

*Pacific
Journal of
Mathematics*

THE RESIDUAL SPECTRUM OF INNER FORMS OF $\mathrm{Sp}(2)$

TAKANORI YASUDA

Volume 232 No. 2

October 2007

THE RESIDUAL SPECTRUM OF INNER FORMS OF $\mathrm{Sp}(2)$

TAKANORI YASUDA

Let G be the unitary group of the hyperbolic hermitian space with rank two over a quaternion division algebra over a totally real number field. We determine the irreducible decomposition of the residual discrete spectrum of G . Finally we give expected description of Arthur parameters associated to these representations.

1. Introduction

Let k be a number field and A its adèle ring. Let G be a semisimple group defined over k . We write $L^2(G(k)\backslash G(A))$ for the Hilbert space of square-integrable functions on $G(k)\backslash G(A)$. The space $L_0^2(G(k)\backslash G(A))$ of cuspidal elements of $L^2(G(k)\backslash G(A))$ is contained in the discrete spectrum of $L^2(G(k)\backslash G(A))$, and the orthogonal complement of $L_0^2(G(k)\backslash G(A))$ in the discrete spectrum is called the *residual spectrum*. In this paper we take as G the unitary group of the two-dimensional hyperbolic hermitian space over a quaternion division algebra D over k . It is an inner form of $\mathrm{Sp}(2)$. We determine the irreducible decomposition of its residual spectrum, as a first example of a nonquasisplit group.

For $G = \mathrm{Sp}(2)$, the irreducible decomposition of its residual spectrum has been determined by Kim [1995]. Kon-no [1994] described it using theta correspondence in the case that k is totally real. We have a decomposition:

$$L^2(\mathrm{Sp}(2, k)\backslash \mathrm{Sp}(2, A)) = L^2(\mathrm{Sp}(2)) \oplus L^2(P_0) \oplus L^2(P_1) \oplus L^2(P_2),$$

along constant terms [Mœglin and Waldspurger 1995, Prop. II.2.4]. Here P_0 is a Borel subgroup, P_1 and P_2 are Siegel and non-Siegel maximal parabolic subgroups, respectively. Then the residual spectrum can be described as

$$L_d^2(P_0) \oplus L_d^2(P_1) \oplus L_d^2(P_2).$$

Here $L_d^2(\cdot)$ denotes the discrete spectrum of $L^2(\cdot)$. Similarly, the residual spectrum of our G coincides with $L_d^2(P)$, where P is a proper parabolic subgroup of G . P corresponds to P_1 via the inner twist. Therefore we can make considerable use of the technique of decomposition of $L_d^2(P_1)$. Generally, the residual spectrum

MSC2000: primary 11F70, 11F72; secondary 11F27.

Keywords: residual spectrum, hermitian Morita theory, theta correspondence.

is spanned by residues of the Eisenstein series associated with the cuspidal representations of the Levi factors of parabolic subgroups. A calculation of the poles of such Eisenstein series reduces to a calculation of normalization factors of certain intertwining operators. In the case of $\mathrm{Sp}(2)$ the Langlands–Shahidi normalization was used, where its normalization factors are written by automorphic L -functions. Therefore we can define the normalization factors for P as an analogy of the case of P_1 .

The residual spectrum for G has a further decomposition,

$$L_d^2(P)_\infty \oplus L_d^2(P)_1,$$

where $L_d^2(P)_\infty$ (resp. $L_d^2(P)_1$) is the space spanned by the residues of the Eisenstein series of infinite dimensional (resp. one-dimensional) cuspidal representations of a Levi factor M of P . The irreducible decomposition of $L_d^2(P)_\infty$ is obtained by the Langlands classification (Theorem 4.1(2)); it is written in the form of the unique irreducible quotient of $\mathrm{Ind}_{P(A)}^{G(A)}\pi$ with an infinite dimensional cuspidal representation π of M satisfying some conditions. Then the representation replacing π with the Jacquet–Langlands correspondence of π appears in $L_d^2(P_1)$ [Kim 1995, Th. 3.3]. $L_d^2(P)_1$ contains the trivial representation (Theorem 4.1(1)). Every other irreducible constituent is described by the theta lift of the trivial representation of the unitary group of a (-1) -hermitian space over D (Theorem 4.1(3)). This representation has a local component contained in $\mathrm{Ind}_{P(k_v)}^{G(k_v)}(\chi_v|\cdot|^{-1/2}) \circ \nu$ for any place v of k , where χ_v is a quadratic character of k_v^\times and ν is the reduced norm of D . On the other hand, any irreducible constituent of $L_d^2(P_0)$ except for the trivial representation is the theta lift of the trivial representation of an orthogonal group; see [Kon-no 1994]. And this representation has a local component contained in $\mathrm{Ind}_{P_1(k_v)}^{\mathrm{Sp}(2,k_v)}(\chi_v|\cdot|^{-1/2}) \circ \det$ for any v . It is interesting that the representations of Theorem 4.1(3) do not have a multiplicity of one, unlike the case of $\mathrm{Sp}(2)$.

By Arthur’s conjecture, an irreducible constituent of the residual spectrum should give the corresponding Arthur parameter. I expect that the Arthur parameters for the residual spectrum of G are contained in those for the residual spectrum of $\mathrm{Sp}(2)$. It seems that the Arthur parameters for $L_d^2(P)_\infty$ appear for $L_d^2(P_1)$, and those for $L_d^2(P)_1$ appear for $L_d^2(P_0)$. I give the expected description of the Arthur parameters in Section 5.

2. Preliminaries

Let k be a number field with adèle ring $A = A_k$. We write A_∞, A_f for the infinite and finite components of A , while $|\cdot|_A$ denotes the idele norm of A^\times . For any place v of k we write k_v for the completion of k at v , and $|\cdot|_v$ for the v -adic norm. If v is finite, \mathbb{O}_v denotes the maximal compact subring of k_v . If ψ is a nontrivial

character of A , trivial on k , and v is a place of k , ψ_v denotes the v -component of ψ .

Let D be a quaternion division algebra over k . We write ν , τ , and ι for the reduced norm, the reduced trace, and the main involution of D , respectively. We write $D_- = \{x \in D \mid \tau(x) = 0\}$. Also we write S_D for the set of places v of k at which D is ramified, and s_D for the number of its elements, which is finite and even. We denote by $\mathbb{M}(n, R)$ the algebra of all $n \times n$ -matrices over a ring R . Let $W = D^{\oplus 2}$ be the free left module over D with rank two, and we equip it with the hermitian form $\langle \cdot, \cdot \rangle$ given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 \iota y_2 + y_1 \iota x_2 \quad (x_1, x_2, y_1, y_2 \in D).$$

Writing $X := \{(*, 0) \in D^{\oplus 2}\}$, $Y := \{(0, *) \in D^{\oplus 2}\}$ we obtain a polarization of W . Let G be the unitary group of this form, so that

$$G(R) = \left\{ g \in \mathrm{GL}(2, D \otimes_k R) \mid g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^* g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

for any commutative k -algebra R . Here we write ${}^*(a_{i,j}) = ({}^{\iota}a_{j,i})$ for $(a_{i,j}) \in \mathbb{M}(2, D \otimes_k R)$. G is nonquasisplit and is an inner form of $\mathrm{Sp}(2)$ with respect to a quadratic extension k' of k such that all $v \in S_D$ do not split fully in k'/k . Fix a k -parabolic subgroup P and its Levi factor M as

$$P = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in G \right\}, \quad M = \left\{ m(x) := \begin{pmatrix} x & 0 \\ 0 & ({}^{\iota}x)^{-1} \end{pmatrix} \mid x \in D^\times \right\},$$

where D^\times is regarded as an algebraic group over k in the natural way. P is the unique proper parabolic subgroup of G up to $G(k)$ -conjugate. We write again ν for the character of M corresponding to the reduced norm via m . U denotes the unipotent radical of P , so that

$$U = \left\{ u(y) := \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mid y \in D_- \right\}.$$

Here D_- is also regarded as an algebraic group over k in the natural way. The k -split component of the center of M is

$$A = \left\{ a(t) := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbf{G}_m \right\}.$$

The element α of the character group $X^*(A)$ of A is defined by $\alpha(a(t)) = t^2$. It is the unique positive root of A with respect to P , and $\alpha^\vee = a$ is the attached coroot. The Weyl group W_G of A in G is equal to $\{1, w_0\}$, where

$$w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v$ of $G(\mathbf{A})$ such that

$$\mathbf{K}_v = \begin{cases} G(\mathbb{C}_v) & \text{if } v \notin S_D \text{ and } v \text{ is nonarchimedean,} \\ O(4) \cap G(\mathbb{R}) & \text{if } v \notin S_D \text{ and } v \text{ is real,} \\ U(4) \cap G(\mathbb{C}) & \text{if } v \text{ is complex,} \\ O(v^{\oplus 2}) \cap G(\mathbb{R}) & \text{if } v \in S_D \text{ and } v \text{ is real,} \end{cases}$$

and \mathbf{K}_v is an A -good maximal compact subgroup if $v \in S_D$ and v is nonarchimedean. Here $O(v^{\oplus 2})$ is the orthogonal group of a quadratic form $W \otimes \mathbb{R} \ni (x_1, x_2) \mapsto v(x_1) + v(x_2)$. Then we have the Iwasawa decomposition $G(\mathbf{A}) = P(\mathbf{A})\mathbf{K}$. We write \mathbf{K}_∞ for $\prod_{v|\infty} \mathbf{K}_v \subset G(\mathbf{A}_\infty)$.

