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We prove existence results that give information about the space of minimal immersions of 2-tori into S^3. More specifically, we show:

- **For every positive integer n, there are countably many real n-dimensional families of minimally immersed 2-tori in S^3. Every linearly full minimal immersion $T^2 \to S^3$ belongs to exactly one of these families.**
- **Let $\mathscr{A}$ be the space of rectangular 2-tori. There is a countable dense subset $\mathscr{B}$ of $\mathscr{A}$ such that every torus in $\mathscr{B}$ can be minimally immersed into S^3.**

Mainly, we find minimal immersions that satisfy *periodicity conditions* and hence obtain maps of tori, rather than simply immersions of the plane. This work uses a correspondence, established by Hitchin, between minimal tori in S^3 and algebraic curve data.

1. Introduction

We show that there is an abundance of minimally immersed 2-tori in S^3; in fact they come in families of every real dimension. We use results of Hitchin [1990], namely his description of harmonic maps $f : T^2 \to S^3 \simeq \mathrm{SU}(2)$ in terms of algebraic curve data. This transforms a problem in analysis to one in algebraic geometry, hence rendering it vulnerable to new techniques. A conformal map is harmonic if and only if it is minimal, and so this reformulation applies to minimal immersions. The challenge of finding *doubly-periodic* minimal immersions (that is, immersions of tori) hence becomes a problem in algebraic geometry, and it is this problem that we solve. Our approach builds upon work of Ercolani, Knörrer and Trubowitz [1993], in which they proved it is possible to obtain families of constant mean curvature tori in $\mathbb{R}^3$ of every even dimension. Hitchin's correspondence excludes minimal branched immersions that are not linearly full, that is, which map into a totally geodesic $S^2 \subset S^3$. We henceforth assume that our minimal immersions are linearly full. We also partially address the interesting question of which tori can be minimally immersed into the 3-sphere. In particular, we show that a countable dense subset $\mathscr{B}$ of the space of rectangular tori does allow such immersions.

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Exploiting the algebro-geometric viewpoint allows us to obtain not only strong existence results but also detailed information about the *space $\mathcal{M}$ of linearly full minimal immersions of $T^2 \to S^3$* $\cong \mathrm{SU}(2)$. These immersions give basic examples of harmonic maps from a torus to a Lie group or symmetric space and provide a model for the general situation. One can also associate a spectral curve to harmonic immersions of tori in other Lie groups and symmetric spaces, although the general theory is incomplete (see for example [Griffiths 1985; Burstall 1992; Ferus et al. 1992; Burstall et al. 1993; McIntosh 1995; 1996]). There has been an explosion of interest in these maps over the last 30 years, both due to their geometric interest and their strong connections with the Yang–Mills equations. These results can be viewed as a preliminary step in the very interesting general program of obtaining information about the spaces of these harmonic maps.

We now explain how the algebro-geometric approach elucidates the structure of $\mathcal{M}$, and why it is a useful for studying these (and other) maps. Hitchin [1990] established an explicit bijection between harmonic maps $T^2 \to S^3$ and *spectral curve data*, which includes a hyperelliptic curve Σ (called a spectral curve) and a line bundle $E(0) \to \Sigma$. (See Definition 2.1 and Theorem 2.2 for details.) This introduces an important new invariant of the harmonic map, namely the arithmetic genus g of Σ, or *spectral genus*. The space $\mathcal{M}$ decomposes into strata $\mathcal{S}_g$, where $\mathcal{S}_g$ consists of maps of spectral genus g. We show that $\mathcal{S}_g$, for $g > 2$, is composed of (at least) countably many $(g-2)$-dimensional families of immersions. From [Hitchin 1990], $\mathcal{S}_1$ is composed of the S^1-symmetric examples, and $\mathcal{S}_0$ is composed of the finite covers of the Clifford torus.

It is easy to write down minimal immersions of the plane to S^3, but finding ones that are doubly periodic is a difficult problem. Hitchin's correspondence transforms this analytic problem into a similarly nontrivial algebro-geometric one. Namely, it becomes the requirement that Σ supports a pair Θ, Ψ of meromorphic differentials of the second kind, whose periods, together with their integrals over certain open curves C_1 and C_{-1} are all integers. These *periodicity conditions* place very strong restrictions on the spectral data; they demand that certain transcendental functions are integer valued, and it is not at all clear, a priori, that solutions exist for each g.

Hitchin proved the existence of harmonic maps $T^2 \to S^3$ of spectral genera $g \leq 3$ and of minimal tori with $g \leq 2$. Our main result is Theorem 3.1, which gives the algebro-geometric statement necessary for our conclusions regarding the space $\mathcal{M}$ of minimally immersed tori:

Theorem 3.1. *For each integer $g > 0$, there are countably many spectral curves of arithmetic genus g giving rise to minimal immersions from rectangular tori to S^3.*

Using Hitchin's spectral curve correspondence, we conclude:

Corollary 3.2. *For each integer $n \geq 0$, there are countably many real n-dimensional families of minimal immersions from rectangular tori to S^3. Each family consists of maps from a fixed torus.*

Bryant [1982] has proven that all compact Riemann surfaces can be minimally immersed into S^4, but the question of which Riemann surfaces or indeed which tori admit minimal immersions into S^3 is both open and very interesting. The area of a surface is invariant under conformal transformations, and so one only needs a conformal structure on a surface to have a notion of minimal immersions of it. Thus when we ask "which tori?" we mean "which conformal classes of tori?"

Using conformal equivalence and taking covers, we straightforwardly get

Corollary 3.3. *Let $\mathscr{A}$ denote the space of rectangular 2-tori, and call a torus admissible if it admits a linearly full minimal immersion into S^3. Write $\mathscr{C}$ for the space of admissible rectangular 2-tori and $\mathscr{C}_g$ for those that possess a minimal immersion of spectral genus g. Then for each $g \geq 0$, $\mathscr{C}_g$ is (at least) countable and dense in $\mathscr{A}$.*

We now briefly discuss the methods by which we prove Theorem 3.1. We impose additional symmetry conditions on our spectral data to ensure that the resulting tori are rectangular. Using these and other symmetries, one can define a smooth map ϕ from an open subset of the space of spectral curves of genus g to a space of the same dimension in which the periodicity conditions are satisfied on a countable dense subset. The technical heart of the problem is then to show that for each g, there is a curve of genus g at which the differential of this map ϕ is invertible.

Our methods extend those of Ercolani, Knörrer and Trubowitz [1993], who proved the existence of rectangular constant mean curvature tori in $\mathbb{R}^3$ for every even spectral genus $g \geq 0$ (there is also a spectral curve description of such tori). The additional symmetry that forces the resulting tori to be rectangular breaks the proof naturally into odd and even genus cases. Our main contribution is the odd genus case, which is more complicated and is presented in detail. For even genera, the proof is reminiscent of [Ercolani et al. 1993] and is briefly indicated. We remark that Jaggy [1994] showed the existence of constant mean curvature tori in $\mathbb{R}^3$ for all spectral genera, but we do not pursue analogous methods since they lead to less refined information concerning the conformal type. It is by imposing this additional symmetry that we are able to obtain the conformal type information stated in Corollary 3.3.

We now mention some open questions that this paper motivates.

- We prove that a countable dense subset of the space of rectangular tori can be minimally immersed in S^3. Our proof shows that in an open subset of the space $\mathscr{M}$ of linearly full minimal immersions of $T^2 \to S^3$, these are the only rectangular tori that occur, but this result is only local. The spectral curve viewpoint suggests that the space $\mathscr{C}$ of admissible rectangular tori is only countable and hence that, in contrast to the situation for S^4, not all tori can be minimally immersed into S^3. One could prove this by showing that the differential of ϕ is everywhere invertible.

- The question of exactly which tori lie in $\mathscr{C}$ is difficult but very interesting. Further, are the $\mathscr{C}_g$ different for different g? It is not even known whether the square torus lies in all $\mathscr{C}_g$.
- These questions have natural analogues for general (not necessarily rectangular) tori.
- Can we prove similar existence results for "nonrectangular" tori? Some care needs to be exercised in stating this problem, since we can always replace a given torus by a conformally equivalent one; we are seeking tori that are not conformally equivalent to any rectangular torus. Methods similar those used here will yield nonrectangular tori (see [Jaggy 1994]). However, it is not clear whether these tori will be conformally equivalent to rectangular tori; the task is to show that for each g, there are some which are not.

We now outline the rest of this paper. In Section 2, we outline Hitchin's spectral curve correspondence and give our geometric reformulation of the periodicity conditions. In Section 3, we state our main results and explain the strategy for their proof. Section 4, the technical heart of this paper, carries out the proofs.

2. Background: spectral curve correspondence

Here, we first explain how a harmonic map from a surface to a Lie group with biinvariant metric can be described in terms of a family of flat connections. We then outline Hitchin's spectral curve construction for harmonic maps from a 2-torus to $S^3 \cong \mathrm{SU}(2)$, which further reduces the harmonic map equations to a linear flow in a complex torus, namely the Jacobian of the spectral curve.

Let M be a compact Riemann surface, G be a compact Lie group with biinvariant metric, and $\mathfrak{g}$ be its Lie algebra. Writing Φ for the $(1, 0)$ part of $f^{-1}df$,

$$f^{-1}df = \Phi - \Phi^*.$$

Here Φ is valued in the complex Lie algebra with real form $\mathfrak{g}$, and $-\Phi^*$ is its image under the corresponding real involution. For G a unitary group, $\Phi^* = \bar{\Phi}^t$.

A smooth map $f : M \to G$ is harmonic if and only if

$$d\phi_\lambda + \tfrac{1}{2}[\phi_\lambda, \phi_\lambda] = 0 \quad \text{for all } \lambda \in \mathbb{C}^*, \tag{2-1}$$

where

$$\phi_\lambda := \tfrac{1}{2}(1 - \lambda^{-1})\Phi - \tfrac{1}{2}(1 - \lambda)\Phi^* \quad \text{for } \lambda \in \mathbb{C}^*.$$

Let P be a trivial principal SU(2)-bundle over M, with trivial connection ∇. Writing

$$\nabla_\lambda := \nabla + \phi_\lambda, \tag{2-2}$$

f gauges the trivial connection $\nabla_1 = \nabla$ to ∇_{-1}, and (2-1) states that ∇_λ is flat for each $\lambda \in \mathbb{C}^*$.

Conversely, take a principal SU(2)-bundle P, a fixed connection ∇ and $\Phi \in \Omega^{1,0}(M, Ad(P) \otimes \mathbb{C})$ such that the connections ∇_λ defined as above are flat. Then locally one recovers a harmonic map $f : U \to G$ by

$$g_{-1}(z) = f(z) g_1(z), \tag{2-3}$$

where g_1 and g_{-1} are sections of P over U parallel with respect to ∇_1 and ∇_{-1}, respectively; patching gives a harmonic section of P. The condition for a global $f : M \to G$ is that both the left connection $\nabla_1 = \nabla$ and the right connection ∇_{-1} are trivial, and then f is determined up to left and right actions of G.

