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Let G be a subgroup of $\mathrm{PL}^+(I)$. Then the stable commutator length of every element of $[G, G]$ is zero.

1. Introduction

This note proves a vanishing theorem for stable commutator length in groups of PL homeomorphisms of the interval. For convenience, we restrict attention to subgroups of the group of orientation preserving PL homeomorphisms, which we denote by $\mathrm{PL}^+(I)$, where I is the unit interval [0, 1]. By a theorem of Bavard (see Section 2), vanishing stable commutator length is equivalent to the injectivity of the map from bounded to ordinary cohomology in two dimensions.

Since the dimension of the second bounded cohomology of a nonabelian free group is uncountable, this gives a new proof of the celebrated result of Brin and Squier [1985] that $\mathrm{PL}^+(I)$ does not contain a nonabelian free subgroup.

At least two other important classes of groups are known to have vanishing stable commutator length:

- Irreducible lattices in semisimple Lie groups of rank at least two. This follows from more general work of Burger and Monod [2002].
- Amenable groups. In this case, the bounded cohomology with real coefficients vanishes in every dimension by Trauber's theorem (see Section 2), and therefore the map is trivially injective.

An important open question is whether Thompson's group $F < \mathrm{PL}^+(I)$ — which consists of homeomorphisms with dyadic rational slopes and break points — is amenable. More generally, no counterexamples are known to the conjecture that a finitely-presented torsion-free group with the property that every subgroup has vanishing stable commutator length is amenable (this should perhaps be thought of as a kind of "homological" version of von Neumann's conjecture). These and related problems are some of the main motivations for this paper.

MSC2000: primary 20F12; secondary 20J05, 37C85, 37E05, 57Q99.

2. Background material

Definition 2.1. Let G be a group and $C_*(G)$ the (bar) complex of integral G-chains. Let $C^*(G)\otimes\mathbb{R}$ be the dual complex of real-valued cochains. For each n, let $C_b^n(G)\otimes\mathbb{R}$ denote the vector space of cochains f for which $\sup_\sigma |f(\sigma)|$ is finite, where σ ranges over the generators of $C_*(G)$. The (real) *bounded cohomology* $H_b^*(G;\mathbb{R})$ of G is the cohomology of the complex $C_b^*(G)\otimes\mathbb{R}$.

Note that $H_b^n(G;\mathbb{R})$ carries an L^∞ pseudo-norm for each n.

Bounded cohomology behaves well under amenable extensions:

Theorem 2.2 (Trauber). *Let*

$$1 \to H \to G \to A \to 1$$

be a short exact sequence of groups, where A is amenable. Then the natural homomorphisms $H_b^*(G;\mathbb{R}) \to H_b^*(H;\mathbb{R})$ are isometric injections.

For a proof, see for example [Gromov 1982, page 39].

Definition 2.3. Let G be a group and $[G,G]$ its commutator subgroup. For any $g \in [G,G]$, the *commutator length* $\ell'(g)$ of g is the minimal number of commutators whose product is equal to g. The *stable commutator length* $\ell(g)$ is defined by

$$\ell(g) = \liminf_{n\to\infty} \frac{\ell'(g^n)}{n}.$$

By including bounded cochains in all cochains, one obtains canonical homomorphisms from bounded cohomology to ordinary cohomology. There is a fundamental relationship between stable commutator length and bounded cohomology:

Theorem 2.4 [Bavard 1991]. *Let G be a group. Then the canonical map from bounded cohomology to ordinary cohomology $H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ is injective if and only if the stable commutator length vanishes on* $[G,G]$.

Bavard's theorem uses the notion of quasimorphisms:

Definition 2.5. Let G be a group. A (homogeneous) *quasimorphism* on G is a map $f : G \to \mathbb{R}$ for which there is some smallest $\epsilon(f) \geq 0$ (called the *error* or *defect* of f) such that

$$f(a^n) = nf(a) \quad \text{and} \quad |f(a)+f(b)-f(ab)| \leq \epsilon(f), \qquad \text{for all } a,b \in G.$$

A homogeneous quasimorphism is necessarily a class function. The set $Q(G)$ of all homogeneous quasimorphisms on G has the structure of a vector space. Quasimorphisms with error 0 are homomorphisms. There is an exact sequence

$$1 \to H^1(G;\mathbb{R}) \to Q(G) \xrightarrow{\delta} H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R}),$$

where δ denotes the coboundary map. See [Bavard 1991] for a proof.

We may interpret Bavard's theorem as saying that if G is a group, the quotient $Q(G)/H^1(G;\mathbb{R})$ is zero exactly when the stable commutator length of every element of $[G, G]$ vanishes.

