ON THE ESTIMATE OF THE FIRST EIGENVALUE OF A
SUBLAPLACIAN ON A PSEUDOHERMITIAN 3-MANIFOLD

SHU-CHENG CHANG AND HUNG-LIN CHIU

Volume 232 No. 2 October 2007
ON THE ESTIMATE OF THE FIRST EIGENVALUE OF A
SUBLAPLACIAN ON A PSEUDOHERMITIAN 3-MANIFOLD

SHU-CHENG CHANG AND HUNG-LIN CHIU

In this paper, we study a lower bound estimate of the first positive eigenvalue of the sublaplacian on a three-dimensional pseudohermitian manifold. S.-Y. Li and H.-S. Luk derived the lower bound estimate under certain conditions for curvature tensors bounded below by a positive constant. By using the Li–Yau gradient estimate, we are able to get an effective lower bound estimate under a general curvature condition. The key is the discovery of a new CR version of the Bochner formula which involves the CR Paneitz operator.

1. Introduction

Let $M$ be a closed 3-manifold with an oriented contact structure $\xi$. There always exists a global contact form $\theta$ obtained by patching together local ones with a partition of unity. The characteristic vector field of $\theta$ is the unique vector field $T$ such that $\theta(T) = 1$ and $\mathcal{L}_T \theta = 0$ or $d\theta(T, \cdot) = 0$. A CR structure compatible with $\xi$ is a smooth endomorphism $J : \xi \to \xi$ such that $J^2 = -\text{Id}$. A pseudohermitian structure compatible with $\xi$ is a CR-structure $J$ compatible with $\xi$ together with a global contact form $\theta$. The CR structure $J$ can extend to $\mathbb{C} \otimes \xi$ and decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$, which are eigenspaces of $J$ with respect to $i$ and $-i$, respectively.

Let $\{T, Z_1, Z_\bar{1}\}$ be a frame of $TM \otimes \mathbb{C}$, where $T$ is the characteristic vector field, $Z_1$ is any local frame of $T_{1,0}$, and $Z_\bar{1} = \overline{Z_1} \in T_{0,1}$. Then $\{\theta, \theta^1, \theta^{\bar{1}}\}$, the coframe dual to $\{T, Z_1, Z_\bar{1}\}$, satisfies

$$d\theta = ih_{1\bar{1}} \theta^1 \wedge \theta^{\bar{1}},$$

for some positive function $h_{1\bar{1}}$. Actually we can always choose $Z_1$ such that $h_{1\bar{1}} = 1$; hence, throughout this paper, we assume $h_{1\bar{1}} = 1$.

MSC2000: primary 32V05, 32V20; secondary 53C56.

Keywords: eigenvalue, gradient estimate, pseudohermitian manifold, Tanaka–Webster curvature, pseudohermitian torsion, CR Paneitz operator, sublaplacian, Carnot–Carathéodory distance, diameter.

Research supported in part by the NSC of Taiwan.
The Levi form \( \langle \, \, \rangle_{L_\theta} \) is the Hermitian form on \( T_{1,0} \) defined by
\[
\langle Z, W \rangle_{L_\theta} = -i \langle d\theta, Z \wedge \bar{W} \rangle.
\]
We can extend \( \langle \, \, \rangle_{L_\theta} \) to \( T_{0,1} \) by defining \( \langle Z, W \rangle_{L_\theta} = \langle Z, W \rangle_{L_\theta} \) for all \( Z, W \in T_{1,0} \).
The Levi form induces naturally a Hermitian form on the dual bundle \( \langle \, \, \rangle_{L_\theta^*} \) of \( T_{1,0} \)
and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over \( M \) with respect to the volume form \( dV = \theta \wedge d\theta \), we get
an inner product on the space of sections of each tensor bundle. We denote the inner product by \( \langle \, \, \rangle \). For example,
\[
\langle \varphi, \psi \rangle = \int_M \varphi \bar{\psi} \, dV,
\]
for functions \( \varphi \) and \( \psi \).

The pseudohermitian connection of \( (J, \theta) \) is the connection \( \nabla \) on \( TM \otimes \mathbb{C} \) (and extended to tensors) given in terms of a local frame \( Z_1 \in T_{1,0} \) by
\[
\nabla Z_1 = \theta_1 \otimes Z_1, \quad \nabla Z_\bar{1} = \theta_\bar{1} \otimes Z_\bar{1}, \quad \nabla T = 0,
\]
where \( \theta_1 \) is the 1-form uniquely determined by
\[
d\theta_1 = \theta_1 \wedge \theta_1 + \theta \wedge \tau_1,
\]
\[
\tau_1 \equiv 0 \mod \theta_\bar{1},
\]
\[
0 = \theta_1 + \theta_\bar{1},
\]
where \( \tau_1 \) is the pseudohermitian torsion. Put \( \tau_1 = A_1 \theta_\bar{1} \). The structure equation for the pseudohermitian connection is
\[
d\theta_1 = R \theta_1 \wedge \theta_\bar{1} + 2i \text{Im}(A_1 \theta_1 \wedge \theta),
\]
where \( R \) is the Tanaka–Webster curvature.

