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We show that a widely believed conjecture concerning rigidity of genus-zero and genus-one holomorphic curves in Calabi–Yau threefolds implies a relation between the genus-one GW-invariants of a quintic threefold in $\mathbb{P}^4$ and the genus-zero and genus-one GW-invariants of $\mathbb{P}^4$. This relation is a special case of a general formula for the genus-one GW-invariants of complete intersections obtained in a previous paper. In contrast to the general case, this paper's derivation is more geometric and makes direct use of the rigidity property. Thus, it provides further evidence for the rigidity conjecture in low genera. On the other hand, this paper also suggests a potential way of disapproving the less commonly believed generalization of the rigidity conjecture to arbitrary genus.

1. Introduction

Suppose $\gamma \to \mathbb{P}^n$ is the tautological line bundle, $a \in \mathbb{Z}^+$, and

$$\mathfrak{L} = \gamma^{*\otimes a}.$$

If $s \in H^0(\mathbb{P}^n; \mathfrak{L})$ is a generic holomorphic section,

$$Y \equiv s^{-1}(0)$$

is a smooth hypersurface in $\mathbb{P}^n$. It has long been known how to express the genus-zero Gromov–Witten invariants of Y in terms of the genus-zero GW-invariants of $\mathbb{P}^n$; see (1-1) below for a special case. The latter can be computed using the classical localization theorem of [Atiyah and Bott 1984]. In [Li and Zinger 2005], we prove a genus-one analogue of (1-1) for an arbitrary hypersurface Y. The proof itself is rather simple. However, it relies on the constructions of `reduced` genus-one GW-invariants in [Zinger 2005b] and of Euler classes of certain natural cones in a setting more general than in [Zinger 2007]. The latter in fact constitutes most of [Li and Zinger 2005].

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In this paper, we rederive a genus-one analogue of (1-1) for a quintic threefold Y in $\mathbb{P}^4$, i.e. for $a = 5$ in the above notation, in a direct, albeit more laborious, way from a certain rigidity property for genus-zero and genus-one J-holomorphic curves in a quintic threefold; see the next subsection. While it is not known whether the rigidity property is satisfied, it is widely believed to be the case for curves of low genus (at least 0 and 1 and likely up to genus 2 or 3). Our derivation generalizes to arbitrary Calabi–Yau complete-intersection threefolds in projective spaces. It can be used for Fano complete-intersection threefolds as well, but in such cases it can be obtained by taking $\nu = 0$ in Section 2.2 of [Li and Zinger 2005]. In the Calabi–Yau cases, this cannot be done and this paper's derivation is different from the one given there.

Quintic threefolds, as well as other Calabi–Yau manifolds, play a prominent role in theoretical physics. As a result physicists have made a number of important predictions concerning CY-manifolds. Some of these predictions have been verified mathematically; others have not. This paper indicates that one of them fits in nicely with known mathematical facts.

If X is a Kähler manifold, g and k are nonnegative integers, and $A \in H_2(X;\mathbb{Z})$, we denote by $\overline{\mathfrak{M}}_{g,k}(X, A)$ the moduli space of (equivalence classes of) stable holomorphic maps from genus-g curves with k marked points in the homology class A. Let

$$\overline{\mathfrak{M}}_g(X, A) = \overline{\mathfrak{M}}_{g,0}(X, A).$$

If $\iota : Y \to \mathbb{P}^n$ is an inclusion and ℓ is the homology class of a line in $\mathbb{P}^n$, let

$$\overline{\mathfrak{M}}_{g,k}(Y, d) = \bigcup_{\iota_* A = d\ell} \overline{\mathfrak{M}}_{g,k}(Y, A).$$

If Y is a Calabi–Yau threefold, the virtual, or expected, dimension of $\overline{\mathfrak{M}}_g(Y, d)$ is zero. The virtual degree of $\overline{\mathfrak{M}}_g(Y, d)$ is the genus-g degree-d GW-invariant of Y. If Y is a quintic threefold, we denote this invariant by $N_g(d)$.

Let

$$\pi_g^d : \mathfrak{U}_g(\mathbb{P}^n, d) \to \overline{\mathfrak{M}}_g(\mathbb{P}^n, d) \quad \text{and} \quad \mathrm{ev}_g^d : \mathfrak{U}_g(\mathbb{P}^n, d) \to \mathbb{P}^n$$

be the semiuniversal family and the natural evaluation map. In other words, the fiber of π_g^d over $[\mathcal{C}, u]$ is the curve $\mathcal{C}$, while

$$\mathrm{ev}_g^d\big([\mathcal{C}, u; z]\big) = u(z) \quad \text{if } z \in \mathcal{C}.$$

We define a section s_g^d of the sheaf $\pi_{g*}^d \mathrm{ev}_g^{d*}\mathfrak{L} \to \overline{\mathfrak{M}}_g(\mathbb{P}^n, d)$ by

$$s_g^d\big([\mathcal{C}, u]\big) = [s \circ u].$$

If $Y = s^{-1}(0)$, $\overline{\mathfrak{M}}_g(Y, d)$ is the zero set of this section.