In terms of quasimorphisms, Bavard proves a sharper statement:

Theorem 2.6 (Bavard). *Let G be a group, and $g \in [G, G]$. Then*

$$\ell(g) = \frac{1}{2} \sup_{f \in Q(G)/H^1(G)} \frac{\|f(g)\|}{\epsilon(f)}.$$

Theorems 2.4 and 2.2 together imply that $\ell(g) = 0$ for any $g \in [G, G]$ whenever G is an amenable group.

3. Subgroups of $\mathrm{PL}^+(I)$

Given a subgroup $G < \mathrm{PL}^+(I)$, we denote by $\mathrm{fix}(G)$ the set of common fixed points of all elements of G.

Definition 3.1. The *endpoint homomorphism* is the homomorphism

$$\eta : \mathrm{PL}^+(I) \to \mathbb{R} \oplus \mathbb{R}$$

defined by

$$\eta(g) = (\log g'(0), \log g'(1)).$$

Given $G < \mathrm{PL}^+(I)$, we denote by G_0 the kernel of η restricted to G.

Observe that every $g \in G_0$ fixes a neighborhood of both 0 and 1.

Theorem A. *Let G be a subgroup of* $\mathrm{PL}^+(I)$. *Then the stable commutator length of every element of* $[G, G]$ *is zero.*

Proof. <u>*Case 1:* $\mathrm{fix}(G) = \{0, 1\}$.</u>

Let G_0 be the kernel of $\eta : G \to \mathbb{R} \oplus \mathbb{R}$. Let $K = [G_0, G_0]$, and let $g \in K$. Then we can write

$$g = [a_1, b_1][a_2, b_2] \cdots [a_m, b_m],$$

for some integer m and a_i, b_i in G_0. Let J be the smallest interval that contains the support of all the a_i, b_i and g. Then J is properly contained in $(0, 1)$. Since $\mathrm{fix}(G)$ contains no interior points, there is some $j \in G$ with $j(J) \cap J = \varnothing$, and therefore $j^n(J) \cap J = \varnothing$ for all nonzero n.

Let $G_0(J)$ be the subgroup of G_0 consisting of elements with support contained in J. For each n we define a diagonal monomorphism

$$\Delta_n : G_0(J) \to G_0$$

by

$$\Delta_n(c) = \prod_{i=0}^{n} c^{j^i},$$

where the superscript notation denotes conjugation. Define

$$g' = \prod_{i=0}^{n} (g^{i+1})^{j^i}.$$

Then

$$[g', j] = \Delta_n(g)(g^{-n-1})^{j^{n+1}}.$$

On the other hand,

$$\begin{aligned}\Delta_n(g) &= \Delta_n([a_1, b_1][a_2, b_2] \cdots [a_m, b_m]) \\ &= [\Delta_n(a_1), \Delta_n(b_1)] \cdots [\Delta_n(a_m), \Delta_n(b_m)],\end{aligned}$$

and therefore g^{n+1} can be written as a product of at most $m + 1$ commutators in elements of G. Since m is fixed but n is arbitrary, it follows that the stable commutator length of g is zero, and hence $f([G_0, G_0]) = 0$ for every quasimorphism $f \in Q(G)/H^1(G)$.

Now, let $g \in [G, G]$. Observe that $[G_0, G_0]$ is normal in G so we can form the quotient $H = G/[G_0, G_0]$, which is two-step solvable and, therefore, amenable. Let $\varphi : G \to H$ be the quotient homomorphism. By Trauber's Theorem 2.2 and Bavard's Theorem 2.4, $\ell(\varphi(g)) = 0$ in H. This means that we can write

$$g^n = [a_1, b_1] \cdots [a_m, b_m]c,$$

where $c \in [G_0, G_0]$, n is arbitrarily big, and m/n is as small as we like. Let f be a quasimorphism of defect at most 1. By the above, we have $f(c) = 0$. Therefore $f(g^n) \le 2m + 1$, and $f(g) \le (2m + 1)/n$. Since n is arbitrarily big and m/n is as small as we like, $f(g) = 0$. Since f and g were arbitrary, $Q(G)/H^1(G) = 0$.

Applying Theorem 2.4, this proves the theorem when $\mathrm{fix}(G) = \{0, 1\}$.