We will denote components of covariant derivatives with indices preceded by comma; thus we write \( A_1 \theta_1 \wedge \theta \). The indices \( \{0, 1, \bar{1}\} \) indicate derivatives with respect to \( \{T, Z_1, Z\bar{1}\} \). For derivatives of a scalar function, we will often omit the comma, for instance, \( \varphi_1 = Z_1 \varphi, \ \varphi_\bar{1} = Z_\bar{1} \varphi - \theta_\bar{1}(Z_1 \varphi), \ \varphi_0 = T \varphi \) for a (smooth) function.

For a real function \( \varphi \), the subgradient \( \nabla_\varphi \) is defined by \( \nabla_\varphi \varphi \in \xi \), and \( \langle Z, \nabla_\varphi \varphi \rangle_{L_\theta} = d\varphi(Z) \) for all vector fields \( Z \) tangent to contact plane. Locally, \( \nabla_\varphi \varphi = \varphi_1 Z_1 + \varphi_\bar{1} Z_\bar{1} \).

We can use the connection to define the subhessian as the complex linear map
\[
(\nabla^H)^2 \varphi : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1},
\]
and
\[
(\nabla^H)^2 \varphi(Z) = \nabla_Z \nabla_\varphi \varphi.
\]
The sublaplacian $\Delta_b$ is defined as $-1$ times the trace of the subhessian, that is, $\Delta_b \phi = -\text{Tr} \left( (\nabla^H)^2 \phi \right) = -(\phi_{\bar{1}\bar{1}} + \phi_{11})$. For all $Z = x^1 Z_1 \in T_{1,0}$, define

$$\text{Ric}(Z, Z) = R x^1 x^{\bar{1}} = R |Z|_L^2,$$

$$\text{Tor}(Z, Z) = 2 \Re i A_{1\bar{1}} x^{\bar{1}} x^{\bar{1}}.$$

Greenleaf [1985] proved the lower bound $(n/n+1)k_0$ of the first positive eigenvalue $\lambda_1$ of the sublaplacian for a pseudohermitian manifold $M^{2n+1}$ with $n \geq 3$ under a condition on the Webster curvature and the torsion. Li and Luk [2004] proved the same result for $n = 1$ and $n = 2$. However, for $n = 1$, they needed a condition depending not only on the Webster curvature and the torsion, but also on a covariant derivative of the torsion.

The same result was proved in [Chiu 2006] under a more geometric condition which involved the positivity of the CR Paneitz operator $P_0$ (see Section 2 for a definition) with respect to $(J, \theta)$.

**Proposition 1.1** [Chiu 2006]. Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold with nonnegative Paneitz operator $P_0$. Suppose that

$$\text{Ric}_m(Z, Z) - \text{Tor}_m(Z, Z) \geq k_0 \langle Z, Z \rangle_{L_0},$$

for all $m \in M$, $Z \in T_{1,0}$, and for some positive constant $k_0$. Let $\lambda_1$ be the first positive eigenvalue of $\Delta_b$. Then

$$\lambda_1 \geq \frac{k_0}{2} > 0.$$

Let $(S^3, J, \theta)$ be a 3-sphere with the induced CR structure from $\mathbb{C}^2$ and the standard contact form $\theta$. One can show that [Chang et al. 2005; Chiu 2006]

$$\lambda_1 = \frac{k_0}{2}.$$

Here $k_0$ is the positive, constant Webster curvature of $S^3$. Thus we get a sharp estimate of $\lambda_1$ on the standard sphere $(S^3, J, \theta)$.

**Conjecture 1.2.** Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold. Suppose that

$$\lambda_1 = \frac{k_0}{2}.$$

We conjecture that $(M, J, \theta)$ is the standard CR 3-sphere due to the theorems of Lichnerowicz [1958] and Obata [1962] in the Riemannian case. In fact, here we have (see the proof of Theorem 1.5)

(i) $\text{Ric}_m(Z, Z) - \text{Tor}_m(Z, Z) = k_0 \langle Z, Z \rangle_{L_0},$

(ii) $\ker (\Delta_b - \lambda_1 I) \subset \ker P_0,$
(iii) $\varphi_{11} = 0$ for $\varphi \in \ker (\Delta_b - \lambda_1 I)$.