If $a=5$, it has long been known that

$$N_0(d)=\big\langle e\big(\pi^d_{0*}\mathrm{ev}_0^{d*}\mathfrak{L}\big),[\overline{\mathfrak{M}}_0(\mathbb{P}^4,d)]\big\rangle. \tag{1-1}$$

The moduli space $\overline{\mathfrak{M}}_0(\mathbb{P}^4,d)$ is a smooth orbivariety and

$$\pi^d_{0*}\mathrm{ev}_0^{d*}\mathfrak{L}\to\overline{\mathfrak{M}}_0(\mathbb{P}^4,d) \tag{1-2}$$

is a locally free sheaf, i.e. a vector bundle. Furthermore,

$$\dim_{\mathbb{C}}\overline{\mathfrak{M}}_0(\mathbb{P}^4,d)=5d+1 \qquad\text{and}\qquad \mathrm{rk}_{\,\mathbb{C}}\,\pi^d_{0*}\mathrm{ev}_0^{d*}\mathfrak{L}=5d+1.$$

Thus, the right-hand side of (1-1) is well-defined. It can be computed via the classical localization theorem of [Atiyah and Bott 1984]. The complexity of this computation increases quickly with the degree d, but a closed formula has been obtained in [Bertram 2000], [Gathmann 2002], [Givental 1999], [Lee 2001], and [Lian et al. 1997].

If $g>0$, the sheaf $\pi^d_{g*}\mathrm{ev}_g^{d*}\mathfrak{L}\to\overline{\mathfrak{M}}_g(\mathbb{P}^4,d)$ is not locally free and does not define an Euler class. Thus, the right-hand side of (1-1) does not even make sense if 0 is replaced by $g>0$. Instead one might try to generalize (1-1) as

$$N_g(d)\overset{?}{=}\big\langle e\big(R^0\pi^d_{g*}\mathrm{ev}_g^{d*}\mathfrak{L}-R^1\pi^d_{g*}\mathrm{ev}_g^{d*}\mathfrak{L}\big),[\overline{\mathfrak{M}}_g(\mathbb{P}^4,d)]^{vir}\big\rangle, \tag{1-3}$$

where $R^i\pi^d_{g*}\mathrm{ev}_g^{d*}\mathfrak{L}\to\overline{\mathfrak{M}}_g(\mathbb{P}^4,d)$ is the i-th direct image sheaf. The right-hand side of (1-3) can be computed via the virtual localization theorem of [Graber and Pandharipande 1999]. However,

$$N_1(d)\neq\big\langle e\big(R^0\pi^d_{1*}\mathrm{ev}_1^{d*}\mathfrak{L}-R^1\pi^d_{1*}\mathrm{ev}_1^{d*}\mathfrak{L}\big),[\overline{\mathfrak{M}}_1(\mathbb{P}^4,d)]^{vir}\big\rangle,$$

according to low-degree checks by T. Graber and R. Pandharipande and independently by S. Katz (personal communications).

Let

$$\mathfrak{M}^0_g(\mathbb{P}^4,d)=\big\{[\mathcal{C},u]\in\overline{\mathfrak{M}}_g(\mathbb{P}^4,d):\mathcal{C}\text{ is smooth}\big\}.$$

We denote by $\overline{\mathfrak{M}}^0_g(\mathbb{P}^4,d)$ the closure of $\mathfrak{M}^0_g(\mathbb{P}^4,d)$ in $\overline{\mathfrak{M}}_g(\mathbb{P}^4,d)$. If $g>0$, then $\overline{\mathfrak{M}}^0_g(\mathbb{P}^4,d)$ is one of the many irreducible components of the moduli space $\overline{\mathfrak{M}}_g(\mathbb{P}^4,d)$.

Theorem 1.1. *If d is a positive integer*, $\mathfrak{L}=\gamma^{*\otimes 5}\to\mathbb{P}^4$,

$$\pi^d_1:\mathfrak{U}_1(\mathbb{P}^4,d)\to\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d)\quad\textit{and}\quad\mathrm{ev}^d_1:\mathfrak{U}_1(\mathbb{P}^4,d)\to\mathbb{P}^4$$

are the semiuniversal family and the natural evaluation map, *respectively*, *then the Euler class of the sheaf*

$$\pi^d_{1*}\mathrm{ev}_1^{d*}\mathfrak{L}\to\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d)$$

is well-defined. Furthermore, if Y satisfies the Rigidity Assumption (page 422), *then*

$$N_1(d) = \tfrac{1}{12}N_0(d) + \big\langle e(\pi^d_{1*}\mathrm{ev}_1^{d*}\mathfrak{L}), [\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d)]\big\rangle. \tag{1-4}$$

