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MÖBIUS ISOPARAMETRIC HYPERSURFACES WITH THREE DISTINCT PRINCIPAL CURVATURES

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Dedicated to Professor Anmin Li on his sixtieth birthday

We investigate Möbius isoparametric hypersurfaces in the $(n+1)$-Euclidean unit sphere $\mathbb{S}^{n+1}$ with three distinct Möbius principal curvatures. As direct consequence of our main result, we establish the complete classification for all such hypersurfaces in $\mathbb{S}^6$.

1. Introduction

Let $x : M^n \to \mathbb{S}^{n+1}$ be a hypersurface in the $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$ without umbilic point. We choose a local orthonormal basis $\{e_1, \cdots, e_n\}$ with respect to the induced metric $I = dx \cdot dx$ and the dual basis $\{\theta_1, \cdots, \theta_n\}$. Let $h = \sum_{i,j} h_{ij}\theta_i \otimes \theta_j$ be the second fundamental form of x, with squared length $\|h\|^2 = \sum_{i,j}(h_{ij})^2$ and mean curvature $H = \frac{1}{n}\sum_i h_{ii}$, respectively. Define $\rho^2 = n/(n-1)\cdot(\|h\|^2 - nH^2)$. Then the positive definite form $g = \rho^2 dx \cdot dx$ is a Möbius invariant and is called the Möbius metric of $x : M^n \to \mathbb{S}^{n+1}$. The Möbius second fundamental form $\mathbf{B}$, another basic Möbius invariant of x, together with g determine completely a hypersurface of $\mathbb{S}^{n+1}$ up to Möbius equivalence; see Theorem 2.2.

An important class of hypersurfaces for Möbius differential geometry consists of the so-called Möbius isoparametric hypersurfaces in $\mathbb{S}^{n+1}$. Recall that, according to [Li et al. 2002], a Möbius isoparametric hypersurface of $\mathbb{S}^{n+1}$ is an umbilic-free hypersurface of $\mathbb{S}^{n+1}$ such that the Möbius invariant 1-form

$$\Phi = -\rho^{-1}\sum_i \Big\{e_i(H) + \sum_j (h_{ij} - H\delta_{ij})e_j(\log\rho)\Big\}\theta_i$$

vanishes and all of its Möbius principal curvatures are constant. Note that by Möbius principal curvatures, we mean the so-called eigenvalues of the Möbius

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shape operator $\Psi := \rho^{-1}(\mathbf{S} - H \text{ id})$ with respect to g; here $\mathbf{S}$ denotes the shape operator of $x : M^n \to \mathbb{S}^{n+1}$. This definition of Möbius isoparametric hypersurfaces is meaningful when compared with that of (Euclidean) isoparametric hypersurfaces in $\mathbb{S}^{n+1}$. The images of all hypersurfaces of the sphere with constant mean curvature and constant scalar curvature under Möbius transformation satisfy $\Phi \equiv 0$, and the Möbius invariant operator Ψ plays the same role in Möbius geometry as $\mathbf{S}$ does in Euclidean geometry (see Theorem 2.2). Standard examples of Möbius isoparametric hypersurfaces are the images of (Euclidean) isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ under Möbius transformations, but there are other examples that cannot be obtained in this way. For example, they occur in our classification for hypersurfaces of $\mathbb{S}^{n+1}$ with *parallel* Möbius second fundamental form; this means that the Möbius second fundamental form is parallel with respect to the Levi-Civita connection of the Möbius metric g. See [Li et al. 2002; Hu and Li 2004] for details. On the other hand, it was proved in [Li et al. 2002] that any Möbius isoparametric hypersurface is in particular a Dupin hypersurface, which implies from [Thorbergsson 1983] that for a compact Möbius isoparametric hypersurface embedded in $\mathbb{S}^{n+1}$, the number γ of distinct principal curvatures can only take the values $\gamma = 2, 3, 4, 6$.

Li et al. [2002] classified locally all Möbius isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ with $\gamma = 2$. By relaxing the restriction of $\gamma = 2$, Hu and Li [2005a] and Hu et al. [2007] classified all Möbius isoparametric hypersurfaces in $\mathbb{S}^4$ and $\mathbb{S}^5$, respectively. To be precise, they showed that a Möbius isoparametric hypersurface in $\mathbb{S}^4$ is either of parallel Möbius second fundamental form or Möbius equivalent to the Euclidean isoparametric hypersurface in $\mathbb{S}^4$ with three distinct principal curvatures, that is, a tube of constant radius over a standard Veronese embedding of RP^2 into $\mathbb{S}^4$. However, a hypersurface in $\mathbb{S}^5$ is Möbius isoparametric if and only if it satisfies either of two properties: First, it has parallel Möbius second fundamental form or is Möbius equivalent to the preimage of the stereographic projection of the cone $\tilde{x} : N^3 \times \mathbb{R}^+ \to \mathbb{R}^5$ defined by $\tilde{x}(x, t) = tx$, where $t \in \mathbb{R}^+$ and $x : N^3 \to \mathbb{S}^4 \hookrightarrow \mathbb{R}^5$ is the Cartan isoparametric immersion in $\mathbb{S}^4$ with three principal curvatures. Second, it is Möbius equivalent to the Euclidean isoparametric hypersurfaces in $\mathbb{S}^5$ with four distinct principal curvatures. All these results remind us of their counterparts in Dupin hypersurfaces; see [Thorbergsson 1983; Pinkall 1985; Niebergall 1991; 1992; Cecil and Jensen 1998; 2000].

Because all Möbius isoparametric hypersurfaces have been classified both for $n \leq 4$ in [Hu and Li 2005a; Hu et al. 2007] and for all $n \geq 2$ with two distinct Möbius principal curvatures in [Li et al. 2002], we will consider here Möbius isoparametric hypersurfaces M^n in $\mathbb{S}^{n+1}$ with three distinct Möbius principal curvatures for all $n \geq 5$. For further background, we note that the classification of all such hypersurfaces under the Möbius transformation group equivalence can be compared with that of the Dupin hypersurfaces with three principal curvatures under the Lie sphere transformation group equivalence established by Cecil and Jensen [1998]. We find it interesting that the Lie sphere transformation group contains the Möbius transformation group in $\mathbb{S}^{n+1}$ as a subgroup and the dimension difference is $n+3$. Therefore, the Möbius differential geometry for hypersurfaces in spheres seems essentially different from the Lie sphere geometry and therefore merits more attention.

Nevertheless, for simplicity, we focus on when one of the three distinct Möbius principal curvatures has multiplicity one. For Möbius isoparametric hypersurfaces with nonparallel Möbius second fundamental form and three distinct Möbius principal curvatures — all of which have multiplicity not smaller than two — the classification is much more involved and will be given in a forthcoming paper.

We will establish these classification results:

Main Theorem. *Let $x : M^n \to \mathbb{S}^{n+1}$ ($n \geq 5$) be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures such that one of them is simple. Then x is Möbius equivalent to an open part of one of these hypersurfaces in $\mathbb{S}^{n+1}$:*

(i) *The preimage of the stereographic projection of the warped product embedding*

$$\tilde{x} : \mathbb{S}^p(a) \times \mathbb{S}^q(\sqrt{1-a^2}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}^{n+1},$$

with $p \geq 1$, $q \geq 1$, $p+q \leq n-1$, and $0 < a < 1$, defined by

$$\tilde{x}(u', u'', t, u''') = (tu', tu'', tu'''),$$

where $u' \in \mathbb{S}^p(a)$, $u'' \in \mathbb{S}^q(\sqrt{1-a^2})$, $t \in \mathbb{R}^+$, and $u''' \in \mathbb{R}^{n-p-q-1}$.

(ii) *Minimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^3 \times \mathbb{H}^{n-3}\Big(-\frac{n-1}{6n}\Big) \to \mathbb{S}^{n+1},$$

where $\tilde{x}_1 = y_1/y_0$, $\tilde{x}_2 = y_2/y_0$, with $y_0 \in \mathbb{R}^+$, $y_1 \in \mathbb{R}^5$, $y_2 \in \mathbb{R}^{n-3}$. Here $y_1 : N^3 \to \mathbb{S}^4\big(\sqrt{6n/(n-1)}\big) \hookrightarrow \mathbb{R}^5$ is Cartan's minimal isoparametric hypersurface with vanishing scalar curvature and principal curvatures $\pm\sqrt{(n-1)/2n}$, 0. Also $(y_0, y_2) : \mathbb{H}^{n-3}(-(n-1)/6n) \hookrightarrow \mathbb{L}^{n-2}$ is the standard embedding of the hyperbolic space of sectional curvature $-(n-1)/6n$ into the $(n-2)$-dimensional Lorentz space with $-y_0^2 + y_2^2 = -6n/(n-1)$.