In this paper, we will try to place a good lower bound on the first positive eigenvalue when the curvature satisfies

$$\text{Ric}_m(Z, Z) - \text{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z \rangle_{L_\theta}$$

for some nonnegative constant $k_0$.

**Definition 1.3.** On a closed pseudohermitian 3-manifold $(M, J, \theta)$, we call the Paneitz operator $P_0$ with respect to $(J, \theta)$ essentially positive if there exists a constant $\Lambda > 0$ such that

$$\int_M P\varphi \cdot \varphi \, d\mu \geq \Lambda \int_M \varphi^2 \, d\mu$$

for all real $C^\infty$ smooth functions $\varphi \in (\ker P_0)^\perp$ (that is, those perpendicular to the kernel of $P_0$ in the $L^2$ norm with respect to the volume form $d\mu = \theta \wedge d\theta$).

**Remark 1.4.** The essential positivity of $P_0$ is a CR invariant in the sense that it is independent of the choice of the contact form $\theta$. Actually, if $\tilde{\theta} = e^{2\lambda} \theta$ is another contact form, then we have $d\tilde{V} = \tilde{\theta} \wedge d\tilde{\theta} = e^{4\lambda} \theta \wedge d\theta$ and the transformation law $\tilde{P}_0 = e^{-4\lambda} P_0$ of the CR Paneitz operator [Hirachi 1993]. Therefore, we have

$$\int \tilde{P}_0 \varphi \cdot \varphi \, d\tilde{V} = \int P_0 \varphi \cdot \varphi \, dV.$$

Firstly, by using the same method as in [Chiu 2006], we are able to prove:

**Theorem 1.5.** Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold with essentially positive Paneitz operator $P_0$. Suppose:

(i) For some nonnegative constant $k_0$,

$$\text{Ric}_m(Z, Z) - \text{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z \rangle_{L_\theta}.$$

(ii) $\ker (\Delta_b - \lambda_1 I) \cap (\ker P_0)^\perp \neq \emptyset$.

Then

$$\lambda_1 \geq \frac{-k_0 + \sqrt{k_0^2 + 6\Lambda}}{4} > 0.$$

However if the torsion is zero, then the corresponding Paneitz operator is essentially positive [Chang et al. 2005]. Therefore,

**Corollary 1.6.** Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold. Suppose:

(i) For some nonnegative constant $k_0$,

$$R \geq -k_0 \quad \text{and} \quad A_{11} = 0.$$

(ii) $\ker (\Delta_b - \lambda_1 I) \cap (\ker P_0)^\perp \neq \emptyset$. 

Then

\[ \lambda_1 \geq \frac{-k_0 + \sqrt{k_0^2 + 6\Lambda}}{4} > 0. \]

**Definition 1.7.** We say that \((M, J)\) has a transversal symmetry if \(M\) admits a one-parameter group of CR automorphisms transverse to the holomorphic tangent bundle.

For example, \((M, J, \theta)\) has a transversal symmetry if \(A_{11} = 0\). For details, we refer to \cite{Graham and Lee 1988} and \cite{Hirachi 1993}.

**Definition 1.8.** A piecewise smooth curve \(\gamma : [0, 1] \to M\) is said to be horizontal if \(\gamma'(t) \in \xi\) whenever \(\gamma'(t)\) exists. The length of \(\gamma\) is then defined by

\[ l(\gamma) = \int_0^1 dt \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{L_\theta}}. \]

The Carnot–Carathéodory distance between two points \(p, q \in M\) is

\[ d(p, q) = \inf \{ l(\gamma) | \gamma \in C_{p,q} \}, \]

where \(C_{p,q}\) is the set of all horizontal curves joining \(p\) and \(q\). By the Chow connectivity theorem \cite{1939}, there always exists a horizontal curve joining \(p\) and \(q\), so the distance is finite. The diameter \(d\) is defined by

\[ d = \sup \{ d(p, q) | p, q \in M \}. \]

Note that there is a minimizing geodesic joining \(p\) and \(q\) so that its length is equal to the distance \(d(p, q)\).

Next, by using the Li–Yau gradient estimates \cite{Yau 1975; Li and Yau 1980}, we have:

**Theorem 1.9.** Let \((M, J, \theta)\) be a closed three-dimensional pseudohermitian manifold that has a transversal symmetry. Suppose:

(i) For some nonnegative constant \(k_0\),

\[ \text{Ric}_m(Z, Z) - \text{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z \rangle_{L_\theta}. \]

(ii) \(\ker (\Delta_b - \lambda_1 I) \cap \ker P_0 \neq \phi\).