The moduli space $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d)$ is not a smooth orbifold. Nevertheless, it determines a fundamental class in $H_{10d}(\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d); \mathbb{Q})$, as its singularities are fairly simple. The sheaf

$$\pi^d_{1*}\mathrm{ev}_1^{d*}\mathfrak{L} \to \overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d) \tag{1-5}$$

is not locally free. Nevertheless, its Euler class is well-defined. In other words, the Euler class of every desingularization of this sheaf is the same, in the sense described in Section 1.2 of [Zinger 2007]. The last expression in (1-4) can be computed via the classical localization theorem. Of course, the singularities of the space $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d)$ cause additional complications. However, since these singularities can be understood, these complications can be handled. A desingularization of $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d)$, i.e. a smooth orbifold $\widetilde{\mathfrak{M}}^0_1(\mathbb{P}^4, d)$ and a map

$$\tilde{\pi} : \widetilde{\mathfrak{M}}^0_1(\mathbb{P}^4, d) \to \overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d),$$

which is biholomorphic onto $\mathfrak{M}^0_1(\mathbb{P}^4, d)$, is constructed in [Vakil and Zinger 2006]. This desingularization of $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d)$ comes with a desingularization of the sheaf (1-5), i.e., a vector bundle

$$\tilde{\mathcal{V}} \to \widetilde{\mathfrak{M}}^0_1(\mathbb{P}^4, d) \qquad \text{such that} \qquad \tilde{\pi}_*\tilde{\mathcal{V}} = \pi^d_{1*}\mathrm{ev}_1^{d*}\mathfrak{L}.$$

In particular,

$$\big\langle e(\pi^d_{1*}\mathrm{ev}_1^{d*}\mathfrak{L}), [\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d)]\big\rangle = \big\langle e(\tilde{\mathcal{V}}), [\widetilde{\mathfrak{M}}^0_1(\mathbb{P}^4, d)]\big\rangle.$$

The localization theorem of [Atiyah and Bott 1984] is directly applicable to the right-hand side of this equality.

Using Theorem 1.1 and the desingularization constructed in [Vakil and Zinger 2006], we have computed the numbers $N_1(d)$ for $d = 1, 2, 3, 4$. The results agree with those predicted in [Bershadsky et al. 1993]; see Section 1.3 in [Li and Zinger 2005] for more details.

From the point of view of symplectic topology as described in [Fukaya and Ono 1999] and [Li and Tian 1998], the numbers $N_g(d)$ can be interpreted as the Euler class of a vector bundle, albeit of an infinite-rank vector bundle over a space of the "same" dimension. As in the finite-dimensional case, this Euler class is the number of zeros, counted with appropriate multiplicities, of a transverse (multivalued, admissible) section.

In brief, we prove (1-4) by slightly perturbing the complex structure J_0 on $\mathbb{P}^4$, then expressing each of the three terms appearing in (1-4) as the number of zeros of a transverse section of a vector bundle and comparing the results for the two sides of (1-4). There are a vector bundle $\mathfrak{F} \to \mathfrak{X}$, possibly of infinite rank, and a section φ of $\mathfrak{F}$ associated to each of three terms. The zero set of φ is easy to describe. However, φ is not transverse to the zero set. We determine the number $\mathcal{C}_{\mathfrak{Z}}(\varphi)$ of zeros of $\varphi+\varepsilon$, for a small generic multisection ε, that lie near each stratum $\mathfrak{Z}$ of $\varphi^{-1}(0)$. These numbers in turn determine the contribution of each J-holomorphic curve in Y to the three numbers in (1-4). We will see that every such curve contributes equally to the two sides of (1-4).

Theorem 1.1 follows immediately from Propositions 1.3–1.5 and separately from Propositions 2.1–2.3. The first three propositions are easier to state and can be deduced from the last three propositions. While the statements of Propositions 2.1–2.3 are more technical, they are easier to prove.

Formula (1-4) also follows from Theorem 1.3 in [Li and Zinger 2005] and Theorem 1.1 in [Zinger 2005b], whether or not the Rigidity Assumption holds. Thus, Theorem 1.1 provides additional evidence for the genus-zero and genus-one cases of the Rigidity Conjecture; see the next subsection. On the other hand, the methods of [Li and Zinger 2005] and [Zinger 2005b] should extend to give relations between higher-genera GW-invariants of complete intersections and of projective spaces without any assumptions, while the methods of this paper should extend to deduce such relations directly from the Rigidity Conjecture. If the two formulas disagree in some genus g, the Rigidity Conjecture must be false in genus g and in all higher genera.

1A. ***Rigidity properties.*** Throughout the rest of the paper, Y will denote a quintic threefold in $\mathbb{P}^4$. If J is an almost complex structure on Y, (Σ, j) is a Riemann surface, and $u:\Sigma\to Y$ is a J-holomorphic map, let

$$D_{J,u}:\Gamma(\Sigma;u^*TY)\to\Gamma\big(\Sigma;\Lambda^{0,1}_{J,j}T^*\Sigma\otimes u^*TY\big)$$

be the linearization of $\bar\partial_J$-operator at u; see Section 2A.

Definition 1.2. An almost complex structure J on Y *satisfies the genus-g rigidity property* if for every smooth connected genus-g Riemann surface (Σ, j) and non-constant J-holomorphic map $u:\Sigma\to Y$

(J_Y1) $u(\Sigma)$ is a smooth curve;

(J_Y2) $\ker D_{J,u}\subset\Gamma(\Sigma;u^*Tu(\Sigma))$.

If J satisfies the genus-g rigidity property, all genus-g J-holomorphic curves in Y are smooth and isolated. We denote by $\mathcal{J}(Y)$ the space of all C^1-smooth almost complex structures on Y, with the C^1-topology, and by $\mathcal{J}^g_{\mathrm{rig}}(Y)\subset\mathcal{J}(Y)$ the subspace of almost complex structures that satisfy the genus-g rigidity property.