3. Decomposition of the L^2 -inner product

We will make use of the results of [Mœglin and Waldspurger 1995], henceforth abbreviated [MW]. Let (M, \mathfrak{P}_1) and (M, \mathfrak{P}_2) be two cuspidal pairs [MW, II.1.1]. For their Paley–Wiener sections $\phi_1 \in P_{(M, \mathfrak{P}_1)}$ and $\phi_2 \in P_{(M, \mathfrak{P}_2)}$ [MW, II.1.2], θ_{ϕ_1} and θ_{ϕ_2} denote the pseudo-Eisenstein series, respectively [MW, II.1.10]. They are elements of $L^2(G(k)\backslash G(\mathbf{A}))$ and if (M, \mathfrak{P}_1) and (M, \mathfrak{P}_2) are not $G(k)$ -conjugate, they are orthogonal. If (M, \mathfrak{P}_1) and (M, \mathfrak{P}_2) are $G(k)$ -conjugate then the L^2 -inner product between them is described as follows [MW, Theorem II.2.1].

$$(3-1) \quad \langle \theta_{\phi_1}, \theta_{\phi_2} \rangle = \int_{\substack{\pi \in \mathfrak{P}_1 \\ \text{Re}(\pi) = \lambda_0}} A(\phi_1, \phi_2)(\pi) d\pi,$$

where

$$A(\phi_1, \phi_2)(\pi) = \sum_{w \in W((M, \mathfrak{P}_1), (M, \mathfrak{P}_2))} \langle M(w, \pi)\phi_1(\pi), \phi_2(-w(\bar{\pi})) \rangle.$$

Here all notation follows [MW]. $W((M, \mathfrak{P}_1), (M, \mathfrak{P}_2))$ is a subset of W_G and $M(1, \pi)$ is the identity, and

$$(M(w_0, \pi)\phi_1(\pi))(g) = \int_{U(\mathbf{A})} \phi_1(\pi)(w_0^{-1}ug) du \quad (g \in G(\mathbf{A})).$$

This integral converges absolutely at $\pi \in \mathfrak{P}_1$ such that $\langle \text{Re}(\pi), \alpha^\vee \rangle \gg 0$ and λ_0 is chosen in this area. We have to extend $M(w_0, \pi)$ to $\langle \text{Re}(\pi), \alpha^\vee \rangle \geq 0$ analytically and rewrite Equation (3-1).

For a cuspidal pair (M, \mathfrak{P}) , $\pi \in \mathfrak{P}$, and $\phi \in P_{(M, \mathfrak{P})}$, $\phi(\pi)$ belongs to the space $A(U(\mathbf{A})M(k)\backslash G(\mathbf{A}))_\pi$ (defined in [MW, II.1.1]). Here π is decomposed into a restricted tensor product $\otimes_v \pi_v$ as a $(\text{Lie } M(\mathbf{A}_\infty) \otimes_{\mathbb{R}} \mathbb{C}, M(\mathbf{A}_\infty) \cap \mathbf{K}_\infty) \times M(\mathbf{A}_f)$ -module. If π is infinite dimensional then it has a multiplicity of one in the space $A_0(M(k)\backslash M(\mathbf{A}))$ of cuspidal forms on $M(k)\backslash M(\mathbf{A})$ by Jacquet–Langlands theory.

Also if π is one-dimensional it has a multiplicity of one clearly. Thus we have

$$A(U(A)M(k)\backslash G(A))_\pi = \text{Ind}_{K \cap M(A)}^K \pi = \bigotimes_v \text{Ind}_{P(k_v)}^{G(k_v)} \pi_v$$

as $(\text{Lie } G(\mathbf{A}_\infty) \otimes_{\mathbb{R}} \mathbb{C}, \mathbf{K}_\infty) \times G(\mathbf{A}_f)$ -modules. We have a decomposition:

$$M(w_0, \pi) = \otimes_v M(w_0, \pi_v),$$

where $M(w_0, \pi_v)$ is defined by

$$[M(w_0, \pi_v)\phi_v](g) = \int_{U(k_v)} \phi_v(w_0^{-1}ug) du \quad (g \in G(k_v), \phi_v \in \text{Ind}_{P(k_v)}^{G(k_v)} \pi_v).$$

Their elementary properties are found in [Knapp and Stein 1980] for the archimedean case and [Silberger 1979] for the nonarchimedean case. This allows us to deduce analytic properties of $M(w_0, \pi)$ from those of the local components $M(w_0, \pi_v)$. We define the normalization factor for $M(w_0, \pi_v)$ by

$$(3-2) \quad r(w_0, \pi_v, \psi_v) = \frac{L_{w_0}(0, \pi_v)}{L_{w_0}(1, \pi_v)\varepsilon_{w_0}(0, \pi_v, \psi_v)},$$

$$L_{w_0}(s, \pi_v) = L(s, \pi_v)L_{k_v}(2s, \omega_{\pi_v}),$$

$$\varepsilon_{w_0}(s, \pi_v, \psi_v) = \varepsilon(s, \pi_v, \psi_v)\varepsilon_{k_v}(2s, \omega_{\pi_v}, \psi_v),$$

where $L(s, \pi_v)$, $\varepsilon(s, \pi_v, \psi_v)$ denote the L and ε -factor defined by Godement and Jacquet [1972], and $L_{k_v}(s, \omega_{\pi_v})$ and $\varepsilon_{k_v}(s, \omega_{\pi_v}, \psi_v)$ denote the Hecke L and ε -factor. For any $v \notin S_D$, this normalization factor coincides with the Langlands–Shahidi normalization factor which is considered in [Shahidi 1990]. Let $\text{Re}(\pi_v)$ denote the local analogue of $\text{Re}(\pi)$.

Lemma 3.1. *For any v , the normalized intertwining operator*

$$N(w_0, \pi_v, \psi_v) = r(w_0, \pi_v, \psi_v)^{-1}M(w_0, \pi_v)$$

is holomorphic for $\langle \text{Re}(\pi_v), \alpha^\vee \rangle \geq 0$.

Proof. If $v \notin S_D$ and π_v is infinite dimensional then the lemma has already been shown by Kim [1995, p. 133–134]. If $v \in S_D$, then π_v is square-integrable modulo the center, and therefore $M(w_0, \pi_v)$ is holomorphic and nonzero for $\langle \text{Re}(\pi_v), \alpha^\vee \rangle \geq 0$. Since $r(w_0, \pi_v, \psi_v)$ does not vanish for $\langle \text{Re}(\pi_v), \alpha^\vee \rangle \geq 0$, the lemma follows. Assume that $v \notin S_D$ and $\pi_v = \chi_v \circ \nu$, where χ_v is a quasi-character of k_v^\times . We write B for the Borel subgroup of $\text{GL}(2)$ consisting of upper triangular matrices. Identifying $M(k_v)$ with $\text{GL}(2, k_v)$, π_v is the unique irreducible subrepresentation of $\text{Ind}_{B(k_v)}^{M(k_v)} (\chi_v |\cdot|^{-1/2} \otimes \chi_v |\cdot|^{1/2})$ [Jacquet and Langlands 1970]. We fix a set of positive roots of the root data for $\text{Sp}(2)$. r_1 and r_2 denote the reflection attached

to the short and long simple roots, respectively. We may assume that $w_0 = r_2 r_1 r_2$. By the functional equation we have

$$M(w_0, \rho[s]) = M(r_2, r_1 r_2 \rho[s]) \circ M(r_1, r_2 \rho[s]) \circ M(r_2, \rho[s]),$$

where $\rho[s] := \chi_v \cdot |v|^{-1/2+s} \otimes \chi_v \cdot |v|^{1/2+s}$. It is known that

$$\frac{L_{k_v}(s+1, \chi_v \cdot |v|^{-1/2}) \varepsilon_{k_v}(s, \chi_v \cdot |v|^{-1/2}, \psi_v)}{L_{k_v}(s, \chi_v \cdot |v|^{-1/2})} M(r_2, r_1 r_2 \rho[s]),$$

$$\frac{L_{k_v}(2s+1, \chi_v^2) \varepsilon_{k_v}(2s, \chi_v^2, \psi_v)}{L_{k_v}(2s, \chi_v^2)} M(r_1, r_2 \rho[s]),$$

and

$$\frac{L_{k_v}(s+1, \chi_v \cdot |v|^{1/2}) \varepsilon_{k_v}(s, \chi_v \cdot |v|^{1/2}, \psi_v)}{L_{k_v}(s, \chi_v \cdot |v|^{1/2})} M(r_2, \rho[s]),$$

are holomorphic in the region $\text{Re}(s) \geq 0$. Since $N(w_0, \rho[s], \psi_v)$ is the composition of these three maps we obtain the lemma. □