Then

\[ \lambda_1 \geq \left( 1 + \sqrt{1 + 2(k_0 + \tau_0)d^2} \right) e^{-\left(1 + \sqrt{1 + 2(k_0 + \tau_0)d^2}\right)}. \]

Here \(\tau_0 = \max |A_{11}| \) and \(d\) is the diameter of \(M\).

**Corollary 1.10.** Let \((M, J, \theta)\) be a closed three-dimensional pseudohermitian manifold. Suppose:
For some nonnegative constant $k_0$,

\[ R \geq -k_0 \quad \text{and} \quad A_{11} = 0. \]

(ii) $\ker (\Delta_b - \lambda_1 I) \cap \ker P_0 \neq \emptyset$.

Then

\[ \lambda_1 \geq \frac{(1 + \sqrt{1 + 2k_0 d^2})}{6d^2} e^{-\left(1 + \sqrt{1 + 2k_0 d^2}\right)} \cdot \frac{-k_0 + \sqrt{k_0^2 + 6\Lambda}}{4}. \]

Combining Theorem 1.5 and Theorem 1.9, we can prove:

**Theorem 1.11.** Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold which has a transversal symmetry. Suppose:

(i) For some nonnegative constant $k_0$,

\[ \text{Ric}_m(Z, Z) - \text{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z \rangle_L. \]

(1) $\Delta_b \ker P_0 \subset \ker P_0$.

Then

\[ \lambda_1 \geq \max \left\{ \frac{(1 + \sqrt{1 + 2(k_0 + \tau_0)d^2})}{6d^2} e^{-\left(1 + \sqrt{1 + 2(k_0 + \tau_0)d^2}\right)}; \frac{-k_0 + \sqrt{k_0^2 + 6\Lambda}}{4} \right\}. \]

Here $\tau_0 = \max |A_{11}|$ and $d$ is the diameter of $M$.

In particular, if $A_{11} = 0$, then $(M, J, \theta)$ has a transversal symmetry and we also have $\Delta_b \ker P_0 \subset \ker P_0$. Therefore, as a consequence of Theorem 1.5 and Theorem 1.11,

**Corollary 1.12.** Let $(M, J, \theta)$ be a closed three-dimensional pseudohermitian manifold with $A_{11} = 0$. Suppose

\[ R \geq -k_0, \]

for some nonnegative constant $k_0$. Then

\[ \lambda_1 \geq \max \left\{ \frac{(1 + \sqrt{1 + 2k_0 d^2})}{6d^2} e^{-\left(1 + \sqrt{1 + 2k_0 d^2}\right)}; \frac{-k_0 + \sqrt{k_0^2 + 6\Lambda}}{4} \right\}. \]

That is, there is a positive constant $C(k_0, d, \Lambda)$ such that

\[ \lambda_1 \geq C(k_0, d, \Lambda). \]
We briefly describe the methods used in our proofs. In Section 2, we first derive the CR version of Bochner formula which involves the CR Paneitz operator. This formula, involving a term that has no analogue in the Riemannian case, is hard to control. A key step is that we relate this extra term to a third-order operator $P$ that characterizes CR-pluriharmonic functions [Lee 1988]. After integrating by parts, we get the CR Paneitz operator.

Section 3 contains the second crucial step. By using the Li–Yau gradient estimate [Yau 1975; Li and Yau 1980], we are able to prove the main Theorem 1.11.

2. The Bochner formula and CR Paneitz operator

We define an operator through

$$P \varphi = (\varphi \overline{\theta}^1 + i A_{11} \varphi^1) \theta^1 = P \varphi = (P_1 \varphi) \theta^1,$$

which characterizes the CR-pluriharmonic functions. Here $P_1 \varphi = \varphi \overline{\theta}^1 + i A_{11} \varphi^1$, and $\overline{P} \varphi = (\overline{P}_\theta) \theta^1$ is the conjugate of $P$. Now define $\delta_b$ as the divergence operator that takes $(1, 0)$-forms to functions by

$$\delta_b(\sigma \theta^1) = \sigma,$$

and similarly define $\overline{\delta}_b$ through $\overline{\delta}_b(\sigma \overline{\theta}^1) = \sigma$. The CR Paneitz operator $P_0$ is then defined through

$$P_0 \varphi = 4 \left( \delta_b(P \varphi) + \overline{\delta}_b(\overline{P} \varphi) \right).$$

We observe that

$$\int \langle P \varphi + \overline{P} \varphi, d_b \varphi \rangle_{L^2} \, dV = -\frac{1}{4} \int P_0 \varphi \cdot \varphi \, dV. \quad (1)$$

One can check that $P_0$ is self-adjoint, that is, $\langle P_0 \varphi, \psi \rangle = \langle \varphi, P_0 \psi \rangle$ for all smooth functions $\varphi$ and $\psi$. For more details about these operators, read [Lee 1988; Graham and Lee 1988; Hirachi 1993; Gover and Graham 2003; Fefferman and Hirachi 2003].