If $n = 5$, then hypersurfaces with three distinct Möbius principal curvatures trivially satisfy the assumption that at least one of the principal curvatures is simple. Hence we have immediately:

Corollary 1.1. *Let $x : M^5 \to \mathbb{S}^6$ be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures. Then x is Möbius equivalent to an open part of one of these hypersurfaces in $\mathbb{S}^6$:*

(i) *The preimage of the stereographic projection of the warped product embedding*

$$\tilde{x} : \mathbb{S}^p(a) \times \mathbb{S}^q(\sqrt{1-a^2}) \times \mathbb{R}^+ \times \mathbb{R}^{4-p-q} \to \mathbb{R}^6,$$

with $p \geq 1$, $q \geq 1$, $p+q \leq 4$, and $0 < a < 1$, defined by

$$\tilde{x}(u', u'', t, u''') = (tu', tu'', tu'''),$$

where $u' \in \mathbb{S}^p(a)$, $u'' \in \mathbb{S}^q(\sqrt{1-a^2})$, $t \in \mathbb{R}^+$, $u''' \in \mathbb{R}^{4-p-q}$.

(ii) *Minimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^5 = N^3 \times \mathbb{H}^2\big(-\tfrac{2}{15}\big) \to \mathbb{S}^6$$

with $\tilde{x}_1 = y_1/y_0$, $\tilde{x}_2 = y_2/y_0$, where $y_0 \in \mathbb{R}^+$, $y_1 \in \mathbb{R}^5$, $y_2 \in \mathbb{R}^2$. Here $y_1 : N^3 \to \mathbb{S}^4(\sqrt{30}/2 \hookrightarrow \mathbb{R}^5$ is Cartan's minimal isoparametric hypersurface with vanishing scalar curvature and principal curvatures $\pm\sqrt{10}/5$, 0, and $(y_0, y_2) : \mathbb{H}^2(-2/15) \hookrightarrow \mathbb{L}^3$ is the standard embedding of the hyperbolic space of sectional curvature $-2/15$ into the 3-dimensional Lorentz space with $-y_0^2 + y_2^2 = -15/2$.

Remark 1.2. The hypersurfaces in (i) consist of two families both of which are of parallel Möbius second fundamental form, whereas (ii) has only one hypersurface whose Möbius second fundamental form is not parallel.

This paper has four more sections. In Section 2, we first review some elementary facts of Möbius geometry for hypersurfaces in $\mathbb{S}^{n+1}$, and then we present classifications for hypersurfaces of $\mathbb{S}^{n+1}$ with parallel Möbius second fundamental form and for those with two distinct constant Blaschke eigenvalues. These classifications have been achieved in [Hu and Li 2004] and [Li and Zhang 2007], respectively. In Section 3, by investigating Möbius isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ with nonparallel Möbius second fundamental form and having three distinct Möbius principal curvatures, one of which is simple, we show its Möbius principal curvature must be zero with multiplicity $n-2$. Furthermore, we prove Theorem 3.7, which gives a preliminary classification for such hypersurfaces. In Section 4, we prove Proposition 4.2, Proposition 4.5 and Proposition 4.6 by calculating the Möbius invariants of the hypersurfaces appearing in Theorem 3.7. Finally, in Section 5, we complete the proof of Main Theorem.

2. Möbius invariants for hypersurfaces in $\mathbb{S}^{n+1}$

Here, we define Möbius invariants and recall structure equations for hypersurfaces in $\mathbb{S}^{n+1}$. For more detail, see [Wang 1998]. Let $\mathbb{L}^{n+3}$ be the Lorentz space, namely $\mathbb{R}^{n+3}$ with inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, w\rangle = -x_0 w_0 + x_1 w_1 + \cdots + x_{n+2} w_{n+2}$$

for $x = (x_0, x_1, \cdots, x_{n+2})$ and $w = (w_0, w_1, \cdots, w_{n+2}) \in \mathbb{R}^{n+3}$.

Let $x : M^n \to \mathbb{S}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ be an immersed umbilic-free hypersurface of $\mathbb{S}^{n+1}$. We define the Möbius position vector field $Y : M^n \to \mathbb{L}^{n+3}$ of x by $Y = \rho(1, x)$, where

$$\rho^2 = \frac{n}{n-1}\big(\|h\|^2 - nH^2\big) > 0.$$

Then we have this classical result:

Theorem 2.1 [Wang 1998]. *Two hypersurfaces $x, \tilde{x} : M^n \to \mathbb{S}^{n+1}$ are Möbius equivalent if and only if there exists T in the Lorentz group $\mathrm{O}(n+2, 1)$ in $\mathbb{L}^{n+3}$ such that $Y = \tilde{Y}T$.*

It follows immediately that $g = \langle dY, dY\rangle = \rho^2 dx \cdot dx$ is a Möbius invariant, and it is defined as the *Möbius metric* of $x : M^n \to \mathbb{S}^{n+1}$.

Let Δ be the Beltrami–Laplace operator of g. Defining

$$N = -\frac{1}{n}\,\Delta\, Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle Y, \tag{2-1}$$

one can show that

$$\begin{aligned} &\langle \Delta Y, Y\rangle = -n, \qquad &&\langle \Delta Y, dY\rangle = 0, \qquad &&\langle \Delta Y, \Delta Y\rangle = 1 + n^2 R,\\ &\langle Y, Y\rangle = 0, &&\langle N, Y\rangle = 1, &&\langle N, N\rangle = 0, \end{aligned} \tag{2-2}$$

where R is the normalized scalar curvature of g and is called the normalized Möbius scalar curvature of $x : M^n \to \mathbb{S}^{n+1}$.

Let $\{E_1, \cdots, E_n\}$ be a local orthonormal basis for (M^n, g) with dual basis $\{\omega_1, \cdots, \omega_n\}$, and write $Y_i = E_i(Y)$. Then it follows from (2-1) and (2-2) that

$$\langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0, \qquad \langle Y_i, Y_j\rangle = \delta_{ij}, \qquad 1 \le i, j \le n.$$

Let V be the orthogonal complement to the subspace $\operatorname{span}\{Y, N, Y_1, \cdots, Y_n\}$ in $\mathbb{L}^{n+3}$. Then along M we have the orthogonal decomposition

$$\mathbb{L}^{n+3} = \operatorname{span}\{Y, N\} \oplus \operatorname{span}\{Y_1, \cdots, Y_n\} \oplus V,$$

where V is called the Möbius normal bundle of $x : M^n \to \mathbb{S}^{n+1}$. A local unit vector basis $E = E_{n+1}$ for V can be written as

$$E = E_{n+1} := (H, Hx + e_{n+1}).$$

Then, along M^n, $\{Y, N, Y_1, \cdots, Y_n, E\}$ forms a moving frame in $\mathbb{L}^{n+3}$. Unless otherwise stated, we will henceforth use the range $1 \le i, j, k, l, t \le n$ for indices.

We can write the structure equations as:

$$\begin{aligned} dY &= \sum_i Y_i \omega_i, \\ dN &= \sum_{i,j} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i E, \\ dY_i &= -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_i B_{ij}\omega_j E, \\ dE &= -\sum_i C_i \omega_i Y - \sum_{i,j} B_{ij}\omega_j Y_i, \end{aligned} \tag{2-3}$$

where ω_{ij} is the connection form of the Möbius metric g and is defined by the structure equations $d\omega_i = \sum_j \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$. The tensors $\mathbf{A} = \sum_{i,j} A_{ij}\omega_i \otimes \omega_j$, $\Phi = \sum_i C_i \omega_i$, and $\mathbf{B} = \sum_{i,j} B_{ij}\omega_i \otimes \omega_j$ are respectively called the Blaschke tensor, the Möbius form, and the Möbius second fundamental form of $x : M^n \to \mathbb{S}^{n+1}$. The relations between Φ, $\mathbf{B}$, $\mathbf{A}$ and the Euclidean invariants of x are given by [Wang 1998]:

$$\begin{aligned} C_i &= -\rho^{-2}\Big(e_i(H) + \sum_j (h_{ij} - H\delta_{ij}) e_j(\log\rho)\Big), \\ B_{ij} &= \rho^{-1}(h_{ij} - H\delta_{ij}), \\ A_{ij} &= -\rho^{-2}\Big(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}\Big) \\ &\qquad - \tfrac{1}{2}\rho^{-2}\Big(|\nabla \log\rho|^2 - 1 + H^2\Big)\delta_{ij}, \end{aligned} \tag{2-4}$$

where Hess_{ij} and ∇ are the Hessian matrix and the gradient operator with respect to the orthonormal basis $\{e_i\}$ of $dx \cdot dx$.

The covariant derivatives of C_i, A_{ij}, B_{ij} are defined by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j \omega_{ji}, \tag{2-5}$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2-6}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2-7}$$

The integrability conditions for the structure equations (2-3) are

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \tag{2-8}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - A_{ik}B_{kj}), \tag{2-9}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2-10}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{2-11}$$

where

$$R_{ij} := \sum_k R_{ikjk} = -\sum_k B_{ik}B_{jk} + (\operatorname{tr} A)\delta_{ij} + (n-2)A_{ij}, \tag{2-12}$$

$$\sum_i B_{ii} = 0, \qquad \sum_{i,j}(B_{ij})^2 = \frac{n-1}{n}, \qquad \operatorname{tr} A = \sum_i A_{ii} = \frac{1}{2n}(1+n^2R). \tag{2-13}$$

Here R_{ijkl} is the curvature tensor of g defined by the structure equations

$$d\omega_{ij} - \sum_k \omega_{ik}\wedge\omega_{kj} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \tag{2-14}$$

and $R = \frac{1}{n(n-1)}\sum_{i,j} R_{ijij}$ is the normalized Möbius scalar curvature of $x : M^n \to \mathbb{S}^{n+1}$.