We outline Hitchin's [1990] algebro-geometric description of harmonic maps $f : T^2 \to \mathrm{SU}(2)$. We can consider the flat connections ∇_λ as living in a rank two complex vector bundle V with a symplectic form ω and a quaternionic structure. f is determined from the connections ∇_λ up to the action of $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ on S^3. The restriction to tori is essential, as the construction requires the compact Riemann surface to have abelian (and nontrivial) fundamental group. It is assumed throughout that f is not a conformal harmonic map to a totally geodesic $S^2 \subset S^3$; such maps do not admit a spectral curve. First, one gives an algebro-geometric description of families of flat connections (2-2) or, equivalently, of harmonic sections f of P.

Given a family (2-2) of flat connections on a marked torus T^2, let $H_z(\lambda)$, $K_z(\lambda) \in SL(2, \mathbb{C})$ denote the holonomy of ∇_λ for the chosen basis $[z, z+1]$, $[z, z+\tau]$ of $\pi_1(T^2, z)$, and let $\mu(\lambda)$, $\nu(\lambda)$ be their eigenvalues.

Define the spectral curve Σ:

$$\eta^2 = p(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$$

as follows. For generic $\lambda \in \mathbb{C}^*$, $H_z(\lambda)$ is not $\pm I$, and so has two eigenvectors $v_1(\lambda)$, $v_2(\lambda)$, which may coincide. λ is a zero of p of order k if $v_1(\lambda), v_2(\lambda) \in V_z$ agree to order k, as measured by the order of vanishing of the symplectic form ω. If $H_z(\lambda) = \pm I$, one can use holomorphic continuation to choose $v_1(\lambda)$ and $v_2(\lambda)$, and then apply the same criterion. See [Hitchin 1990] for a more careful treatment. If f is conformal ($\det \Phi = 0$), then p has odd order and has a simple zero at 0, while if f is nonconformal, then p has even order and is nonzero at 0.

Since the eigenspaces of the holonomy matrices $H_z(\lambda)$ for different z are related by parallel transport, Σ is independent of z. It follows because $\pi_1(T^2)$ is abelian that Σ is also independent of the choice of generator of the fundamental group and, crucially, it is a (finite genus) algebraic curve [Hitchin 1990].

Clearly the eigenvalue functions μ and ν are well-defined regular functions on $\Sigma - \lambda^{-1}(0, \infty)$.

$$\Theta := \frac{1}{2\pi i} d \log \mu \quad \text{and} \quad \Psi := \frac{1}{2\pi i} d \log \nu$$

are differentials of the second kind on Σ. Their only singularities are double poles at $\pi^{-1}\{0, \infty\}$. They have no residues and their periods are integers.

For each $z \in T^2$, $\Sigma - \pi^{-1}\{0, \infty\}$ supports the eigenspace line bundle

$$(E(z))_{(\lambda,\eta)} \subseteq \ker(H(\lambda, z) - \mu(\lambda, \eta)) \subseteq V_z,$$

which extends to a holomorphic line bundle $E(z)$ on Σ, and the map $z \to E(z)$ is linear. Σ has a hyperelliptic involution σ, and the SU(2) structure also induces an antiholomorphic involution ρ, covering $\lambda \mapsto \bar{\lambda}^{-1}$.

Thus by explicit construction, Hitchin [1990, Theorem 8.1] associates the following spectral data $(\Sigma, \lambda, \rho, \Theta, \Psi, E(0))$ to the family (2-2):

Definition 2.1. By spectral data we mean the following setup.

(1) Σ is a hyperelliptic curve $\eta^2 = p(\lambda)$ of arithmetic genus g, and $\pi : \Sigma \to \mathrm{CP}^1$ is the projection $\pi(\lambda, \eta) = \lambda$.

(2) $p(\lambda)$ is real with respect to the real structure $\lambda \mapsto \bar{\lambda}^{-1}$ on CP^1.

(3) $p(\lambda)$ has no real zeros (that is, no zeros on the unit circle $\lambda = \bar{\lambda}^{-1}$).

(4) $p(\lambda)$ has at most simple zeros at $\lambda = 0$ and $\lambda = \infty$.

(5) Θ and Ψ are meromorphic differentials on Σ whose only singularities are double poles at $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$ and which have no residues. Their principal parts are linearly independent over $\mathbb{R}$, and they satisfy

$$\sigma^*\Theta = -\Theta, \quad \sigma^*\Psi = -\Psi, \quad \rho^*\Theta = \bar{\Theta}, \quad \rho^*\Psi = \bar{\Psi},$$

where σ is the hyperelliptic involution $(\lambda, \eta) \mapsto (\lambda, -\eta)$ and ρ is the real structure induced from $\lambda \mapsto \bar{\lambda}^{-1}$.

(6) The periods of Θ and Ψ are all integers.

(7) $E(0)$ is a line bundle of degree $g+1$ on Σ, which is quaternionic with respect to the real structure $\sigma\rho$.

Conversely, from spectral data $(\Sigma, \lambda, \rho, \Theta, \Psi, E(0))$, one can recover a marked torus T^2, an SU(2) principal bundle P, and a family of flat connections of the form

(2-2) unique up to gauge transformations and the operation of tensoring the associated complex vector bundle V by a flat $\mathbb{Z}_2$-bundle; see [Hitchin 1990, Theorem 8.1].

The existence of differentials Θ, Ψ with integral periods places a stringent constraint on Σ. It corresponds to the double periodicity of a harmonic section of $P = G \times G / G$.

The following is essentially [Hitchin 1990, Theorem 8.20]. The additional periodicity conditions guarantee that the connections ∇_1, ∇_{-1} are trivial and hence the harmonic section (2-3) of P gives a map $T^2 \to S^3$.

Theorem 2.2. *Let* $(\Sigma, \lambda, \rho, \Theta, \Psi, E(0))$ *be spectral data*, μ *and* ν *be functions on* $\Sigma - \lambda^{-1}\{0, \infty\}$ *satisfying*

$$\Theta = \frac{1}{2\pi i} d \log \mu, \quad \Psi = \frac{1}{2\pi i} d \log \nu, \quad \mu\sigma^*\mu = 1, \quad \nu\sigma^*\nu = 1.$$

(*Such functions exist by the periodicity conditions on* Θ, Ψ.) *Let* T^2 *be the marked torus with generators* $[z, z+1]$, $[z, z+\tau]$ *for* $\pi_1(T^2, z)$, *where* (*in some local coordinate system*)

$$\tau = \frac{\textit{principal part}_{P_0}\Psi}{\textit{principal part}_{P_0}\Theta} \qquad \textit{and} \qquad P_0 \in \lambda^{-1}(0).$$

Then

(1) $(\Sigma, \lambda, \rho, \mu, \nu, E(0))$ *determines a harmonic map* $f : T^2 \to \mathrm{SU}(2)$ *if and only if* $\mu(\lambda, \eta) = \nu(\lambda, \eta) = 1$ *for all* $(\lambda, \eta) \in \lambda^{-1}\{1, -1\}$.

(2) f *is conformal if and only if* $P(0) = 0$.

(3) *The harmonic map is uniquely determined by* $(\Sigma, \lambda, \rho, \mu, \nu, E(0))$ *modulo the action of* $\mathrm{SO}(4)$ *on* S^3.

Let C_1 be a curve in Σ joining the two points in $\lambda^{-1}(1)$, and let C_{-1} be a curve joining the two points in $\lambda^{-1}(-1)$.

Periodicity Conditions 2.3. The existence of functions μ, ν as above is equivalent to these periodicity conditions (μ and ν are then determined up to sign):

(1) The periods of Θ, Ψ are all integers.

(2) $\int_{C_{\pm 1}} \Theta$, $\int_{C_{\pm 1}} \Psi \in \mathbb{Z}$.

3. Statement of results

Here we state our main algebro-geometric Theorem 3.1 and explain how it enables us to obtain information regarding the space of minimally immersed tori in S^3 (Corollaries 3.2, 3.3). We also outline our approach for proving this theorem.

Theorem 3.1. *For each integer $g > 0$, there are countably many spectral curves of arithmetic genus g giving rise to minimal immersions from rectangular tori to S^3.*

Corollary 3.2. *For each integer $n \geq 0$, there are countably many real n-dimensional families of linearly full minimal immersions from rectangular tori to S^3. Each family consists of maps from a fixed torus.*

Proof. Given spectral data $(\Sigma, \lambda, \rho, \Theta, \Psi)$, condition (iii) of Hitchin's spectral data (Definition 2.1) implies that the quaternionic structure $\rho\sigma$ has no fixed points and so by [Atiyah 1971] there is a real g-dimensional family of quaternionic bundles of degree $g + 1$. Factoring out by reparameterizations of the domain torus, for $g > 2$ we obtain a real $(g - 2)$-dimensional family of minimal immersions from a fixed torus. □

Corollary 3.3. *Let $\mathscr{A}$ denote the space of rectangular 2-tori, and call a torus admissible if it admits a linearly full minimal immersion into S^3. Write $\mathscr{C}$ for the space of admissible rectangular 2-tori and $\mathscr{C}_g$ for those that possess a minimal immersion of spectral genus g. Then for each $g \geq 0$, $\mathscr{C}_g$ is (at least) countable and dense in $\mathscr{A}$.*

Proof. $\mathscr{C}_0$ contains the square torus (Clifford torus), and, as shown above, each $\mathscr{C}_g$, $g > 0$ is nonempty. Let τ be such that the torus with sides $[0, 1]$ and $[0, \tau]$ is in $\mathscr{C}_g$. Then for each $q \in \mathbb{Q}$, the torus with sides $[0, 1]$ and $[0, q\tau]$ is conformally equivalent to a finite cover of the original torus, and hence is also in $\mathscr{C}_g$. □

To prove Theorem 3.1, one wishes, for each positive integer g, to find spectral data satisfying the conditions of Hitchin's correspondence (Definition 2.1 and Periodicity Conditions 2.3). In particular, the periodicity conditions require that the periods of the meromorphic differentials Θ and Ψ, as well as their integrals over the open curves $C_{\pm 1}$, are all integers. It suffices to show that the required integrals of Θ and Ψ each give rational points in CP^{2g+1}, as then an appropriate multiple of each differential will satisfy the periodicity conditions. To guarantee that our tori are rectangular, we will demonstrate the existence of spectral curves having the additional symmetry $\lambda \mapsto 1/\lambda$, where $\pi : \Sigma \to \mathrm{CP}^1$, $(\lambda, \eta) \mapsto \lambda$. Using this and other symmetries of the spectral curve, we observe that every spectral curve Σ supports differentials Θ, Ψ such that some of these integrals vanish, while the remaining ones are real. Indeed, by assigning to each spectral curve these integrals, one obtains a smooth map ϕ from an open subset of the space of spectral curves of genus g to a product of real projective spaces such that the domain and range of this map have the same dimension. Our task then reduces to showing that for each g, there is a curve of genus g at which the differential of this map is invertible. An application of the inverse function theorem then yields Theorem 3.1. All linearly full branched minimal immersions $T^2 \to S^3$ are in fact minimal immersions

[Hitchin 1990], which is why we are able to conclude that our maps do not have branch points.