We first derive the following new CR version of the Bochner formula:

**Lemma 2.1.** For a real function $\varphi$,

$$\frac{1}{2} \Delta_b |\nabla_b \varphi|^2 = -|\nabla^H \varphi|^2 + 3\langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle_{L^2}$$

$$- (2 \text{Ric} - 3 \text{Tor})(\langle \nabla_b \varphi \rangle_C, (\nabla_b \varphi)_C)$$

$$+ 4\langle P \varphi + \overline{P} \varphi, d_b \varphi \rangle_{L^2}.$$

Here $(\nabla_b \varphi)_C = \varphi \overline{Z}_1$ is the corresponding complex $(1, 0)$-vector field of $\nabla_b \varphi$ and $d_b \varphi = \varphi_1 \theta^1 + \varphi_\overline{1} \overline{\theta}^1$. 

Proof. From [Greenleaf 1985], we have for a real function $\varphi$

$$(2) \quad \Delta_b |\nabla b \varphi|^2 = -2|\nabla^H \varphi|^2 + 2\langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle_{L^2} - (4 \text{Ric} + 2 \text{Tor}) \langle \nabla_b \varphi, \nabla_b \varphi \rangle_{L^2}.$$

Lemma 2.1 follows from this and

$$(3) \quad \langle J \nabla_b \varphi, \nabla_b \varphi \rangle_{L^2} = -\langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle_{L^2} - 2 \text{Tor}(\nabla_b \varphi, \nabla_b \varphi)_{\mathcal{C}},$$

which we next prove. The commutation relation $i\varphi_0 = \varphi_{1\bar{1}} - \varphi_{\bar{1}1}$ [Lee 1988] gives $\varphi_{1\bar{1}1} - \varphi_{\bar{1}11} = i\varphi_0$. Thus

$$(4) \quad \langle J \nabla_b \varphi, \nabla_b \varphi/2 \rangle_{L^2} = i(\varphi_{1\bar{1}} \varphi_{\bar{1}1} - \varphi_{1\bar{1}} \varphi_{\bar{1}1})$$

$$= \varphi_1 (\varphi_{1\bar{1}1} - \varphi_{\bar{1}11}) + \varphi_1 (\varphi_{1\bar{1}1} - \varphi_{\bar{1}11}).$$

On the other hand,

$$(5) \quad \langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle_{L^2} = \varphi_1 (\Delta_b \varphi)_{1\bar{1}} + \varphi_1 (\Delta_b \varphi)_{\bar{1}1}$$

$$= -\varphi_1 (\varphi_{1\bar{1}1} + \varphi_{\bar{1}11}) - \varphi_1 (\varphi_{1\bar{1}1} + \varphi_{\bar{1}11}).$$

It follows from (4) and (5) that

$$\langle J \nabla_b \varphi, \nabla_b \varphi/2 \rangle_{L^2} + \langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle_{L^2} = -2\varphi_1 (\varphi_{1\bar{1}1} + \varphi_{\bar{1}11})$$

$$= -2\varphi_1 (P_1 \varphi - iA_{1\bar{1}} \varphi) - 2\varphi_1 (\bar{P}_1 \varphi + iA_{\bar{1}1} \varphi)$$

$$= -2 \text{Tor}(\nabla_b \varphi, \nabla_b \varphi)_{\mathcal{C}}.$$ \hfill\(\square\)

Proof of Theorem 1.5. Let $\varphi \in (\ker P_0)^\perp$ be an eigenfunction of $\Delta_b$ having the first positive eigenvalue $\lambda_1$. By definition,

$$(6) \quad \int_M \varphi P_0 \varphi dV \geq \Lambda \int_M \varphi^2 dV.$$

By integrating (2), (3) and using (1), we have

$$(7) \quad \int_M |(\nabla^H)^2 \varphi |_{L^2}^2 dV = \int_M (\Delta_b \varphi)^2 dV + 2 \int \varphi_0^2 dV$$

$$- \int (2 \text{Ric} + \text{Tor}) \langle \nabla_b \varphi, \nabla_b \varphi \rangle_{\mathcal{C}} dV,$$

and

$$(8) \quad \int \varphi_0^2 dV = \int (\Delta_b \varphi)^2 dV + 2 \int \text{Tor}(\nabla_b \varphi, \nabla_b \varphi)_{\mathcal{C}} dV - \frac{1}{2} \int P_0 \varphi \cdot \varphi dV.$$