Take a $\phi(\pi) = \bigotimes_v \phi(\pi)_v \in P_{(M, \mathfrak{P})}$. Let S be a finite set of places of k including all the archimedean places such that at every $v \notin S$, D, π , and ψ are unramified and $\phi(\pi)_v$ is the K_v -fixed vector with $\phi(\pi)_v(1) = 1$. For any $v \notin S$ we have the Gindikin–Karpelevich formula [Langlands 1971, p.45]:

$$M(w, \pi_v) \phi(\pi)_v = r(w, \pi_v, \psi_v) \phi(\pi)_v.$$

Therefore

$$\begin{aligned} M(w_0, \pi) \phi(\pi) &= \bigotimes_{v \in S} r(w_0, \pi_v, \psi_v) N(w_0, \pi_v, \psi_v) \phi(\pi)_v \otimes \bigotimes_{v \notin S} r(w_0, \pi_v, \psi_v) \phi(\pi)_v \\ &= r(w_0, \pi) N(w_0, \pi) \phi(\pi), \end{aligned}$$

where

$$\begin{aligned} r(w_0, \pi) &:= \prod_v r(w_0, \pi_v, \psi_v), \\ N(w_0, \pi) \phi(\pi) &:= \bigotimes_{v \in S} N(w_0, \pi_v, \psi_v) \phi(\pi)_v \otimes \bigotimes_{v \notin S} \phi(\pi)_v. \end{aligned}$$

From Lemma 3.1, $M(w_0, \pi)$ is continued for $\langle \text{Re}(\pi), \alpha^\vee \rangle \geq 0$ and the poles of $M(w_0, \pi)$ coincide with those of $r(w_0, \pi)$ in this region.

Proposition 3.2. *Suppose that $M(w_0, \pi)$ has a pole π' for $\langle \text{Re}(\pi), \alpha^\vee \rangle \geq 0$.*

- (i) *If \mathfrak{P} consists of infinite dimensional representations, π' is equal to $\pi_0|v|_A^{1/2}$, where π_0 is an irreducible self-dual cuspidal representation of $M(\mathbf{A})$ whose standard L -function $L(s, \pi_0)$ does not vanish at $s = 1/2$. It is simple.*

- (ii) If \mathfrak{P} consists of one-dimensional representations then π' is equal to $|v|_A^{3/2}$ or $(\omega_{k'/k} \cdot | \cdot |_A^{1/2}) \circ v$ where k'/k is a quadratic extension such that all $v \in S_D$ do not split fully in k'/k . It is simple.

Proof. (i) From Shahidi's nonvanishing theorem [1981, Theorem 5.1] and the Jacquet–Langlands theory, the only pole of $r(w_0, \pi)$ for $\langle \mathrm{Re}(\pi), \alpha^\vee \rangle \geq 0$ is $\pi_0 |v|_A^{1/2}$ where ω_{π_0} is trivial and $L(s, \pi_0)$ does not vanish at $s = 1/2$. Note that $\omega_{\pi_0} = \mathbf{1}_{D^\times(A)}$ implies π_0 is self-dual.

(ii) Let $\pi = \chi \circ v$. $L(s, \pi_v)$ is described as follows [Jacquet and Langlands 1970].

$$L(s, \pi_v) = \begin{cases} L_{k_v}(s + 1/2, \chi_v) L_{k_v}(s - 1/2, \chi_v) & \text{if } v \notin S_D, \\ L_{k_v}(s + 1/2, \chi_v) & \text{if } v \in S_D \text{ and } v \text{ is finite,} \\ 2(2\pi)^{-(s+1/2)} \Gamma(s + 1/2) & \text{if } v \in S_D \text{ and } v \text{ is real.} \end{cases}$$

Thus a factor of $r(w_0, \pi)$ related to the poles is

$$\left(\prod_{v \in S_D} L_{k_v}(-1/2, \chi_v) \right)^{-1} \cdot L_k(1/2, \chi) \cdot L_k(0, \chi^2) \cdot L_k(1, \chi^2)^{-1}.$$

Here $L_k(\cdot, \cdot)$ denotes the Hecke L -function. We can calculate its poles easily. \square

Write $\mathfrak{S}(\pi_0) = \pi_0 |v|_A^{1/2}$ for π_0 satisfying the condition of Proposition 3.2(i). Similarly write $\mathfrak{S}_1 = |v|_A^{3/2}$ and $\mathfrak{S}_\Theta(k') = (\omega_{k'/k} \cdot | \cdot |_A^{1/2}) \circ v$ for $\omega_{k'/k}$ satisfying the condition of Proposition 3.2(ii). From [Harish-Chandra 1968, Lemma 101], $M(w_0, \pi)$ is bounded on any region of the form

$$\{\pi \in \mathfrak{P} \mid 0 \leq \langle \mathrm{Re}(\pi), \alpha^\vee \rangle \leq R\}, \quad 0 < R \in \mathbb{R}.$$

Thus we can apply the residue theorem to (3-1). From Proposition 3.2 we have the following.

Theorem 3.3. *Let (M, \mathfrak{P}_1) and (M, \mathfrak{P}_2) be cuspidal pairs, and θ_{ϕ_1} ($\phi_1 \in P_{(M, \mathfrak{P}_1)}$) and θ_{ϕ_2} ($\phi_2 \in P_{(M, \mathfrak{P}_2)}$) pseudo-Eisenstein series.*

- (i) *If $\mathfrak{P}_2 = w_0(\mathfrak{P}_1)$ and one of $\mathfrak{S}(\pi_0)$, \mathfrak{S}_1 and $\mathfrak{S}_\Theta(k')$ is contained in \mathfrak{P}_1 , which is denoted by \mathfrak{S} , then*

$$(3-3) \quad \langle \theta_{\phi_1}, \theta_{\phi_2} \rangle = \int_{\pi \in \mathfrak{P}_1 | \mathrm{Re}(\pi)=0} A(\phi_1, \phi_2)(\pi) d\pi + c \langle N(w_0, \mathfrak{S})\phi_1(\mathfrak{S}), \phi_2(\mathfrak{S}) \rangle$$

for some nonzero constant c .

- (ii) *Otherwise,*

$$\langle \theta_{\phi_1}, \theta_{\phi_2} \rangle = \int_{\pi \in \mathfrak{P}_1 | \mathrm{Re}(\pi)=0} A(\phi_1, \phi_2)(\pi) d\pi.$$

4. The residual spectrum of G

We are now ready to determine the irreducible decomposition of the residual discrete spectrum of G .

For a (-1) -hermitian right D -space (V, h) , $G(V)$ denotes the unitary group of (V, h) and $G(V_A)$ the group of its adelic points. Set $\mathbb{W} = V \otimes_D W$ and $\langle\langle \cdot, \cdot \rangle\rangle = \frac{1}{2}\tau(h_v \otimes \langle \cdot, \cdot \rangle)$. Then $(\mathbb{W}, \langle\langle \cdot, \cdot \rangle\rangle)$ is a symplectic space over k . We will define the Weil representation $\omega_{\psi, V}$ of $G(V_A) \times G(A)$ in Section 4.1.

Theorem 4.1. *Let k be a totally real number field. The irreducible constituents of the residual spectrum of G consist of the following representations.*

- (i) *The trivial representation $\mathbf{1}_{G(A)}$.*
- (ii) *The unique irreducible quotient $J_P^G(\pi)$ of $\text{Ind}_{P(A)}^{G(A)}(\pi|_V|_A^{1/2})$. Here π runs over infinite dimensional irreducible self-dual cuspidal representations of $M(A)$ whose standard L -functions $L(s, \pi)$ do not vanish at $s = 1/2$.*
- (iii) *The theta lift $R(V)$ from the trivial representation of $G(V)$ under the Weil representation $\omega_{\psi, V}$. Here V runs over local isometry classes of (-1) -hermitian right D -spaces with dimension one.*

In the cases (1) and (2), the multiplicity of each representation is one. In the case (3), the multiplicity of each representation is 2^{s_D-2} .

The proof of the theorem occupies the rest of this section. It is known that the discrete term, which is the second term on the right hand of (3-3), expresses the inner product for the residual spectrum [Mœglin and Waldspurger 1995, VI]. In other words, the residual spectrum of G is isomorphic to the direct sum of the images of intertwining operators for all $\mathfrak{S} = \mathfrak{S}_1, \mathfrak{S}(\pi_0)$ and $\mathfrak{S}_\Theta(k')$ appearing in Proposition 3.2.

