $(i_L(B), \nu_L(B)) \in \mathbb{Z} \times \{0, 1, \dots, n\}$ for a continuous symmetric matrix function $B : [0, 1] \to \mathrm{L}_s(2n)$ (here $\mathrm{L}_s(2n)$ is the

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set of symmetric $2n \times 2n$ matrices). In the Appendix, we give a brief introduction of this index theory.

Theorem 1.1. *Let H satisfy conditions (H_0) and (H_∞). Suppose $JB_1(t) = B_1(t)J$ and $B_0(t) = H''(t, 0)$ satisfying one of the twisted conditions*

$$B_1(t) + kI \le B_0(t), \tag{1–2}$$

$$B_0(t) + kI \le B_1(t), \tag{1–3}$$

for some constant $k \ge \pi$. Then (1–1) *possesses at least one nontrivial solution. If $\nu_L(B_0) = 0$, the system* (1–1) *possesses at least two nontrivial solutions.*

For the periodic solutions of a asymptotically linear Hamiltonian system, we refer to [Chang 1981; Long 1993; Conley and Zehnder 1984; Liu 2005b]. We note that we only need to prove the case

$$L = L_0 = \{0\} \oplus \mathbb{R}^n.$$

The reason is that there is an orthogonal symplectic matrix P such that $PL = L_0$. All the conditions hold in (1–1) after taking $z(t) = Px(t)$ there. We note that the problem (1–1) is related to the Bolza problem (see [Clarke and Ekeland 1982; Ekeland 1990]).

We should briefly review the general study of the problem (1–1). For a general symplectic manifold (M, ω) (usually closed, that is, compact without boundary; an example nonclosed case is the cotangent bundle of a closed Riemannian manifold with the zero section as the Lagrangian submanifold) and a closed Lagrangian submanifold $L \subset M$, the problem (1–1) has been widely studied. The multiplicity problems of Hamiltonian systems on a symplectic manifold with Lagrangain boundary values are related to Arnold's conjecture about Lagrangian intersections. The autonomous case of this problem in $\mathbb{R}^{2n}$ is related to the Arnold chord conjecture. Generally, a Hamiltonian flow starting from a Lagrangian submanifold does not necessary return to the Lagrangian submanifold again. Arnold conjectured that, under some conditions, the Lagrangian intersection number has a lower bound estimated by the sum of all Beti numbers of the Lagrangian submanifold in the nondegenerate case; this sum is in turn estimated by the cup-length of the Lagrangian submanifold (see for example [Conley and Zehnder 1984; Hofer 1988; Floer 1988, 1989; Oh 1995; Ono 1996; Chekanov 1996, 1998; Liu 2005a]). For the Arnold chord conjecture, we mention [Arnold 1986; Mohnke 2001]. The multiplicity of the fixed energy problem (1–1) was studied in [Guo and Liu 2007]. The main differences between this work and the others are that here the symplectic manifold and the Lagrangian submanifold are not compact and all the topological data of the Lagrangian submanifold are trivial.

2. Some further properties of the Maslov type index theory

Liu [2007] developed some important properties of the L-index theory. In this section we study the relation between the L-index of solutions of Hamiltonian systems with L-boundary conditions and the Morse index of the corresponding functional defined via the Galerkin approximation method on the finite-dimensional truncated space at its corresponding critical points. Fei and Qiu [1996] treated the periodic case.

The eigenspace E_k of the operator $A = -Jd/dt$ in the domain

$$W_{L_0}^{1,2}([0,1],\ \mathbb{R}^{2n}) := \{z \in W^{1,2}([0,1],\mathbb{R}^{2n}) : z(0) \in L_0,\ z(1) \in L_0\}$$

can be written as

$$E_k = -J\exp(k\pi tJ)a_k = -J(\cos(k\pi t)I_{2n} + J\sin(k\pi t))a_k,$$
$$a_k = (a_{k1}, \cdots, a_{kn}, 0, \cdots, 0) \in \mathbb{R}^{2n}.$$

We define a Hilbert space

$$\mathrm{W}_{L_0} = W_{L_0}^{1/2,2}([0,1],\mathbb{R}^{2n}) \subset \bigoplus_{k\in\mathbb{Z}} E_k$$

with L_0 boundary conditions

$$\mathrm{W}_{L_0} = \left\{ z \in L^2 \;\middle|\; z(t) = \sum_{k\in\mathbb{Z}} -J\exp(k\pi tJ)a_k,\ \|z\|^2 := \sum_{k\in\mathbb{Z}}(1+|k|)|a_k|^2 < \infty \right\}.$$

We denote its inner product by $\langle\cdot,\cdot\rangle$. By the well-known Sobolev embedding theorem, for any $s \in [1,+\infty)$, there is a constant $C_s > 0$ such that

$$\|z\|_{L^s} \le C_s\,\|z\| \quad \text{for all } z \in \mathrm{W}_{L_0}.$$

For any Lagrangian subspace $L \in \Lambda(n)$, suppose $P \in \mathrm{Sp}(2n) \cap O(2n)$ such that $L = PL_0$. Then we define $\mathrm{W}_L = P\mathrm{W}_{L_0}$. We denote by

$$\mathrm{W}_{L_0}^m = \bigoplus_{k=-m}^{m} E_k = \left\{ z \;\middle|\; z(t) = \sum_{k=-m}^{m} -J\exp(k\pi tJ)a_k \right\}$$

the finite dimensional truncation of W_{L_0}, and $\mathrm{W}_L^m = P\mathrm{W}_{L_0}^m$.

Let $P^m = P_L^m : \mathrm{W}_L \to \mathrm{W}_L^m$ be the orthogonal projection for $m \in \mathbb{N}$. Then $\Gamma = \{P^m;\ m \in \mathbb{N}\}$ is a Galerkin approximation scheme with respect to A defined in (2–2) below, that is,

$$P^m \to I \text{ strongly as } m \to \infty \quad \text{and} P^m A = AP^m.$$

In this section we still consider the problem (1–1), with H satisfying

$$|H''(t,z)| \le a(1+|z|^p) \quad \text{for all } (t,z) \in \mathbb{R}\times\mathbb{R}^{2n} \tag{2–1}$$

and for some $a > 0$, $p > 1$. We consider the functional on W_L

$$f(z) = \int_0^1 \left(\tfrac{1}{2}(-J\dot{z}, z) - H(t, z)\right) dt = \tfrac{1}{2}\langle Az, z\rangle - g(z), \quad z \in W_L. \tag{2–2}$$

A critical point of f on W_L is a solution of (1–1). For a critical point $z = z(t)$, we denote $B(t) = H''(t, z(t))$ and define an operator B on W_L by

$$\langle Bz, w\rangle = \int_0^1 (B(t)z, w)dt.$$

Using the Floquet theory we have

$$\nu_L(B) = \dim\ker(A - B). \tag{2–3}$$

For $\delta > 0$, we denote by $m_\delta^*(\cdot)$, where $* = +, 0, -$, the dimension of the total eigenspace corresponding to the eigenvalue λ belonging to $[\delta, +\infty)$, $(-\delta, \delta)$, $(-\infty, -\delta]$, respectively, and denote by $m^*(\cdot)$, where again $* = +, 0, -$ the dimension of the total eigenspace corresponding to the eigenvalue λ belonging to $(0, +\infty)$, $\{0\}$, $(-\infty, 0)$, respectively. For any adjoint operator L, we define $L^\sharp = (L|_{ImL})^{-1}$, and we also define $P^m L P^m = (P^m L P^m)|_{W_L^m}$. The following result is adapted from [Fei and Qiu 1996], where the periodic boundary condition was considered (see also [Long 1993]).