The second covariant derivatives of B_{ij} are defined through

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k} + \sum_l B_{lj,k}\omega_{li} + \sum_l B_{il,k}\omega_{lj} + \sum_l B_{ij,l}\omega_{lk}. \tag{2-15}$$

By exterior differentiation of (2-7), we have the Ricci identity

$$B_{ij,kl} - B_{ij,lk} = \sum_t B_{tj}R_{tikl} + \sum_t B_{it}R_{tjkl}. \tag{2-16}$$

From the second of (2-4), we see that the Möbius type operator of $x : M^n \to \mathbb{S}^{n+1}$ takes the form

$$\Psi = \rho^{-1}(\mathbf{S} - H\mathrm{id}) = \sum_{i,j} B_{ij}\omega_i E_j,$$

which implies that for an umbilic-free hypersurface in $\mathbb{S}^{n+1}$, the number of distinct Möbius principal curvatures is the same as that of its distinct Euclidean principal curvatures.

One can easily show that all coefficients in (2-3) are determined by $\{g, \Psi\}$ and thus we obtain:

Theorem 2.2 [Akivis and Goldberg 1996; 1997; Wang 1998]. *Two hypersurfaces $x : M^n \to \mathbb{S}^{n+1}$ and $\tilde{x} : \tilde{M}^n \to \mathbb{S}^{n+1}$ $(n \geq 3)$ are Möbius equivalent if and only if there exists a diffeomorphism $F : M^n \to \tilde{M}^n$ that preserves the Möbius metric and the Möbius shape operator.*

Recall that an umbilic-free hypersurface $x : M^n \to \mathbb{S}^{n+1}$ is said to have parallel Möbius second fundamental form if $B_{ij,k} = 0$ for all i, j, k. Hu and Li [2004] established a complete classification for hypersurfaces of $\mathbb{S}^{n+1}$ with parallel Möbius second fundamental form. In particular, we have:

Theorem 2.3 [Hu and Li 2004]. *Let $x : M^n \to \mathbb{S}^{n+1}$ $(n \geq 2)$ be an immersed umbilic-free hypersurface with parallel Möbius second fundamental form and with three distinct principal curvatures. Then x is Möbius equivalent to an open part of the image of σ of the warped product embedding*

$$\tilde{x} : \mathbb{S}^p(a) \times \mathbb{S}^q(\sqrt{1-a^2}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}^{n+1},$$

with $p \geq 1$, $q \geq 1$, $p+q \leq n-1$, and $0 < a < 1$, defined by

$$\tilde{x}(u', u'', t, u''') = (tu', tu'', tu''')$$

with $u' \in \mathbb{S}^p(a)$, $u'' \in \mathbb{S}^q(\sqrt{1-a^2})$, $t \in \mathbb{R}^+$, and $u''' \in \mathbb{R}^{n-p-q-1}$. The conformal diffeomorphism $\sigma : \mathbb{R}^{n+1} \to \mathbb{S}^{n+1} \setminus \{(-1, 0, \cdots, 0)\}$ is the inverse of the stereographic projection and is defined by

$$\sigma(u) = \left(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\right), \quad \textit{for } u \in \mathbb{R}^{n+1}.$$

To prove our main theorem, we also need the following partial classification of umbilic-free hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct Blaschke eigenvalues.

Theorem 2.4 [Li and Zhang 2007]. *Let $x : M^n \to \mathbb{S}^{n+1}(n \geq 3)$ be an immersed umbilic-free hypersurface with two distinct, constant Blaschke eigenvalues and vanishing Möbius form. If x has three distinct principal Möbius curvatures, then it is locally Möbius equivalent to one of these two families of hypersurfaces in $\mathbb{S}^{n+1}$:*

(i) *Minimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^p \times \mathbb{H}^{n-p}(-r^{-2}) \to \mathbb{S}^{n+1},$$
$$\textit{where} \quad \tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0,$$

with $y_0 \in \mathbb{R}^+$, $y_1 \in \mathbb{R}^{p+2}$, $y_2 \in \mathbb{R}^{n-p}$, $2 \leq p \leq n-1$, and $r > 0$. Also, $y_1 : N^p \to \mathbb{S}^{p+1}(r) \hookrightarrow \mathbb{R}^{p+2}$ is an immersed umbilic-free minimal hypersurface in the $(p+1)$-dimensional sphere of radius r with constant scalar curvature

$$\tilde{R}_1 = \frac{np(p-1)-(n-1)r^2}{nr^2}.$$

$(y_0, y_2) : \mathbb{H}^{n-p}(-r^{-2}) \to \mathbb{L}^{n-p+1}$ is the standard embedding of the hyperbolic space of sectional curvature $-r^{-2}$ into the $(n-p+1)$-dimensional Lorentz space with $-y_0^2 + y_2^2 = -r^2$.

(ii) *Nonminimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^p \times \mathbb{S}^{n-p}(r) \to \mathbb{S}^{n+1},$$
$$\textit{where} \quad \tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0,$$

with $y_0 \in \mathbb{R}^+$, $y_1 \in \mathbb{R}^{p+1}$, $y_2 \in \mathbb{R}^{n-p+1}$, $2 \le p \le n-1$, *and* $r > 0$. *Also* $(y_0, y_1) : N^p \to \mathbb{H}^{p+1}(-r^{-2}) \hookrightarrow \mathbb{L}^{p+2}$, *with* $-y_0^2 + y_1^2 = -r^2$, *is an immersed umbilic-free minimal hypersurface in the* $(p+1)$*-dimensional hyperbolic space of sectional curvature* $-r^{-2}$ *and with constant scalar curvature*

$$\tilde{R}_1 = -\frac{np(p-1)+(n-1)r^2}{nr^2}.$$

$y_2 : \mathbb{S}^{n-p}(r) \to \mathbb{R}^{n-p+1}$ *is the standard embedding of* $(n-p)$*-dimensional sphere of radius* r.

3. Möbius isoparametric hypersurfaces with $\gamma = 3$

Here, we consider Möbius isoparametric hypersurfaces $x : M^n \to \mathbb{S}^{n+1}$ of any dimension $n \ge 5$ with three distinct principal curvatures with multiplicities $m_1 \ge m_2 \ge m_3$.

For our choice of the local orthonormal basis $\{E_i\}_{1 \le i \le n}$, that Ψ has constant eigenvalues is equivalent to that the matrix (B_{ij}) has constant eigenvalues. From $\Phi = 0$ and (2-9), we see that, for all i, j,

$$\sum_k (B_{ik}A_{kj} - A_{ik}B_{kj}) = 0. \tag{3-17}$$

This implies that we can choose $\{E_i\}$ to simultaneously diagonalize (A_{ij}) and (B_{ij}). Let us write

$$(B_{ij}) = \operatorname{diag}(b_1, \cdots, b_n), \qquad (A_{ij}) = \operatorname{diag}(A_1, \cdots, A_n), \tag{3-18}$$

where the $\{b_i\}$ are constants. By assumption and without loss of generality, we can put

$$\begin{aligned} b_1 = \cdots = b_{m_1} &= B_1, \\ b_{m_1+1} = \cdots = b_{m_1+m_2} &= B_2, \\ b_{m_1+m_2+1} = \cdots = b_n &= B_3, \end{aligned} \tag{3-19}$$

with distinct B_1, B_2, and B_3. From (2-13), they satisfy the conditions

$$m_1B_1 + m_2B_2 + m_3B_3 = 0, \qquad m_1B_1^2 + m_2B_2^2 + m_3B_3^2 = \frac{n-1}{n}. \tag{3-20}$$

In this section, if not stated otherwise, we use the further index conventions

$$\begin{gathered} 1 \le a, b \le m_1, \\ m_1 + 1 \le p, q \le m_1 + m_2, \\ m_1 + m_2 + 1 \le \alpha, \beta \le m_1 + m_2 + m_3 = n. \end{gathered}$$

Applying the condition $\Phi = 0$ to (2-8) and (2-10), we see that both $B_{ij,k}$ and $A_{ij,k}$ are totally symmetric. As usual we define

$$\omega_{ij} = \sum_k \Gamma^i_{kj}\omega_k, \qquad \Gamma^i_{kj} = -\Gamma^j_{ki}.$$

From this, (2-7), (3-18), and that $\{b_i\}_{1\le i\le n}$ consists of constants, we get

$$B_{ij,k} = (b_i - b_j)\Gamma^i_{kj} = (b_j - b_k)\Gamma^j_{ik} = (b_k - b_i)\Gamma^k_{ji}, \quad \text{for all } i, j, k.$$

Thus we see that

(3-21) $$B_{ii,j} = B_{ij,i} = B_{ab,j} = B_{pq,j} = B_{\alpha\beta,j} = 0, \quad \text{for all } i, j, a, b, p, q, \alpha, \beta,$$

and the only possible nonzero elements in $\{B_{ij,k}\}$ are of the form $B_{\alpha a,p}$.