First assume that $\mathfrak{S} = \mathfrak{S}_1$. We have that at each v , $\text{Im}N(w_0, \mathfrak{S}_v)$ is isomorphic to $\mathbf{1}_{G(k_v)}$ by the Langlands classification. Thus the representation of Theorem 4.1(1) is obtained. Next assume that $\mathfrak{S} = \mathfrak{S}(\pi_0)$. For $v \notin S_D$, the proof of Proposition 3.2 in [Kim 1995] shows that $\text{Ind}_{P(k_v)}^{G(k_v)}\mathfrak{S}_v$ has a unique irreducible quotient and $\text{Im}N(w_0, \mathfrak{S}_v)$ is isomorphic to this quotient. If $v \in S_D$, \mathfrak{S}_v is tempered so that $\text{Im}N(w_0, \mathfrak{S}_v)$ is also isomorphic to the unique irreducible quotient of $\text{Ind}_{P(k_v)}^{G(k_v)}\mathfrak{S}_v$. Therefore, all images of $N(w_0, \mathfrak{S})$ are given by the representations of Theorem 4.1(2). We remark that the above results for $\mathfrak{S} = \mathfrak{S}_1$ and $\mathfrak{S}(\pi_0)$ do not need the fact that k is totally real.

4.1. Construction by theta correspondence. Finally assume that $\mathfrak{S} = \mathfrak{S}_\Theta(k')$. At each place v write $I(s, \chi_v) = \text{Ind}_{P(k_v)}^{G(k_v)}((\chi_v| \cdot |_v^s) \circ \nu)$, where χ_v is a character of k_v^\times and $s \in \mathbb{C}$. Writing $\omega_{k'_v/k_v}$ for the v -component of $\omega_{k'/k}$, $\text{Im}N(w_0, \mathfrak{S}_\Theta(k')_v, \psi_v)$ is a

subrepresentation of $I(-1/2, \omega_{k'_v/k_v})$. We will describe the irreducible constituents of $I(-1/2, \omega_{k'_v/k_v})$ by using the theta correspondence.

(-1)-Hermitian spaces over quaternion algebras. We will review some basic facts about (-1) -hermitian spaces. The main involution ι of D is of the first kind and of symplectic type [Scharlau 1985, p. 304]. For any place v we again write ι for the involution of $D_v := D \otimes_k K_v$ induced by the main involution of D , which also becomes of the first kind and of symplectic type. Therefore, from Remarks (iii) in the same location we can identify the local involution ι with the local main involution.

Let F be a local field and R be a quaternion algebra over F with the main involution ι . For $\Gamma = (\gamma_{i,j}) \in \mathbb{M}(m, R)$ such that $({}^t\gamma_{j,i}) = -\Gamma$ (resp. $({}^t\gamma_{j,i}) = \Gamma$), a (-1) -hermitian (resp. hermitian) form on a right (resp. left) R -module $R^{\oplus m}$ (the set of column (resp. row) vectors) is defined by $((v_i), (v'_j)) \mapsto {}^t(v_i)\Gamma(v'_j)$ (resp. $((w_i), (w'_j)) \mapsto (w_i)\Gamma({}^t(w'_j))$). We denote this form by $\langle \Gamma \rangle$. Similarly for $B \in \mathbb{M}(m, F)$ such that ${}^tA = A$ (resp. ${}^tA = -A$) we can define a quadratic (resp. symplectic) form $\langle B \rangle$ on $F^{\oplus m}$ (the set of column (resp. row) vectors).

- The case $R = \mathbb{M}(2, F)$. To observe (-1) -hermitian modules over $\mathbb{M}(2, F)$ we make use of an available theory which is called hermitian Morita theory. This implies an equivalence between the category of (-1) -hermitian (right) R -modules and the category of quadratic F -spaces. We will describe the correspondence in this theory ([Knus 1991] § I.9, [Scharlau 1985] p.361,362).

Let (V, h) be a (-1) -hermitian right R -module. Set

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}(2, F) = R,$$

and $T_V = T_{(V,h)} := Ve$. A bilinear form $b_V = b_{(V,h)}$ on T_V over F is defined by

$$h(v_e, v'_e) = \begin{pmatrix} 0 & 0 \\ b_{(V,h)}(v_e, v'_e) & 0 \end{pmatrix} \in \mathbb{M}(2, F) \quad \text{for } v, v' \in V.$$

Then (T_V, b_V) is the quadratic space corresponding to (V, h) in hermitian Morita theory. We define $\det V$ as the determinant of (T_V, b_V) . For two (-1) -hermitian modules (V_1, h_1) , (V_2, h_2) and an isometry $\sigma : (V_1, h_1) \rightarrow (V_2, h_2)$, the restriction $\sigma|_{T_{V_1}} : T_{V_1} \rightarrow T_{V_2}$ becomes an isometry from (T_{V_1}, b_{V_1}) to (T_{V_2}, b_{V_2}) as quadratic F -spaces. Therefore, the unitary group of (V, h) is isomorphic to the orthogonal group of (T_V, b_V) .

In particular, consider the case of (-1) -hermitian free module with rank one. For a

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathbb{M}(2, F)_-,$$

we define $\mathcal{M}(\Gamma) \in \text{Sym}_2(F) = \{X \in \mathbb{M}(2, F) \mid {}^tX = X\}$ as

$$\mathcal{M}(\Gamma) = \begin{pmatrix} \gamma & -\alpha \\ -\alpha & -\beta \end{pmatrix} = J\Gamma.$$

For a (-1) -hermitian right R -module, $(V, h) = (R, \langle \Gamma \rangle)$, one has that a quadratic space $(F^{\oplus 2}, \langle \mathcal{M}(\Gamma) \rangle)$ is isometric to (T_V, b_V) . Note that $\det V = \det \mathcal{M}(\Gamma) = \det \Gamma$.

Hermitian Morita theory also implies the equivalence between the category of hermitian left R -modules and the category of symplectic F -spaces. Writing (W_F, h_F) for the hermitian left R -module

$$(R^{\oplus 2}, \langle \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \rangle),$$

the Morita correspondence of (W_F, h_F) is isometric to (eW_F, s_{W_F}) where s_{W_F} is defined by

$$h_F(ew, ew') = \begin{pmatrix} 0 & s_{W_F}(ew, ew') \\ 0 & 0 \end{pmatrix} \in \mathbb{M}(2, F) \quad \text{for } w, w' \in W_F.$$

Also, we have an isomorphism between the unitary group $G(W_F)$ of (W_F, h_F) and $\text{Sp}(eW_F)$ by restriction to eW_F .

- The case that R is the quaternion division algebra over F . For a (-1) -hermitian right R -space (V, h) , the determinant $\det V \in F^\times / (F^\times)^2$ is defined as the reduced norm of a matrix expression of h .

Proposition 4.2 [Scharlau 1985, Theorem 3.6, 3.7].

- (a) *Let F be nonarchimedean.*
 - (i) *The isometry class of any regular (-1) -hermitian right R -space is determined by its dimension and determinant.*
 - (ii) *There exists a regular (-1) -hermitian right R -space with any dimension and determinant except for dimension 1 and determinant -1 .*
- (b) *Let F be real and archimedean.*
 - (i) *The isometry class of any regular (-1) -hermitian right R -space is determined by its dimension.*
 - (ii) *There exists a regular (-1) -hermitian right R -space with any dimension.*

The next proposition is a statement about the local-global property with regard to (-1) -hermitian spaces. For a (-1) -hermitian right D -space (V, h) , the determinant $\det V \in k^\times / (k^\times)^2$ is defined as the reduced norm of a matrix expression of h . For $\gamma \in D_- \setminus \{0\}$, let $k_{D, \gamma}^\times = \{c \in k^\times \mid (\gamma^2, c)_v = 1 \text{ for all } v \notin S_D\}$, where $(,)_v$ is the Hilbert symbol at v . A group homomorphism λ is defined by

$$k_{D, \gamma}^\times \ni c \mapsto \{(\gamma^2, c)_v\}_{v \in S_D} \in \{\pm 1\}^{s_D}.$$

Let $\{\pm 1\}$ be regarded as the subgroup of $\{\pm 1\}^{s_D}$ via the diagonal embedding. Note that the number of elements of $k_{D,\gamma}^\times / \lambda^{-1}(\{\pm 1\})$ is 2^{s_D-2} .

Proposition 4.3 [Scharlau 1985, Theorem 10.4.6, Remark 10.4.6]. *Let (V, h) be a (-1) -hermitian right D -space.*

- (i) *If $\dim_D V = 1$ and $h = \langle \gamma \rangle$ for some $\gamma \in D_- \setminus \{0\}$, then for any $c \in k_{D,\gamma}^\times$, $\langle c\gamma \rangle$ is locally isometric to $\langle \gamma \rangle$. For any $a \in \lambda^{-1}(\{\pm 1\})$, $\langle a\gamma \rangle$ is globally isometric to $\langle \gamma \rangle$. Moreover*

$$\{\langle a\gamma \rangle \mid a \in k_{D,\gamma}^\times / \lambda^{-1}(\{\pm 1\})\}$$

is the set of classes locally isometric to $\langle \gamma \rangle$, so this set contains 2^{s_D-2} elements.