4. Proofs

As described above, we enforce an additional symmetry $\lambda \mapsto 1/\lambda$ upon our spectral data in order to find rectangular tori. This symmetry induces two holomorphic involutions on Σ, and the resulting quotient curves $Y_\pm$ are our basic object of study. We will use proof by induction, in which at each step the genera of $Y_\pm$ increase by one, and hence the genus g of Σ increases by two. Thus the proof divides naturally into the even and odd genus cases. We give in this section a proof for odd genera. In this case, where the quotient curves have different genera from one another, which is a complicating factor. The proof for even genera is both simpler and analogous, and it is similar to the work in [Ercolani et al. 1993]. A full proof is thus unnecessary, and we give instead a brief description.

Odd genera. We begin by describing the algebraic curve data to be studied. We will then turn to proving Theorem 4.1, which states that for each $n \geq 0$, the differential of a certain map ϕ is somewhere invertible. We proceed by induction and at each step pass to the boundary of the moduli space; we then use a further limiting argument. We collect in Lemmata 4.2, 4.3, 4.5 the results of the necessary calculations and derive in Lemma 4.4 an additional assumption necessary for the induction step. We then state Theorem 4.6, which is simply Theorem 4.1 together with this additional condition, and prove it by induction using Lemmata 4.2, 4.3 and the case $n = 0$ (Lemma 4.7).

Let $Y_+ = Y_+(r, x_1, \bar{x_1}, \dots, x_n, \bar{x_n})$ be the curve given by

$$y_+^2 = (x - r) \prod_{i=1}^{n} (x - x_i)(x - \bar{x_i})$$

and $Y_- = Y_-(r, x_1, \bar{x_1}, \dots, x_n, \bar{x_n})$ be that given by

$$y_-^2 = (x - 2)(x + 2)(x - r) \prod_{i=1}^{n} (x - x_i)(x - \bar{x_i}),$$

where we assume that $r \in (-\infty, -2) \cup (2, \infty)$, $x_i \neq 2$ for $i = 1, \dots, 2n$, and $x_i \neq x_j$ for $i \neq j$. Denote by $\pi_\pm : Y_\pm \to \mathrm{CP}^1$ the projections $(x, y_\pm) \mapsto x$. Construct $\pi : \Sigma \to \mathrm{CP}^1$ as the fibre product of $\pi_+ : Y_+ \to \mathrm{CP}^1$ and $\pi_- : Y_- \to \mathrm{CP}^1$, that is, let $\Sigma := \{(p_+, p_-) \in Y_+ \times Y_- : \pi_+(p_+) = \pi_-(p_-)\}$ with the obvious projection π to CP^1. Then Σ is given by

$$\eta^2 = \lambda(\lambda - R)(\lambda - R^{-1}) \prod_{i=1}^{n} (\lambda - \lambda_i)(\lambda - {\lambda_i}^{-1})(\lambda - \bar{\lambda}_i)(\lambda - \bar{\lambda}_i{}^{-1}),$$

where $R + R^{-1} = r$, $\alpha_i + {\alpha_i}^{-1} = x_i$.

Σ has genus $2n+1$ and possesses the holomorphic involutions

$$\begin{array}{rccc} i_\pm : & \Sigma & \longrightarrow & \Sigma \\ & (\lambda, \eta) & \longmapsto & \left(\dfrac{1}{\lambda}, \dfrac{\pm\eta}{\lambda^{2n+1}}\right). \end{array}$$

The curves $Y_\pm$ are the quotients of Σ by these involutions, with quotient maps

$$q_+(\lambda, \eta) = \left(\lambda + \frac{1}{\lambda}\,, \frac{\eta}{\lambda^{n+1}}\right) = (x, y_+),$$
$$q_-(\lambda, \eta) = \left(\lambda + \frac{1}{\lambda}\,, \frac{(\lambda+1)(\lambda-1)\eta}{\lambda^{n+2}}\right) = (x, y_-).$$

$Y_\pm$ possesses a real structure $\rho_\pm$ characterized by the properties that it covers the involution $x \mapsto \bar{x}$ of CP^1 and fixes the points in ${\pi_\pm}^{-1}[-2, 2]$. These real structures are given by

$$\rho_\pm(x, y_\pm) = \begin{cases} (\bar{x}, \mp\bar{y}_\pm) & \text{for } r > 2, \\ (\bar{x}, \pm\bar{y}_\pm) & \text{for } r < -2. \end{cases} \tag{4-1}$$

The cases $r > 2$ and $r < -2$ are similar, but the sign difference carries through to future computations. For simpler exposition we assume henceforth that $r > 2$. Then the corresponding real structure on Σ is given by

$$\rho(\lambda, \eta) = \left(\frac{1}{\bar{\lambda}}\,, \frac{-\bar{\eta}}{\bar{\lambda}^{2n+1}}\right).$$

The curves $Y_\pm$ described above are those that yield spectral curves, but it is easier to work with more general curves that do not have the real structure $\rho_\pm$ and then identify those we are interested in by the fact that they satisfy a reality condition. We thus consider also curves $Y_\pm = Y_\pm(r, x_1, \dots, x_{2n})$ given by

$$y_+^2 = (x-r)\prod_{i=1}^{2n}(x - x_i) \quad \text{and} \quad y_-^2 = (x-2)(x+2)(x-r)\prod_{i=1}^{2n}(x-x_i),$$

respectively, where we assume that $r \in (2, \infty)$, $x_i \neq \pm 2$ for $i = 1, \dots, 2n$, and that the sets $\{x_1, x_2\}, \dots, \{x_{2n-1}, x_{2n}\}$ are mutually disjoint.

Take $(r, x_1, \dots, x_{2n})$ as described above. Let $\tilde{a}_0, \dots, \tilde{a}_n$ be simple closed curves in $\mathrm{CP}^1 - \{r, 2, -2, x_1, \dots, x_{2n}\}$, and let $\tilde{c}_1$ and $\tilde{c}_{-1}$ be simple closed curves in $\mathrm{CP}^1 - \{r, x_1, \dots, x_{2n}\}$ such that (see Figure 1)

(1) $\tilde{a}_0$ has winding number one around 2 and r and winding number zero around the other branch points of Y_-;

(2) for $i = 1, \dots, n$, $\tilde{a}_i$ has winding number one around x_{2i-1} and x_{2i} and winding number zero around the other branch points of $Y_\pm$;

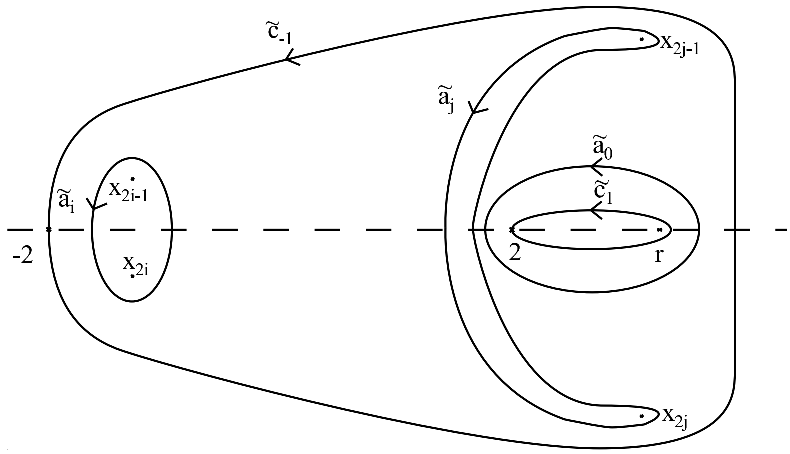

Figure 1. Curves $\tilde{a}_i$, $\tilde{c}_{\pm 1}$ for $p \in M_{n,\mathbb{R}}$.

(3) $\tilde{c}_1$ begins and ends at $x = 2$ and has winding number one around r and zero around x_i for $i = 1, \dots, 2n$;

(4) $\tilde{c}_{-1}$ begins and ends at $x = -2$ and has winding number one around r and each x_i for $i = 1, \dots, 2n$.

Choose lifts of the curves $\tilde{a}_1, \dots, \tilde{a}_n$ to Y_+ and also of $\tilde{a}_0, \dots, \tilde{a}_n$ to Y_-. Let $a_0^-, a_1^\pm, \dots, a_n^\pm \in H_1(Y_\pm, \mathbb{Z})$ denote the homology classes of these lifts. Denote by $b_0^-, b_1^\pm, \dots, b_n^\pm$ the completions to canonical bases of $H_1(Y_\pm, \mathbb{Z})$. Choose open curves c_1, c_{-1} in Y_+ covering the loops $\tilde{c}_1$ and $\tilde{c}_{-1}$.

Denote by M_n the space of $2n+1$-tuples $(r, x_1, \dots, x_{2n})$ as above together with the choices we have described. Let $M_{n,\mathbb{R}}$ denote the subset of M_n such that

(1) $x_{2i} = \bar{x}_{2i-1}$ for $i = 1, \dots, n$;

(2) for $i = 1, \dots, n$, $\tilde{a}_i$ is invariant under conjugation and intersects the real axis exactly twice, both times in the interval $(-2, 2)$;

(3) the lifts of $\tilde{a}_1, \dots, \tilde{a}_n$ to Y_+ are chosen so that the point where $\tilde{a}_i$ intersects the x-axis with positive orientation is lifted to a point where $y_+/\sqrt{-1}$ is negative;

(4) the lifts of $\tilde{a}_0, \dots, \tilde{a}_n$ to Y_- are chosen so that the point where $\tilde{a}_i$ intersects the x-axis with positive orientation is lifted to a point in Y_- where y_- is positive;

(5) c_1 and c_{-1} begin at points with $y_+/\sqrt{-1} < 0$.

For each $p \in M_{n,\mathbb{R}}$, there is a unique canonical basis $A_0, \dots, A_{2n}, B_0, \dots, B_{2n}$ for the homology of Σ such that $A_0, \dots, A_{2n}$ cover the homotopy classes of loops $\tilde{A}_0, \dots, \tilde{A}_{2n}$ shown in Figure 2 and

$$(q_-)_*(A_0) = 2a_0^-, \qquad (q_\pm)_*(A_i) = \mp(q_\pm)_*(A_{n+i}) = a_i^\pm.$$

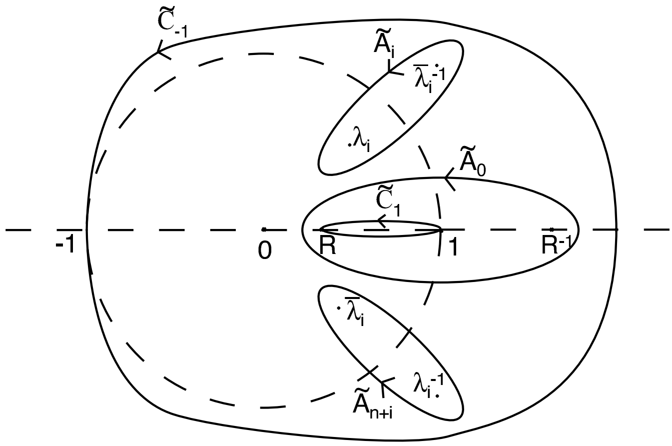

Figure 2. The curves $\tilde{A}_i$ and $\tilde{C}_{\pm 1}$.