From now on in this section, we assume $B_{ij,k} \not\equiv 0$ and $m_3 = 1$. Then we can prove three lemmas:

Lemma 3.1. *The set $\{B_{na,p}\}$ has only one nonzero element.*

Proof. From the calculation

$$\begin{aligned}\sum_i B_{ab,pi}\omega_i &= dB_{ab,p} + \sum_i B_{ib,p}\omega_{ia} + \sum_i B_{ai,p}\omega_{ib} + \sum_i B_{ab,i}\omega_{ip}\\ &= B_{nb,p}\omega_{na} + B_{an,p}\omega_{nb} = \sum_q \big(B_{nb,p}\Gamma^n_{qa} + B_{na,p}\Gamma^n_{qb}\big)\omega_q\\ &= \sum_q \frac{B_{nb,p}B_{na,q} + B_{na,p}B_{nb,q}}{B_3 - B_1}\omega_q,\end{aligned}$$

we have for all a, b, p, i that

(3-22) $$B_{ab,pi} = \frac{B_{nb,p}B_{na,i} + B_{na,p}B_{nb,i}}{B_3 - B_1}, \qquad B_{ab,pp} = \frac{2B_{na,p}B_{nb,p}}{B_3 - B_1}.$$

On the other hand, from the calculation

$$\sum_i B_{pp,ai}\omega_i = 2\sum_i B_{ip,a}\omega_{ip} = 2B_{np,a}\omega_{np} = 2\sum_b B_{np,a}\Gamma^n_{bp}\omega_b = \sum_b \frac{2B_{na,p}B_{nb,p}}{B_3 - B_2}\omega_b,$$

we have for all a, b, p, i that

(3-23) $$B_{pp,ai} = \frac{2B_{na,p}B_{ni,p}}{B_3 - B_2}, \qquad B_{pp,ab} = \frac{2B_{na,p}B_{nb,p}}{B_3 - B_2}.$$

For $a \neq b$, using (3-18), (3-19) and (2-11), we obtain from (2-16) that

(3-24) $$B_{pp,ab} = B_{pa,pb} = B_{pa,bp} = B_{ab,pp}.$$

From (3-22)–(3-24), we obtain

(3-25) $$B_{na,p}B_{nb,p} = 0, \quad \text{for all } a, b, p \text{ with } a \neq b \quad \text{if } m_1 \geq 2.$$

Similarly, we can show

$$B_{nn,ab} = \frac{2\sum_p B_{na,p}B_{nb,p}}{B_2 - B_3}, \quad \text{for all } a, b; \tag{3-26}$$

$$B_{nn,ap} = 0, \quad \text{for all } a, b, p; \tag{3-27}$$

$$B_{pq,ai} = \frac{B_{na,p}B_{ni,q} + B_{na,q}B_{ni,p}}{B_3 - B_2}, \quad \text{for all } a, i, p, q; \tag{3-28}$$

$$B_{na,p}B_{na,q} = 0, \quad \text{for all } a, p, q \text{ with } p \neq q \text{ if } m_2 \geq 2. \tag{3-29}$$

From (3-22), (3-28) and that

$$B_{ab,pq} = B_{pq,ab}, \quad \text{if } a \neq b \text{ or } p \neq q,$$

we obtain

$$B_{na,p}B_{nb,q} + B_{na,q}B_{nb,p} = 0, \quad \text{for all } a, b, p, q \text{ with } (a, p) \neq (b, q). \tag{3-30}$$

As $B_{ij,k} \not\equiv 0$, we can take indices $\bar{a}$ and $\bar{p}$ to satisfy

$$B_{n\bar{a},\bar{p}} \neq 0, \quad \bar{a} \in \{1, \cdots, m_1\},\ \bar{p} \in \{m_1 + 1, \cdots, m_1 + m_2\}.$$

Then from (3-25) and (3-29), we have

$$B_{nb,\bar{p}} = 0, \quad \text{for all } b \neq \bar{a} \text{ if } m_1 \geq 2, \quad B_{n\bar{a},q} = 0, \quad \text{for all } q \neq \bar{p} \text{ if } m_2 \geq 2,$$

and this, together with (3-30), gives $B_{n\bar{a},\bar{p}}B_{nb,q} = 0$ for all $(b, q) \neq (\bar{a}, \bar{p})$. Therefore, we have $B_{nb,q} = 0$ for all $(b, q) \neq (\bar{a}, \bar{p})$. □

Lemma 3.2. *For the indices $\bar{a}$, $\bar{p}$ in Lemma 3.1, the $B_{n\bar{a},\bar{p}}$ are constant.*

Proof. From Lemma 3.1 and (2-15), we have $dB_{n\bar{a},\bar{p}} = \sum_i B_{n\bar{a},\bar{p}i}\omega_i$. Seeing that the four indices in $B_{n\bar{a},\bar{p}i}$ are totally symmetric, using (3-22), (3-27) and (3-28), we get for all b, q that

$$B_{n\bar{a},\bar{p}b} = B_{\bar{a}b,\bar{p}n} = 0, \quad B_{n\bar{a},\bar{p}q} = B_{\bar{p}q,\bar{a}n} = 0, \quad B_{n\bar{a},\bar{p}n} = B_{nn,\bar{a}\bar{p}} = 0. \qquad □$$

Lemma 3.3. *We have these results for the sectional curvature*:

$$R_{apap} = 0, \quad \textit{for all } (a, p) \neq (\bar{a}, \bar{p}); \qquad R_{\bar{a}\bar{p}\bar{a}\bar{p}} = \frac{2B^2_{n\bar{a},\bar{p}}}{(B_1 - B_3)(B_2 - B_3)};$$

$$R_{nana} = 0, \quad \textit{for all } a \neq \bar{a}; \qquad R_{n\bar{a}n\bar{a}} = \frac{2B^2_{n\bar{a},\bar{p}}}{(B_1 - B_2)(B_3 - B_2)};$$

$$R_{npnp} = 0, \quad \textit{for all } p \neq \bar{p}; \qquad R_{n\bar{p}n\bar{p}} = \frac{2B^2_{n\bar{a},\bar{p}}}{(B_2 - B_1)(B_3 - B_1)}.$$

Proof. Now we have the facts

$$\omega_{ap}=0,\quad \text{if } (a,p)\neq(\bar a,\bar p);\qquad \omega_{\bar a\bar p}=\frac{B_{n\bar a,\bar p}}{B_1-B_2}\omega_n;$$
$$\omega_{na}=0,\quad \text{if } a\neq\bar a;\qquad \omega_{n\bar a}=\frac{B_{n\bar a,\bar p}}{B_3-B_1}\omega_{\bar p};$$
$$\omega_{np}=0,\quad \text{if } p\neq\bar p;\qquad \omega_{n\bar p}=\frac{B_{n\bar a,\bar p}}{B_3-B_2}\omega_{\bar a}.$$

According to this, (2-14), and $B_{n\bar a,\bar p}$ being constant, we have, for all $a\neq\bar a$ and $p\neq\bar p$,

$$-\tfrac12\sum_{i,j}R_{apij}\omega_i\wedge\omega_j=d\omega_{ap}-\sum_i\omega_{ai}\wedge\omega_{ip}=0,$$
$$-\tfrac12\sum_{i,j}R_{a\bar pij}\omega_i\wedge\omega_j=d\omega_{a\bar p}-\sum_i\omega_{ai}\wedge\omega_{i\bar p}=-\omega_{a\bar a}\wedge\omega_{\bar a\bar p}=-\frac{B_{n\bar a,\bar p}}{B_1-B_2}\omega_{a\bar a}\wedge\omega_n,$$
$$-\tfrac12\sum_{i,j}R_{\bar apij}\omega_i\wedge\omega_j=d\omega_{\bar ap}-\sum_i\omega_{\bar ai}\wedge\omega_{ip}=-\omega_{\bar a\bar p}\wedge\omega_{\bar pp}=-\frac{B_{n\bar a,\bar p}}{B_1-B_2}\omega_n\wedge\omega_{\bar pp},$$
$$-\tfrac12\sum_{i,j}R_{naij}\omega_i\wedge\omega_j=d\omega_{na}-\sum_i\omega_{ni}\wedge\omega_{ia}=-\omega_{n\bar a}\wedge\omega_{\bar aa}=-\frac{B_{n\bar a,\bar p}}{B_3-B_1}\omega_{\bar p}\wedge\omega_{\bar aa},$$
$$-\tfrac12\sum_{i,j}R_{npij}\omega_i\wedge\omega_j=d\omega_{np}-\sum_i\omega_{ni}\wedge\omega_{ip}=-\omega_{n\bar p}\wedge\omega_{\bar pp}=-\frac{B_{n\bar a,\bar p}}{B_3-B_2}\omega_{\bar a}\wedge\omega_{\bar pp},$$
$$\begin{aligned}-\tfrac12\sum_{i,j}R_{n\bar aij}\omega_i\wedge\omega_j&=d\omega_{n\bar a}-\sum_i\omega_{ni}\wedge\omega_{i\bar a}=d(\Gamma^n_{\bar p\bar a}\omega_{\bar p})-\omega_{n\bar p}\wedge\omega_{\bar p\bar a}\\&=\Gamma^n_{\bar p\bar a}\sum_i\omega_{\bar pi}\wedge\omega_i-\omega_{n\bar p}\wedge\omega_{\bar p\bar a}=\Gamma^n_{\bar p\bar a}(\omega_{\bar p\bar a}\wedge\omega_{\bar a}+\omega_{\bar pn}\wedge\omega_n)-\omega_{n\bar p}\wedge\omega_{\bar p\bar a}\\&=-\frac{2B^2_{n\bar a,\bar p}}{(B_1-B_2)(B_3-B_2)}\omega_n\wedge\omega_{\bar a}.\end{aligned}$$