Rigidity Conjecture. For all g and all Calabi–Yau threefolds Y, $\mathcal{J}^g_{\mathrm{rig}}(Y)$ is dense in $\mathcal{J}(Y)$.

Rigidity Assumption. If Y is a quintic threefold, the closure of $\mathcal{J}^0_{\mathrm{rig}}(Y)\cap\mathcal{J}^1_{\mathrm{rig}}(Y)$ in $\mathcal{J}(Y)$ contains J_0.

We note that $\mathcal{J}^g_{\mathrm{rig}}(Y)$ is open in $\mathcal{J}(Y)$. Thus, the $g=0,1$ cases of the Rigidity Conjecture imply our Rigidity Assumption.

Since the expected dimension of the moduli space $\overline{\mathfrak{M}}_g(Y,d;J)$ of genus-g degree-d J-holomorphic maps into Y is zero, it is easy to show that the property (J_Y1) of Definition 1.2 is satisfied by a generic almost complex structure J. But despite years of attempts, this has not been shown to be the case for (J_Y2), even for $g=0$. Nevertheless, this is believed to be case, though with some hesitation for g above 2 or 3. (If $u:\Sigma\to Y$ is an embedding, denote by $N_u\to\Sigma$ the normal bundle to u. In order to prove the rigidity conjecture, it is sufficient to show that for a generic almost complex structure J on Y, for every J-holomorphic embedding $u:\Sigma\to Y$ of a Riemann surface, and for every branch cover $f:\tilde{\Sigma}\to\Sigma$, the operator $D^{\perp}_{J,u\circ f}$ on f^*N_u induced by $D_{J,u\circ f}$ is injective. If (J_Y2) is to be proved only up to genus g, it is sufficient to consider covers $\tilde{\Sigma}$ of genus up to g. If J is a holomorphic and $N_u\to\Sigma$ is generic (of degree $2g(\Sigma)-2$), then $D^{\perp}_{J,u\circ f}$ is injective for every f if and only if the genus of Σ is 0 or 1. The expectation is that this kind of generic situation can be achieved using a nonintegrable J.)

For each $J\in\mathcal{J}(Y)$, let $\mathcal{S}^d_g(Y;J)$ be the set of J-holomorphic genus-g degree-d (simple) *curves* in Y. If $J\in\mathcal{J}^g_{\mathrm{rig}}(Y)$, this set is finite. By Propositions 1.3 and 1.4 below, the number of elements in $\mathcal{S}^d_g(Y;J)$, counted with appropriate signs, is independent of $J\in\mathcal{J}^g_{\mathrm{rig}}(Y)$ for $g=0,1$. We denote this number by $n_g(d)$.

If $\sigma\in\mathbb{Z}^+$, let $m(\sigma)$ denote the number of degree-σ covers of an elliptic curve by elliptic curves.

Proposition 1.3. *For all* $d\in\mathbb{Z}^+$,

$$N_0(d)=\sum_{\sigma|d}\frac{n_0(d/\sigma)}{\sigma^3}.$$

Proposition 1.4. *For all* $d\in\mathbb{Z}^+$,

$$N_1(d)=\frac{1}{12}\sum_{\sigma|d}\frac{n_0(d/\sigma)}{\sigma}+\sum_{\sigma|d}m(\sigma)\frac{n_1(d/\sigma)}{\sigma}.$$

Proposition 1.5. *If* d, $\mathfrak{L}$, π^d_1, *and* ev^d_1 *are as in the statement of Theorem 1.1, then*

$$\big\langle e(\pi^d_{1*}\mathrm{ev}^{d*}_1\mathfrak{L}),[\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d)]\big\rangle=\frac{1}{12}\sum_{\sigma|d}\frac{\sigma^2-1}{\sigma^3}n_0(d/\sigma)+\sum_{\sigma|d}m(\sigma)\frac{n_1(d/\sigma)}{\sigma}.$$

We do not prove these three propositions as stated, since this is not necessary for the proof of Theorem 1.1. Instead, we prove the less elegant and more notationally involved Propositions 2.1–2.3, which also imply Theorem 1.1. Propositions 1.3, 1.4, and 1.5 can be derived from Propositions 2.1, 2.2, and 2.3, respectively; see the end of Section 2B.

2. Preliminaries

2A. ***Review of key definitions.*** In this subsection, we give geometric definitions of the three terms that appear in (1-4). The construction of the Gromov–Witten invariants described below is a slight variation on that of [Fukaya and Ono 1999] and [Li and Tian 1998], but it is easy to see the only difference is in the presentation. Below we use the term *multisection*, or *multivalued section*, of a vector orbibundle as defined in Section 3 of [Fukaya and Ono 1999].

If X is a smooth submanifold of $\mathbb{P}^n$, we denote by $\mathfrak{X}_g(X,d)$ the space of equivalence classes of stable degree-d smooth maps from genus-g Riemann surfaces to X. Let $\mathfrak{X}_g^0(X,d)$ be the subset of $\mathfrak{X}_g(X,d)$ consisting of stable maps with smooth domains. The spaces $\mathfrak{X}_g(X,d)$ are topologized using L_1^p-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. Here and throughout the rest of the paper, p denotes a real number greater than two. The spaces $\mathfrak{X}_g(X,d)$ can be stratified by the smooth infinite-dimensional orbifolds $\mathfrak{X}_{\mathcal{T}}(X)$ of stable maps from domains of the same geometric type. The closure of the main stratum, $\mathfrak{X}_g^0(X,d)$, is $\mathfrak{X}_g(X,d)$.