There are also unique curves C_1 and C_{-1} on Σ such that $(q_+)_*(C_{\pm 1}) = c_{\pm 1}$; they project to $\tilde{C}_1$ and $\tilde{C}_{-1}$ of Figure 2. Note that C_1 connects the two points of Σ with $\lambda = 1$, while C_{-1} connects the two points of Σ with $\lambda = -1$.

Denote by $\mathscr{A}^\pm$ the subgroup of $H_1(Y_\pm, \mathbb{Z})$ generated by the $a^\pm$ classes. Then modulo $\mathscr{A}^\pm$,

$$(q_-)_*(B_0) \equiv b_0^-, \quad (q_\pm)_*(B_i) \equiv b_i^\pm, \quad (q_\pm)_*(B_{n+i}) \equiv \mp b_i^\pm.$$

Let $p \in M_n$ and define $\Omega_\pm = \Omega_\pm(p)$ on $Y_\pm(p)$ by:

(1) $\Omega_\pm(p)$ are meromorphic differentials of the second kind. Their only singularities are double poles at $x = \infty$, and they have no residues.

(2) $\int_{a_0^-} \Omega_-(p) = 0$ and $\int_{a_i^\pm} \Omega_\pm(p) = 0$ for $i = 1, \ldots, n$.

(3) As $x \to \infty$, $\Omega_+(p) \to x^n dx/y_+(p)$ and $\Omega_-(p) \to x^{n+1} dx/y_-(p)$.

In view of the defining conditions above, we may write

$$\Omega_\pm = \frac{\prod_{j=l_\pm}^{n} (x - \zeta_j^\pm) dx}{y_\pm}, \quad \text{where } l_+ = 1,\ l_- = 0. \tag{4-2}$$

Let

$$\begin{aligned} I_+(p) &:= \sqrt{-1}\Big(\int_{c_1} \Omega_+(p), \int_{c_{-1}} \Omega_+(p), \int_{b_1^+} \Omega_+(p), \ldots, \int_{b_n^+} \Omega_+(p)\Big) \\ I_-(p) &:= \Big(\int_{b_0^-} \Omega_-(p), \int_{b_1^-} \Omega_-(p), \ldots, \int_{b_n^-} \Omega_-(p)\Big). \end{aligned} \tag{4-3}$$

Then $I_+(p)$ and $I_-(p)$ are real for $p \in M_{n,\mathbb{R}}$, since in this case

$$\rho_\pm^*(\Omega_\pm) = \mp\overline{\Omega_\pm}$$

and

$$(\rho_\pm)_*(b_i^\pm) = b_i^\pm \bmod \mathscr{A}^\pm, \qquad (\rho_+)_*(c_{\pm1}) = c_{\pm1} \bmod \mathscr{A}^+.$$

Given $p \in M_{n,\mathbb{R}}$, there are real numbers s_+ and s_- such that $\sqrt{-1}s_+{q_+}^*(\Omega_+(p))$ and $s_-{q_-}^*(\Omega_-(p))$ are differentials on Σ satisfying the conditions of Hitchin's correspondence (Definition 2.1 and Periodicity Conditions 2.3) if and only if $I_+(p)$ and $I_-(p)$ represent rational elements of $\mathbb{R}P^{n+1}$ and $\mathbb{R}P^n$, respectively.

Theorem 4.1. *For each nonnegative integer n, there exists $p \in M_{n,\mathbb{R}}$ such that*

(1) *$\zeta_j^+(p)$ for $j = 1, \dots, n$ are pairwise distinct, as are $\zeta_j^-(p)$ for $j = 0, \dots, n$.*
(2) *The map*
$$\begin{aligned} \phi:\ M_n &\longrightarrow \quad \mathrm{CP}^{n+1} \times \mathrm{CP}^n \\ p &\longmapsto ([I_+(p)],\ [I_-(p)]). \end{aligned}$$
has invertible differential at p.

This gives that the restriction

$$\phi\big|_{M_{n,\mathbb{R}}} : M_{n,\mathbb{R}} \to \mathbb{R}P^{n+1} \times \mathbb{R}P^n$$

of ϕ to $M_{n,\mathbb{R}}$ also has invertible differential at p. The inverse function theorem then implies that for each positive odd integer g, there are countably many spectral curves Σ of genus g each giving rise to a marked torus (T^2, τ) and a branched minimal immersion $f : (T^2, \tau) \to S^3$. The conformal type of the torus is given by [Hitchin 1990] as

$$\tau = \frac{\sqrt{-1}s_+\mathrm{p.p.}_\infty({q_+}^*(\Omega_+(p)))}{s_-\mathrm{p.p.}_\infty({q_-}^*(\Omega_-(p)))} = \frac{\sqrt{-1}s_+}{s_-},$$

where $\mathrm{p.p.}_\infty({q_\pm}^*(\Omega_\pm))$ denotes the principal part of ${q_\pm}^*(\Omega_\pm)$ at ∞. Thus each torus, (T^2, τ) is rectangular.

In fact we shall prove a slightly stronger result. The extra strength resides in a statement that arises from an attempt to prove Theorem 4.1 by induction on n and enables one to complete the induction step. This statement requires some length to formulate and would appear unmotivated at this juncture. Our approach is thus to present an attempt to prove Theorem 4.1 by induction and to derive the necessary modifications. We direct the reader who wishes to see the modified statement now to Theorem 4.6, page 63.

Proof of Theorem 4.1. Suppose then the differential of ϕ is invertible at $p \in M_{n,\mathbb{R}}$. For $\mu \in (-2, 2)$, $\nu \in \mathbb{R}$, we denote by (p, μ, ν) the point of $M_{n+1,\mathbb{R}}$ such that $x_i(p, \mu, \nu) = x_i(p)$ for $i = 1, \dots, 2n$ and additionally $x_{2n+1}(p, \mu, \nu) = \mu + \sqrt{-1}\nu$

and $x_{2n+2}(p,\mu,\nu)=\mu-\sqrt{-1}\nu$. By considering the boundary case $\nu=0$, we show that for a generic $\mu\in(-2,2)$ and ν sufficiently small, ϕ has an invertible differential at (p,μ,ν). We write $p_0'=p_0'(p,\mu)$ for $(p,\mu,0)$. Let

$$H(\mu):=\begin{pmatrix} I_+(p_0') & 0 \\ 0 & I_-(p_0') \\ \frac{\partial}{\partial r}I_+(p_0') & \frac{\partial}{\partial r}I_-(p_0') \\ \frac{\partial}{\partial x_1}I_+(p_0') & \frac{\partial}{\partial x_1}I_-(p_0') \\ \vdots & \vdots \\ \frac{\partial}{\partial x_{2n}}I_+(p_0') & \frac{\partial}{\partial x_{2n}}I_-(p_0') \\ \frac{\partial}{\partial \mu}I_+(p_0') & \frac{\partial}{\partial \mu}I_-(p_0') \\ \frac{\partial^2}{\partial \nu^2}I_+(p_0') & \frac{\partial^2}{\partial \nu^2}I_-(p_0') \end{pmatrix}, \tag{4-4}$$

and

$$h(\mu):=\det H(\mu). \tag{4-5}$$

h is a real-analytic function of $\mu\in(-2,2)$, and for each $\epsilon\in(0,\min_{i=1,\dots,n}|x_i+2|)$, we may use (4-5) to define it as a real-analytic function h_ϵ of μ on the curve L_ϵ shown in Figure 3.

We will show that $h(\mu)\not\to 0$ as $\mu\to\infty$ along L_ϵ by computing asymptotics for each of the vectors in $H(\mu)$. We will also prove that

$$\frac{\partial}{\partial\nu}\big(I_+(p_0'),I_-(p_0')\big)=0.$$

Then for generic $\mu\in(-2,2)$ and ν sufficiently small, $d\phi_{(p,\mu,\nu)}$ is invertible.

A simplification provided by choosing $\nu=0$ is that $Y_\pm(p)$ are the respective normalizations of $Y_\pm(p_0')$, with normalization maps

$$\begin{aligned}\Psi_\pm:\quad Y_\pm(p) &\longrightarrow Y_\pm(p_0')\\ (x,y_\pm(p)) &\longmapsto (x,(x-\mu)y_\pm(p)),\end{aligned}$$

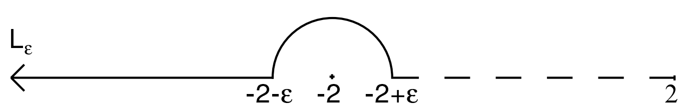

Figure 3. h_ϵ is a function of $\mu\in L_\epsilon$.

and that

$$I_\pm(p_0') = \Big(I_\pm(p), \sqrt{\mp 1}\int_{b^\pm_{n+1}} \Omega_\pm(p_0')\Big). \tag{4-6}$$

For each point $q \in M_n$, let $u_\pm(q)$ be local coordinates on $Y_\pm(q)$ near ${\pi_\pm}^{-1}(\infty)$ such that $u_\pm(q)^2 = {x_\pm}^{-1}$ and $y_\pm(q) = u_\pm(q)^{2m_\pm+1} + O(u_+(q)^{2m_\pm})$ as $x_\pm \to \infty$, where $m_+ = n$ and $m_- = n+1$. Then, for x near ∞,

$$\Omega_\pm(q) = (u_\pm(q) + D_\pm(q)u_\pm(q)^3 + O(u_\pm(q)^5))dx, \tag{4-7}$$

where

$$D_\pm(q) := \frac{1}{2}r(q) + \sum_{i=1}^{2n} x_i(q) - \sum_{j=l_\pm}^{n} \zeta_j^\pm(q), \quad l_+ = 1, l_- = 0. \tag{4-8}$$

Lemma 4.2. *As $\mu \to \infty$ along L_ϵ, these asymptotic expressions hold*:

(1) $I_\pm(p_0') = \big(I_\pm(p), 4\sqrt{\mp 1}\mu^{1/2} - 4\sqrt{\mp 1}D_\pm(p)\mu^{-1/2} + O(\mu^{-3/2})\big)$.

(2) $\dfrac{\partial}{\partial r} I_\pm(p_0') = \Big(\dfrac{\partial}{\partial r} I_\pm(p), \sqrt{\mp 1}\Big(-2 + \sum_{j=l_\pm}^{n} \dfrac{\partial \zeta_j^\pm}{\partial r}\Big)\mu^{-1/2} + O(\mu^{-3/2})\Big)$.

(3) *For $i = 1, \dots, 2n$,*
$$\frac{\partial}{\partial x_i} I_\pm(p_0') = \Big(\frac{\partial}{\partial x_i} I_\pm(p), \sqrt{\mp 1}\Big(-2 + \sum_{j=l_\pm}^{n} \frac{\partial \zeta_j^\pm}{\partial x_i}\Big)\mu^{-1/2} + O(\mu^{-3/2})\Big).$$

(4) $\dfrac{\partial}{\partial \mu} I_\pm(p_0') = \big(0, 2\sqrt{\mp 1}\mu^{-1/2} + 2\sqrt{\mp 1}D_\pm(p)\mu^{-3/2} + O(\mu^{-5/2})\big)$.