- (ii) *In general, for every dimension there exists exactly 2^{s_D-2} classes locally isometric to (V, h) .*

Local theta correspondence. Let (V, h) be a (-1) -hermitian right D -module with dimension m . We denote by $G(V)$ the unitary group of V and, if R is a k -algebra, by $G(V_R)$ the group of R -valued points of $G(V)$. For a place v , let (V_v, h_v) be a (-1) -hermitian space over D_v . We define $G(V_v)$ similarly, but we often use $G(V_v)$ for the group of k_v -valued points of $G(V_v)$ by an abuse of notation. Let $(W_v, \langle \cdot, \cdot \rangle_v)$ denote the completion of the hermitian space $(W, \langle \cdot, \cdot \rangle)$ over D at v . We define the Weil representation of $G(V_v) \times G(k_v)$ as follows.

Let $\mathbb{W}_v := V_v \otimes_{D_v} W_v$ and $\langle \langle \cdot, \cdot \rangle \rangle_v := \frac{1}{2} \tau(h_v \otimes \langle \cdot, \cdot \rangle_v)$. Then $(\mathbb{W}_v, \langle \langle \cdot, \cdot \rangle \rangle_v)$ is a symplectic space over k_v of dimension $8m$.

- The case $v \notin S_D$. A homomorphism

$$\mathbb{W}_v = V_v \otimes W_v \ni v \otimes w \mapsto ve'e \otimes ee'w + ve \otimes ew \in V_v e \otimes_{k_v} eW_v$$

becomes an isometry between $(\mathbb{W}_v, \langle \langle \cdot, \cdot \rangle \rangle_v)$ and $(V_v e \otimes_{k_v} eW_v, \frac{1}{2} b_{V_v} \otimes (-s_{W_v}))$. On the other hand, a homomorphism

$$O(V_{2e}, \frac{1}{2} b_{V_v}) \times \mathrm{Sp}(eW_v, -s_{W_v}) \rightarrow \mathrm{Mp}(V_v e \otimes eW_v)$$

is given in [Kudla 1994], where $\mathrm{Mp}(V_v e \otimes eW_v)$ denotes the metaplectic group. From this and Section 4.1 we have a homomorphism $G(V_v) \times G(k_v) \rightarrow \mathrm{Mp}(\mathbb{W}_v)$. Therefore the Weil representation ω_{ψ_v} of $\mathrm{Mp}(\mathbb{W}_v)$ induces a representation $\omega_{V_v} = \omega_{\psi_v, V_v}$ of $G(V_v) \times G(k_v)$, which is realized on the space $\mathcal{S}(V_v)$ of Schwartz–Bruhat functions on $V_v = V_v \otimes X_v$.

- The case $v \in S_D$. A homomorphism $G(V_v) \times G(k_v) \rightarrow \mathrm{Mp}(\mathbb{W}_v)$ is given in [Kudla 1994]. The Weil representation ω_{ψ_v} of $\mathrm{Mp}(\mathbb{W}_v)$ induces a representation $\omega_{V_v} = \omega_{\psi_v, V_v}$ of $G(V_v) \times G(k_v)$, which is realized on $\mathcal{S}(V_v) = \mathcal{S}(V_v \otimes X_v)$.

For all places v we have defined the Weil representation ω_{V_v} of $G(V_v) \times G(k_v)$. Some explicit formulae involving ω_{V_v} are as follows. Let $\phi \in \mathcal{S}(V_v)$ and $v \in V_v$.

- $\omega_{V_v}\left(\begin{pmatrix} a & 0 \\ 0 & {}_v a^{-1} \end{pmatrix}\right)\phi(v) = ((-1)^m \det V_v, {}_v(a))_v |a|_{k_v}^m \phi(va) \quad (a \in D^\times(k_v))$
- $\omega_{V_v}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\phi(v) = \psi_v(\frac{1}{4}\tau(bh_v(v)))\phi(v) \quad (b \in D_{v,-})$
- $\omega_{V_v}(h)\phi(v) = \phi(h^{-1}v) \quad (h \in G(V_v))$

We will describe constituents of $I(-1/2, \chi_v)$. Let (V_v, h_v) be a (-1) -hermitian (free) right module over D_v with rank m . χ_{V_v} denotes the quadratic character of k_v^\times defined by $\chi_{V_v}(x) = ((-1)^m \det V_v, x)_v$. For $v \notin S_D$ the description of the irreducible constituents of $I(-1/2, \chi_v)$ has been obtained by Kudla, Rallis, and Soudry [Kudla et al. 1992]. Therefore we will restrict to $v \in S_D$.

- The case of nonarchimedean $v \in S_D$. Consider the reducible points of $I(s, \chi_v)$, where χ_v is a quadratic character of k_v^\times .

Lemma 4.4. (i) *If $\chi_v = 1$ then $I(\pm 3/2, \chi_v)$ is reducible. $I(s, \chi_v)$ is irreducible for real $s \neq \pm 3/2$.*

(ii) *If $\chi_v \neq 1$ then $I(\pm 1/2, \chi_v)$ is reducible. $I(s, \chi_v)$ is irreducible for real $s \neq \pm 1/2$.*

Proof. The local Jacquet–Langlands correspondence of $\chi_v \circ \nu$ is $\sigma_0 = \chi_v \circ \det \otimes \delta_{S_t}$, where δ_{S_t} denotes the Steinberg representation of $GL(2, k_v)$. By Proposition 2.1 in [Muić and Savin 2000] the Plancherel measure of χ_v coincides with the Plancherel measure of $\chi_v \circ \det \otimes \delta_{S_t}$. Therefore, the poles and zeros of $\mu(s, \chi_v \circ \nu)$ coincide with the poles and zeros of

$$\frac{L(1-s, \sigma_0^\vee)L_k(1-2s, \omega_{\sigma_0}^{-1})L(1+s, \sigma_0)L_k(1+2s, \omega_{\sigma_0})}{L(-s, \sigma_0^\vee)L_k(-2s, \omega_{\sigma_0}^{-1})L(s, \sigma_0)L_k(2s, \omega_{\sigma_0})},$$

respectively. Since $\mu(0, \chi_v \circ \nu) = 0$, the reducible points of $I(s, \chi_v)$ coincide with the poles of $\mu(s, \chi_v \circ \nu)$ [Silberger 1979]. All the poles are $s = \pm 3/2$ if $\chi_v = 1$ and $s = \pm 1/2$ if $\chi_v \neq 1$. □

In Proposition 3.2(ii), the local component $\omega_{k'_v/k_v}$ at v of $\omega_{k'/k}$ is not trivial. Therefore we want the description of constituents of $I(-1/2, \chi_v)$ with a nontrivial chi_v , using theta correspondence. We write $\mathcal{S}(V_v)_{G(V_v)}$ for the $G(V_v)$ -coinvariant space of $S(V_v)$. Let $R(V_v)$ denote the image of the map

$$\mathcal{S}(V_v) \ni \phi \mapsto [G(k_v) \ni g \mapsto \omega_{V_v}(g)\phi(0)] \in I(m - 3/2, \chi_{V_v}).$$

This map induces an isomorphism $\mathcal{S}(V_v)_{G(V_v)} \simeq R(V_v)$; see [Mœglin et al. 1987, Chap. 3 IV Th. 7]. Let (V'_v, h'_v) be a 1-dimensional (-1) -hermitian space with

$\det V'_v \neq -1$. (V''_v, h''_v) denotes the 2-dimensional (-1) -hermitian space given by $\det V''_v = -\det V'_v$. Proposition 4.2 guarantees the existence and uniqueness of (V'_v, h'_v) and (V''_v, h''_v) .

Proposition 4.5. (i) $R(V'_v)$ and $R(V''_v)$ are the unique irreducible subrepresentations of $I(-1/2, \chi_{V'_v})$ and $I(1/2, \chi_{V'_v})$, respectively.

(ii) $N(w_0, (\chi_{V'_v} | \cdot |_v^{1/2}) \circ \nu, \psi_\nu)$ induces an isomorphism $I(1/2, \chi_{V'_v})/R(V''_v) \simeq R(V'_v)$.

Proof. The Jacquet module $I(-1/2, \chi_{V'_v})_P$ of $I(-1/2, \chi_{V'_v})$ along P is

$$(\chi_{V'_v} | \cdot |_v^{1/2}) \circ \nu + (\chi_{V'_v} | \cdot |_v^{-1/2}) \circ \nu$$

in the Grothendieck group. On the other hand, $R(V'_v)_P \simeq (\chi_{V'_v} | \cdot |_v^{-1/2}) \circ \nu$. Since $I(-1/2, \chi_{V'_v})$ has at most one proper $G(k_v)$ -invariant space, the former of (i) follows. Similarly the latter of (i) is proved. From the Langlands classification we obtain (ii). \square

• The case of real archimedean $v \in S_D$. We have $G(k_v) \simeq \mathrm{Sp}(1, 1)$. The reducible points of $I(s, \mathbf{1})$ have been obtained, as follows.