On the other hand, it is easy to verify that

$$| (\nabla^H)^2 \varphi |_{L^2}^2 = 2 |\varphi_{1\bar{1}}|^2 + \frac{1}{2} (\Delta_b \varphi)^2 + \frac{1}{2} \varphi_0^2.$$
Substituting this into the left hand of (7) and combining with (8), we get
\[ 2 \int |\varphi_{11}|^2 \, dV = 2 \int (\Delta_b \varphi)^2 \, dV - 2 \int (\text{Ric} - \text{Tor})((\nabla_b \varphi)_{\mathbb{C}}, (\nabla_b \varphi)_{\mathbb{C}}) \, dV - \frac{3}{4} \int P_0 \varphi \cdot \varphi \, dV. \]

By combining with (6), we get
\[ 0 \geq -2 \int (\Delta_b \varphi)^2 \, dV + 2 \int (\text{Ric} - \text{Tor})((\nabla_b \varphi)_{\mathbb{C}}, (\nabla_b \varphi)_{\mathbb{C}}) \, dV + \frac{3}{4} \int P_0 \varphi \cdot \varphi \, dV \]
\[ \geq \int (2\lambda_1 - k_0) |\nabla_b \varphi|^2_{L_0} \, dV \]
\[ \geq \int \left( -2\lambda_1 - k_0 + \frac{3\Lambda}{4\lambda_1} \right) |\nabla_b \varphi|^2_{L_0} \, dV. \]

This holds if and only if
\[ -2\lambda_1 - k_0 + \frac{3\Lambda}{4\lambda_1} \leq 0, \]
and Theorem 1.5 follows immediately. \( \square \)

3. The Li–Yau gradient estimate

Let \((M, J, \theta)\) be a closed three-dimensional pseudohermitian manifold. In the case that \(\ker(\Delta_b - \lambda_1 I) \cap (\ker P_0) \neq \phi\), then, by using the so-called Li–Yau gradient estimate [Yau 1975; Li and Yau 1980], one can place a lower bound on the positive first eigenvalue of a sublaplacian \(\Delta_b\).

**Lemma 3.1.** Let \(\varphi = \ln f\) for \(f > 0\). Then
\[ 4 \langle P \varphi + \tilde{P} \varphi, d_b \varphi \rangle_{L^2_0} = 4 \frac{\langle Pf + \tilde{P} f, d_b f \rangle_{L^2_0}}{f^2} - 4 \langle \nabla_b \varphi, \nabla_b |\nabla_b \varphi|^2 \rangle_{L_0} \]
\[ + 2 \frac{\Delta_b f}{f} |\nabla_b \varphi|^2. \]

**Proof:** Let \(Q(x) = |\nabla_b \varphi|^2 (x)\). We compute
\[ \nabla_b Q = Q_\tau Z_1 + Q_1 Z_\tau = 2\nabla_b (\varphi_1 \varphi_\tau) \]
\[ = \frac{f^2 f_1 f_{\Pi} + f^2 f_\tau f_1 - 2 f f_\tau^2 f_1}{f^4} Z_1 + \text{complex conjugate}. \]
It follows that
\[
P_1 \varphi = \varphi_1 + i A_1 \varphi_1 = \frac{f^3 f_{111} - f^2 f_1 f_{11} - 2 f^2 f_1 f_{11} + 2 f f_1^2 f_1 + i A_1 f_1}{f^4}
\]
\[
= \frac{P_1 f}{f} - \frac{f_1 f_{11}}{f^2} = \frac{P_1 f}{f} - Q_1 - \varphi_1 f_{11} \frac{1}{f}.
\]
Thus
\[
4 \langle P \varphi + \bar{P} \varphi, d_b \varphi \rangle_{L^2} = 4 \left( (P_1 \varphi) \theta^1 + (\bar{P}_1 \varphi) \bar{\theta}^1, \varphi_1 \theta_1^1 + \bar{\varphi}_1 \bar{\theta}_1^1 \right)_{L^2}\]
\[
= 4 \left[ (P_1 \varphi) \varphi_1 + (\bar{P}_1 \varphi) \bar{\varphi}_1 \right] = 4 \left( \frac{P_1 f}{f} - Q_1 - \varphi_1 f_{11} \frac{1}{f} \right) \varphi_1 + \text{complex conjugate}
\]
\[
= 4 \left\{ (P f + \bar{P} f, d_b f)_{L^2} \right\} - 4 \left\{ \nabla_b \varphi, \nabla_b |\nabla_b \varphi|^2 \right\} + 2 \left( \frac{\Delta_b f}{f} - |\nabla_b \varphi|^2 \right).
\]
This implies the lemma. \(\square\)

The next lemma will ready us to show Theorem 1.9.