(5) $\dfrac{\partial}{\partial \nu} I_\pm(p_0') = 0$.

Proof. In (1)–(4), all but the last component are applications of Equation (4-6). The second components involve integrals over the curves ${b^\pm}_{n+1}(p_0')$. Let Γ denote the circle $|x| = \mu$, traversed clockwise. For μ sufficiently large,

$$\int_{b^+_{n+1}(p_0')} \Omega_+(p_0') = -\int_\Gamma \frac{\prod_{j=1}^n (x - \zeta_j^+)dx}{\sqrt{(x-r)\prod_{i=1}^n (x - x_i)(x - \bar{x}_i)}} \quad \text{(see Figure 4)}$$
$$= 4\mu^{1/2} - 4D_\pm(p)\mu^{-1/2} + O(\mu^{-3/2});$$

a similar result holds for $\int_{b^-_{n+1}(p_0')} \Omega_-(p_0')$, which gives the remainder of (1)–(4).

To prove (5), we write $\dot{f}$ for $\partial f/\partial\nu\big|_{\nu=0}$. We will work only with $\Omega_+(p_0')$, but similar arguments apply to $\Omega_-(p_0')$. For $i = 1, \dots, n+1$,

$$\int_{a_i^+} \dot{\Omega}_+(p_0') = 0. \tag{4-9}$$

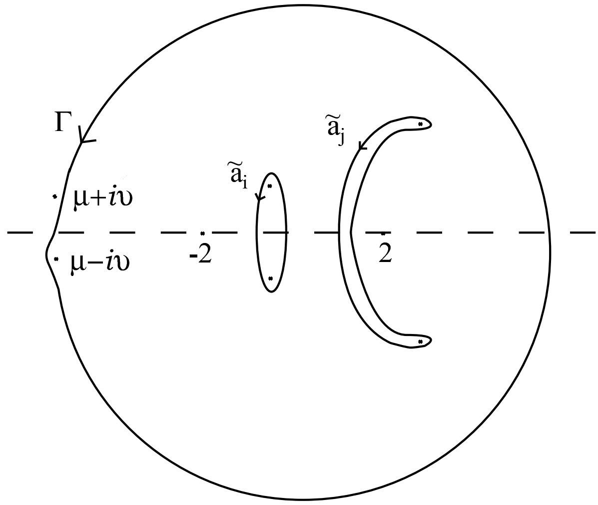

Figure 4. We can take a representative of $b^+_{n+1}(p'_0)$ that projects to a circle.

For small ν, we may write

$$\Omega_+(p,\mu,\nu) = \frac{\prod_{j=1}^{n+1}(x-\zeta_j^+(p,\mu,\nu))dx}{y_+(p,\mu,\nu)},$$

where ζ_j^+ are analytic functions satisfying

$$\zeta_j^+(p'_0) = \begin{cases} \zeta_j(p) & \text{for } j = 1,\dots,n, \\ \mu & \text{for } j = n+1. \end{cases}$$

(Due to this accordance, we write simply $\zeta_j^\pm$ for $\zeta_j^\pm(p)$ or $\zeta_j^\pm(p'_0)$.) Then

$$\dot{\Omega}_+(p'_0) = \sum_{j=1}^{n+1}\Big(\frac{-\dot{\zeta}_j^+}{x-\zeta_j^+}\Big)\frac{\prod_{k=1}^{n+1}(x-\zeta_k)dx}{y_+}.$$

If μ coincides with one of the ζ_j for $j = 1,\dots,n$, then clearly the lift of $\dot{\Omega}_+$ to the normalization $Y_+(p)$ of $Y_+(p'_0)$ is holomorphic. If μ does not equal any of the ζ_j^+, then from (4-9), $\dot{\Omega}_+(p'_0)$ has zero residue at $x = \mu$, and hence $\dot{\zeta}_{n+1} = 0$. Then again $\dot{\Omega}_+(p'_0)$ lifts to a holomorphic differential. This lift has zero a-periods and so is itself zero. For $i = 1,\dots,n$ then,

$$\frac{\partial}{\partial\nu}\Big|_{\nu=0}\int_{b_i^+}\Omega_+(p'_0) = \int_{b_i^+}\dot{\Omega}_+(p'_0) = 0.$$

To compute $\partial/\partial\nu\big|_{\nu=0}\int_{b_{n+1}^+}\Omega_+(p_0')$ we use reciprocity with the holomorphic differential $\omega(p,\mu,\nu)$ on $Y_+(p,\mu,\nu)$ defined by

$$\int_{a_i^+}\omega(p,\mu,\nu)=\begin{cases}0 & \text{for } i=1,\dots,n,\\ 2\pi\sqrt{-1} & \text{for } i=n+1.\end{cases}$$

To simplify our notation, we write p' for (p,μ,ν). Reciprocity gives that

$$\int_{b_{n+1}^+}\Omega_+(p')=4\kappa(p'), \tag{4-10}$$

where

$$\omega(p')=\frac{\kappa(p')\prod_{j=1}^n(x-\beta_j(p'))dx}{y_+(p')}. \tag{4-11}$$

Since

$$\int_{a_i^+}\dot\omega=\frac{\partial}{\partial\nu}\Big|_{\nu=0}\int_{a_i^+}\omega(p')=0 \quad \text{for } i=1,\dots,n+1,$$

we have $\dot\omega=0$. Now

$$\dot\omega=\left(\dot\kappa-\kappa(p_0')\sum_{j=1}^n\frac{\dot\beta_j}{x-\beta_j(p_0')}\right)\frac{\prod_{j=1}^n(x-\beta_j^+(p_0'))dx}{y_+(p_0')},$$

so $\dot\kappa=0$ (and $\dot\beta_j=0$ for $j=1,\dots,n$), and thus by (4-10),

$$\frac{\partial}{\partial\nu}\bigg|_{\nu=0}\int_{b_{n+1}^+}\Omega_+=0. \qquad \square$$

Due to the last equation, we can now compute $\partial^2/\partial\nu^2\big|_{\nu=0}\int_{b_{n+1}^+}\Omega_+$.

Using $\dot\Omega_+(p_0')=0$, a calculation shows that

$$\ddot\Omega_+(p_0')=-\left(\sum_{j=1}^n\frac{\ddot\zeta_j^+}{x-\zeta_j^+}-\frac{\ddot\zeta_{n+1}^+}{x-\mu}-\frac{1}{(x-\mu)^2}\right)\Omega_+(p_0'). \tag{4-12}$$

$\ddot\Omega_+(p_0')$ has zero residue at $x=\mu$. Write $\Omega_+(p_0')=k(x)dx$; then

$$\ddot\zeta_{n+1}^+k(\mu)=-\frac{dk}{dx}(\mu),$$

and hence using Equations (4-7) and (4-8),

$$\ddot\zeta_{n+1}^+=\frac{1}{2}\mu^{-1}+D_+(p)\mu^{-2}+O(\mu^{-3}). \tag{4-13}$$

Represent the homology classes a_i^+ and b_i^+ by loops that project to a fixed compact region $K \subset \mathrm{CP}^1$ that is independent of μ. For μ sufficiently large and $x \in K$,

$$(x-\mu)^{-1} = -\mu^{-1} - x\mu^{-2} + O(\mu^{-3}), \tag{4-14}$$

$$(x-\mu)^{-2} = \mu^{-2} + x\mu^{-3} + O(\mu^{-4}). \tag{4-15}$$

Substituting these, together with (4-13), into (4-12), we obtain that in K

$$\ddot{\Omega}_+(p_0') = \Big(-\sum_{j=1}^{n} \frac{\ddot{\zeta}_j^+}{x-\zeta_j^+} - \frac{1}{2\mu^2} + \frac{2D_+(p)-3x}{2\mu^3} + O\Big(\frac{1}{\mu^4}\Big)\Big)\Omega_+(p_0'). \tag{4-16}$$

Note that both $\Omega_+(p_0')$ and $\ddot{\Omega}_+(p_0')$ have trivial a-periods, so

$$\frac{3}{2}\int_{a_i(p)} x\Omega_+(p) + \ddot{\zeta}_j^+\mu^3 \sum_{j=1}^{n} \int_{a_i^+(p)} \frac{\Omega_+(p)}{x-\zeta_j^+} = O\Big(\frac{1}{\mu}\Big) \quad \text{for } i = 1, \dots, n.$$

Motivated by this, we define $c_j^\pm(p)$ by the equations

$$\frac{3}{2}\int_{a_i^\pm} x\Omega_\pm(p) + \sum_{j=l_\pm}^{n} c_j^\pm(p) \int_{a_i^\pm} \frac{\Omega_\pm(p)}{x-\zeta_j^\pm} = 0 \quad \text{for } i = l_\pm, \dots, n, \tag{4-17}$$

where $l_+ = 1$ and $l_- = 0$. We let

$$\widehat{\Omega}_\pm\,(p) := \frac{3}{2}x\Omega_\pm(p) + \sum_{j=l_\pm}^{n} c_j^\pm(p)\frac{\Omega_\pm(p)}{x-\zeta_j^\pm}. \tag{4-18}$$

We assumed the $\zeta_j^\pm$ are pairwise distinct; thus the differentials $\Omega_\pm(p)/(x-\zeta_j^\pm)$ form a basis for the holomorphic differentials on $Y_\pm(p)$, and $c_j^\pm(p)$ are well-defined.

Notice that $\ddot{\zeta}_j^+ = c_j^+(p)\mu^{-3} + O(\mu^{-4})$. From this and (4-16), we have

$$\Psi_+^*\ddot{\Omega}_+ = -\tfrac{1}{2}\mu^{-2}\Omega_+ + \mu^{-3}\big(D_+(p)\Omega_+ + \widehat{\Omega}_+\big) + O(\mu^{-4}). \tag{4-19}$$

To compute $\partial^2/\partial\nu^2\big|_{\nu=0}\int_{b_{n+1}^+}\Omega_+$, we again use reciprocity with $\omega(p_0')$. Differentiating (4-11), we get

$$\ddot{\omega} = \Big(\frac{\ddot{\kappa}}{\kappa(p_0')} - \frac{1}{(x-\mu)^2} - \sum_{j=1}^{n}\frac{\ddot{\beta}_j}{x-\beta_j(p_0')}\Big)\omega(p_0'). \tag{4-20}$$

Taking residues at $x=\mu$, and using $\mathrm{res}_{x=\mu}\,\omega(p_0') = 1$ and $\mathrm{res}_{x=\mu}\,\ddot{\omega} = 0$, this gives

$$\frac{\ddot{\kappa}}{\kappa(p_0')} - \sum_{j=1}^{n}\frac{\ddot{\beta}_j}{x-\beta_j(p_0')} = \mathrm{res}_{x=\mu}\frac{\omega(p_0')}{(x-\mu)^2}. \tag{4-21}$$