Case 2: fix(G) *is arbitrary.*

Suppose $f \in Q(G)$ has defect at most 1, and suppose $f(g) \neq 0$, where $g \in [G, G]$. Let H be a finitely generated subgroup of G such that $g \in [H, H]$. Then $\mathrm{fix}(H)$ equals the intersection of $\mathrm{fix}(h_i)$ for the generators h_i. The fixed set of any element of $\mathrm{PL}^+(I)$ is a union of finitely many points and intervals, and so the same is true for $\mathrm{fix}(H)$. Hence $I \backslash \mathrm{fix}(H)$ consists of finitely many open intervals, whose closures we denote by $I_1, I_2, \ldots I_n$.

Let $\rho : H \to \mathbb{R}^{2n}$ denote the product of the endpoint homomorphisms for each i, and let H_0 denote the kernel. We will show that f vanishes on $[H, H]$, contrary to the fact that $f(g) \neq 0$ and $g \in [H, H]$.

Let $r \in [H_0, H_0]$, and suppose we have an expression

$$r = [a_1, b_1] \cdots [a_m, b_m],$$

where all the r, a_i, b_i have support in the union $\bigcup_i J_i$ in which $J_i \subset \mathrm{int}(I_i)$ is an interval. For each i there is a $j_i \in H$ such that $j_i(J_i) \cap J_i = \varnothing$. Note that this implies $j_i^n(J_i) \cap J_i = \varnothing$ for all nonzero n.

However, we claim that we can construct a single element $j \in H$ such that $j(J_i) \cap J_i = \varnothing$ for all i.

The case $n = 1$ is trivial; for better exposition, we detail the $n = 2$ case before treating the general case.

Without loss of generality, we may assume j_1 moves J_1 to the right. Now, let J_2' be the smallest interval that contains both J_2 and $j_1^{-1}(J_2)$, and let j_2 be such that $j_2(J_2') \cap J_2' = \varnothing$. After replacing j_2 by j_2^{-1}, if necessary, we may also assume that j_2 moves the leftmost point J_1^- of J_1 to the right. We similarly use the notation J_1^+ to denote the rightmost point of J_1, that is,

$$j_1(J_1^-) > J_1^+, \ j_2(J_1^-) \geq J_1^-.$$

Then

$$j_1 j_2(J_1^-) > J_1^+,$$

and therefore

$$j_1 j_2(J_1) \cap J_1 = \varnothing.$$

Moreover,

$$j_1 j_2(J_2) \cap J_2 = j_1(j_2(J_2) \cap j_1^{-1}(J_2)) \subset j_1(j_2(J_2') \cap J_2') = \varnothing.$$

Now we treat the general case. As before, without loss of generality, we assume j_1 moves J_1 to the right. For all $i > 1$ we let $J_i' \subset I_i$ denote the smallest interval containing both J_i and $j_1^{-1}(J_i)$. By induction, we assume that there is some j with $j(J_i') \cap J_i' = \varnothing$ for all $i > 1$. After replacing j with j^{-1}, if necessary, we may assume that j moves the leftmost point of J_1 to the right. Then the argument above shows that

$$j_1 j(J_i) \cap J_i = \varnothing$$

for all i. Therefore we have proved the claim.

Now the proof that $\ell(r) = 0$ follows exactly as in Case 1, since for any m there is a diagonal monomorphism

$$\Delta_m : G_0\big(\bigcup_i J_i\big) \to G_0$$

defined by

$$\Delta_m(c) = \prod_{i=0}^{m} c^{j^i},$$

where now j moves every J_i off itself simultaneously. Since r was arbitrary, it follows that the stable commutator length vanishes on $[H_0, H_0]$, and since $H/[H_0, H_0]$

is amenable, f must vanish on all of $[H, H]$ by Theorem 2.4, contrary to the definition of H. This contradiction implies that $Q(G)/H^1(G) = 0$, and the theorem follows. $\square$

Remark 3.2. The "diagonal trick" is a variation on Mather's argument [1971] for proving that $\mathrm{Homeo}_0(\mathbb{R}^n)$ is acyclic. Matsumoto and Morita [1985] modified this argument to prove that the bounded cohomology of $\mathrm{Homeo}_0(\mathbb{R}^n)$ vanishes in every degree. One significant difference between $\mathrm{PL}^+(I)$ and $\mathrm{Homeo}_0(\mathbb{R}^n)$ is that every finitely generated subgroup G of $\mathrm{Homeo}_0(\mathbb{R}^n)$ is contained in an *unrestricted* wreath product with $\mathbb{Z}$ (that is, a product of the form $\prod_i G \rtimes \mathbb{Z}$), whereas, in a PL group, only *restricted* wreath products (that is, products of the form $\oplus_i G \rtimes \mathbb{Z}$) with infinite groups are possible.

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