If J is an almost complex structure on $\mathbb{P}^n$, let

$$\Gamma_g^{0,1}(X,d;J) \to \mathfrak{X}_g(X,d)$$

be the bundle of (TX,J)-valued $(0,1)$-forms. In other words, the fiber of the space $\Gamma_g^{0,1}(X,d;J)$ over a point $[b]=[\Sigma,j;u]$ in $\mathfrak{X}_g(X,d)$ is

$$\Gamma_g^{0,1}(X,d;J)\big|_{[b]} = \Gamma^{0,1}(b;TX;J)\big/\mathrm{Aut}(b),$$

where

$$\Gamma^{0,1}(b;TX;J) = \Gamma\big(\Sigma;\Lambda_{J,j}^{0,1}T^*\Sigma\otimes u^*TX\big).$$

Here j is the complex structure on Σ, the domain of the smooth map u. The bundle $\Lambda_{J,j}^{0,1}T^*\Sigma\otimes u^*TX$ over Σ consists of (J,j)-antilinear homomorphisms:

$$\Lambda_{J,j}^{0,1}T^*\Sigma\otimes u^*TX = \big\{\alpha\in\mathrm{Hom}(T\Sigma,u^*TX):\alpha\circ j=-J\circ\alpha\big\}.$$

The total space of the bundle $\Gamma_g^{0,1}(TX,d;J)\to\mathfrak{X}_g(X,d)$ is topologized using L^p-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The restriction of $\Gamma_g^{0,1}(TX,d;J)$ to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ is a smooth vector orbibundle of infinite rank.

We define a continuous section of the bundle $\Gamma^{0,1}_g(TX,d;J) \to \mathfrak{X}_g(X,d)$ by

$$\bar\partial_J\big([\Sigma,j;u]\big) = \tfrac{1}{2}\big(du + J\circ du\circ j\big).$$

By definition, the zero set of this section is the moduli space $\overline{\mathfrak{M}}_g(X,d;J)$ of equivalence classes of stable J-holomorphic degree-d maps from genus-g curves into X. The restriction of $\bar\partial_J$ to each stratum of $\mathfrak{X}_g(X,d)$ is smooth. For each element $[b]=[\Sigma,j,u]$ of $\mathfrak{X}_g(X,d)$, we put

$$D_{J,b}\xi = \tfrac{1}{2}\big(\nabla^X\xi + J\circ\nabla^X\xi\circ j\big) + \tfrac{1}{2}(\nabla^X_\xi J)\circ du\circ j \quad \text{if } \xi\in\Gamma(b;TX)\equiv\Gamma(\Sigma;u^*TX),$$

where ∇^X denotes the Levi-Civita connection of a J-compatible metric on X. The linear operator $D_{J,b}$ describes the restriction of a linearization of $\bar\partial_J$ at $[b]$ to a finite-codimensional subspace of the tangent bundle of the stratum $\mathfrak{X}_{\mathcal{T}}(X)$ of $\mathfrak{X}_g(X,d)$ containing $[b]$.

The section $\bar\partial_J : \mathfrak{X}_g(X,d) \to \Gamma^{0,1}_g(X,d;J)$ is Fredholm; that is, its linearization at every point of $\bar\partial_J^{-1}(0)$ has finite-dimensional kernel and cokernel. The index of $\bar\partial_J$ at a point of $\mathfrak{X}^0_g(X,d)$ is the expected dimension of the moduli space $\overline{\mathfrak{M}}_g(X,d;J)$. If $X=Y$, this expected dimension is 0. By definition,

$$N_g(d) = {}^{\pm}\big|\{\bar\partial_J+\varepsilon\}^{-1}(0)\big|, \tag{2-1}$$

where ε is a small multivalued perturbation such that $\bar\partial_J+\varepsilon$ is transverse to the zero set along each stratum $\mathfrak{X}_{\mathcal{T}}(Y)$ of $\mathfrak{X}_g(Y,d)$ and

$${}^{\pm}\big|\{\bar\partial_J+\varepsilon\}^{-1}(0)\big|$$

is the number of elements in the finite set $\{\bar\partial_J+\varepsilon\}^{-1}(0)$, counted with appropriate multiplicities. By the transversality condition,

$$\{\bar\partial_J+\varepsilon\}^{-1}(0)\subset\mathfrak{X}^0_g(Y,d).$$

The smallness condition implies in particular that the set $\{\bar\partial_J+\varepsilon\}^{-1}(0)$ is close to $\bar\partial_J^{-1}(0)$. Since the set $\bar\partial_J^{-1}(0)$ is compact, it follows that the set $\{\bar\partial_J+\varepsilon\}^{-1}(0)$ is also compact. Let $\mathcal{A}^d_g(\bar\partial_J)$ denote the set of all perturbations ε of $\bar\partial_J$ that satisfy the two conditions above. Such perturbations will be called $\bar\partial_J$-admissible. Below we will refer to the number in (2-1) as the Euler class of the tuple

$$\mathcal{V}^d_g(\bar\partial;J)\equiv\big(\mathfrak{X}_g(Y,d),\Gamma^{0,1}_g(Y,d;J),\pi;\bar\partial_J,\mathcal{A}^d_g(\bar\partial_J)\big).$$

This Euler class depends on the Fredholm homotopy class of the section $\bar\partial_J$.