Theorem 2.1. *For any $B(t) \in C([0, 1], L_s(\mathbb{R}^{2n}))$ having the pair of L indexes $(i_L(B), \nu_L(B))$ and any constant $0 < \delta \leq \frac{1}{4}\|(A - B)^\sharp\|$, there exists $m_0 > 0$ such that for $m \geq m_0$, we have*

$$\begin{aligned} m_\delta^+(P^m(A - B)P^m) &= mn - i_L(B) - \nu_L(B), \\ m_d^-(P^m(A - B)P^m) &= mn + i_L(B) + n, \\ m_\delta^0(P^m(A - B)P^m) &= \nu_L(B). \end{aligned} \tag{2–4}$$

Proof. We follow the ideas of [Fei and Qiu 1996].

Step 1. There is an $m_1 > 0$ such that for $m \geq m_1$

$$\dim\ker(P^m(A - B)P^m) \leq \dim\ker(A - B). \tag{2–5}$$

In fact, by contradiction it is easy to show that there is a constant $m_2 > 0$ such that for $m \geq m_2$

$$\dim P^m \ker(A - B) = \dim\ker(A - B). \tag{2–6}$$

Since B is compact, there is $m_1 \geq m_2$ such that for $m \geq m_1$

$$\|(I - P^m)B\| \leq 2\delta.$$

Take $m \geq m_1$, and let $\mathrm{W}_L^m = P^m \ker(A-B) \oplus Y^m$. Then $Y^m \subset \mathrm{Im}(A-B)$. For $y \in Y^m$ we have

$$y = (A-B)^\sharp (A-B) y = (A-B)^\sharp (P^m (A-B) P^m y + (P^m - I) B y).$$

This implies

$$\|y\| \leq \frac{1}{2\delta} \left\| P^m (A-B) P^m y \right\| \quad \text{for all } y \in Y^m. \tag{2–7}$$

By (2–6) and (2–7) we have (2–5).

Step 2. We distinguish two cases.

Case 1: $\nu_L(B) = 0$. By (2–3) and step 1 we obtain for $m \geq m_1$ that

$$m^0(P^m(A-B)P^m) = \dim \ker(A-B) = 0.$$

Since B is compact, there exists $m_3 \geq m_1$ such that, for $m \geq m_3$,

$$\|(I - P^m)B\| \leq \tfrac{1}{2} \|(A-B)^\sharp\|^{-1}.$$

Then $P^m(A-B)P^m = (A-B)P^m + (I-P^m)BP^m$ implies that

$$\|P^m(A-B)P^m z\| \geq \tfrac{1}{2} \|(A-B)^\sharp\|^{-1} \|z\| \quad \text{for all } z \in \mathrm{W}_L^m.$$

Thus the eigen-subspace $M_\delta^*(P^m(A-B)P^m)$ with eigenvalue λ belonging to the intervals $m_\delta^*(P^m(A-B)P^m)$ and the eigen-subspace $M^*(P^m(A-B)P^m)$ satisfy

$$M_\delta^*(P^m(A-B)P^m) = M^*(P^m(A-B)P^m) \quad \text{for } * = +, 0, -.$$

By Equation (A.5), there is $m_0 \geq m_3$ such that for $m \geq m_0$ the relation (2–4) holds.

Case 2: $\nu_L(B) > 0$. By step 1, it is easy to show that there exists $m_4 > 0$ such that for $m \geq m_4$

$$m_\delta^0(P^m(A-B)P^m) \leq \nu_L(B). \tag{2–8}$$

Let $\gamma \in \mathrm{P}(2n)$ be the fundamental solution of the linear Hamiltonian system

$$\dot{z} = JB(t)z.$$

Let γ_s, $0 \leq s \leq 1$ be the perturbed path defined by Equation (A.4). Define

$$B_s(t) = -J\dot{\gamma}_s(t)\gamma_s(t)^{-1},\ t \in [0,1].$$

Let B_s be the compact operator defined as B corresponding to $B_s(t)$. For $s \neq 0$, there holds $m^0(A - B_s) = 0$ and $\|B_s - B\| \to 0$ as $s \to 0$. If $s \in (0,1]$, we have

$$i_L(\gamma_s) - i_L(\gamma_{-s}) = \nu_L(\gamma) = \nu_L(B),\ i_L(\gamma_{-s}) = i_L(B) = i_L(\gamma). \tag{2–9}$$

Choose $0 < s < 1$ such that $\|B - B_{\pm s}\| \le \delta/2$. By case 1, (2–8), (2–9) and that

$$P^m(A - B_\pm)P^m = P^m(A - B)P^m + P^m(B - B_\pm)P^m,$$

there exists $m_0 \ge m_4$ such that for $m \ge m_0$

$$m_\delta^+(P^m(A - B)P^m) \le m^+(P^m(A - B_s)P^m) = mn - i_L(B) - \nu_L(B),$$

$$\begin{aligned} m_\delta^+(P^m(A - B)P^m) &\ge m^+(P^m(A - B_{-s})P^m) - m_\delta^0(P^m(A - B)P^m) \\ &\ge mn - i_L(B) - \nu_L(B). \end{aligned}$$

Hence, $m_\delta^0(P^m(A - B)P^m) = \nu_L(B)$ and

$$m_\delta^+(P^m(A - B)P^m) = mn - i_L(B) - \nu_L(B).$$

Note that $\dim \mathrm{W}_L^m = (2m + 1)n$, so

$$m_\delta^-(P^m(A - B)P^m) = mn + n + i_L(B). \qquad \square$$

Corollary 2.2. *Let* $B_j(t) \in C([0, 1], \mathrm{L}_s(\mathbb{R}^{2n}))$, $j = 1, 2$. *Assume* $B_1(t) < B_2(t)$, *that is*, $B_2(t) - B_1(t)$ *is positive definite for all* $t \in [0, 1]$. *Then there holds*

$$i_L(B_1) + \nu_L(B_1) \le i_L(B_2).$$

Proof. Just as in Theorem 2.1, corresponding to $B_j(t)$ we have the operator B_j. Let $\Gamma = \{P^m\}$ be the approximation scheme with respect to the operator A. Then by (2–4), there exists $m_0 > 0$ such that if $m \ge m_0$ there holds

$$m_\delta^-(P^m(A - B_1)P^m) = mn + n + i_L(B_1),$$

$$m_\delta^-(P^m(A - B_2)P^m) = mn + n + i_L(B_2),$$

where we choose $0 < \delta < \|B_2 - B_1\|/2$. Since $A - B_2 = (A - B_1) - (B_2 - B_1)$ and $B_2 - B_1$ is positive definite in $\mathrm{W}_L^m = P^m \mathrm{W}_L$ and $\langle (B_2 - B_1)x, x\rangle \ge 2\delta\|x\|$, we have $\langle (P^m(A - B_2)P^m)x, x\rangle \le -\delta\|x\|$ with

$$x \in M_\delta^-(P^m(A - B_1)P^m) \oplus M_\delta^0(P^m(A - B_1)P^m).$$

This implies that $mn + n + i_L(B_1) + \nu_L(B_1) \le mn + n + i_L(B_2)$. $\square$

Remark. From the proof of Corollary 2.2, it is easy to show that if $B_1(t) \le B_2(t)$ for all $0 \le t \le 1$,

$$i_L(B_1) \le i_L(B_2), \quad i_L(B_1) + \nu_L(B_1) \le i_L(B_2) + \nu_L(B_2).$$

Definition 2.3. For any two matrix functions $B_j \in C([0, 1], \mathrm{L}_s(\mathbb{R}^{2n}))$, $j = 0, 1$ with $B_0(t) < B_1(t)$ for all $t \in \mathbb{R}$, we define

$$I_L(B_0, B_1) = \sum_{s \in [0,1)} \nu_L((1 - s)B_0 + sB_1).$$

Theorem 2.4. *For any two matrix functions $B_j \in C([0,1], \mathrm{L}_s(\mathbb{R}^{2n}))$ with $B_0(t) < B_1(t)$ for all $t \in \mathbb{R}$, we have*

$$I_L(B_0, B_1) = i_L(B_1) - i_L(B_0). \tag{2–10}$$

So we call $I_L(B_0, B_1)$ the relative L-index of the pair (B_0, B_1).