Similarly,

$$-\tfrac12\sum_{i,j}R_{n\bar pij}\omega_i\wedge\omega_j=d\omega_{n\bar p}-\sum_i\omega_{ni}\wedge\omega_{i\bar p}=-\frac{2B^2_{n\bar a,\bar p}}{(B_2-B_1)(B_3-B_1)}\omega_n\wedge\omega_{\bar p},$$
$$-\tfrac12\sum_{i,j}R_{\bar a\bar pij}\omega_i\wedge\omega_j=d\omega_{\bar a\bar p}-\sum_i\omega_{\bar ai}\wedge\omega_{i\bar p}=-\frac{2B^2_{n\bar a,\bar p}}{(B_1-B_3)(B_2-B_3)}\omega_{\bar a}\wedge\omega_{\bar p}.$$

From the equations above, we come to the conclusion immediately. □

Now, we are ready to prove:

Proposition 3.4. *Let $x: M^n\to\mathbb{S}^{n+1}$ $(n\geq 5)$ be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures of multiplicities $m_1\geq m_2\geq m_3$. If the Möbius second fundamental form is not parallel and $m_3=1$, then*

$m_2 = 1$ *and the Möbius principal curvatures are* $B_1 = 0$ *with multiplicity* $n-2$ *and* $B_2 = -B_3 = \pm\sqrt{(n-1)/2n}$.

Proof. From Lemma 3.3 and the Gauss Equation (2-11), we obtain the equations

$$R_{a\bar{p}a\bar{p}} = B_1B_2 + A_a + A_{\bar{p}} = 0, \qquad a \neq \bar{a}, \tag{3-31}$$

$$R_{\bar{a}p\bar{a}p} = B_1B_2 + A_{\bar{a}} + A_p = 0, \qquad p \neq \bar{p} \tag{3-32}$$

$$R_{apap} = B_1B_2 + A_a + A_p = 0, \qquad a \neq \bar{a}, \quad p \neq \bar{p}, \tag{3-33}$$

$$R_{nana} = B_1B_3 + A_a + A_n = 0, \qquad a \neq \bar{a}, \tag{3-34}$$

$$R_{npnp} = B_2B_3 + A_p + A_n = 0, \qquad p \neq \bar{p}, \tag{3-35}$$

$$R_{n\bar{a}n\bar{a}} = B_1B_3 + A_{\bar{a}} + A_n = \frac{2B^2_{n\bar{a},\bar{p}}}{(B_1-B_2)(B_3-B_2)}, \tag{3-36}$$

$$R_{n\bar{p}n\bar{p}} = B_2B_3 + A_{\bar{p}} + A_n = \frac{2B^2_{n\bar{a},\bar{p}}}{(B_2-B_1)(B_3-B_1)}, \tag{3-37}$$

$$R_{\bar{a}\bar{p}\bar{a}\bar{p}} = B_1B_2 + A_{\bar{a}} + A_{\bar{p}} = \frac{2B^2_{n\bar{a},\bar{p}}}{(B_1-B_3)(B_2-B_3)}. \tag{3-38}$$

If $m_2 \geq 2$, then we can form (3-31)+(3-36)−(3-34)−(3-38) and (3-32)+(3-37)−(3-35)−(3-38), which give $(B_2-B_3)(2B_1-B_2-B_3) = 0$ and $(B_1-B_3)(2B_2-B_1-B_3) = 0$, respectively. Therefore we have

$$2B_1 - B_2 - B_3 = 2B_2 - B_1 - B_3 = 0,$$

which implies $B_1 = B_2 = B_3$. This contradiction thus means we should have $m_2 = 1$.

Now for $m_2 = 1$, we form (3-31) + (3-36) − (3-34) − (3-38) once again to obtain $2B_1 = B_2 + B_3$. On the other hand, the first equation of (3-20) now reads $(n-2)B_1 + B_2 + B_3 = 0$. These imply that $B_1 = B_2 + B_3 = 0$. Now applying the second equation of (3-20), we obtain

$$B_2 = -B_3 = \pm\sqrt{\frac{n-1}{2n}}. \qquad \square$$

In the rest of this section, we assume $B_{ij,k} \neq 0$ and $m_2 = m_3 = 1$. Without loss of generality, we assume that

$$B_1 = 0, \quad B_2 = \sqrt{\frac{n-1}{2n}}, \quad B_3 = -\sqrt{\frac{n-1}{2n}} \quad B_{\bar{a}\bar{p},n} \neq 0, \tag{3-39}$$

where, for simplicity, we use the notation $\bar{p} = n-1$, $\bar{a} = n-2$.

Lemma 3.5. *With the assumptions above, we have for all $a \neq \bar{a}$ that*

$$\begin{aligned} \omega_{a\bar{a}} &= \omega_{a\bar{p}} = \omega_{an} = 0, & R_{a\bar{p}a\bar{p}} &= R_{anan} = 0, \\ \omega_{\bar{a}\bar{p}} &= \frac{B_{n\bar{a},\bar{p}}}{B_1 - B_2}\omega_n, & R_{\bar{a}\bar{p}\bar{a}\bar{p}} &= \frac{2B^2_{n\bar{a},\bar{p}}}{(B_1 - B_3)(B_2 - B_3)}, \\ \omega_{\bar{a}n} &= \frac{B_{n\bar{a},\bar{p}}}{B_1 - B_3}\omega_{\bar{p}}, & R_{\bar{a}n\bar{a}n} &= \frac{2B^2_{n\bar{a},\bar{p}}}{(B_1 - B_2)(B_3 - B_2)}, \\ \omega_{\bar{p}n} &= \frac{B_{n\bar{a},\bar{p}}}{B_2 - B_3}\omega_{\bar{a}}, & R_{\bar{p}n\bar{p}n} &= \frac{2B^2_{n\bar{a},\bar{p}}}{(B_2 - B_1)(B_3 - B_1)}. \end{aligned}$$

Proof. These are direct consequences of (3-21), Lemma 3.3, and the equations that finish its proof. To make it clear that $\omega_{a\bar{a}} = 0$ for all $a \neq \bar{a}$, we note

$$\omega_{a\bar{a}} \wedge \omega_{\bar{p}} = \omega_{a\bar{a}} \wedge \omega_n = 0, \quad \text{for all } a \neq \bar{a},$$

which are also implied by the second and fourth equations in the second group in the proof of Lemma 3.3.

Lemma 3.6. *In our situation, we have for all $a \neq \bar{a}$*

$$A_a = -A_{\bar{a}} = -A_{\bar{p}} = -A_n = -\frac{n-1}{12n}, \qquad B_{n\bar{a},\bar{p}} = \pm\frac{n-1}{6n}\sqrt{3}, \tag{3-40}$$

$$R_{a\bar{a}a\bar{a}} = 0, \quad R_{\bar{a}\bar{p}\bar{a}\bar{p}} = \frac{n-1}{6n}, \quad R_{\bar{a}n\bar{a}n} = \frac{n-1}{6n}, \quad R_{\bar{p}n\bar{p}n} = -\frac{n-1}{3n}.$$

Furthermore, also for $a \neq \bar{a}$, the first structure equations can be written as

$$\begin{aligned} d\omega_a &= \sum_{b\neq\bar{a}} \omega_{ab} \wedge \omega_b, & d\omega_{\bar{a}} &= \pm\sqrt{\frac{2(n-1)}{3n}}\omega_{\bar{p}} \wedge \omega_n, \\ d\omega_{\bar{p}} &= \mp\sqrt{\frac{n-1}{6n}}\omega_{\bar{a}} \wedge \omega_n, & d\omega_n &= \pm\sqrt{\frac{n-1}{6n}}\omega_{\bar{a}} \wedge \omega_{\bar{p}}. \end{aligned}$$