Lemma 3.2 [Graham and Lee 1988; Hirachi 1993]. Let \((M, J, \theta)\) be a closed three-dimensional pseudohermitian manifold with a transversal symmetry, and let \(\theta\) be any pseudohermitian structure on \(M\). Then a smooth real-valued function \(f\) satisfies \(P_0 f = 0\) on \(M\) if and only if \(P_1 f = 0\) on \(M\), that is, \(f\) is CR-pluriharmonic.

Proof of Theorem 1.9. Let \(f\) be an eigenfunction of \(\Delta_b\) with eigenvalue \(\lambda_1\), that is, \(\Delta_b f = \lambda_1 f\). Also suppose \(P_0 f = 0\). Since
\[
\lambda_1 \int_M f = \int_M \Delta_b f = 0,
\]
f must change sign. We may normalize \(f\) to satisfy \(\min f = -1\) and \(\max f \leq 1\). Let us consider the function \(\varphi = \ln (a + f)\), for some constant \(a > 1\). Then the function \(\varphi\) satisfies
\[
\Delta_b \varphi = \frac{\Delta_b f}{a + f} - \left( \nabla_b \left( \frac{1}{a + f} \right), \nabla_b (a + f) \right)_{L^2}
\]
\[
= \frac{\Delta_b f}{a + f} + \frac{|\nabla_b f|^2}{(a + f)^2} = \frac{\lambda_1 f}{a + f} + |\nabla_b \varphi|^2.
\]
Since \(\|\nabla^2 \varphi\|_{L^2}^2 = 2 |\varphi_1|^2 + \frac{1}{2} (\Delta_b \varphi)^2 + \frac{1}{2} \varphi_0^2\), we have
\[
- \|\nabla^2 \varphi\|_{L^2}^2 \leq - \frac{1}{2} (\Delta_b \varphi)^2 \leq - \frac{1}{2} |\nabla_b \varphi|^4 - \frac{\lambda_1 f}{a + f} |\nabla_b \varphi|^2.
\]
On the other hand, we have
\[
\langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle_{L_0} = \left\langle \nabla_b \varphi, \nabla_b \left( \frac{\lambda_1 f}{a + f} + |\nabla_b \varphi|^2 \right) \right\rangle_{L_0} \\
= \langle \nabla_b \varphi, \nabla_b |\nabla_b \varphi|^2 \rangle_{L_0} + \frac{\lambda_1 a}{a + f} |\nabla_b \varphi|^2.
\]

Because $\text{Ric}_m(Z, Z) - \text{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z \rangle_{L_0}$, we have
\[
(9) \quad 2 \text{Ric}_m(Z, Z) - 2 \text{Tor}_m(Z, Z) \geq -2k_0 \langle Z, Z \rangle_{L_0}.
\]

On the other hand, put $\tau_0 = \max |A_{11}|$. Then from
\[
-2|A_{11}| \langle Z, Z \rangle_{L_0} \leq -\text{Tor}(Z, Z) \leq 2|A_{11}| \langle Z, Z \rangle_{L_0},
\]
we have
\[
(10) \quad -2\tau_0 \langle Z, Z \rangle_{L_0} \leq -\text{Tor}(Z, Z) \leq 2\tau_0 \langle Z, Z \rangle_{L_0}.
\]
Combining (9) and (10), one has
\[
2 \text{Ric}_m(Z, Z) - 3 \text{Tor}_m(Z, Z) \geq -2(k_0 + \tau_0) \langle Z, Z \rangle_{L_0}.
\]