Lemma 4.6 [Johnson 1990, Corollary of Lemma 5.4]. *If $s = \pm(2n + 1)/2$ ($n \in \mathbb{N}$) then $I(s, \mathbf{1})$ is reducible, otherwise $I(s, \mathbf{1})$ is irreducible. In particular $I(\pm 1/2, \mathbf{1})$ are irreducible.*

Since $\nu(D_v^\times) = \mathbb{R}_+^\times$, we need not take the signed character in this lemma. Let $\mathcal{G}^0(V_v)$ denote the subspace of $\mathcal{G}(V_v)$ of functions of the form $e^{-\pi\nu(x)}P(x)$ where P is a polynomial on $V_v (\cong k_v^{\oplus 4m})$. This space is a $(\mathfrak{g}_v, \mathbf{K}_v)$ -module where $\mathfrak{g}_v = \mathrm{Lie} G(k_v) \otimes_{\mathbb{R}} \mathbb{C}$. Fix an isomorphism $V_v \cong D_v^{\oplus m}$. We set $\mathfrak{h}_v = \mathrm{Lie} O^*(2m) \otimes_{\mathbb{R}} \mathbb{C}$ and a maximal compact subgroup $L_v = O(\nu^{\oplus m}) \cap O^*(2m)$ of $O^*(2m)$ where $O(\nu^{\oplus m})$ is the orthogonal group of $\nu^{\oplus m}$ (m times direct sum of the norm form on D_v). We write $\mathcal{G}^0(V_v)_{(\mathfrak{h}_v, L_v)}$ for the (\mathfrak{h}_v, L_v) -coinvariant space of $\mathcal{G}^0(V_v)$. $R(V_v)$ denotes the image of the map

$$\mathcal{G}^0(V_v) \ni \phi \mapsto [G(k_v) \ni g \mapsto \omega_{V_v}(g)\phi(0)] \in I(m - 3/2, \chi_{V_v}).$$

This map induces an isomorphism $\mathcal{G}^0(V_v)_{(\mathfrak{h}_v, L_v)} \simeq R(V_v)$ as $(\mathfrak{g}_v, \mathbf{K}_v)$ -modules [Zhu 1992, Theorem II]. Let (V'_v, h'_v) be a (-1) -hermitian space with dimension one. By Proposition 4.2 it is determined uniquely. From the above lemma and the Langlands classification we have the following.

Proposition 4.7. $N(w_0, | \cdot |_v^{1/2}, \psi_\nu)$ induces an isomorphism $I(1/2, \mathbf{1}) \simeq R(V'_v) (= I(-1/2, \mathbf{1}))$.

We go back to a general place v . Let (V_v, h_v) be a (-1) -hermitian right D_v -module with rank m . For $\beta \in D_{v,-}$, let $\Omega_\beta = \{v \in V_v \mid h_v(v, v) = \beta\}$. Also we write $\psi_{v,\beta}$ for the character of $U(k_v)$ given by $\psi_{v,\beta}(u(b)) = \psi_v(\frac{1}{4}\tau(b\beta))$.

Suppose k_v is nonarchimedean. Set

$$\mathcal{S}(V_v)_\beta = \mathcal{S}(V_v) / \text{Span}\{\omega_{V_v}(u)\varphi - \psi_{v,\beta}(u)\varphi \mid \varphi \in \mathcal{S}(V_v), u \in U(k_v)\}.$$

The next two lemmas are shown by an argument similar to the proof of [Rallis 1987, Lemma 4.2].

Lemma 4.8. *If $\Omega_\beta = \phi$ then $R(V_v)_\beta = 0$. If $v(\beta) \neq 0$ and $\Omega_\beta \neq \phi$ then $\dim R(V_v)_\beta = 1$.*

Suppose that k_v is real. $I^\infty(s, \chi_{V_v})$ denotes the smooth induced representation including $I(s, \chi_{V_v})$ with its Fréchet topology. Let the topology of $\mathcal{S}(V_v)$ be given by the usual one. The map $i : \mathcal{S}^0(V_v) \rightarrow R(V_v) \subset I((m-3)/2, \chi_{V_v})$ extends to a continuous map

$$i : \mathcal{S}(V_v) \rightarrow R^\infty(V_v) \subset I^\infty((m-3)/2, \chi_{V_v}),$$

where $R^\infty(V_v)$ is the image of $\mathcal{S}(V_v)$. For $\beta \in D_{v,-}$, let $R^\infty(V_v)'_\beta$ be the space of all continuous linear functionals μ on $R^\infty(V_v)$ such that $\mu(r(X)\Phi) = d\psi_{v,\beta}(X)\mu(\Phi)$ for all $X \in \mathfrak{u} = \text{Lie } U(\mathbb{R})$ and all $\Phi \in R^\infty(V_v)$. Here $d\psi_{v,\beta} : \mathfrak{u} \rightarrow \mathbb{C}$ is the differential of $\psi_{v,\beta}$ and r denotes the action on $R^\infty(V_v)$.

Lemma 4.9. *If $\Omega_\beta = \phi$ then $R^\infty(V_v)'_\beta = 0$. If $v(\beta) \neq 0$ and $\Omega_\beta \neq \phi$ then $\dim R^\infty(V_v)'_\beta = 1$.*

Global theta correspondence. For a (-1) -hermitian right D -module (V, h) we write $\omega_V = \omega_{\psi,V}$ for the Weil representation of $G(V_A) \times G(\mathbf{A})$ on the space $\mathcal{S}(V_A)$ of Schwartz–Bruhat functions on V_A . For any $\beta \in D_-$, we define a character ψ_β of $U(\mathbf{A})$ by

$$\psi_\beta(u(b)) = \prod_v \psi_{v,\beta}(u(b_v)) \quad (u(b) = (u(b_v)) \in U(\mathbf{A})).$$

Let $\{(V_v, h_v)\}$ be a collection of (-1) -hermitian right D_v -modules with rank one for all v , and let $\Pi = \bigotimes_v R(V_v)$ (if it can be defined). Π is regarded as a representation of $(\text{Lie } G(\mathbf{A}_\infty) \otimes_{\mathbb{R}} \mathbb{C}, \mathbf{K}_\infty) \times G(\mathbf{A}_f)$, whose action is written by r . We write $\mathcal{W}_\beta(\Pi)$ for the space of linear functionals μ on Π which satisfy

$$(4-1) \quad \begin{aligned} \mu(r(u)f) &= \psi_\beta(u)\mu(f) & (f \in \Pi, u \in U(\mathbf{A}_f)), \\ \mu(r(X)f) &= d\psi_\beta(X)\mu(f) & (X \in \text{Lie } U(\mathbf{A}_\infty)), \end{aligned}$$

where $d\psi_\beta$ is the differential of the restriction of the character ψ_β to $U(\mathbf{A}_\infty)$. For each archimedean place v , $R^\infty(V_v)$ denotes the closure of $R(V_v)$ in $I^\infty(1/2, \chi_{V_v})$

as in the previous subsection. Let

$$\Pi^\infty = \left(\bigotimes_{v:\text{arch}} R^\infty(V_v) \right) \otimes \left(\bigotimes_{v:\text{nonarch}} R(V_v) \right),$$

which is a representation of $G(\mathbf{A})$. Let ${}^{\mathcal{W}}W_\beta^\infty(\Pi^\infty)$ be the space of all linear functionals μ on Π^∞ which satisfy (4-1) and whose restrictions to $R^\infty(V_v)$ lie in $R^\infty(V_v)'_\beta$. Let ${}^{\mathcal{W}}W_\beta^\infty(\Pi)$ be the subspace of ${}^{\mathcal{W}}W_\beta(\Pi)$ spanned by the restrictions of functionals in ${}^{\mathcal{W}}W_\beta^\infty(\Pi^\infty)$. We have the following proposition which is the same as Proposition 2.1 in [Kudla et al. 1992].

Proposition 4.10. *Suppose that $\{(V_v, h_v)\}$ is given by the completions at all v of some (-1) -hermitian right D -module (V, h) with rank one and $\det V \neq 0$. We write $\Pi(V) = \bigotimes_v R(V_v)$. Let $\mathbb{O}_V = \{\beta \in D_- \setminus \{0\} \mid h(x, x) = c\beta \text{ for some } x \in V \text{ and some } c \in k_{D, \beta}^\times\}$.*

- (i) *If $\beta \notin \mathbb{O}_V$ and $\beta \neq 0$ then ${}^{\mathcal{W}}W_\beta^\infty(\Pi(V)) = 0$.*
- (ii) *If $\beta \in \mathbb{O}_V$ then $\dim {}^{\mathcal{W}}W_\beta^\infty(\Pi(V)) = 1$.*

$A(G(k) \backslash G(\mathbf{A}))$ denotes the space of automorphic forms on $G(k) \backslash G(\mathbf{A})$. For $f \in A(G(k) \backslash G(\mathbf{A}))$, the β -th Fourier coefficient $W_\beta(f)$ is defined by

$$W_\beta(f)(g) = \int_{U(k) \backslash U(\mathbf{A})} f(ug) \psi_\beta(u^{-1}) du.$$

Denote by \overline{W}_β the linear functional on $A(G(k) \backslash G(\mathbf{A}))$ defined by $f \mapsto W_\beta(f)(1)$. It satisfies

$$\overline{W}_\beta(r(u)f) = \psi_\beta(u) \overline{W}_\beta(f) \quad (u \in U(\mathbf{A})).$$

Let (V, h) be a (-1) -hermitian space over D . We write $A^\infty(G(k) \backslash G(\mathbf{A}))$ for the space of smooth automorphic forms without the \mathbf{K} -finiteness condition. Let r again denote the right regular action on $A(G(k) \backslash G(\mathbf{A}))$ or on $A^\infty(G(k) \backslash G(\mathbf{A}))$.