Thus to obtain an asymptotic expression for $\ddot{\kappa}$, we must first compute expressions for $\kappa(p_0')$, $\beta_j(p_0')$ and $\ddot{\beta}_j$. We assume throughout that $x \in K$. We will write $\omega(p_0')$ for $\Psi_+{}^*(\omega(p_0'))$. From (4-11) and (4-14),

$$\int_{a_i^+} \frac{\prod_{j=1}^n (x-\beta_j(p_0'))dx}{y_+(p)} = O\Big(\frac{1}{\mu}\Big).$$

The integrand (with dx) is moreover a differential of the second kind on $Y_+(p)$ whose only singularity is a double pole at $x=\infty$, and it approaches $x^n/y_+(p)$ as $x \to \infty$. Hence

$$\begin{aligned} \frac{\prod_{j=1}^n (x-\beta_j(p_0'))dx}{y_+(p)} + O\Big(\frac{1}{\mu}\Big) &= \Omega_+(p), \\ \beta_j(p_0') = \zeta_j + O\Big(\frac{1}{\mu}\Big) &\quad \text{for } j=1,\dots,n, \\ \kappa(p_0') = \mu^{1/2} - D_+(p)\mu^{-1/2} &+ O(\mu^{-3/2}), \end{aligned} \tag{4-22}$$

and therefore

$$\omega(p_0') = \mu^{-1/2}\Omega_+(p) + O(\mu^{-3/2}). \tag{4-23}$$

From (4-20), (4-23) and (4-14) we obtain

$$\sum_{j=1}^n \ddot{\beta}_j \int_{a_i^+} \frac{\omega(p_0')}{x-\beta_j(p_0')} = \frac{2}{\mu^{7/2}} \int_{a_i^+} x\Omega_+(p_0') + O\Big(\frac{1}{\mu^{9/2}}\Big) \quad \text{for } i=1,\dots n.$$

For each i, $\int_{a_i^+} x\Omega_+(p_0')$ is independent of μ, and so

$$\sum_{j=1}^n \ddot{\beta}_j \mu^{1/2} \int_{a_i^+} \frac{\omega(p_0')}{x-\beta_j(p_0')} = O\Big(\frac{1}{\mu^3}\Big). \tag{4-24}$$

But by (4-22) and (4-23),

$$\Big(\mu^{1/2}\int_{a_i^+} \frac{\omega(p_0')}{x-\beta_j(p_0')}\Big)_j^i = \Big(\int_{a_i^+} \frac{\Omega_+(p)}{x-\zeta_j}\Big)_j^i + O\Big(\frac{1}{\mu}\Big),$$

and the matrix on the right is invertible and independent of μ. Thus

$$\ddot{\beta}_j(p_0') = O\left(\mu^{-3}\right) \quad \text{for } j=1,\dots,n.$$

We now have the asymptotics for $\kappa(p_0')$, $\beta_j(p_0')$ and $\ddot{\beta}_j$ that we desired earlier; substituting them into (4-21) gives

$$\ddot{\kappa} = \tfrac{3}{8}\mu^{-3/2} + \tfrac{9}{8}D_+(p)\mu^{-5/2} + O\left(\mu^{-7/2}\right). \tag{4-25}$$

Set

$$\widehat{I}_+(p) := \sqrt{-1}\bigg(\int_{c_1}\widehat{\Omega}_+(p), \int_{c_{-1}}\widehat{\Omega}_+(p), \int_{b_1^+}\widehat{\Omega}_+(p), \dots, \int_{b_n^+}\widehat{\Omega}_+(p)\bigg), \tag{4-26}$$

$$\widehat{I}_-(p) := \bigg(\int_{b_0^-}\widehat{\Omega}_-(p), \int_{b_1^-}\widehat{\Omega}_-(p), \dots, \int_{b_n^-}\widehat{\Omega}_-(p)\bigg). \tag{4-27}$$

Using (4-10), (4-25) and (4-19), we then obtain:

Lemma 4.3.

$$\frac{\partial^2}{\partial \nu^2} I_\pm(p_0') = \Big(\big(-\frac{1}{2}\mu^{-2} + D_\pm(p)\mu^{-3}\big)I_\pm(p) - \mu^{-3}\widehat{I}_\pm(p) + O(\mu^{-4}),$$
$$\frac{3\sqrt{\mp 1}}{2}\mu^{-3/2} + \frac{9\sqrt{\mp 1}}{2}D_\pm(p)\mu^{-5/2} + O(\mu^{-7/2})\Big)$$

We now have asymptotic expressions for each row of the $(2n+5)\times(2n+5)$ matrix $H(\mu)$ in (4-4), which we wish to show is nonsingular in the limit as $\mu\to\infty$ along L_ϵ of Figure 3. The inductive assumption and Lemmata 4.2, 4.3 tell us that columns $1,\dots,n+2, n+4,\dots,2n+4$ of the first $2n+3$ rows of $H(\mu)$ are linearly independent and that its $(2n+4)$-th row is

$$\Big(\frac{\partial}{\partial\mu}I_+(p_0'); \frac{\partial}{\partial\mu}I_-(p_0')\Big) = \Big(\underbrace{0,\dots,0}_{n+2 \text{ zeros}}, 2\sqrt{-1}(\mu^{-1/2} + D_+(p)\mu^{-3/2}) + O(\mu^{-5/2});$$
$$\underbrace{0,\dots,0}_{n+1 \text{ zeros}}, 2(\mu^{-1/2} + D_-(p)\mu^{-3/2}) + O(\mu^{-5/2})\Big).$$

Note that the two nonzero entries in this row have leading terms differing only by the factor $\sqrt{-1}$. We find a linear combination of the rows of $H(\mu)$ that equals

$$\underbrace{0,\dots,0}_{n+2 \text{ zeros}}, \sqrt{-1}\mu^{-5/2}(4\delta^+(p) - 5D_+(p)) + O(\mu^{-7/2});$$
$$\underbrace{0,\dots,0}_{n+1 \text{ zeros}}, \mu^{-5/2}(4\delta^-(p) - 5D_-(p)) + O(\mu^{-7/2})),$$

where $\delta^\pm(p)$ are defined in (4-29).

Lemma 4.4. *The matrix $H(\mu)$ is nonsingular if*

$$\lim_{\mu\to\infty} 4\delta^+(p) - 5D_+(p) \neq \lim_{\mu\to\infty} 4\delta^-(p) - 5D_-(p),$$

where the limits are taken along L_ϵ.

Proof. This is the extra condition referred to earlier, and we will modify the statement we prove by induction to ensure that it is satisfied. First, we find the linear combination yielding this condition.

From Lemmata 4.2 and 4.3, we have

$$\text{(4-28)}\quad \Big(\frac{\partial^2}{\partial\nu^2}I_+(p_0'); \frac{\partial^2}{\partial\nu^2}I_-(p_0')\Big) =$$
$$\Big(\big(D_+(p)\mu^{-3}-\tfrac{1}{2}\mu^{-2}\big)I_+(p)-\mu^{-3}\widehat{I}_+(p)+o_{-4}, \tfrac{3\sqrt{-1}}{2}\big(\mu^{-3/2}+3D_+\mu^{-5/2}+o_{-7/2}\big);$$
$$(D_-(p)\mu^{-3}-\tfrac{1}{2}\mu^{-2})I_-(p)-\mu^{-3}\widehat{I}_-(p)+o_{-4}, \tfrac{3}{2}\mu^{-3/2}+\tfrac{9}{2}D_-\mu^{-5/2}+o_{-7/2}\Big),$$

where we have put $o_n = O(\mu^n)$. By the induction hypothesis, there are unique $\delta^{\pm}(p)$, $\chi(p)$, and $\xi_i(p)$ for $i = 1, \dots 2n$ such that

$$\text{(4-29)}\quad (\widehat{I}_+(p); \widehat{I}_-(p)) = \delta^+(p)(I_+(p); 0) + \delta^-(p)(0; I_-(p))$$
$$+\chi(p)\frac{\partial}{\partial r}(I_+(p); I_-(p)) + \sum_{i=1}^{2n}\xi_i(p)\frac{\partial}{\partial x_i}(I_+(p); I_-(p)).$$

Thus by (4-28) and Lemmata 4.2, 4.3, we know there are $\tilde{\delta}^{\pm}(p) = \delta^{\pm}(p)+O(\mu^{-1})$, $\tilde{\chi}(p) = \chi(p) + O(\mu^{-1})$, and $\tilde{\xi}_i(p) = \xi_i(p) + O(\mu^{-1})$ satisfying

$$\text{(4-30)}\quad \Big(\frac{\partial^2}{\partial\nu^2}I_+(p_0'); \frac{\partial^2}{\partial\nu^2}I_-(p_0')\Big) + \big(\tfrac{1}{2}\mu^{-2} + (\tilde{\delta}^+(p) - D_+(p))\mu^{-3}\big)\,\big(I_+(p_0'); 0\big)$$
$$+\big(\tfrac{1}{2}\mu^{-2} + (\tilde{\delta}_-(p) - D_+(p))\mu^{-3}\big)\,\big(0; I_-(p_0')\big)$$
$$+\Big(\tilde{\chi}(p)\mu^{-3}\frac{\partial}{\partial r} + \sum_{i=1}^{2n}\tilde{\xi}_i(p)\mu^{-3}\frac{\partial}{\partial x_i}\Big)\,(I_+(p); I_-(p))$$
$$= (\underbrace{0, \dots, 0}_{n+2 \text{ zeros}}, g_+(p_0'); \underbrace{0, \dots, 0}_{n+1 \text{ zeros}}, g_-(p_0')),$$

where

$$g_{\pm}(\mu) = \sqrt{\mp 1}\big(\tfrac{7}{2}\mu^{-3/2} - \tfrac{3}{2}D_{\pm}(p)\mu^{-5/2} + 4\delta^{\pm}(p)\mu^{-5/2} + O(\mu^{-7/2})\big)$$

Denote by $\tilde{l}(p_0')$ the linear combination appearing in (4-30). Define

$$l(p_0') := \tilde{l}(p_0') - \tfrac{7}{4}\mu^{-1}\frac{\partial}{\partial\mu}(I_+(p_0'), I_-(p_0'));$$

then

$$\text{(4-31)}\quad l(p_0') = \big(\underbrace{0, \dots, 0}_{n+2 \text{ zeros}}, \sqrt{-1}(4\delta^+(p) - 5D_+(p))\mu^{-5/2} + O(\mu^{-7/2});$$
$$\underbrace{0, \dots, 0}_{n+1 \text{ zeros}}, (4\delta^-(p) - 5D_-(p))\mu^{-5/2} + O(\mu^{-7/2})\big). \quad \square$$

We are therefore led to modify the statement we prove by induction to include the assumption that

$$\lim_{\mu\to\infty} 4\delta^+(p) - 5D_+(p) \neq \lim_{\mu\to\infty} 4\delta^-(p) - 5D_-(p),$$

where the limits are taken along L_ϵ. Of course this modification needs to be such that it is preserved under the induction step. With this in mind, we define $\delta^\pm(p_0')$ by the condition that

$$(4\text{-}32)\quad (\widehat{I}_+(p_0'); \widehat{I}_-(p_0')) - \delta^+(p_0')(I_+(p_0'); 0) + \delta^-(p_0')(0; I_-(p_0')) \in \operatorname{span}\left\{\frac{\partial}{\partial r}I_{+-}, \frac{\partial}{\partial x_i}I_{+-}, \frac{\partial}{\partial \mu}I_{+-}, \frac{\partial^2}{\partial \nu^2}I_{+-}\right\},$$

where $I_{+-} = (I_+(p_0'); I_-(p_0'))$, and we then calculate the relationship between $\delta^+(p_0') - \delta^-(p_0')$ and $\delta^+(p) - \delta^-(p)$.