We now describe the last term in (1-4) in a similar way. If $\mathfrak{L} \to \mathbb{P}^4$ is as in Theorem 1.1, let $\Gamma_g(\mathfrak{L}, d) \to \mathfrak{X}_g(\mathbb{P}^4, d)$ be the cone such that the fiber of $\Gamma_g(\mathfrak{L}, d)$ over $[b] = [\Sigma, j; u]$ in $\mathfrak{X}_g(\mathbb{P}^4, d)$ is the Banach space

$$\Gamma_g(\mathfrak{L}, d)\big|_{[b]} = \Gamma(b; \mathfrak{L})\big/\mathrm{Aut}(b), \quad \text{where } \Gamma(b; \mathfrak{L}) = L_1^p(\Sigma; u^*\mathfrak{L}),$$

and the topology on $\Gamma_g(\mathfrak{L}, d)$ in defined analogously to the topology on $\Gamma_g(\mathbb{P}^4, d)$. Let ∇ denote the hermitian connection in the line bundle $\mathfrak{L} \to \mathbb{P}^4$ induced from the standard connection on the tautological line bundle over $\mathbb{P}^4$. If (Σ, j) is a Riemann surface and $u : \Sigma \to \mathbb{P}^4$ is a smooth map, let

$$\nabla^u : \Gamma(\Sigma; u^*\mathfrak{L}) \to \Gamma\big(\Sigma; T^*\Sigma \otimes u^*\mathfrak{L}\big)$$

be the pullback of ∇ by u. If $b = (\Sigma, j; u)$, we define the corresponding $\bar\partial$-operator by

$$\text{(2-2)} \quad \bar\partial_{\nabla,b} : \Gamma(\Sigma; u^*\mathfrak{L}) \to \Gamma\big(\Sigma; \Lambda^{0,1}_{\mathfrak{i},j} T^*\Sigma \otimes u^*\mathfrak{L}\big), \quad \bar\partial_{\nabla,b}\xi = \tfrac{1}{2}\big(\nabla^u\xi + \mathfrak{i}\nabla^u\xi \circ j\big),$$

where $\mathfrak{i}$ is the complex multiplication in the bundle $u^*\mathfrak{L}$. Let

$$\mathcal{V}_g^d = \big\{[b, \xi] \in \Gamma_g(\mathfrak{L}, d) : [b] \in \mathfrak{X}_g(\mathbb{P}^4, d),\ \xi \in \ker \bar\partial_{\nabla,b} \subset \Gamma_g(b; \mathfrak{L})\big\} \subset \Gamma_{g,k}(\mathfrak{L}, d).$$

The cone $\mathcal{V}_g^d \to \mathfrak{X}_{g,k}(\mathbb{P}^4, d)$ inherits its topology from $\Gamma_g(\mathfrak{L}, d)$.

Let $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J) \subset \overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ denote the closed subset containing the set

$$\mathfrak{M}_1^0(\mathbb{P}^4, d; J) = \big\{[\mathcal{C}, u] \in \overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J) : \mathcal{C} \text{ is smooth}\big\},$$

defined in [Zinger 2004b]. If the almost complex structure J is close enough to J_0, $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ is the closure of $\mathfrak{M}_1^0(\mathbb{P}^4, d; J)$ in $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$. In such a case, $\mathfrak{M}_1^0(\mathbb{P}^4, d; J)$ is a smooth orbifold of dimension $10d$, while $\partial\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ is a finite union of smooth orbifolds of dimension at most $10d-2$. On the other hand, $\mathcal{V}_1^d|_{\mathfrak{M}_1^0(\mathbb{P}^4,d;J)}$ is a complex vector orbibundle of rank $5d$. The last term in (1-4) is the number of zeros, counted with appropriate multiplicities, of any continuous multisection φ of the cone $\mathcal{V}_1^d$ over $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ such that $\varphi^{-1}(0)$ is contained in $\mathfrak{M}_1^0(\mathbb{P}^4, d; J)$ and $\varphi|_{\mathfrak{M}_1^0(\mathbb{P}^4,d;J)}$ is smooth and transverse to the zero set; see Sections 1.2 and 1.3 in [Zinger 2007]. Proposition 3.1 in the same reference guarantees that a section φ satisfying the two conditions exists. In our case, it is more convenient to think of φ as $s_1^d + \varepsilon$, where ε is a multivalued perturbation of s_1^d. We denote by $\mathcal{A}_1^d(s; J)$ the set of all perturbations ε of s_1^d such that $s_1^d + \varepsilon$ satisfies the two conditions above. Such perturbations ε will be called s_1^d-admissible. Let

$$\mathcal{V}_1^d(s; J) \equiv \big(\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J), \mathcal{V}_1^d, \pi; s_1^d, \mathcal{A}_1^d(s)\big).$$

This tuple will be the focus of Section 4.