By a parallel argument to the first part of the proof of Theorem 2.2 in [Kudla et al. 1992], we have that if $A, B : \Pi(V) \rightarrow A(G(k) \backslash G(\mathbf{A}))$ are two intertwining operators then A and B both extend to continuous $G(\mathbf{A})$ -intertwining operators $A^\infty, B^\infty : \Pi(V)^\infty \rightarrow A^\infty(G(k) \backslash G(\mathbf{A}))$. Since \overline{W}_β can extend to a continuous linear functional on $A^\infty(G(k) \backslash G(\mathbf{A}))$, we conclude that both $A_\beta = \overline{W}_\beta \circ A$ and $B_\beta = \overline{W}_\beta \circ B$ lie in ${}^{\mathcal{W}}W_\beta^\infty(\Pi(V))$. As in the observation just before Lemma 2.5 in [Kudla et al. 1992], we have the following results. If $\beta \notin \mathbb{O}_V$ and $\beta \neq 0$, then $A_\beta = B_\beta = 0$ by Proposition 4.10(i). If $\beta \in \mathbb{O}_V$, there is a $c_\beta \in \mathbb{C}$ such that $A_\beta = c_\beta B_\beta$ by Proposition 4.10(ii). Moreover, if both β_1 and β_2 lie in the same orbit in \mathbb{O}_V then $c_{\beta_1} = c_{\beta_2}$.

Proposition 4.11. *Suppose that $\{(V_v, h_v)\}$ is as in the assumption of Proposition 4.10. Then $\dim \mathrm{Hom}(\Pi(V), A(G(k) \backslash G(\mathbf{A}))) \leq 2^{s_D-2}$.*

Proof. From the definition of \mathbb{O}_V and Proposition 4.3(i), \mathbb{O}_V has 2^{s_D-2} orbits. By the preceding observation and the next lemma, the proposition is proved. \square

Lemma 4.12. *(It is not necessary to impose the assumption of Proposition 4.10.) If $E : \Pi \rightarrow A(G(k)\backslash G(A))$ is an intertwining operator satisfying $\overline{W}_\beta \circ E = 0$ for all $\beta \in D_- \setminus \{0\}$ then $E = 0$.*

This is shown by the same argument as [Kudla et al. 1992, Lemma 2.5].

Next consider the equality of the relation in Proposition 4.11. For $\phi \in \mathcal{S}(V_A)$ and $g \in G(A)$, we let

$$I_\phi(g) = \int_{G(V_k)\backslash G(V_A)} \theta(g, l; \phi) dl,$$

with the usual theta kernel

$$\theta(g, l; \phi) = \sum_{x \in V(k)} \omega_V(g)\phi(l^{-1}x).$$

Since $G(V)$ is anisotropic, $I_\phi(g)$ is well-defined and $I_\phi \in A(G(k)\backslash G(A))$. This defines an intertwining operator I_V from $\mathcal{S}(V_A)$ to $A(G(k)\backslash G(A))$. It is $G(V_v)$ -invariant for all nonarchimedean places and (\mathfrak{h}_v, L_v) -invariant for archimedean places. Therefore the image $\text{Im}I_V$ of the intertwining operator is isomorphic to a quotient of $\Pi(V)$. Since $\Pi(V)$ is irreducible, $\text{Im}I_V$ is isomorphic to $\Pi(V)$. Applying the square integrability criterion, and by Proposition 6.9 in [Kudla et al. 1992], we have that $\text{Im}I_V \subset L^2(G(k)\backslash G(A))$. For $\beta \in D_- \setminus \{0\}$,

$$\begin{aligned} W_\beta(I_\phi)(g) &= \int_{U(k)\backslash U(A)} I_\phi(ug)\psi_\beta(u^{-1}) du \\ &= \int_{U(k)\backslash U(A)} du \int_{G(V_k)\backslash G(V_A)} \sum_{x \in V(k)} \omega_V(ug)\phi(l^{-1}x)\psi_\beta(u^{-1}) dl \\ &= \int_{G(V_k)\backslash G(V_A)} \sum_{h(x,x)=\beta} \omega_V(g)\phi(l^{-1}x) dl. \end{aligned}$$

In particular, if β is not represented by h then $W_\beta(I_\phi) = 0$. If $\beta = h(x_0, x_0)$ for some $x_0 \in V \setminus \{0\}$ then

$$W_\beta(I_\phi)(g) = \int_{G(V_A)} \omega_V(g)\phi(l^{-1}x_0) dl.$$

Choosing a suitable ϕ we have $W_\beta(I_\phi) \neq 0$. In particular, if V_1, \dots, V_n are not isometric to each other then I_{V_1}, \dots, I_{V_n} are linearly independent. Thus we have obtained the following theorem.

Theorem 4.13. *Suppose that $\{(V_v, h_v)\}$ is as in the assumption of Proposition 4.10. Then $\dim \text{Hom}(\Pi(V), A(G(k)\backslash G(A))) = 2^{s_D-2}$.*

Proposition 4.14. *Suppose that there are no global (-1) -hermitian space (V, h) such that $\{(V_v, h_v)\}_v$ is given by the completions of (V, h) . Then*

$$\mathrm{Hom}(\Pi, A(G(k)\backslash G(\mathbf{A}))) = 0.$$

Proof. Let $E : \Pi \rightarrow A(G(k)\backslash G(\mathbf{A}))$ be a nonzero intertwining operator. There exists $\beta \in D_- \setminus \{0\}$ such that $E_\beta = \overline{W}_\beta \circ E$ is nonzero by Lemma 4.12. Therefore for any v , the restriction of E_β to $R(V_v)$ is nonzero. By Lemma 4.8 and 4.9 $\langle \beta \rangle \simeq V_v$. This contradicts the assumption for $\{(V_v, h_v)\}$. \square

Let $\{(V_v, h_v)\}$ be given by the completions at all v for some (-1) -hermitian right D -module (V, h) with rank one and $\det V \neq 0$. We define the quadratic character χ_V of A^\times/k^\times by $\chi_V = \prod_v \chi_{V_v}$. Since

$$-v(D_- \setminus \{0\}) = \{\alpha \in k^\times \mid \alpha \notin (k_v^\times)^2 \text{ for all } v \in S_D\},$$

any $\omega_{k'/k}$ appearing in Proposition 3.2(ii) can be written by the form χ_V for some V . By the definition of $R(V_v)$ we have

$$\bigotimes_v R(V_v) \subset \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}((\chi_V | \cdot |^{-1/2}) \circ \nu).$$

On the other hand, by Proposition 1.1, 1.2 in [Kudla et al. 1992] and Proposition 4.5, 4.7, any irreducible constituent of $\mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}((\chi_V | \cdot |^{-1/2}) \circ \nu)$ is written by the form $\bigotimes_v R(\tilde{V}_v)$ where $\{\tilde{V}_v\}_v$ is a collection of (-1) -hermitian right D_v -modules with rank one and $\det \tilde{V}_v = \det V_v$ for all v . Therefore Proposition 4.14 and Theorem 4.13 conclude that the representations obtained by the images of $N(w_0, (\chi_V | \cdot |_{\mathbf{A}}^{1/2}) \circ \nu)$ are exhausted by the representations given by Theorem 4.1(3).

5. Arthur parameters for residual spectrum

Here we give an expectation about the Arthur parameters for the residual spectrum of G . The dual group \widehat{G} of G equals $SO(5, \mathbb{C})$, which we realize as

$$\widehat{G} = \left\{ g \in \mathrm{SL}(5, \mathbb{C}) \mid g \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ -1 & & & & \\ 1 & & & & \end{pmatrix} {}^t g = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ -1 & & & & \\ 1 & & & & \end{pmatrix} \right\}.$$

For each v we denote by W_{k_v} the Weil group of k_v and write ${}^L G_v$ for the L -group $\widehat{G} \times W_{k_v}$ of $G_v = G \otimes_k k_v$. We write \mathcal{L}_v for the Langlands group introduced in

[Kottwitz 1984]:

$$\mathcal{L}_v := \begin{cases} W_{k_v} & \text{if } k_v \text{ is archimedean,} \\ W_{k_v} \times SU(2, \mathbb{R}) & \text{if } k_v \text{ is nonarchimedean.} \end{cases}$$

A local Arthur parameter for G is a continuous homomorphism

$$\psi : \mathcal{L}_v \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G_v$$

such that (i) $\psi|_{W_{k_v}}$ is semisimple and has bounded image, (ii) the composition

$$W_{k_v} \xrightarrow{\psi} {}^L G_v \xrightarrow{\mathrm{pr}_2} W_{k_v}$$

is the identity, and (iii) ψ restricted to $\mathrm{SL}(2, \mathbb{C})$ or $SU(2) \times \mathrm{SL}(2, \mathbb{C})$ is analytic.