Lemma 4.5. *As $\mu \to \infty$ along L_ϵ,*

$$\widehat{I}_\pm(p_0') = \left(\widehat{I}_\pm(p), \sqrt{\mp 1}(2\mu^{3/2} + (6D_\pm(p) - 4\delta^\pm(p))\mu^{1/2} + O(\mu^{-1/2}))\right)$$

Proof. For μ sufficiently large, we may assume that $\mu \neq \zeta_j^\pm$ for $j = 1, \dots, n$. Then (again arguments for Ω_- are similar to those for Ω_+)

$$\widehat{\Omega}_+(p_0') := \tfrac{3}{2}x\Omega_+(p_0') + \sum_{j=1}^{n+1} \frac{c_j^+(p_0')\Omega_+(p_0')}{x - \zeta_j^+},$$

where the $c_j^+(p_0')$ for $j = 1, \dots, n+1$ are determined by the (nonsingular) system of equations

$$\frac{3}{2}\int_{a_i^\pm} x\Omega_\pm(p_0') + \sum_{j=1}^{n+1} c_j^\pm(p_0') \int_{a_i^\pm} \frac{\Omega_\pm(p_0')}{x - \zeta_j^\pm} = 0 \quad \text{for } i = 1, \dots, n+1.$$

Taking $i = n+1$, we see $c_{n+1}^+(p_0') = 0$ and $c_j^+(p_0') = c_j^+(p)$ for $j = 1, \dots, n$, so

$$\Psi_+{}^*(\widehat{\Omega}_+(p_0')) = \widehat{\Omega}_+(p),$$

which proves all but the last part of the lemma. For this, we again let Γ denote the circle $|x| = \mu$, traversed clockwise. For μ sufficiently large,

$$\begin{aligned}\int_{b_{n+1}^+} \widehat{\Omega}_+(p_0') &= -\int_\Gamma \left(\frac{3}{2}x + \sum_{j=1}^n \frac{c_j^+(p)}{x - \zeta_j^+}\right) \frac{\prod_{k=1}^n (x - \zeta_k^+)dx}{\sqrt{(x-r)\prod_{i=1}^{2n}(x - x_i)}} \\ &= 2\sqrt{-1}\mu^{3/2} + (6\sqrt{-1}D_+(p) - 4\sqrt{-1}\delta^+(p))\mu^{1/2} + O(\mu^{-1/2}).\end{aligned}$$

A similar result holds for $\int_{b_{n+1}^-(p_0')} \widehat{\Omega}_-(p_0')$. $\square$

From Lemmata 4.2, 4.3, and 4.5,

$$\begin{aligned}(4\text{-}33)\quad & (\widehat{I}_+(p_0'); \widehat{I}_-(p_0')) - \delta^+(p)(I_+(p_0'); 0) - \delta^-(p)(0; I_-(p_0')) \\ & \qquad - \Big(\chi(p)\frac{\partial}{\partial r} + \sum_{i=1}^{2n} \xi_i(p)\frac{\partial}{\partial x_i}\Big)(I_+(p_0'); I_-(p_0')) \\ &= \big(0, 2\sqrt{-1}\mu^{3/2} + (6\sqrt{-1}D_+(p) - 4\sqrt{-1}\delta^+(p))\mu^{1/2} + O(\mu^{-1/2}); \\ & \qquad 0, 2\mu^{3/2} + (6D_-(p) - 4\delta^-(p))\mu^{1/2} + O(\mu^{-1/2})\big) \\ &= \Lambda\frac{\partial}{\partial\mu}(I_+(p_0'); I_-(p_0')) + \Upsilon l(p_0'),\end{aligned}$$

where $l(p_0')$ is defined in (4-31) and Λ and Υ are defined by the equations

$$\begin{pmatrix} \frac{2\sqrt{-1}}{\mu^{1/2}}\Big(1+\frac{D_+(p)}{\mu}\Big) + O(\mu^{-5/2}) & \frac{\sqrt{-1}(4\delta^+(p)-5D_+(p))}{\mu^{5/2}} + O(\mu^{-7/2}) \\ \frac{2}{\mu^{1/2}}\Big(1+\frac{D_-(p)}{\mu}\Big) + O(\mu^{-5/2}) & \frac{4\delta^-(p)-5D_-(p)}{\mu^{5/2}} + O(\mu^{-7/2}) \end{pmatrix}\begin{pmatrix}\Lambda \\ \Upsilon\end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{-1}\mu^{3/2} + (6\sqrt{-1}D_+(p) - 4\sqrt{-1}\delta^+(p))\mu^{1/2} + O(\mu^{-1/2}) \\ 2\mu^{3/2} + (6D_-(p) - 4\delta^-(p))\mu^{1/2} + O(\mu^{-1/2}) \end{pmatrix}.$$

Hence using (4-31) and (4-33),

$$\delta^+(p_0') - \delta^-(p_0') = \frac{(D_+(p) - D_-(p))\big(3(\delta^+(p) - \delta_-(p)) - 4(D_+(p) - D_-(p))\big)}{4(\delta^+(p) - \delta_-(p)) - 5(D_+(p) - D_-(p))}.$$

Defining T_p to be the linear fractional transformation

$$T_p : x \mapsto (D_+(p) - D_-(p))\frac{3x - 4(D_+(p) - D_-(p))}{4x - 5(D_+(p) - D_-(p))},$$

we have

$$T_p(\delta^+(p) - \delta^-(p)) = \delta^+(p_0') - \delta^-(p_0').$$

Moreover, we know that $D_\pm(p_0') = D_\pm(p)$ and thus $T_{p_0'} = T_p$. To conclude the proof of Theorem 4.1, it suffices then to show:

Theorem 4.6. *For each positive integer m and integer n with $0 \le n \le m$ there exists $p \in M_{n,\mathbb{R}}$ such that*

(1) *$\zeta_j^+(p)$ for $j = 1, \dots, n$ are pairwise distinct, as are $\zeta_j^-(p)$ for $j = 0, \dots, n$;*

(2) *$\mathbb{R}^{2n+3}$ is spanned by the vectors*

$$(I_+(p), 0),\ (0, I_-(p)),\ \frac{\partial}{\partial r}(I_+(p), I_-(p)),\ \frac{\partial}{\partial x_i}(I_+(p), I_-(p))$$

for $i = 1, \dots, 2n$;

(3) *$5(D_+(p) - D_-(p)) + 4T_p^k(\delta_+(p) - \delta_-(p)) \ne 0$ for $0 \le k \le m - n$.*

$M_{n,\mathbb{R}}$ is defined on page 51, $\zeta^{\pm}$, I_+, I_-, $D_{\pm}$ are defined in Equations (4-2), (4-3), (4-8) and $\delta^{\pm}$ is given by Equations (4-17), (4-18), (4-26), (4-27) and (4-32).

Proof. Fix m, and for $n < m$ suppose $p \in M_{n,\mathbb{R}}$ satisfies the conditions of Theorem 4.6. By the arguments above, the set of $\mu \in (-2, 2)$ such that

(i) $h_\epsilon(\mu) \neq 0$ for all $\epsilon \in (0, \min_{i=1,\dots,n} |x_i + 2|)$ (see (4-5)),

(ii) $\mu \neq \zeta_j^+$ for $j = 1, \dots n$, and

(iii) $\mu \neq \zeta_j^-$ for $j = 0, \dots n$

is dense in $(-2, 2)$.

Take such a μ. Then $p_0' = (p, \mu, 0)$ satisfies Theorem 4.6, where in (ii) we replace $\partial/\partial x_{2n+2}(I_+(p_0'); I_-(p_0'))$ by $\partial^2/\partial \nu^2(I_+(p_0'); I_-(p_0'))$. Then for ν small,

$$\delta^{\pm}(p, \mu, \nu) = \delta^{\pm}(p_0') + O(\nu), \quad D_{\pm}(p, \mu, \nu) = D_{\pm}(p_0') + O(\nu),$$

and $T_{(p,\mu,\nu)} = T_{p_0'} + O(\nu)$; we conclude that (p, μ, ν) satisfies Theorem 4.6. It remains to show the existence of $p \in M_{0,\mathbb{R}}$ verifying (i) and (ii) of Theorem 4.6 and such that for no $k \geq 0$ do we have $5\,(D_+(p) - D_-(p)) + 4T_p^k\,(\delta_+(p) - \delta_-(p)) = 0$.
Genus one ($n = 0$). We consider pairs $Y_+ = Y_+(r)$ and $Y_- = Y_-(r)$ given by

$$y_+^2 = (x - r) \quad \text{and} \quad y_-^2 = (x + 2)(x - 2)(x - r),$$

respectively, where $r > 2$. Writing $\pi_{\pm} : (x, y_{\pm}) \mapsto x$ for the projections to CP^1, the fibre product of these is the genus one curve $\Sigma = \Sigma(r)$, given by

$$\eta^2 = \lambda(\lambda - R)\Big(\lambda - \frac{1}{R}\Big), \quad \text{where } R + \frac{1}{R} = r.$$

Lemma 4.7. *There exists a $p \in M_{0,\mathbb{R}}$ such that*

(1) *$\mathbb{R}^3$ is spanned by the vectors $(I_+(p), 0)$, $(0, I_-(p))$ and $\partial/\partial r(I_+(p), I_-(p))$,*

(2) *$5(D_+(p) - D_-(p)) + 4T^k(p)(\delta^+(p) - \delta^-(p)) \neq 0$ for all $k \geq 0$,*

Proof. The natural limit to consider is $R \to 1$, that is, $r = R + 1/R \to 2$, which suggests setting $\zeta := x + 2$, $t := r - 2$. Then $Y_+(t)$ and $Y_-(t)$ are given by

$$y_+^2 := \zeta - t, \quad y_-^2 := \zeta(\zeta - t)(\zeta + 4).$$

For each $t > 0$, choose $c_1(t), c_{-1}(t)$, and $a^-(t)$ as shown in Figure 5. We write $\Omega_-(t) = (\zeta - s(t))d\zeta/y_-$, where $s(t)$ is defined by the condition $\int_{a^-(t)} \Omega_-(t) = 0$. Since

$$\int_{a^-(t)} \Omega_-(t) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\int_{a^-(t)} \Omega_-(t) = 0,$$

we have

$$s(t) = t + O(t^2), \tag{4-34}$$

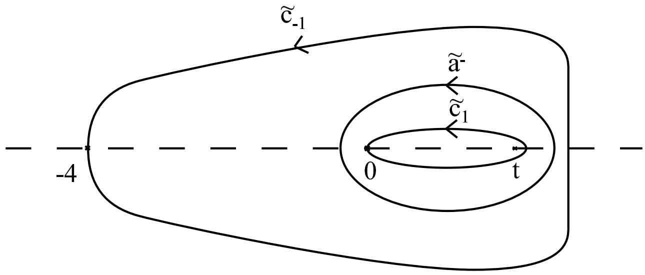

Figure 5. The curves $\tilde{a}^-$ and $\tilde{c}_{\pm 1}$, for $n = 0$.

and so $I_-(t) = 8 + O(t)$. Now

$$I_+(t) = \sqrt{-1}\Big(\int_{c_1(t)} \Omega_+(t),\ \int_{c_{-1}(t)} \Omega_+(t)\Big),$$

where $c_1(t)$ is a path in $Y_+(t)$ joining the two points with $\zeta = 0$, and $c_{-1}(t)$ is one joining the two points with $\zeta = -4$; both begin at points with $y_+/\sqrt{-1} < 0$. Then

$$I_+(t) = (4t^{1/2}, 4(4+t)^{1/2}) \quad \text{and} \quad \frac{\partial I_+(t)}{\partial t} = (2t^{-1/2}, 2(4+t)^{-1/2}),$$

so we see that condition (i) of Lemma 4.7 is satisfied for all $r > 2$.