Proof. Step 1. By Corollary 2.2, if we denote $i_L(\lambda) = i_L((1-\lambda)B_0 + \lambda B_1)$, $\nu_L(\lambda) = \nu_L((1-\lambda)B_0 + \lambda B_1)$, there holds

$$i_L(\lambda_2) \geq i_L(\lambda_1) + \nu_L(\lambda_1), \text{ for } \lambda_2 > \lambda_1. \tag{2–11}$$

So the function $i_L(\lambda)$ is a monotone function in $[0, 1]$.

Step 2. We prove that for any $\lambda \in [0, 1)$ there holds

$$i_L(\lambda + 0) = i_L(\lambda) + \nu_L(\lambda),$$

where $i_L(\lambda + 0)$ is the right limit of $i_L(s)$ at λ. In fact, by (2–11), we have $i_L(\lambda) + \nu_L(\lambda) \leq i_L(\lambda + 0)$. We now use the saddle point reduction methods to prove the opposite inequality $i_L(\lambda) + \nu_L(\lambda) \geq i_L(\lambda + 0)$. Define $B_\lambda(t) = (1-\lambda)B_0(t) + \lambda B_1(t)$. We define in $L^2([0, 1], \mathbb{R}^{2n})$

$$f_\lambda(x) = \int_0^1 [(-J\dot{x}(t), x(t)) - (B_\lambda(t)x(t), x(t))]\, dt \quad \text{for all } x \in \operatorname{dom}(A) = W_L.$$

Then by the saddle point reduction methods (see Equation (A.5)), we can reduce the functional f_λ in $L^2([0, 1], \mathbb{R}^{2n})$ to a finite-dimensional subspace X of $L^2([0, 1], \mathbb{R}^{2n})$ by $a_\lambda(x) = f_\lambda(u_\lambda(x))$, where $u_\lambda : X \to L^2([0, 1], \mathbb{R}^{2n})$ is injective, and a_λ is continuous in λ. Denote the Morse indices of a_λ on X at $x = 0$ by m_λ^-, m_λ^0 and m_λ^-. If $\dim X = 2d + n$ large enough, we have from (A.5)

$$m_\lambda^- = d + n + i_L(\lambda), \quad m_\lambda^0 = \nu_L(\lambda), \quad m_\lambda^+ = d - i_L(\lambda) - \nu_L(\lambda). \tag{2–12}$$

For any fixed $\lambda \in [0, 1)$, choosing $\mu \in (\lambda, 1) \cup [0, \lambda)$ sufficiently close to λ, we obtain

$$m_\lambda^\pm \leq m_\mu^\pm \leq m_\lambda^\pm + \nu_L(\lambda).$$

Then by (2–12), we have $i_L(\lambda) \leq i_L(\mu)$ and $i_L(\lambda) + \nu_L(\lambda) \geq i_L(\mu)$. This implies $i_L(\lambda) + \nu_L(\lambda) \geq i_L(\lambda + 0)$ and $i_L(\lambda) \leq i_L(\lambda - 0)$. But by (2–11), we have $i_L(\lambda) \geq i_L(\lambda - 0)$, so $i_L(\lambda) = i_L(\lambda - 0)$. That is to say, the function $i_L(\lambda)$ is left continuous at $(0, 1]$. Moreover if $m_\lambda^0 = m^0$ is constant in some interval $[\lambda_1, \lambda_2]$, then $m_\lambda^- = m^-$ and $m_\lambda^+ = m^+$ are constant in this interval. Thus the function $i_L(\lambda)$ is locally constant at its continuous points, its discontinuous points are those with with $\nu_L(\lambda) > 0$, and there holds

$$i_L(1) = i_L(0) + \sum_{0 \leq \lambda < 1} \nu_L(\lambda),$$

which is exactly (2–10). □

Corollary 2.5. *If* $\gamma \in \mathrm{P}(2n)$ *is the fundamental solution of the linear Hamiltonian system with respect to* $B(t) > 0$, *there holds*

$$i_L(\gamma) = \sum_{0<t<1} \dim(\gamma(t)L \cap L). \tag{2–13}$$

Thus we can understand the index $i_L(\gamma)$ *as a kind of intersection number of the two Lagrangian paths* $w(t) = \gamma(t)L$ *and* $w_0(t) = L$.

Proof. We take $B_1(t) = B(t)$ and $B_0(t) = 0$ in Theorem 2.4. We note that the fundamental solution corresponding to $B_0(t) = 0$ is the constant path I. We have

$$I_L(0, B) = i_L(\gamma) - i_L(I).$$

But $i_L(I) = i_{L_0}(I) = -n$ and $B_s(t) = (1-s)B_0(t) + sB_1(t) = sB(t)$. The corresponding fundamental solution corresponding to $B_s(t) = sB(t)$ is $\gamma(st)$. Thus

$$I_L(0, B) = \sum_{s\in[0,1)} \nu_L(sB) = \sum_{s\in[0,1)} \dim[(\gamma(s)L) \cap L].$$

But $\dim[(\gamma(0)L) \cap L] = \dim L = n$, so we have (2–13). □

3. Dual index theory for linear Hamiltonian systems

Let $B \in C([0, 1], \mathrm{L}_s(\mathbb{R}^{2n}))$. Recall that $\mathrm{L}_s(\mathbb{R}^{2n})$ is the set of symmetric $2n \times 2n$ metrics. Consider the linear Hamiltonian system

$$\dot{z} = JB(t)z, \quad z \in \mathbb{R}^{2n}. \tag{3–1}$$

We consider in this section the dual Morse index theory of system (3–1) with Lagrangian boundary condition. The dual Morse index theory for periodic boundary condition was studied by Girardi and Matzeu [1991] for the cases of superquadatic Hamiltonian systems, and by the author in [Liu 2001] for the subquadratic Hamiltonian systems. This theory is an application of the Morse–Ekeland index theory [Ekeland 1990]. The dual action principal in Hamiltonian framework was first established by Clarke [1978; 1979; 1981] and Clarke and Ekeland [1978; 1980], and has since been adapted by many mathematicians to the study of various variational problems. The index theory for convex Hamiltonian systems was established by I. Ekeland (see for example [1990]), whose works are of fundamental importance in the study of convex Hamiltonian systems.

Let W_L be the Hilbert space defined by

$$W_L = \{z = (x, y)^T \in W^{1,2}([0, 1], \mathbb{R}^{2n}) \mid z(0),\ z(1) \in L\} \subset L^2.$$

The embedding $j : W_L \to \mathrm{L} = L^2([0,1], \mathbb{R}^{2n})$ is compact. Denote by $\langle \cdot , \cdot \rangle$ and $\langle \cdot , \cdot \rangle_2$ the respective inner products on W_L and L. We define an operator $A : \mathrm{L} \to \mathrm{L}$ with domain W_L by $A = -J d/dt$. The spectrum of A is isolated, and in fact, $\sigma(A) = \pi\mathbb{Z}$. Let $k \notin \sigma(A)$ be so large such that $B(t) + kI > 0$. Then the operator $\Lambda_k = A + kI : W_L \to \mathrm{L}$ is invertible, and its inverse is compact. We define a quadratic form in L by

$$Q^*_{k,B}(v,u) = \int_0^1 \big((C_k(t)v(t), u(t)) - (\Lambda_k^{-1} v(t), u(t))\big)\, dt \quad \text{for all } v, u \in \mathrm{L},$$

where $C_k(t) = (B(t) + kI)^{-1}$. Define $Q^*_{k,B}(v) = Q^*_{k,B}(v,v)$. Then

$$\langle C_k v, v\rangle_2 = \int_0^1 (C_k(t)v(t), v(t))\, dt$$

defines a Hilbert structure in L. $C_k^{-1}\Lambda_k^{-1}$ is a self-adjoint and compact operator under this inner product. By the spectral theory, there exists a basis e_j, $j \in \mathbb{N}$ of L, and an eigenvalue sequence $\lambda_j \to 0$ in $\mathbb{R}$ such that

$$\begin{aligned} &\langle C_k e_i, e_j\rangle_2 = \delta_{ij}, \\ &\langle \Lambda_k^{-1} e_j, v\rangle_2 = \langle C_k \lambda_j e_j, v\rangle_2 \quad \text{for all } v \in \mathrm{L}. \end{aligned}$$