Proof. Assuming always that $a \neq \bar{a}$, from (3-39), Lemma 3.5, and the Gauss Equation (2-11), we now have

$$\begin{aligned} R_{a\bar{p}a\bar{p}} &= A_a + A_{\bar{p}} = R_{anan} = A_a + A_n = 0, \\ R_{\bar{a}\bar{p}\bar{a}\bar{p}} &= A_{\bar{a}} + A_{\bar{p}} = R_{\bar{a}n\bar{a}n} = A_{\bar{a}} + A_n = \frac{2n}{n-1}B^2_{n\bar{a},\bar{p}}, \\ R_{\bar{p}n\bar{p}n} &= -\frac{n-1}{2n} + A_{\bar{p}} + A_n = -\frac{4n}{n-1}B^2_{n\bar{a},\bar{p}}. \end{aligned}$$

From these, we obtain

$$A_a = -A_{\bar p} = -A_n = -\frac{n-1}{4n} + \frac{2n}{n-1} B^2_{n\bar a,\bar p},$$
$$A_{\bar a} = -\frac{n-1}{4n} + \frac{4n}{n-1} B^2_{n\bar a,\bar p}.$$

On the other hand, because $\omega_{a\bar a} = \omega_{a\bar p} = \omega_{an} = 0$, we obtain from (2-14) that $R_{a\bar a a\bar a} = 0$, According to $R_{a\bar a a\bar a} = B_1^2 + A_a + A_{\bar a}$, we find $A_a = -A_{\bar a}$. Putting this into the above, we find $B^2_{n\bar a,\bar p} = ((n-1)/n)^2/12$. Then our conclusion follows immediately from Lemma 3.5. □

Lemma 3.6 shows that, in this case, the Blaschke tensor has exactly two distinct constant eigenvalues. Therefore, as an application of Theorem 2.4, we have proved:

Theorem 3.7. *Let $x : M^n \to \mathbb{S}^{n+1}$ $(n \ge 5)$ be a Möbius isoparametric hypersurface with nonparallel Möbius second fundamental form and three distinct Möbius principal curvatures, one of which is simple. Then locally x can only be Möbius equivalent to one of these two families of hypersurfaces in $\mathbb{S}^{n+1}$:*

$(\mathfrak{C}_1)$ *Minimal hypersurfaces defined by*

$$\tilde x = (\tilde x_1, \tilde x_2) : \tilde M^n = N^p \times \mathbb{H}^{n-p}(-r^{-2}) \to \mathbb{S}^{n+1},$$
$$\text{where } \tilde x_1 = y_1/y_0, \quad \tilde x_2 = y_2/y_0,$$

with $y_0 \in \mathbb{R}^+$, $y_1 \in \mathbb{R}^{p+2}$, $y_2 \in \mathbb{R}^{n-p}$, $2 \le p \le n-1$, and $r > 0$. Also, $y_1 : N^p \to \mathbb{S}^{p+1}(r) \hookrightarrow \mathbb{R}^{p+2}$ is an immersed umbilic-free minimal hypersurface in the $(p+1)$-dimensional sphere of radius r and with constant scalar curvature

$$\tilde R_1 = \frac{np(p-1) - (n-1)r^2}{nr^2}. \tag{3-41}$$

$(y_0, y_2) : \mathbb{H}^{n-p}(-r^{-2}) \to \mathbb{L}^{n-p+1}$ is the standard embedding of the hyperbolic space of sectional curvature $-r^{-2}$ into the $(n-p+1)$-dimensional Lorentz space with $-y_0^2 + y_2^2 = -r^2$.

$(\mathfrak{C}_2)$ *Nonminimal hypersurfaces defined by*

$$\tilde x = (\tilde x_1, \tilde x_2) : \tilde M^n = N^p \times \mathbb{S}^{n-p}(r) \to \mathbb{S}^{n+1},$$
$$\text{where} \quad \tilde x_1 = y_1/y_0, \quad \tilde x_2 = y_2/y_0,$$

with $y_0 \in \mathbb{R}^+$, $y_1 \in \mathbb{R}^{p+1}$, $y_2 \in \mathbb{R}^{n-p+1}$, $2 \le p \le n-1$, and $r > 0$. Also $(y_0, y_1) : N^p \to \mathbb{H}^{p+1}(-r^{-2}) \hookrightarrow \mathbb{L}^{p+2}$, with $-y_0^2 + y_1^2 = -r^2$, is an immersed umbilic-free minimal hypersurface in the $(p+1)$-dimensional hyperbolic space of *sectional curvature $-r^{-2}$ and with constant scalar curvature*

$$\tilde{R}_1 = -\frac{np(p-1)+(n-1)r^2}{nr^2}. \tag{3-42}$$

$y_2 : \mathbb{S}^{n-p}(r) \to \mathbb{R}^{n-p+1}$ is the standard embedding of $(n-p)$-dimensional sphere of radius r.

To answer which of the hypersurfaces in $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are Möbius isoparametric, we need to calculate their Möbius invariants, and we do this next.

4. Möbius invariants of hypersurfaces in $\mathfrak{C}_1$ and $\mathfrak{C}_2$

Keeping in mind that some of the hypersurfaces in $\mathfrak{C}_1$ and $\mathfrak{C}_2$ might be not Möbius isoparametric, we will sort them out by direct calculation.

Example 4.1. Calculation for hypersurfaces in $\mathfrak{C}_1$. Compare with [Li and Zhang 2007].

For $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ as defined under $\mathfrak{C}_1$ in Theorem 3.7, we have

$$d\tilde{x} = -\frac{dy_0}{y_0^2}(y_1, y_2) + \frac{1}{y_0}(dy_1, dy_2), \tag{4-43}$$

and then its Euclidean induced metric is given by

$$\tilde{I} = d\tilde{x} \cdot d\tilde{x} = y_0^{-2}(-dy_0^2 + dy_1^2 + dy_2^2)\big|_{\tilde{M}^n}.$$

Let ξ_1 be the unit normal vector field of $y_1 : N^p \to \mathbb{S}^{p+1}(r) \hookrightarrow \mathbb{R}^{p+2}$. Then $\xi = (\xi_1, 0) \in \mathbb{R}^{n+2}$ is a unit normal vector field of $\tilde{x}$. Consequently, by (4-43), the (Euclidean) second fundamental form $\tilde{h}$ of $\tilde{x}$ is related to the (Euclidean) second fundamental form $\tilde{h}^*$ of y_1 by

$$\tilde{h} = -d\xi \cdot d\tilde{x} = -y_0^{-1}(d\xi_1 \cdot dy_1) = y_0^{-1}\tilde{h}^*.$$

Let $\{\tilde{E}_i\}_{1\le i\le p}$ and $\{\tilde{E}_i\}_{p+1\le i\le n}$ be the local orthonormal bases on (N^p, dy_1^2) and $\mathbb{H}^{n-p}(-r^{-2})$, respectively. Then $\{\tilde{E}_i\}_{1\le i\le n}$ form a local orthonormal basis on $\tilde{M}^n$ with respect to the metric $(-dy_0^2 + dy_1^2 + dy_2^2)\big|_{\tilde{M}^n} = y_0^2\tilde{I}$.

Put $\tilde{e}_i = y_0\tilde{E}_i$, $1 \le i \le n$. Then $\{\tilde{e}_i\}_{1\le i\le n}$ is a local orthonormal basis on $\tilde{M}^n$ with respect to the metric $\tilde{I}$. Thus

$$\begin{aligned} \tilde{h}_{ij} &= \tilde{h}(\tilde{e}_i, \tilde{e}_j) = y_0^2\tilde{h}(\tilde{E}_i, \tilde{E}_j) \\ &= y_0\tilde{h}^*(\tilde{E}_i, \tilde{E}_j) = y_0\tilde{h}^*_{ij} \qquad \text{if } 1 \le i, j \le p, \\ \tilde{h}_{ij} &= 0 \qquad \text{if } i > p \text{ or } j > p. \end{aligned} \tag{4-44}$$

From this and the minimality of y_1, we see that $\tilde{x} : \tilde{M}^n \to \mathbb{S}^{n+1}$ is also minimal, that is, $\tilde{H} = 0$. Therefore, by definition, the Möbius factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\tilde{\rho}^2 = \frac{n}{n-1}\Big(\sum_{i,j=1}^{n} \tilde{h}_{ij}^2 - n\tilde{H}^2\Big) = \frac{n}{n-1} y_0^2 \sum_{i,j=1}^{p} (\tilde{h}_{ij}^*)^2 = y_0^2,$$

where in the last equality, we use that $\sum_{i,j=1}^{p} (\tilde{h}_{ij}^*)^2 = (n-1)/n$, which is implied by (3-41) and the Gauss equation of y_1. Hence, the Möbius position vector of $\tilde{x}$ is $\tilde{Y} = \tilde{\rho}(1, \tilde{x}) = (y_0, y_1, y_2) \in \mathbb{L}^{n+3}$ and the Möbius metric of $\tilde{x}$ is

$$\tilde{g} = \langle d\tilde{Y}, d\tilde{Y}\rangle = \big(-dy_0^2 + dy_1^2 + dy_2^2\big)\big|_{\tilde{M}^n} = y_0^2 \tilde{I}. \tag{4-45}$$