Now we define $Q(x) = |\nabla_b \varphi|^2$. Then, by Lemma 2.1 and Lemma 3.1, we see that the sublaplacian satisfies
\[
\frac{1}{2} \Delta_b Q + \langle \nabla_b \varphi, \nabla_b Q \rangle \leq -\frac{1}{2} Q^2 - \left( \lambda_1 - \frac{2(k_0 + \tau_0)}{2} - \frac{4\lambda_1 a}{a + f} - \frac{2\lambda_1 f}{a + f} \right) Q \\
\leq -\frac{1}{2} Q^2 - \left( \lambda_1 - \frac{2(k_0 + \tau_0)}{2} - \frac{6\lambda_1 a}{a - 1} \right) Q.
\]
If $x_0 \in M$ is a point where $Q$ achieves its maximum, we have
\[
0 \leq \frac{1}{2} \Delta_b Q(x_0) + \langle \nabla_b \varphi, \nabla_b Q \rangle (x_0).
\]
Hence
\[
\frac{1}{2} Q^2(x_0) + \left( \lambda_1 - \frac{2(k_0 + \tau_0)}{2} - \frac{6\lambda_1 a}{a - 1} \right) Q(x_0) \leq 0
\]
which implies that
\[
Q(x) \leq Q(x_0) \leq -2 \left( \lambda_1 - \frac{2(k_0 + \tau_0)}{2} - \frac{6\lambda_1 a}{a - 1} \right) \leq \frac{12a}{a - 1} \lambda_1 + 2(k_0 + \tau_0),
\]
for all \( x \in M \). Integrating \( Q^1 = |\nabla_b \varphi| = |\nabla_b \ln (a + f)| \) along a minimal horizontal geodesic \( \gamma \) joining the points at which \( f = -1 \) and \( f = \max f \), it follows that
\[
\ln \frac{a}{a - 1} \leq \ln \left( \frac{a + \max f}{a - 1} \right) = \ln (a + \max f) - \ln (a - 1)
\leq \int_\gamma |\nabla_b \ln (a + f)| \leq d \sqrt{\frac{12a}{a - 1} \lambda_1 + 2(k_0 + \tau_0)},
\]
for all \( a > 1 \). Setting \( t = (a - 1)/a \), we have
\[
12 \lambda_1 \geq \left( \frac{1}{d^2} \left( \ln \frac{1}{t} \right)^2 - 2(k_0 + \tau_0) \right) t
\]
for all \( 0 < t < 1 \). Maximizing the right hand side as a function of \( t \) by setting \( t = \exp(-1 - \sqrt{1 + 2(k_0 + \tau_0)d^2}) \), we obtain the estimate
\[
\lambda_1 \geq \frac{1}{12} \left( \frac{(1 + \sqrt{1 + 2(k_0 + \tau_0)d^2})^2}{d^2} - 2(k_0 + \tau_0) \right) e^{(-1 - \sqrt{1 + 2(k_0 + \tau_0)d^2})} - \frac{1}{6d^2} e^{(-1 - \sqrt{1 + 2(k_0 + \tau_0)d^2})}.
\]

**Proof of Theorem 1.11.** If \((M, J, \theta)\) is a closed three-dimensional pseudohermitian manifold that has a transversal symmetry, then there exists a torsion free pseudohermitian contact structure \( \widetilde{\theta} = e^{2f} \theta \) for some real smooth function \( f \). Therefore \( \widetilde{P}_0 \), the CR Paneitz operator with respect to \( \widetilde{\theta} \), is essentially positive. But \( \widetilde{P}_0 = e^{-4f} P_0 \). It follows that \( P_0 \) is essentially positive.

On the other hand, suppose that the CR Paneitz operator \( P_0 \) and the sublaplacian \( \Delta_b \) satisfy \( \Delta_b (\ker P_0) \subset \ker P_0 \). Hence we have the following decomposition (see [Chang et al. 2005, Section 5] for details):
\[
\ker (\Delta_b - \lambda_1 I) = E_K \oplus_{P_0} E_K^\perp,
\]
where \( E_K \subset \ker P_0 \) and \( E_K^\perp \subset (\ker P_0)^\perp \).

Let \( f \) be an eigenfunction of \( \Delta_b \) with respect to the first positive eigenvalue \( \lambda_1 \). \( P_0 \) decomposes \( f \) as
\[
f = f^\perp \oplus f_{\ker},
\]
whence
\[
\Delta_b f^\perp = \lambda_1 f^\perp \quad \text{and} \quad \Delta_b f_{\ker} = \lambda_1 f_{\ker}.
\]

Theorem 1.11 then follows directly from Theorem 1.5 and Theorem 1.9. \( \square \)
Acknowledgments

The first author would like to express his thanks to Professor S.-T. Yau for constant encouragement and Professor J.-P. Wang for valuable discussions during his visit at NCTS, Hsinchu, Taiwan.

References


Received April 25, 2006.

SHU-CHENG CHANG  
DEPARTMENT OF MATHEMATICS  
NATIONAL TSING HUA UNIVERSITY  
HSINCHU 30013  
TAIWAN  
scchang@math.nthu.edu.tw

HUNG-LIN CHIU  
DEPARTMENT OF APPLIED MATHEMATICS  
NATIONAL CENTRAL UNIVERSITY  
CHUNG-LI 32054  
TAIWAN  
hlchiu@math.ncu.edu.tw