For any $v \in \mathrm{L}$ with $v = \sum_{j=1}^\infty \xi_j e_j$, there holds

$$Q^*_{k,B}(v) = -\int_0^1 (\Lambda_k^{-1} v(t), v(t)) - (C_k(t)v(t), v(t))\, dt = \sum_{j=1}^\infty (1-\lambda_j)\xi_j^2.$$

Define

$$\mathrm{L}_k^-(B) = \Big\{ \sum_{j=1}^\infty \xi_j e_j \;\Big|\; \xi_j = 0 \text{ if } 1-\lambda_j \ge 0 \Big\},$$

$$\mathrm{L}_k^0(B) = \Big\{ \sum_{j=1}^\infty \xi_j e_j \;\Big|\; \xi_j = 0 \text{ if } 1-\lambda_j \ne 0 \Big\},$$

$$\mathrm{L}_k^+(B) = \Big\{ \sum_{j=1}^\infty \xi_j e_j \;\Big|\; \xi_j = 0 \text{ if } 1-\lambda_j \le 0 \Big\}.$$

Observe that $\mathrm{L}_k^-(B)$, $\mathrm{L}_k^0(B)$ and $\mathrm{L}_k^+(B)$ are $Q^*_{k,B}$-orthogonal, and also that $\mathrm{L} = \mathrm{L}_k^-(B) \oplus \mathrm{L}_k^0(B) \oplus \mathrm{L}_k^+(B)$. Since $\lambda_j \to 0$ as $j \to \infty$, both $\mathrm{L}_k^-(B)$ and $\mathrm{L}_k^0(B)$ are finite subspaces. We define the k-dual Morse index of B by

$$i_k^*(B) = \dim \mathrm{L}_k^-(B), \qquad \nu_k^*(B) = \dim \mathrm{L}_k^0(B).$$

Theorem 3.1. *There holds*

$$i_k^*(B) = i_L(B) + n + n\left[\frac{k}{\pi}\right], \quad \nu_k^*(B) = \nu_L(B), \tag{3–2}$$

where $[a]=\max\{j\in\mathbb{Z}\mid j\le a\}$.

Proof. We only prove (3–2) for the special case $L=L_0$. We first define a functional on

$$W^m=\Big\{x\;\Big|\;x(t)=\sum_{j=-m}^{m}-J\exp(j\pi tJ)a_j,\ a_j\in\mathbb{R}^n\oplus\{0\}\subset\mathbb{R}^{2n}\Big\}$$

by

$$\begin{aligned}Q_m(x)&=\int_0^1[(\Lambda_k x(t),x(t))-(C_k^{-1}(t)x,x)]\,dt\\&=\int_0^1[(-J\dot{x}(t),x(t))-(B(t)x(t),x(t))]\,dt\quad\text{for all } x\in W^m.\end{aligned}$$

We define two linear operators A_k and B_k from W^m onto its dual space $W^{m*}\cong W^m$ such that

$$\begin{aligned}\langle A_k x,y\rangle_2&=\int_0^1(\Lambda_k x(t),y(t))\,dt&&\text{for all } x,y\in W^m,\\ \langle B_k x,y\rangle_2&=\int_0^1((B(t)+kI)x(t),y(t))\,dt&&\text{for all } x,y\in W^m.\end{aligned}$$

Next $\langle\cdot,\cdot\rangle_m:=\langle B_k\cdot,\cdot\rangle_2$ is a inner product in W^m. We consider the eigenvalues $\mu_j\in\mathbb{R}$ of A_k with respect to this inner product, that is,

$$A_k x_j=\mu_j B_k x_j$$

for some $x_j\in W^m\setminus\{0\}$. Suppose $\mu_1\le\mu_2\le\cdots\le\mu_l$ with $l=\dim W^m=2mn+n$ (each eigenvalue is counted with its multiplicity), and construct a basis in W^m of eigenvectors $v_1,\dots,v_l$ such that, for $i,j=1,2,\dots,l$,

$$\begin{aligned}\langle v_i,v_j\rangle_m&=\delta_{ij},\\ \langle A_m v_i,v_j\rangle_m&=\mu_i\delta_{ij},\\ Q_m(v_i,v_j)&=(\mu_i-1)\delta_{ij}.\end{aligned}$$

The Morse indexes $m^-(Q_m)$, $m^0(Q_m)$ and $m^+(Q_m)$ of Q_m satisfy

$$\begin{aligned}m^-(Q_m)&=^\sharp\{\mu_j\mid 1\le j\le l,\ \mu_j<1\},\\ m^+(Q_m)&=^\sharp\{\mu_j\mid 1\le j\le l,\ \mu_j>1\},\\ m^0(Q_m)&=^\sharp\{\mu_j\mid 1\le j\le l,\ \mu_j=1\}.\end{aligned}$$

By Theorem 2.1, we have for $m>0$ large enough

$$m^-(Q_m)=mn+n+i_L(B),\qquad m^0(Q_m)=\nu_L(B).\tag{3–3}$$

We denote by $Q^*_{k,m}$ the restriction of the quadratic Q^*_k to the subspace W^m, and define $i^*_{k,m}(B) = m^-(Q^*_{k,m})$, $\nu^*_{k,m}(B) = m^0(Q^*_{k,m})$. By an argument from [Girardi and Matzeu 1991], we have $i^*_{k,m}(B) \to i^*_k(B)$ and $\nu^*_{k,m}(B) \to \nu^*_k(B)$ as $m \to \infty$. Let $v'_j = A_m v_j$ for $j = 1, 2, \ldots, l$. It is a basis of W^m and

$$Q^*_{k,m}(v'_i, v'_j) = \begin{cases} 0, & \text{for } i \neq j, \\ \mu_j(\mu_j - 1), & \text{for } i = j. \end{cases}$$

$Q^*_{k,m}(v'_j)$ is negative if and only if $0 < \mu_j < 1$. We now deduce the total multiplicity of the negative eigenvalues $\mu_j < 0$. If one replaces the inner product $\langle \cdot, \cdot \rangle_m$ by the usual one, that is , one replaces the matrix B_k by the identity I, the eigenvalues μ_j should be replaced by the eigenvalues η_j of A_m with respect to the standard inner product. It is easy to check that μ_j and η_j have the same signs. So the total multiplicity of negative μ_j's equals the total multiplicity of negative η_h's. But we have

$$\eta_h = h\pi + k, \quad -m \leq h \leq m,$$

and each has multiplicity n. Therefore, the total multiplicity of the negative η_h is $n(m - [k/\pi])$. So the total multiplicity of $\mu_j \in (0, 1)$ is $m^-(Q_m) - n(m - [k/\pi])$. By definition we have

$$i^*_{k,m}(B) = m^-(Q_m) - n(m - [k/\pi]).$$

So for $m > 0$ large enough, from (3–3) we get (3–2). □

Corollary 3.2. 3.2 *Under the condition of Equation* (2–3), *there holds*

$$I_L(B_0, B_1) = i^*_k(B_1) - i^*_k(B_0).$$

4. Proof of Theorem 1.1 and some consequences

Lemma 4.1 [Chang 1981, Theorem 5.1, Corollary II.5.2]. *Let $f \in C^2(\mathrm{L}, \mathbb{R})$ satisfy the* (*PS*) *condition $f'(0) = 0$ and suppose there exists*

$$r \notin [m^-(f''(0)), m^-(f''(0)) + m^0(f''(0))]$$

with $H_q(\mathrm{L}, f_a; \mathbb{R}) \cong \delta_{q,r}\mathbb{R}$. Then f has at least one nontrivial critical point $u_1 \neq 0$. Moreover, if $m^0(f''(0)) = 0$ and $m^0(f''(u_1)) \leq |r - m^-(f''(0))|$, then f has one more nontrivial critical point $u_2 \neq u_1$.