Therefore, $\{\tilde{E}_i\}_{1\le i\le n}$ is in fact a local orthonormal basis of the Möbius metric $\tilde{g}$. Furthermore, the Möbius second fundamental form of $\tilde{x}$ is

$$\tilde{\mathbf{B}} = \tilde{\rho}^{-1} \sum_{i,j=1}^{n} (\tilde{h}_{ij} - \tilde{H}\delta_{ij})\tilde{\omega}_i\tilde{\omega}_j = \sum_{i,j=1}^{p} \tilde{h}_{ij}^* \tilde{\omega}_i\tilde{\omega}_j, \tag{4-46}$$

where $\{\tilde{\omega}_i\}_{1\le i\le n}$ is the dual basis of $\{\tilde{E}_i\}_{1\le i\le n}$ on $\tilde{M}^n$. Note that (4-46) is equivalent to

$$\tilde{B}_{ij} = \tilde{h}_{ij}^*, \quad \text{for } 1 \le i, j \le p; \qquad \tilde{B}_{ij} = 0, \quad \text{for } i > p \text{ or } j > p. \tag{4-47}$$

Since (4-45) shows that $(\tilde{M}^n, \tilde{g})$ is the Riemannian *direct* product

$$(\tilde{M}^n, \tilde{g}) = (N^p, dy_1^2) \times \mathbb{H}^{n-p}(-r^{-2}),$$

we can use the Gauss equation to write down the Ricci tensor of $\tilde{g}$ with respect to $\{\tilde{E}_i\}_{1\le i\le n}$ as:

$$\tilde{R}_{ij} = \begin{cases} \dfrac{p-1}{r^2}\delta_{ij} - \sum\limits_{k=1}^{p} \tilde{h}_{ik}^*\tilde{h}_{kj}^*, & \text{if } 1 \le i, j \le p; \\ -\dfrac{n-p-1}{r^2}\delta_{ij}, & \text{if } p+1 \le i, j \le n; \\ 0, & \text{for all other cases,} \end{cases} \tag{4-48}$$

which implies that the normalized scalar curvature $\tilde{R}$ of $\tilde{g}$ satisfies

$$\begin{aligned} n(n-1)\tilde{R} &= \frac{p(p-1) - (n-p)(n-p-1)}{r^2} - \sum_{i,j=1}^{p} \big(\tilde{h}_{ij}^*\big)^2 \\ &= \frac{p(p-1) - (n-p)(n-p-1)}{r^2} - \frac{n-1}{n}. \end{aligned} \tag{4-49}$$

Thus

$$\frac{1}{2n}(1+n^2\tilde{R})=\frac{p(p-1)-(n-p)(n-p-1)}{2(n-1)r^2}. \tag{4-50}$$

From (2-12), (2-13), and (4-47)–(4-50), it follows easily that the Blaschke tensor of $\tilde{x}$ is given by $\tilde{\mathbf{A}}=\sum_{i,j=1}^{n}\tilde{A}_{ij}\tilde{\omega}_i\tilde{\omega}_j$, where the $\tilde{A}_{ij}$ form a diagonal matrix with entries $\tilde{A}_{ii}=1/(2r^2)$ if $1\le i\le p$, $\tilde{A}_{ii}=-1/(2r^2)$ if $p+1\le i\le n$, and 0 elsewhere.

For the Möbius form $\tilde{\Phi}=\sum_{i=1}^{n}\tilde{C}_i\tilde{\omega}_i$ of $\tilde{x}$, from (2-4), (4-44) and that $\tilde{H}=0$, $\tilde{\rho}=y_0$, we see, for $1\le i\le n$, that

(4-51)

$$\tilde{C}_i=-\tilde{\rho}^{-2}\Big(\tilde{e}_i(\tilde{H})+\sum_{j=1}^{n}(\tilde{h}_{ij}-\tilde{H}\delta_{ij})\tilde{e}_j(\log\tilde{\rho})\Big)=-y_0^{-1}\sum_{j=1}^{p}\tilde{h}^*_{ij}\tilde{e}_j(\log y_0)=0.$$

Therefore, we have $\tilde{\Phi}=0$. To summarize the above calculation, we present:

Proposition 4.2. *A hypersurface $\tilde{x}$ in $\mathfrak{C}_1$ is Möbius isoparametric if and only if it satisfies*:

(i) $p=3$;

(ii) $r=\sqrt{6n/(n-1)}$;

(iii) $y_1: N^3\to\mathbb{S}^4\big(\sqrt{6n/(n-1)}\big)$ *is a minimal isoparametric hypersurface with vanishing scalar curvature*; *moreover*, *it has three distinct principal curvatures with values* $\pm\sqrt{(n-1)/(2n)}$, 0.

Proof. From (3-40) and the expression for $\tilde{\mathbf{A}}$, we see that $p=3$ and $r=\sqrt{6n/(n-1)}$. From (3-41) we find that $y_1: N^3\to\mathbb{S}^4(r)$ has vanishing scalar curvature. From (4-46), we know that all the nonzero Möbius principal curvatures of $\tilde{x}$ are equal to the nonzero Euclidean principal curvatures of y_1. From (3-39), we then deduce that the principal curvatures of y_1 are exactly $\pm\sqrt{(n-1)/(2n)}$, 0. □

Remark 4.3. According to E. Cartan [1939], minimal isoparametric hypersurfaces in $\mathbb{S}^4(r)$, $r=\sqrt{6n/(n-1)}$, with three distinct principal curvatures do exist, and they are unique. More precisely, the hypersurface is the tube of constant radius over the standard Veronese embedding of $\mathbb{R}P^2$ into $\mathbb{S}^4(r)$ with principal curvatures $\pm\sqrt{(n-1)/(2n)}$, 0.

Example 4.4. Calculation for hypersurfaces in $\mathfrak{C}_2$. Compare with [Li and Zhang 2007].

For $\tilde{x}=(\tilde{x}_1,\tilde{x}_2)$ as defined under $\mathfrak{C}_2$ in Theorem 3.7, we have

$$d\tilde{x}=-\frac{dy_0}{y_0^2}(y_1,y_2)+\frac{1}{y_0}(dy_1,dy_2), \tag{4-52}$$

and then its Euclidean induced metric is

$$\tilde{I} = d\tilde{x}\cdot d\tilde{x} = y_0^{-2}(-dy_0^2 + dy_1^2 + dy_2^2)\big|_{\tilde{M}^n}.$$

Let (ξ_0, ξ_1) be the unit normal vector field of

$$\tilde{y} := (y_0, y_1) : N^p \to \mathbb{H}^{p+1}(-r^{-2}) \hookrightarrow \mathbb{L}^{p+2},$$

where $\xi_0 \in \mathbb{R}^+$, $\xi_1 \in \mathbb{R}^{p+1}$. Then we can easily verify that

$$\xi = (\xi_1, 0) - \xi_0\tilde{x} \in \mathbb{R}^{n+2}$$

is a unit normal vector field of $\tilde{x}$, and the Euclidean second fundamental form $\tilde{h}$ of $\tilde{x}$ is given by

$$\begin{aligned}\tilde{h} &= -d\xi\cdot d\tilde{x} = \xi_0 d\tilde{x}\cdot d\tilde{x} - (d\xi_1, 0)\cdot d\tilde{x} = -y_0^{-1}(-d\xi_0 dy_0 + d\xi_1\cdot dy_1) + \xi_0\tilde{I}\\ &= -y_0^{-1}\langle d(\xi_0,\xi_1), d(y_0, y_1)\rangle + \xi_0\tilde{I} = y_0^{-1}\tilde{h}^* + \xi_0\tilde{I},\end{aligned}$$

where $\tilde{h}^*$ denotes the *Euclidean* second fundamental form of $\tilde{y} : N^p \to \mathbb{H}^{p+1}(-r^{-2})$ and, in the third equality, we have used $d\xi_1\cdot y_1 = y_0 d\xi_0$ which is implied by

$$-\xi_0 y_0 + \xi_1\cdot y_1 = -\xi_0 dy_0 + \xi_1\cdot dy_1 = 0.$$

Let $\{\tilde{E}_i\}_{1\le i\le p}$ and $\{\tilde{E}_i\}_{p+1\le i\le n}$ be the local orthonormal bases on $(N^p, d\tilde{y}^2)$ and $\mathbb{S}^{n-p}(r)$, respectively. Then $\{\tilde{E}_i\}_{1\le i\le n}$ form a local orthonormal basis on $\tilde{M}^n$ with respect to the metric $(-dy_0^2 + dy_1^2 + dy_2^2)\big|_{\tilde{M}^n} = y_0^2\tilde{I}$.