Remark. Since Y is a semipositive symplectic manifold, one can define the numbers $N_g(d)$ without using the infinite-rank orbibundles $\Gamma_g^{0,1}(Y,d;J)$; see [Ruan and Tian 1997]. However, there would be no effect on the proofs of Propositions 1.3–1.5 and 2.1–2.3, and the construction described above appears more natural in the present context, even though it involves more complicated objects.

2B. ***Components of the proof.*** We now set up additional notation that allows us to state more notationally involved, but also easier-to-prove, versions of Propositions 1.3–1.5.

By Theorems 1.6 and 2.3 in [Zinger 2004b], there exists $\delta(d)\in\mathbb{R}^+$ with the property that if J is an almost complex structure on $\mathbb{P}^4$ such that $\|J-J_0\|_{C^1}\le\delta(d)$, then J is genus-one $d\ell$-regular in the sense of Definition 1.4 in [Zinger 2004b]. This regularity condition implies that the moduli spaces $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4,d;J)$ and $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^4,d;J)$ have the same stratification structure as the moduli spaces

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4,d)\equiv\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4,d;J_0)\quad\text{and}\quad\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^4,d)\equiv\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^4,d;J_0),$$

respectively. In addition, by Theorem 1.2 in [Zinger 2007], $\delta(d)\in\mathbb{R}^+$ can be chosen so that the Euler class of the cone

$$\mathcal{V}_1^d\to\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J)$$

is well defined and

$$\big\langle e(\mathcal{V}_1^d),[\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J)]\big\rangle=\big\langle e(\mathcal{V}_1^d),[\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d)]\big\rangle,\tag{2-3}$$

if $\|J-J_0\|_{C^1}\le\delta(d)$.

If J is an almost complex structure on $\mathbb{P}^4$ and $[\Sigma,u]\in\overline{\mathfrak{M}}_1(\mathbb{P}^4,d;J)$, we put

$$s_1^d\big([\Sigma,u]\big)=[s\circ u]\in\Gamma(\mathfrak{L},d)\big|_{[\Sigma,u]}.$$

If J is ∇s*-equivalent* to J_0, i.e.

$$\nabla s\circ J_0=\nabla s\circ J\in\Gamma\big(\mathbb{P}^4;\mathrm{Hom}_{\mathbb{R}}(T\mathbb{P}^4,\mathfrak{L})\big),$$

then $s_1^d([\Sigma,u])\in\mathcal{V}_1^d|_{[\Sigma,u]}$. Thus, in such a case, we obtain a continuous section of the cone

$$\mathcal{V}_1^d\to\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J),$$

which restricts to a smooth section over each stratum of $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J)$. Note that

$$\big\{s_1^d|_{\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J)}\big\}^{-1}(0)=\overline{\mathfrak{M}}_1^0(Y,d;J)=\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J)\cap\overline{\mathfrak{M}}_1(Y,d;J).\tag{2-4}$$

Since the (∇,J_0)-holomorphic section s of page 417 is transverse to the zero set in $\mathfrak{L}$, the $(\mathfrak{i},J_0)$-linear map

$$\nabla s:T\mathbb{P}^4\to\mathfrak{L}$$

does not vanish along $Y = s^{-1}(0)$. Let U_s be a small neighborhood of Y in $\mathbb{P}^4$ such that ∇s does not vanish over U_s. The kernel of ∇s over U_s is then a rank-three complex subbundle of $(T\mathbb{P}^4, J_0)|_{U_s}$, which restricts to TY along Y. We denote this subbundle by $\tilde{T}Y$. If J is an almost complex structure on $\mathbb{P}^4$ such that

(J1) $J = J_0$ on $\mathbb{P}^4 - U_s$ and

(J2) $J(\tilde{T}Y) = \tilde{T}Y$ and $J = J_0$ on $T\mathbb{P}^4|_{U_s}/\tilde{T}Y$,

then J_0 and J are ∇s-equivalent. Thus, every almost complex structure J_Y on Y extends to an almost complex structure J on $\mathbb{P}^4$ which is ∇s-equivalent to J_0. Furthermore, such an extension can be chosen so that

$$\|J - J_0\|_{C^1} \le 2\big\|J_Y - J_0|_{TY}\big\|_{C^1}. \tag{2-5}$$

We denote by $\mathcal{J}_{\mathrm{rig}}(s)$ the set of almost complex structures J on $\mathbb{P}^4$ such that J is ∇s-equivalent to J_0 and $J_Y \equiv J|_{TY}$ is an element of $\mathcal{J}^0_{\mathrm{rig}}(Y) \cap \mathcal{J}^1_{\mathrm{rig}}(Y)$. By the above and the Rigidity Assumption (page 422), the C^1-closure of $\mathcal{J}_{\mathrm{rig}}(s)$ in $\mathcal{J}(\mathbb{P}^4)$ contains J_0.