Theorem 1.1. Without loss any generality we can suppose $H(t, 0) = 0$ and $L = L_0$. By the condition (H_∞) and the remark after Equation (2–1), we get that $i_L(B_1) + \nu_L(B_1) \leq i_L(B_2) + \nu_L(B_2)$, and so we have $\nu_L(B_1) = 0$. We shall first prove that under the above conditions (1–2) or (1–3), there holds

$$i_L(B_1) \notin [i_L(B_0), i_L(B_0) + \nu_L(B_0)].$$

More clearly, under the condition (1–2), it is claimed

$$i_L(B_1) = i_L(B_1) + \nu_L(B_1) < i_L(B_0), \tag{4–1}$$

and under the condition (1–3), it is claimed

$$i_L(B_0) + \nu_L(B_0) < i_L(B_1). \tag{4–2}$$

We first prove (4–1). By Equation (2–1) and condition (1–2), we have

$$i_L(B_1) \le i_L(B_1 + kI) \le i_L(B_0).$$

We shall prove

$$i_L(B_1) < i_L(B_1 + kI).$$

In fact, suppose

$$\gamma_1(t) = \begin{pmatrix} S_1(t) & V_1(t) \\ T_1(t) & U_1(t) \end{pmatrix} \in \mathrm{P}(2n)$$

is a symplectic path that is the fundamental solution of the linear Hamiltonian system associated with the matrix function $B_1(t)$. Since $JB_1(t) = B_1(t)J$, one can show that $\exp(Jkt)\gamma_1(t)$ is the fundamental solution of the linear Hamiltonian system

$$\dot{z} = J(B_1(t) + kI)z.$$

One has

$$\exp(Jkt)\gamma_1(t) = \begin{pmatrix} S_1(t)\cos kt - T_1(t)\sin kt & V_1(t)\cos kt - U_1(t)\sin kt \\ S_1(t)\sin kt + T_1(t)\cos kt & V_1(t)\sin kt + U_1(t)\cos kt \end{pmatrix}.$$

The associated unitary $n \times n$ matrix $\mathrm{Q}(t)$ defined by (2–2) with respect to the above matrix is

$$\begin{aligned} \mathrm{Q}(t) &= [U_1(t) - \sqrt{-1}V_1(t)][U_1(t) + \sqrt{-1}V_1(t)]^{-1} \exp(2k\sqrt{-1}t) \\ &= \mathrm{Q}_1(t)\exp(2k\sqrt{-1}t). \end{aligned}$$

In Equation (A.6), $\Delta_j = \theta_j(1) - \theta_j(0)$ and $\Delta_j^1 = \theta_j^1(1) - \theta_j^1(0)$, associated respectively to $\mathrm{Q}(t)$ and $\mathrm{Q}_1(t)$, satisfy

$$\Delta_j = \theta_j(1) - \theta_j(0) = \Delta_j^1 + 2k = \theta_j^1(1) - \theta_j^1(0) + 2k.$$

Since $k \ge \pi$, there holds

$$i_L(B_1) + n \le i_L(B_1 + kI). \tag{4–3}$$

Thus we have proved (4–1), and (4–2) can be proved similarly.

By the condition (H_∞), $H''(t, x)$ is bounded and there exist $\mu_1,\ \mu > 0$ such that

$$I \le H''(t, x) + \mu I \le \mu_1 I \quad \text{for all } (t, x). \tag{4–4}$$

We define a convex function $N(t,x) = H(t,x) + \mu|x|^2/2$. Its Fenchel dual defined by

$$N^*(t,x) = \sup_{y\in\mathbb{R}^{2n}} \{(x,y) - N(t,y)\}$$

satisfies (see [Ekeland 1990])

$$N^* \in C^2([0,1]\times\mathbb{R}^{2n},\mathbb{R}),$$
$$N^{*\prime\prime}(t,y) = N''(t,x)^{-1} \quad \text{for } y = N'(t,x).$$

From (4–4) we have

$$\mu_1^{-1} I \le N^{*\prime\prime}(t,y) \le I \quad \text{for all } (t,y). \tag{4–5}$$

So we have $|x| \to \infty$ if and only if $|y| \to \infty$ with $y = N'(t,x)$. Thus there exists $r_1 > 0$ such that

$$(B_2(t) + \mu I)^{-1} \le N^{*\prime\prime}(t,y) \le (B_1(t) + \mu I)^{-1} \tag{4–6}$$

for all t, y with $|y| \ge r_1$. We choose $\mu > 0$ satisfying (4–4) and $\mu \notin \sigma(A)$. We recall that $(\Lambda_\mu x)(t) = -J\dot{x}(t) + \mu x(t)$. We consider the functional

$$f(u) = -\frac{1}{2}\int_0^1 \left[(\Lambda_\mu^{-1}u(t), u(t)) - N^*(t,u(t))\right] dt \quad \text{for } u \in \mathrm{L}.$$

It is easy to see that $f \in C^2$ and satisfies (PS) condition (see [Ekeland 1990]). There is a one to one correspondence from the critical points of f to the solutions of Hamiltonian systems (1–1). We note that 0 is a trivial critical point of f and $N^{*\prime}(t,0) = 0$. At every critical point u_0, the second variation of f defines a quadratic form on L by

$$(f''(u_0)u,u) = -\int_0^1 \left[(\Lambda_\mu^{-1}u(t),u(t)) - (N^{*\prime\prime}(t,u_0(t))u(t),u(t))\right] dt \quad \text{for } u \in \mathrm{L}.$$

Its Morse index and nullity are both finite we denote by $(i_\mu^*(u_0), \nu_\mu^*(u_0))$ the index pair. The critical point u_0 corresponds to a solution $x_0 = \Lambda_\mu^{-1}u_0$ of (1–1), and $N^{*\prime\prime}(t,u_0(t)) = N''(t,x_0(t))^{-1}$. So by Theorem 3.1, we have

$$i_\mu^*(u_0) = i_L(x_0) + n + n\left[\frac{\mu}{\pi}\right], \ \nu_\mu^*(u_0) = \nu_L(x_0).$$

The index pair $(i_L(x_0), \nu_L(x_0))$ is the L-index of the linear Hamiltonian system

$$\dot{y}(t) = JH''(t,x_0(t))y(t).$$

By condition (1–2) and the result (4–3), we have

$$i_L(B_1) + \nu_L(B_1) + n \le i_L(B_0). \tag{4–7}$$

By condition (1–3), similarly we have

$$i_L(B_0)+\nu_L(B_0)+n\le i_L(B_1).$$

From (4–7) and the above inequality, we have that

$$|i_L(B_0)-i_L(B_1)|\ge n \quad\text{and}\quad |i^*_\mu(B_0)-i^*_\mu(B_1)|\ge n. \tag{4–8}$$

In the following, we need to prove that the homology groups satisfy

$$H_q(\mathrm{L},\,f_a;\mathbb{R})\cong\delta_{qr}\mathbb{R},\quad q=0,1,\dots, \tag{4–9}$$

for some $a\in\mathbb{R}$ and $r=i^*_\mu(B_1)$. $f_a=\{x\in\mathrm{L}\mid f(x)\le a\}$ is the level set below a. We follow the ideas of the proof of Lemma II.5.1 in [Chang 1981] to prove (4–9). See [Dong 2005] and [Liu 2005b] for some similar computations.