Put $\tilde{e}_i = y_0\tilde{E}_i$, $1 \le i \le n$. Then $\{\tilde{e}_i\}_{1\le i\le n}$ is a local orthonormal basis on $\tilde{M}^n$ with respect to the metric $\tilde{I}$. Thus, we have, for $1 \le i, j \le p$,

$$\tilde{h}_{ij} = \tilde{h}(\tilde{e}_i, \tilde{e}_j) = y_0^2\tilde{h}(\tilde{E}_i, \tilde{E}_j) = y_0\tilde{h}^*(\tilde{E}_i, \tilde{E}_j) + \xi_0 y_0^2\tilde{I}(\tilde{E}_i, \tilde{E}_j) = y_0\tilde{h}^*_{ij} + \xi_0\delta_{ij}$$

and, for $i > p$ or $j > p$,

$$\tilde{h}_{ij} = y_0^2\tilde{h}(\tilde{E}_i, \tilde{E}_j) = y_0\tilde{h}^*(\tilde{E}_i, \tilde{E}_j) + \xi_0 y_0^2\tilde{I}(\tilde{E}_i, \tilde{E}_j) = \xi_0\delta_{ij}.$$

From this and the minimality of $\tilde{y}$, the mean curvature of $\tilde{x} : \tilde{M}^n \to \mathbb{S}^{n+1}$ is

$$\tilde{H} = \frac{1}{n}\sum_{i=1}^{n}\tilde{h}_{ii} = \frac{y_0}{n}\sum_{i=1}^{p}\tilde{h}^*_{ii} + \xi_0 = \xi_0.$$

Therefore, by definition, the Möbius factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\tilde{\rho}^2 = \frac{n}{n-1}\bigg(\sum_{i,j=1}^{n}\tilde{h}_{ij}^2 - n\tilde{H}^2\bigg) = \frac{n}{n-1}y_0^2\sum_{i,j=1}^{p}\big(h^*_{ij}\big)^2 = y_0^2,$$

where, in the last equality, we use $\sum_{i,j=1}^{p}(\tilde{h}^*_{ij})^2 = (n-1)/n$, which is implied by (3-42) and the Gauss equation of $\tilde{y}$.

Hence, the Möbius position vector of $\tilde{x}$ is $\tilde{Y} = \tilde{\rho}(1, \tilde{x}) = (y_0, y_1, y_2) \in \mathbb{L}^{n+3}$ and the Möbius metric of $\tilde{x}$ is

$$\tilde{g} = \langle d\tilde{Y}, d\tilde{Y} \rangle = \left(-dy_0^2 + dy_1^2 + dy_2^2\right)\big|_{\tilde{M}^n} = y_0^2 \tilde{I}. \tag{4-53}$$

Therefore, $\{\tilde{E}_i\}_{1 \le i \le n}$ is in fact a local orthonormal basis of the Möbius metric $\tilde{g}$. Furthermore, the Möbius second fundamental form of $\tilde{x}$ is

$$\tilde{\mathbf{B}} = \tilde{\rho}^{-1} \sum_{i,j=1}^{n} (\tilde{h}_{ij} - \tilde{H}\delta_{ij})\tilde{\omega}_i\tilde{\omega}_j = \sum_{i,j=1}^{p} \tilde{h}^*_{ij}\tilde{\omega}_i\tilde{\omega}_j,$$

where $\{\tilde{\omega}_i\}_{1 \le i \le n}$ is the dual basis of $\{\tilde{E}_i\}_{1 \le i \le n}$ on $\tilde{M}^n$. This is equivalent to

$$\tilde{B}_{ij} = \begin{cases} \tilde{h}^*_{ij} & 1 \le i, j \le p, \\ 0 & i > p \text{ or } j > p. \end{cases} \tag{4-54}$$

Since (4-53) shows that $(\tilde{M}^n, \tilde{g})$ is the Riemannian direct product

$$(\tilde{M}^n, \tilde{g}) = (N^p, d\tilde{y}^2) \times \mathbb{S}^{n-p}(r),$$

we can use the Gauss equation to write down the Ricci tensor of $\tilde{g}$ with respect to $\{\tilde{E}_i\}_{1 \le i \le n}$ as

$$\tilde{R}_{ij} = \begin{cases} -\dfrac{p-1}{r^2}\delta_{ij} - \displaystyle\sum_{k=1}^{p} \tilde{h}^*_{ik}\tilde{h}^*_{kj} & \text{if } 1 \le i, j \le p, \\ \dfrac{n-p-1}{r^2}\delta_{ij} & \text{if } p+1 \le i, j \le n, \\ 0, & \text{for all other cases,} \end{cases} \tag{4-55}$$

which implies that the normalized scalar curvature $\tilde{R}$ of $\tilde{g}$ satisfies

$$\begin{aligned} n(n-1)\tilde{R} &= \frac{(n-p)(n-p-1) - p(p-1)}{r^2} - \sum_{i,j=1}^{p} \left(\tilde{h}^*_{ij}\right)^2 \\ &= \frac{(n-p)(n-p-1) - p(p-1)}{r^2} - \frac{n-1}{n}. \end{aligned} \tag{4-56}$$

Thus

$$\frac{1}{2n}(1 + n^2\tilde{R}) = \frac{(n-p)(n-p-1) - p(p-1)}{2(n-1)r^2}. \tag{4-57}$$

From (2-12), (2-13) and (4-54)–(4-57), it follows that the Blaschke tensor of $\tilde{x}$ is given by $\tilde{\mathbf{A}} = \sum_{i,j=1}^{n} \tilde{A}_{ij}\tilde{\omega}_i\tilde{\omega}_j$, where the $\tilde{A}_{ij}$ form a diagonal matrix with entries $\tilde{A}_{ii} = -1/(2r^2)$ if $1 \le i \le p$, $\tilde{A}_{ii} = 1/(2r^2)$ if $p+1 \le i \le n$, and 0 elsewhere.

To show that $\tilde{x} : \tilde{M}^n \to \mathbb{S}^{n+1}$ has vanishing Möbius form, we can calculate directly according to its Equation (2-4). Nevertheless, compared with (4-51), it is not easy this time. Here is a simple argument proving $\tilde{\Phi} = 0$: On the Riemannian direct product $(\tilde{M}^n, \tilde{g}) = (N^p, d\tilde{y}^2) \times \mathbb{S}^{n-p}(r)$, the Blaschke tensor has two distinct constant eigenvalues as seen from the expression of $\tilde{\mathbf{A}}$, and their eigendistributions are both integrable with $(N^p, d\tilde{y}^2)$ and $\mathbb{S}^{n-p}(r)$ as their respective integrable manifolds. Then we easily see that the Blaschke tensor is also parallel. Now, according to [Li and Zhang 2006], we have $\tilde{\Phi} = 0$.

Then, similarly to Proposition 4.2, if we compare the expression of $\tilde{\mathbf{A}}$ with (3-40) and (4-54) with (3-39), we can summarize the above calculation:

Proposition 4.5. *If a hypersurface $\tilde{x}$ in $\mathfrak{C}_2$ is Möbius isoparametric, then must necessarily satisfy three conditions*:

(i) $p = n - 3$;

(ii) $r = \sqrt{6n/(n-1)}$;

(iii) $\tilde{y} = (y_0, y_1) : N^{n-3} \to \mathbb{H}^{n-2}(-(n-1)/(6n))$ *is a minimal isoparametric hypersurface with principal curvatures* $\pm\sqrt{(n-1)/(2n)}$ *and* 0 *with multiplicity* $n - 5$.

On the other hand, according to E. Cartan [1938], an isoparametric hypersurface M^n in hyperbolic space $\mathbb{H}^{n+1}$ can have at most two distinct principal curvatures, and M^n must be either totally umbilic or else an open subset of a standard product $\mathbb{S}^k \times \mathbb{H}^{n-k}$ in $\mathbb{H}^{n+1}$; moreover, the later must be nonminimal. From this fact and Proposition 4.5, we have proved:

Proposition 4.6. *There are no Möbius isoparametric hypersurfaces in $\mathfrak{C}_2$. More precisely, any hypersurface in $\mathfrak{C}_2$, if it exists, is of vanishing Möbius form, has two distinct, constant Blaschke eigenvalues, and nevertheless has nonconstant Möbius principal curvatures.*

5. Proof of the Main Theorem

Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures, one of which is simple. Then we have two cases: if x has parallel Möbius second fundamental form, then we apply Theorem 2.3 to obtain that it is locally Möbius equivalent to a hypersurface in (i) of the Main Theorem; if x has nonparallel Möbius second fundamental form, then we can apply Theorem 3.7, Proposition 4.2, Remark 4.3 and Proposition 4.6 to conclude that it is locally Möbius equivalent to the hypersurface in (ii) of the Main Theorem. □

Final Remark. For the general theory of Möbius submanifolds in $\mathbb{S}^{n+p}$ (see [Wang 1998]), the Möbius form Φ is an important invariant. In many interesting situations, we find that $\Phi = 0$ is a very natural condition. For details, we refer to [Guo et al. 2001; Li et al. 2001; Liu et al. 2001; Li et al. 2002; 2003b; Li and Wang 2003a; Hu and Li 2003; 2004; 2005a; 2005b; Hu et al. 2007; Li and Zhang 2006; 2007], a series of nice results established in recent years under the condition $\Phi = 0$.

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