We have $\lim_{t\to 0} D_\pm(t) = \pm 1$ and we proceed to calculate $\lim_{t\to 0} \delta^\pm(t)$.

$$\widehat{\Omega}_+(t) = \tfrac{3}{2}x\Omega_+(t) = \frac{3(\zeta+2)d\zeta}{2\sqrt{\zeta - t}},$$

so

$$\widehat{I}_+(t) = (12t^{1/2} + 8t^{3/2},\ 12(4+t)^{1/2} + 8(4+t)^{3/2}).$$

Recall that $\widehat{\Omega}_-(t) = 3/2(\zeta+2)\Omega_-(t) + c_-(t)\Omega_-(t)/(\zeta - s(t))$, where $c_-(t)$ is defined by

$$\frac{3}{2}\int_{a^-(t)} \frac{(\zeta+2)(\zeta - s(t))d\zeta}{\sqrt{\zeta(\zeta+4)(\zeta-t)}} + c_-(t)\int_{a^-(t)} \frac{d\zeta}{\sqrt{\zeta(\zeta+4)(\zeta-t)}} = 0. \tag{4-35}$$

Residue calculations show that

$$\int_{a^-(t)} \frac{d\zeta}{\sqrt{\zeta(\zeta+4)(\zeta-t)}} = \pi\sqrt{-1} - \frac{\pi\sqrt{-1}}{16}t + O(t^2),$$
$$\int_{a^-(t)} \frac{\zeta\, d\zeta}{\sqrt{\zeta(\zeta+4)(\zeta-t)}} = \pi\sqrt{-1}t + O(t^2),$$
$$\int_{a^-(t)} \frac{\zeta^2 d\zeta}{\sqrt{\zeta(\zeta+4)(\zeta-t)}} = O(t^2).$$

Substituting these and (4-34) into (4-35), gives $c_-(t) = O(t^2)$. so, using (4-34), we find $\widehat{I_-}(t) = -8 + O(t)$. Thus the matrix

$$\begin{pmatrix} I_+(t) & 0 \\ 0 & I_-(t) \\ \dfrac{\partial I_+(t)}{\partial t} & \dfrac{\partial I_-(t)}{\partial t} \\ \widehat{I_+}(t) & \widehat{I_-}(t) \end{pmatrix} = \begin{pmatrix} 4t^{1/2} & 4(4+t)^{1//2} & 0 \\ 0 & 0 & 8+O(t) \\ 2t^{-1/2} & 2(4+t)^{-1/2} & O(1) \\ 12t^{1/2}+8t^{3/2} & 12(4+t)^{1/2}+8(4+t)^{3/2} & -8+O(t) \end{pmatrix}$$

becomes, upon multiplying its third row by t, its first column by $2t^{-1/2}$, and its second column by $2(4+t)^{-1/2}$,

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 8 \\ 1 & 0 & 0 \\ 6 & 22 & -8 \end{pmatrix} + O(t).$$

Since $-11(2, 2, 0) + 1(0, 0, 8) + 16(1, 0, 0) + 1(6, 22, -8) = (0, 0, 0)$, then, recalling that $\delta^{\pm}(t)$ are defined by the condition

$$(\widehat{I_+}(t), \widehat{I_-}(t)) + \delta^+(t)(I_+(t), 0) + \delta^-(t)(0, I_-(t)) \in \operatorname{span}\left\{\frac{\partial}{\partial t}(I_+(t), I_-(t))\right\},$$

we conclude that

$$\lim_{t\to 0} \delta^+(t) = -11 \quad \text{and} \quad \lim_{t\to 0} \delta^-(t) = 1.$$

The linear fractional transformation T_t is defined by

$$T_t : u \mapsto \frac{-(D_-(t) - D_+(t))(3u + 4(D_-(t) - D_+(t)))}{4u + 5(D_-(t) - D_+(t))},$$

and so letting $T := \lim_{t\to 0} T_t$ gives

$$T : u \to \frac{3u-8}{2u-5}.$$

This has a unique fixed point ($u = 2$) and so is conjugate to a translation. In fact, denoting the map $u \mapsto 1/(u-2)$ by S, we have $STS^{-1} : u \mapsto u - 2$. Now

$$4T^k(\lim_{t\to 0}(\delta^-(t) - \delta^+(t))) = 5(\lim_{t\to 0}(D_+(t) - D_-(t))),$$

which is seen to be equivalent to, successively, $T^k(12) = \frac{5}{2}$, $(STS^{-1})^k(\frac{1}{10}) = 2$, and $\frac{1}{10} - 2k = 2$, and this last condition fails for all integers $k \geq 0$. Thus for $t > 0$ sufficiently small, the lemma holds. □

This concludes the proof of Theorem 4.6, and hence also that of Theorem 4.1. □

Even genera. We now briefly indicate how to prove Theorem 3.1 for even genera. In this case, the quotient curves have the same genus, and the proof is both simpler than in the odd genus case and similar to the proof in [Ercolani et al. 1993]. For these reasons, we content ourselves with describing the appropriate even genus analogue of Theorem 4.1.

One is now interested in spectral curves Σ of the form

$$\eta^2 = \lambda \prod_{i=1}^{n} (\lambda - \lambda_i)(\lambda - \lambda_i{}^{-1})(\lambda - \bar{\lambda}_i)(\lambda - \bar{\lambda}_i{}^{-1}).$$

These possess a real structure

$$\rho : (\lambda, \eta) \mapsto (\bar{\lambda}^{-1}, \bar{\eta}\bar{\lambda}^{-(2n+1)})$$

and holomorphic involutions

$$\begin{array}{rccc} i_{\pm}: & \Sigma & \longrightarrow & \Sigma \\ & (\lambda, \eta) & \longmapsto & \left(\dfrac{1}{\lambda}, \dfrac{\pm\eta}{\lambda^{2n+1}}\right). \end{array}$$

The quotients of Σ by these involutions have the same genus and are given by

$$y_{\pm}^2 = (x \pm 2) \prod_{i=1}^{2n} (x - x_i),$$

with quotient maps

$$q_{\pm}(\lambda, \eta) = \left(\lambda + \frac{1}{\lambda}, \frac{(\lambda \pm 1)\eta}{\lambda^{n+1}}\right) = (x, y_{\pm}).$$

The real structure on Σ induces real structures $\rho_{\pm}(x, y_{\pm}) = (\bar{x}, \pm\bar{y}_{\pm})$ on $Y_{\pm}$. Let A be the subspace of $H_1(\Sigma, \mathbb{Z})$ generated by the A_j. One can define a standard homology basis $A_1, \ldots, A_{2n}, B_1, \ldots, B_{2n}$ for Σ and open curves C_1, C_{-1}, where $C_{\pm 1}$ joins the two points on Σ with $\lambda = \pm 1$, such that

(1) $(q_{\pm})_*(A_i) = \pm(q_{\pm})_*(A_{n+i})$ for $i = 1, \ldots, n$,

(2) $\rho_*(B_i) \cong B_i \bmod \mathcal{A}$ and $\rho_*(C_{\pm}) \cong C_{\pm} \bmod \mathcal{A}$;

see Figure 6.

Set

$$a_i^{\pm} = (q_{\pm})_*(A_i) \quad \text{and} \quad b_i^{\pm} = (q_{\pm})_*(B_i),$$

and define differentials $\Omega_{\pm} = \Omega_{\pm}(p)$ on $Y_{\pm}(p)$ by

(1) $\Omega_{\pm}(p)$ are meromorphic differentials of the second kind: their only singularities are double poles at $x = \infty$, and they have no residues;

(2) $\int_{a_i^{\pm}} \Omega_{\pm}(p) = 0$ for $i = 1, \ldots, n$;

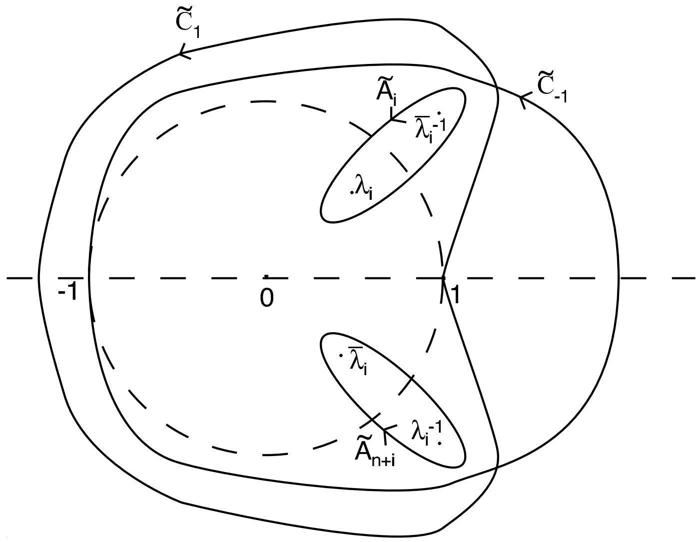

Figure 6. Projections $\tilde{A}_i$ and $\tilde{C}_{\pm 1}$ of A_i and C_{pm1} onto the λ-plane.

(3) as $x \to \infty$, $\Omega_\pm(p) \to x^n dx/y_\pm(p)$.

Then $\rho_\pm^*(\Omega_\pm) = \pm\overline{\Omega_\pm}$, so

$$I_\pm(p) := \sqrt{\pm 1}\Big(\int_{c_{\pm 1}} \Omega_\pm(p), \int_{b_1^\pm} \Omega_\pm(p), \ldots, \int_{b_n^\pm} \Omega_\pm(p)\Big)$$

are real. If they give rational elements of RP^n, then certain real multiples of $\sqrt{\pm 1}(q_\pm)^*(\Omega_\pm)$ will satisfy the Periodicity Conditions 2.3.

Thus the natural analogue of Theorem 4.1 is the statement that for each n, there is a point where the map

$$(x_1, \ldots, x_{2n}) \mapsto ([I_+], [I_-])$$

has invertible differential. This can again be proven by induction, although again additional conditions are required to yield the induction step.

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