Step 1. Under the condition (H_∞), there holds

$$\mathrm{L}=\mathrm{L}^-_\mu(B_1)\oplus\mathrm{L}^+_\mu(B_2),$$

where $\mathrm{L}^*_\mu(B)$ for $*=\pm,0$ is defined in Section 3. In fact, it is clear that $\mathrm{L}^-_\mu(B_1)\cap\mathrm{L}^+_\mu(B_2)=\{0\}$. By $\nu^*_\mu(B_2)=\nu_L(B_2)=0$, we have $\mathrm{L}=\mathrm{L}^-_\mu(B_2)\oplus\mathrm{L}^+_\mu(B_2)$. By condition (H_∞), we have $i^*_\mu(B_1)=i^*_\mu(B_2)=r$. Suppose $\xi_1,\xi_2,\cdots,\xi_r$ is a basis in $\mathrm{L}^-_\mu(B_1)$. Decompose ξ_j by $\xi_j=\xi^-_j+\xi^+_j$ with $\xi_j\in\mathrm{L}^\pm_\mu(B_2)$. It is clear that $\xi^-_1,\cdots,\xi^-_r$ are linear independent, so it is a basis for $\mathrm{L}^-_\mu(B_2)$. For any $\xi\in\mathrm{L}$, there holds $\xi=\xi^-+\xi^+$ with $\xi^\pm\in\mathrm{L}^\pm_\mu(B_2)$. Suppose $\xi^-=a_1\xi^-_1+\cdots+a_r\xi^-_r$. Then

$$\xi=\sum_{j=1}^r a_j\xi_j+\Big(\xi^+-\sum_{j=1}^r a_j\xi^+_j\Big)=\xi_1+\xi_2$$

with $\xi_1\in\mathrm{L}^-_\mu(B_1)$ and $\xi_2\in\mathrm{L}^+_\mu(B_2)$.

Step 2. For sufficiently small $s>0$, from the structure of the symplectic group and the definition of the Maslov-type index, we know that $\nu_L(B_1-sI)=\nu_L(B_1)=0$, and $\nu_L(B_2+sI)=\nu_L(B_2)=0$, and so $i_L(B_1-sI)=i_L(B_1)=i_L(B_2)=i_L(B_2+sI)$. Denote the so-called deformation space by

$$D_R=\mathrm{L}^-_\mu(B_1-sI)\oplus\{u\in\mathrm{L}^+_\mu(B_2+sI)\mid \|u\|\le R\}.$$

For $R>0$ and $-a>0$ large, we have the deformation result

$$H_q(\mathrm{L},\,f_a;\mathbb{R})=H_q(D_R,\,D_R\cap f_a;\mathbb{R}). \tag{4–10}$$

The proof of (4–10) is standard in the Morse theory [Bott 1982]. We only need to use the negative flow to deform $(\mathrm{L},\,f_a)$ to $(D_R,\,D_R\cap f_a)$. For any $u=u_1+u_2\in\mathrm{L}$

with $u_1 \in \mathrm{L}^-_\mu(B_1 - sI)$ and $u_2 \in \mathrm{L}^+_\mu(B_2 + sI)$, by the self-adjointness, we have

$$
\begin{aligned}
(f'(u), u_2 - u_1) &= -\int_0^1 dt \left[(\Lambda^{-1}u, u_2 - u_1) - (N^{*\prime}(t,u), u_2 - u_1)\right] \\
&= \int_0^1 dt \ \left[(\Lambda^{-1}u_1, u_1) - (\Lambda^{-1}u_2, u_2)\right] \\
&\quad + \int_0^1 dt \left(\int_0^1 d\tau \ N^{*\prime\prime}(t, \tau u)(u_1 + u_2), u_2 - u_1\right) \\
&= \int_0^1 dt \ (\Lambda^{-1}u_1, u_1) - \int_0^1 dt \left(\int_0^1 d\tau \ N^{*\prime\prime}(t, \tau u)u_1, u_1\right) \\
&\quad - \int_0^1 dt \ (\Lambda^{-1}u_2, u_2) + \int_0^1 dt \left(\int_0^1 d\tau \ N^{*\prime\prime}(t, \tau u)u_2, u_2\right).
\end{aligned}
$$

By (4–5) and (4–6), we have

$$
\begin{aligned}
&\int_0^1 dt \left(\int_0^1 d\tau \ N^{*\prime\prime}(t, \tau u)u_1, u_1\right) \\
&\quad = \int_0^1 dt \int_0^{h(t,u)} d\tau \ (N^{*\prime\prime}(t, \tau u)u_1, u_1) + \int_0^1 dt \int_{h(t,u)}^1 d\tau \ (N^{*\prime\prime}(t, \tau u)u_1, u_1) \\
&\quad \le c_0\|u\| + \int_0^1 dt \ ((B_1(t) + \mu I - sI)u_1, u_1),
\end{aligned}
$$

where $h(t,u) = r_1/|u(t)|$. Similarly,

$$
\begin{aligned}
\int_0^1 dt \left(\int_0^1 d\tau \ N^{*\prime\prime}(t, \tau u)u_2, u_2\right) &\ge \int_0^1 dt \int_{h(t,u)}^1 d\tau \ (N^{*\prime\prime}(t, \tau u)u_2, u_2) \\
&\ge \int_0^1 dt \ ((B_2(t) + \mu I + sI)u_2, u_2) - c\|u\|
\end{aligned}
$$

for some $c > 0$. So by the last three relations, we have

$$(f'(u), u_2 - u_1) \ge c_1\|u_1\|^2 + c_2\|u_2\|^2 - c_3(\|u_1\| + \|u_2\|).$$

Thus for large R with $\|u_1\| \ge R$ or $\|u_2\| \ge R$, we have

$$(-f'(u), u_2 - u_1) < -1. \tag{4–11}$$

We know from (4–11) that f has no critical point outside D_R, and that $-f'(u)$ points inward to D_R on ∂D_R. So we can define the deformation by negative flow. In fact, for any $u = u_1 + u_2 \notin D_R$, let $\sigma(\theta, u) = e^\theta u_1 + e^{-\theta}u_2$, and $d_u = \log\|u_2\| - \log R$. We define the deformation map $\eta : [0,1] \times \mathrm{L} \to \mathrm{L}$ by

$$
\eta(\theta, u_1 + u_2) = \begin{cases} u_1 + u_2, & \|u_2\| \le R, \\ \sigma(d_u\theta, u), & \|u_2\| > R. \end{cases}
$$

The map η satisfies the properties

$$\eta(0,\cdot)=\mathrm{id},\quad \eta(1,\mathrm{L})\subset D_R,\quad \eta(1,f_a)\subset D_R\cap f_a$$
$$\eta(\theta,f_a)\subset f_a,\quad \eta(\theta,\cdot)|_{D_R}=id|_{D_R}.$$

Thus the pair $(D_R, D_R\cap f_a)$ is a deformation retract of the pair (L, f_a).

Step 3. For large $R, -a>0$, there holds

$$H_q(D_R, D_R\cap f_a)\cong \delta_{q,r}\mathbb{R}.$$

In fact, similarly to the above computation, for large $m>0$, we have

$$\begin{aligned}
&\int_0^1 dt\, N^*(t,u(t))\\
&=\int_0^1 dt\Big(N^*(t,0)+\iint\limits_{[0,1]\times[0,1]} d\tau\, ds\, \tau(N^{*\prime\prime}(t,\tau s u(t))u(t),u(t))\Big)\\
&\le \int_{|u(t)|\ge mr_1} dt \iint\limits_{[0,1]\times[0,1]} d\tau\, ds\, \tau(N^{*\prime\prime}(t,\tau s u(t))u(t),u(t))+c_m\\
&\le \int\limits_{|u(t)|\ge mr_1} dt \iint\limits_{|s\tau u(t)|\ge r_1,\ \tau,s\in[0,1]} d\tau\, ds\, \tau(N^{*\prime\prime}(t,\tau s u(t))u(t),u(t))\\
&\qquad + \int\limits_{|u(t)|\ge mr_1} dt \iint\limits_{|s\tau u(t)|\le r_1,\ \tau,s\in[0,1]} d\tau\, ds\, \tau(N^{*\prime\prime}(t,\tau s u(t))u(t),u(t))+c_m\\
&\le \frac{1}{2}\int_0^1 dt\, ((B_1(t)+\mu I)^{-1}u(t),u(t))+k_m\|u\|+c_m,
\end{aligned}$$

where c_m and k_m are constants depending only on m and $k_m\to 0$ as $m\to+\infty$. So for the small s in the step 2 above, we can choose a large number m such that