Expectation 5.1. Let $\pi = \bigotimes_v \pi_v$ be a residual discrete representation appearing in Theorem 4.1. The Arthur parameter ψ_{π_v} associated to π_v is given by the following.

(1) For $\mathbf{1}_{G(A)}$ we have $\psi_{\mathbf{1}_{G(A)}}|_{\mathcal{L}_v} = 1_5 \times \mathrm{pr}_{W_{k_v}}$ and

$$\psi_{\mathbf{1}_{G(A)}}\left(\begin{pmatrix} a & x \\ y & -a \end{pmatrix}\right) = \begin{pmatrix} 4a & x & & & \\ 2y & 2a & x & & \\ & 2y & 0 & x & \\ & & 2y & -2a & x \\ & & & 2y & -4a \end{pmatrix} \in \mathrm{Lie} \widehat{G}.$$

(2) For $J_P^G(\pi)$ we have

$$\psi_{J_P^G(\pi)}|_{\mathcal{L}_v} = \left(\begin{array}{c|c|c} \varphi_{\pi_v} & & \\ \hline & 1 & \\ \hline & & \mathrm{Ad}(J)^t \varphi_{\pi_v}^{-1} \end{array} \right) \times \mathrm{pr}_{W_{k_v}}$$

and

$$\psi_{J_P^G(\pi)}\left(\begin{pmatrix} a & x \\ y & -a \end{pmatrix}\right) = \begin{pmatrix} a & & x \\ \hline a & & x \\ \hline & 0 & \\ \hline y & & -a \\ y & & -a \end{pmatrix} \in \mathrm{Lie} \widehat{G}.$$

Here φ_{π_v} is the Langlands parameter for π_v . Since π_v is self-dual the image of φ_{π_v} is contained in $\mathrm{SL}(2, \mathbb{C}) \times W_{k_v}$.

(3) For $R(V)$ we have

$$\psi_{R(V_v)}|_{\mathcal{L}_v} = \begin{pmatrix} \chi_{V_v} & & & & \\ & c_1 & & c_2 & \\ & & \chi_{V_v} & & \\ & c_3 & & c_4 & \\ & & & & \chi_{V_v} \end{pmatrix} \times \mathrm{pr}_{W_{k_v}}$$

and

$$\psi_{R(V_v)}\left(\begin{pmatrix} a & x \\ y & -a \end{pmatrix}\right) = \left(\begin{array}{cc|c|cc} 2a & 0 & x & & \\ 0 & 0 & 0 & & \\ \hline 2y & 0 & 0 & 0 & -x \\ \hline & & 0 & 0 & 0 \\ & & -2y & 0 & -2a \end{array} \right) \in \mathrm{Lie}\widehat{G},$$

where

$$\begin{pmatrix} c_1(w) & c_2(w) \\ c_3(w) & c_4(w) \end{pmatrix} = \begin{cases} 1_2 & \text{if } w \in W_{k'_v}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } w \in W_{k_v} \setminus W_{k'_v}, \end{cases}$$

for the quadratic extension k' of k attached to χ_V .

Remark 5.2. The (global) Arthur parameter of $\mathbf{1}_{G(A)}$ should coincide with that of $\mathbf{1}_{\mathrm{Sp}(2,A)}$. For $J_P^G(\pi)$ the unique irreducible quotient $J_{P_1}^{\mathrm{Sp}(2)}(JL(\pi))$ of

$$\mathrm{Ind}_{P_1(A)}^{\mathrm{Sp}(2,A)}(JL(\pi) | \det |_A^{1/2})$$

occurs in the residual spectrum of $\mathrm{Sp}(2)$ where P_1 is the Siegel parabolic subgroup of $\mathrm{Sp}(2)$ and $JL(\pi)$ is the Jacquet–Langlands correspondence of π [Kim 1995]. The Arthur parameters of $J_P^G(\pi)$ and $J_{P_1}^{\mathrm{Sp}(2)}(JL(\pi))$ should coincide. For $R(V)$ the theta lift of the trivial representation of the orthogonal group of a 2-dimensional quadratic space with determinant $\det V$ occurs in the residual spectrum of $\mathrm{Sp}(2)$ [Kon-no 1994] and its Arthur parameter should coincide with that of $R(V)$.

Acknowledgment

The author thanks Professor Takuya Kon-no for much helpful advice and encouragement.

References

[Godement and Jacquet 1972] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics **260**, Springer, Berlin, 1972. MR 49 #7241 Zbl 0244.12011

[Harish-Chandra 1968] Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Lecture Notes in Mathematics **62**, Springer, Berlin, 1968. MR 38 #1216 Zbl 0186.04702

[Jacquet and Langlands 1970] H. Jacquet and R. P. Langlands, *Automorphic forms on $\mathrm{GL}(2)$* , Lecture Notes in Mathematics **114**, Springer, Berlin, 1970. MR 53 #5481 Zbl 0236.12010

[Johnson 1990] K. D. Johnson, “Degenerate principal series and compact groups”, *Math. Ann.* **287**:4 (1990), 703–718. MR 91h:22028 Zbl 0687.22003

[Kim 1995] H. H. Kim, “The residual spectrum of Sp_4 ”, *Compositio Math.* **99**:2 (1995), 129–151. MR 97c:11056 Zbl 0877.11030

[Knapp and Stein 1980] A. W. Knap and E. M. Stein, “Intertwining operators for semisimple groups, II”, *Invent. Math.* **60**:1 (1980), 9–84. MR 82a:22018 Zbl 0454.22010

- [Knus 1991] M.-A. Knus, *Quadratic and Hermitian forms over rings*, Grundlehren der Math. Wissenschaften **294**, Springer, Berlin, 1991. MR 92i:11039 Zbl 0756.11008
- [Kon-no 1994] T. Kon-no, “The residual spectrum of $\mathrm{Sp}(2)$ ”, *Proc. Japan Acad. Ser. A Math. Sci.* **70**:6 (1994), 204–207. MR 96a:11048 Zbl 0833.11018
- [Kottwitz 1984] R. E. Kottwitz, “Stable trace formula: cuspidal tempered terms”, *Duke Math. J.* **51**:3 (1984), 611–650. MR 85m:11080 Zbl 0576.22020
- [Kudla 1994] S. S. Kudla, “Splitting metaplectic covers of dual reductive pairs”, *Israel J. Math.* **87**:1-3 (1994), 361–401. MR 95h:22019 Zbl 0840.22029
- [Kudla et al. 1992] S. S. Kudla, S. Rallis, and D. Soudry, “On the degree 5 L -function for $\mathrm{Sp}(2)$ ”, *Invent. Math.* **107**:3 (1992), 483–541. MR 93b:11061 Zbl 0776.11028
- [Langlands 1971] R. P. Langlands, *Euler products*, Yale Mathematical Monographs **1**, Yale University Press, New Haven, Conn., 1971. MR 54 #7387 Zbl 0231.20016
- [Mœglin and Waldspurger 1995] C. Mœglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Tracts in Mathematics **113**, Cambridge Univ. Press, Cambridge, 1995. MR 97d:11083 Zbl 0846.11032
- [Mœglin et al. 1987] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p -adique*, Lecture Notes in Mathematics **1291**, Springer, Berlin, 1987. MR 91f:11040 Zbl 0642.22002
- [Muić and Savin 2000] G. Muić and G. Savin, “Complementary series for Hermitian quaternionic groups”, *Canad. Math. Bull.* **43**:1 (2000), 90–99. MR 2001g:22019 Zbl 0945.22005
- [Rallis 1987] S. Rallis, *L -functions and the oscillator representation*, Lecture Notes in Mathematics **1245**, Springer, Berlin, 1987. MR 89b:11046 Zbl 0605.10016
- [Scharlau 1985] W. Scharlau, *Quadratic and hermitian forms*, Grundlehren der Mathematischen Wissenschaften **270**, Springer, Berlin, 1985. MR 86k:11022 Zbl 0584.10010
- [Shahidi 1981] F. Shahidi, “On certain L -functions”, *Amer. J. Math.* **103**:2 (1981), 297–355. MR 82i:10030 Zbl 0467.12013
- [Shahidi 1990] F. Shahidi, “A proof of Langlands’ conjecture on Plancherel measures; complementary series for p -adic groups”, *Ann. of Math. (2)* **132**:2 (1990), 273–330. MR 91m:11095 Zbl 0780.22005
- [Silberger 1979] A. J. Silberger, *Introduction to harmonic analysis on reductive p -adic groups*, Math. Notes **23**, Princeton Univ. Press, Princeton, N.J., 1979. MR 81m:22025 Zbl 0458.22006
- [Zhu 1992] C.-b. Zhu, “Invariant distributions of classical groups”, *Duke Math. J.* **65**:1 (1992), 85–119. MR 92k:22022 Zbl 0764.22009

Received March 19, 2006. Revised August 28, 2006.

TAKANORI YASUDA
GRADUATE SCHOOL OF MATHEMATICS
KYUSHU UNIVERSITY 33
FUKUOKA 812-8581
JAPAN
tyasuda@math.kyushu-u.ac.jp