$$\int_0^1 dt\, N^*(t,u(t))\le \frac{1}{2}\int_0^1 dt\, ((B_1(t)+\mu I-sI)^{-1}u(t),u(t))+C\quad \text{for all } u\in \mathrm{L}$$

for some constant $C>0$. Thus for any $u=u_1+u_2$ with $u_1\in \mathrm{L}_\mu^-(B_1-sI)$ and $u_2\in \mathrm{L}_\mu^+(B_2+sI)$ with $\|u_2\|\le R$, there holds

$$f(u)\le -C_1\|u_1\|^2+C_2\|u_1\|+C_3,$$

where $C_j,\ j=1,2,3$ are constants and $C_1>0$. It implies that $f(u)\to-\infty$ if and only if $\|u_1\|\to\infty$ uniformly for $u_2\in \mathrm{L}_\mu^+(B_2+sI)$ with $\|u_2\|\le R$. In the following we denote by $B_r=\{x\in \mathrm{L}|\ \|x\|\le r\}$ the ball with radius r in L. Therefore for $-a_1>-a_2$ sufficiently large, there exist three numbers with $R<R_1<R_2<R_3$

satisfying

$$\begin{aligned}&(\mathrm{L}_\mu^+(B_2+sI)\cap B_{R_3})\oplus(\mathrm{L}_\mu^-)(B_1-sI)\setminus B_{R_2})\subset f_{a_1}\cap D_{R_3}\\ &\subset(\mathrm{L}_\mu^+(B_2+sI)\cap B_{R_3})\oplus(\mathrm{L}_\mu^-)(B_1-sI)\setminus B_{R_1})\subset f_{a_2}\cap D_{R_3}.\end{aligned}$$

Recall that $\sigma(\theta,u)=e^\theta u_1+e^{-\theta}u_2$. By definition, we have $f(\sigma(0,u))=f(u)>a_1$ and $f(\sigma(\theta,u))\to-\infty$ as $\theta\to\infty$ if $u=u_1+u_2\in D_{R_3}\cap(f_{a_2}\setminus f_{a_1})$. It implies that there exists $\theta_0=\theta_0(u)>0$ such that $f(\sigma(\theta_0,u))=a_1$. But by (4–11),

$$\frac{d}{d\theta}f(\sigma(\theta,u))\le-1 \quad \text{at any point } \theta>0.$$

By the implicit function theorem, $\theta_0(u)$ is continuous in u. We define another deformation map $\eta_0:[0,1]\times f_{a_2}\cap D_{R_3}\to f_{a_2}\cap D_{R_3}$ by

$$\eta_0(\theta,u)=\begin{cases}u & u\in f_{a_1}\cap D_{R_3},\\ \sigma(\theta_0(u)\theta,u), & u\in D_{R_3}\cap(f_{a_2}\setminus f_{a_1}).\end{cases}$$

It is clear that η_0 is a deformation from $f_{a_2}\cap D_{R_3}$ to $f_{a_1}\cap D_{R_3}$. We now define

$$\tilde\eta(u)=d(\eta_0(1,u)) \quad \text{with } d(u)=\begin{cases}u, & \|u_1\|\ge R_1,\\ u_2+\dfrac{u_1}{\|u_1\|}R_1, & 0<\|u_1\|<R_1.\end{cases}$$

This map defines a strong deformation retract:

$$\tilde\eta: D_{R_3}\cap d_{a_2}\to\big(\mathrm{L}_\mu^+(B_2+sI)\cap B_{R_3}\big)\oplus\big(\mathrm{L}_\mu^-(B_1-sI)\cap\{u\in\mathrm{L}|\ \|u\|\ge R_1\}\big).$$

Now we can compute the homology groups

$$\begin{aligned}&H_q(D_{R_3},D_{R_3}\cap f_{a_2};\mathbb{R})\\ &\quad\cong H_q(D_{R_3},(\mathrm{L}_\mu^+(B_2+sI)\cap B_{R_3})\oplus(\mathrm{L}_\mu^-(B_1-sI)\cap\{u\in\mathrm{L}|\ \|u\|\ge R_1\});\mathbb{R})\\ &\quad\cong H_q(\mathrm{L}_\mu^-(B_1-sI)\cap B_{R_3},\partial(\mathrm{L}_\mu^-(B_1-sI)\cap B_{R_3});\mathbb{R})\\ &\quad\cong\delta_{qr}\mathbb{R}.\end{aligned}$$

From (4–8), (4–9), and (A.2) below, and by using Equation (4–1), we complete the proof. □

Corollary 4.2. *Let H satisfy the conditions (H_0) and (H_∞), and suppose $B_0(t)=H''(t,0)$ satisfies one of the twisted conditions*:

(i) $B_1(t)<B_0(t)$, *there exists* $\lambda\in(0,1)$ *such that* $\nu_L((1-\lambda)B_1+\lambda B_0)\ne0$;

(ii) $B_0(t)<B_1(t)$, *there exists* $\lambda\in(0,1)$ *such that* $\nu_L((1-\lambda)B_0+\lambda B_1)\ne0$.

Then (1–1) *possesses at least one nontrivial solution. Furthermore, if $\nu_L(B_0)=0$ and in* (*i*) *we replace the second condition by $\sum_{\lambda\in(0,1)}\nu((1-\lambda)B_1+\lambda B_0)\ge n$, or in* (*ii*) *we replace the second condition by $\sum_{\lambda\in(0,1)}\nu((1-\lambda)B_0+\lambda B_1)\ge n$, the Hamiltonian system* (1–1) *possesses at least two nontrivial solutions.*

Proof. It follows from (2–3), the proof of Theorem 1.1 and (4–2). In the first case, we have $r = i_L(B_1) \notin [i_L(B_0), i_L(B_0) + \nu_L(B_0)]$. In the second case we have $|i_L(B_0) - i_L(B_1)| \geq n$. □

The proof of Theorem 1.1 in fact proves this:

Theorem 4.3. *Let H satisfy conditions* (H_0) *and* (H_∞). *Suppose $B_0(t) = H''(t, 0)$ satisfies the twisted conditions*

$$i_L(B_1) \notin [i_L(B_0), i_L(B_0) + \nu_L(B_0)].$$

Then the problem (1–1) *possesses at least one nontrivial solution. Moreover, if $\nu_L(B_0) = 0$ and $|i_L(B_1) - i_L(B_0)| \geq n$, then* (1–1) *possesses at least two nontrivial solutions.*

Remark. The condition $B_1(t) < B_2(t)$ in Theorem 2.4 can be replaced by $B_1(t) \leq B_2(t)$ for all t and $B_2 - B_1 \geq \delta > 0$ for some constant δ as an operator in L. So the conditions in parts (i) and (ii) of Corollary 4.2 can be replaced by this kind of condition. The condition $JB_1(t) = B_1(t)J$ in (H_∞) can be replaced by $JB_0(t) = B_0(t)J$.

Appendix. Maslov-type index for symplectc paths with Lagrangian boundary condition

We give a brief introduction to the Maslov-type index for symplectc paths with Lagrangian boundary condition. The details can be found in [Liu 2007]. We denote the symplectic group by

$$\mathrm{Sp}(2n) = \left\{ M \in \mathrm{L}(\mathbb{R}^{2n}) \mid M^T J M = J \right\},$$

and denote the symplectic path space by

$$\mathrm{P}(2n) = \{\gamma \in C([0, 1], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\}.$$

We write a symplectic path $\gamma \in \mathrm{P}(2n)$, in the form

$$\gamma(t) = \begin{pmatrix} S(t) & V(t) \\ T(t) & U(t) \end{pmatrix}, \tag{A.1}$$

where $S(t), T(t), V(t), U(t)$ are $n \times n$ matrices. The n vectors coming from the rightmost columns of the above matrix are linearly independent and they span a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$. In particular, at $t = 0$, this Lagrangian subspace is $L_0 = \{0\} \oplus \mathbb{R}^n$.

Definition A.1. We define the L_0-nullity of any symplectic path $\gamma \in \mathrm{P}(2n)$ by

$$\nu_{L_0}(\gamma) \equiv \dim \ker_{L_0}(\gamma(1)) := \dim \ker V(1) = n - \mathrm{rank} V(1) \tag{A.2}$$

with the $n \times n$ matrix function $V(t)$ defined in (A.1).

We define two subsets of $\mathrm{P}(2n)$ by

$$\mathrm{P}(2n)^*_{L_0} = \{\gamma \in \mathrm{P}(2n) \mid \nu_{L_0}(\gamma) = 0\},$$
$$\mathrm{P}(2n)^0_{L_0} = \{\gamma \in \mathrm{P}(2n) \mid \nu_{L_0}(\gamma) > 0\}.$$

We note that

$$\operatorname{rank}\begin{pmatrix} V(t) \\ U(t) \end{pmatrix} = n,$$

so the complex matrix $U(t) \pm \sqrt{-1}V(t)$ is invertible. We define a complex matrix function by

$$\text{(A.3)} \qquad \mathrm{Q}(t) = \big(U(t) - \sqrt{-1}V(t)\big)\big(U(t) + \sqrt{-1}V(t)\big)^{-1}.$$

It is easy to see that the matrix $\mathrm{Q}(t)$ is a unitary matrix for any $t \in [0, 1]$. We define

$$M_+ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad M_- = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \quad J_n = \operatorname{diag}(-1, 1, \dots, 1).$$

For a path $\gamma \in \mathrm{P}(2n)^*_{L_0}$, we first adjoin it with a simple symplectic path starting from $J = -M_+$, that is, we define a symplectic path by

$$\tilde{\gamma}(t) = \begin{cases} I\cos(\pi/2)(1-2t) + J\sin(\pi/2)(1-2t), & t \in [0, 1/2]; \\ \gamma(2t-1), & t \in [1/2, 1]. \end{cases}$$

then we choose a symplectic path $\beta(t)$ in $\mathrm{Sp}(2n)^*_{L_0}$ starting from $\gamma(1)$ and ending at M_+ or M_-. We now define a joint path by

$$\bar{\gamma}(t) = \beta * \tilde{\gamma} := \begin{cases} \tilde{\gamma}(2t), & t \in [0, 1/2], \\ \beta(2t-1), & t \in [1/2, 1]. \end{cases}$$

By the definition, we see that the symplectic path $\bar{\gamma}$ starting from $-M_+$ and ending at either M_+ or M_-. As above, we define

$$\text{(A.4)} \qquad \bar{\mathrm{Q}}(t) = \big(\bar{U}(t) - \sqrt{-1}\bar{V}(t)\big)\big(\bar{U}(t) + \sqrt{-1}\bar{V}(t)\big)^{-1}.$$

for $\bar{\gamma}(t) = \begin{pmatrix} \bar{S}(t) & \bar{V}(t) \\ \bar{T}(t) & \bar{U}(t) \end{pmatrix}$. We can choose a continuous function $\bar{\Delta}(t)$ in $[0, 1]$ such that

$$\text{(A.5)} \qquad \det \bar{\mathrm{Q}}(t) = e^{2\sqrt{-1}\bar{\Delta}(t)}.$$

By the above arguments, we see that the number $\frac{1}{\pi}(\bar{\Delta}(1) - \bar{\Delta}(0)) \in \mathbb{Z}$ and it does not depend on the choice of the function $\bar{\Delta}(t)$.

Definition A.2. For a symplectic path $\gamma \in \mathrm{P}(2n)^*_{L_0}$, we define the L_0-index of γ by

$$i_{L_0}(\gamma) = \frac{1}{\pi}(\bar{\Delta}(1) - \bar{\Delta}(0)). \tag{A.6}$$

Definition A.3. For a symplectic path $\gamma \in \mathrm{P}(2n)^0_{L_0}$, we define the L_0-index of γ by

$$i_{L_0}(\gamma) = \inf\left\{i_{L_0}(\tilde{\gamma}) \mid \tilde{\gamma} \in \mathrm{P}(2n)^*_{L_0}, \text{ and } \tilde{\gamma} \text{ is sufficiently close to } \gamma \right\}.$$

We note that $\Lambda(n) = U(n)/O(n)$; this means that for any linear subspace $L \in \Lambda(n)$, there is an orthogonal symplectic matrix

$$P = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with $A \pm \sqrt{-1}B \in U(n)$ such that $PL_0 = L$. P is uniquely determined by L up to an orthogonal matrix $C \in O(n)$. It means that for any other choice P' satisfying above conditions, there exists a matrix $C \in O(n)$ such that

$$P' = P \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

See [McDuff and Salamon 1998, Lemma 2.31]. We define the conjugated symplectic path $\gamma_c \in \mathrm{P}(2n)$ of γ by $\gamma_c(t) = P^{-1}\gamma(t)P$.

Definition A.4. We define the L-nullity of any symplectic path $\gamma \in \mathrm{P}(2n)$ by

$$\nu_L(\gamma) \equiv \dim \ker_L(\gamma(1)) := \dim \ker V_c(1) = n - \operatorname{rank} V_c(1),$$

The $n \times n$ matrix function $V_c(t)$ is defined in (A.1) with the symplectic path γ replaced by γ_c, that is,

$$\gamma_c(t) = \begin{pmatrix} S_c(t) & V_c(t) \\ T_c(t) & U_c(t) \end{pmatrix}.$$

Definition A.5. For a symplectic path $\gamma \in \mathrm{P}(2n)$, we define the L-index of γ by

$$i_L(\gamma) = i_{L_0}(\gamma_c).$$

Theorem A.6. *If $\gamma \in \mathrm{P}(2n)^0_L$, there is a family of paths $\gamma_s \in \mathrm{P}(2n)_L$ depend continuous on $s \in [-1, 1]$ such that $\gamma_0 = \gamma$, $\gamma_s \in \mathrm{P}(2n)^*_L$, $s \neq 0$ and*

$$i_L(\gamma_s) - i_L(\gamma_{-s}) = \nu_L(\gamma) \quad \text{for all } s \in (0, 1],$$

and

$$i_L(\gamma) = i_L(\gamma_{-s}),\ s \in (0, 1].$$

For a symmetric matrix function $B : [0, 1] \to \mathrm{L}_s(2n)$, we consider the functional

$$f(z) = \int_0^1 \left(\tfrac{1}{2}(-J\dot{z}, z) - (B(t)z, z)\right) dt, \quad z \in W_L,$$

where $W_L = \{z = (x, y)^T \in W^{1,2}([0, 1], \mathbb{R}^{2n}) \mid z(0), z(1) \in L\} \subset L^2$. By the saddle point reduction methods (see [Amann 1979; Amann and Zehnder 1980; Long 1993; 2002; Liu 2007]), there exists a finite-dimensional subspace X of W_L with $\dim X = 2d + n$ and an injection map $X \to W_L$, such that the function $a(x) = f(u(x))$ is C^2 and we have:

Theorem A.7. *For any* $L \in \Lambda(n)$,

$$m^-(a) = d + i_L(B) + n,$$
$$m^0(a) = \nu_L(B),$$
$$m^+(a) = d - i_L(B) - \nu_L(B),$$

where $m^*(a)$ *for* $* = +, 0, -$ *are respectively the positive*, *null and the negative Morse indices of the function* $a(x)$ *at the origin.*

Theorem A.8. *For any symplectic path* $\gamma \in \mathrm{P}(2n)$, *there holds*

$$i_{L_0}(\gamma) = \sum_{j=1}^{n} E\left(\frac{\theta_j(1) - \theta_j(0)}{2\pi}\right),$$

where $E(a) = \max\{k \in \mathbb{Z} \mid k < a\}$ *and* $\lambda_j(t) = e^{\sqrt{-1}\theta_j(t)}$ *are the eigenvalues of* $\mathrm{Q}(t)$ *defined in* (A.3).

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CHUN-GEN LIU
SCHOOL OF MATHEMATICS AND LPMC
NANKAI UNIVERSITY
TIANJIN 300071
CHINA

liucg@nankai.edu.cn