From now on, we assume that $J \in \mathcal{J}_{\mathrm{rig}}(s)$ is an almost complex structure on $\mathbb{P}^4$ sufficiently close to J_0. For $g = 0, 1$, we put

$$\mathcal{S}^d_g(Y; J) = \mathcal{S}^d_g(Y; J_Y) \quad \text{for all } d \in \mathbb{Z}^+ \quad \text{and} \quad \mathcal{S}_g(Y; J) = \bigsqcup_{d=1}^{\infty} \mathcal{S}^d_g(Y; J).$$

If $\kappa \in \mathcal{S}_g(Y; J)$, let d_κ denote the degree of κ in $\mathbb{P}^4$. If $\kappa \in \mathcal{S}_0(Y; J)$ and q is a positive integer, let $\mathfrak{M}^q_1(\kappa, d)$ be the subset of $\overline{\mathfrak{M}}_1(\kappa, d)$ consisting of stable maps $[\mathscr{C}, u]$ such that $\mathscr{C}$ is an elliptic curve E with q rational components attached directly to E and $u|_E$ is constant. Figure 1 shows the domain of a typical element of $\mathfrak{M}^3_1(\kappa, d)$, from the points of view of symplectic topology and of algebraic geometry. In the first diagram, each shaded disc represents a sphere; the integer next to each rational component $\mathscr{C}_i$ indicates the degree of $u|_{\mathscr{C}_i}$. In the second diagram, the components of $\mathscr{C}$ are represented by curves, and the pair of integers next to each component $\mathscr{C}_i$ shows the genus of $\mathscr{C}_i$ and the degree of $u|_{\mathscr{C}_i}$. For stability reasons, the restriction of u to each rational component must be nonconstant. We denote by $\overline{\mathfrak{M}}^q_1(\kappa, d)$ the closure of $\mathfrak{M}^q_1(\kappa, d)$ in $\overline{\mathfrak{M}}_1(\kappa, d)$. Note that

$$\dim_{\mathbb{C}} \overline{\mathfrak{M}}^q_1(\kappa, d) = \begin{cases} 2d, & \text{if } q = 0; \\ 2d + 1 - q, & \text{if } q \in \mathbb{Z}^+. \end{cases} \tag{2-6}$$

If $q \in \mathbb{Z}^+$, $\overline{\mathfrak{M}}^q_1(\kappa, d)$ is a smooth orbivariety. In contrast, $\overline{\mathfrak{M}}^0_1(\kappa, d)$ is a singular orbivariety, if $d > 2$; its structure is described in Section 4B.

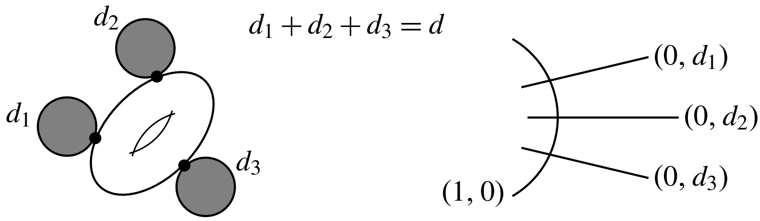


Figure 1. Domain of a typical element of $\mathfrak{M}_1^3(\kappa,d)$.

For each $q\in\mathbb{Z}^+$, let $[q]=\{1,\dots,q\}$. If $\underline{d}=(d_1,\dots,d_q)$ is a q-tuple of positive integers and $\kappa\in\mathcal{S}_0(Y;J)$, we put

$$\begin{aligned}(2\text{-}7)\quad &\overline{\mathfrak{M}}_0(\kappa,\underline{d})\\ &=\Big\{(b_1,\dots,b_q)\in\prod_{i=1}^{i=q}\overline{\mathfrak{M}}_{0,1}(\kappa,d_i):\mathrm{ev}_0(b_i)=\mathrm{ev}_0(b_j)\text{ for all }i,j\in[q]\Big\},\end{aligned}$$

where $\mathrm{ev}_0:\overline{\mathfrak{M}}_{0,1}(\kappa,d_i)\to\kappa$ is the evaluation map corresponding to the marked point. Let

$$\overline{\mathfrak{M}}_0^q(\kappa,d)=\bigsqcup_{\substack{d_i>0\\ \sum d_i=d}}\overline{\mathfrak{M}}_0\big(\kappa,(d_1,\dots,d_q)\big).$$

The spaces $\overline{\mathfrak{M}}_0^q(\kappa,d)$ are smooth orbivarieties. We note that

$$\dim_{\mathbb{C}}\overline{\mathfrak{M}}_0^q(\kappa,d)=2d+1-2q. \tag{2-8}$$

There is a natural node-identifying immersion

$$\iota_q:\overline{\mathcal{M}}_{1,q}\times\overline{\mathfrak{M}}_0^q(\kappa,d)\to\overline{\mathfrak{M}}_1^q(\kappa,d), \tag{2-9}$$

where $\overline{\mathcal{M}}_{1,q}$ is the moduli space of genus-one curves with q marked points. It descends to an immersion

$$\big(\overline{\mathcal{M}}_{1,q}\times\overline{\mathfrak{M}}_0^q(\kappa,d)\big)\big/S_q\to\overline{\mathfrak{M}}_1^q(\kappa,d),$$

where S_q is the q-th symmetric group; the latter immersion restricts to a diffeomorphism on the preimage of $\mathfrak{M}_1^q(\kappa,d)$. The immersion (2-9) is illustrated in Figure 2. In this figure, we represent an entire space of stable maps by the domain of a typical element of the space. We shade the components of the domain on which the maps are nonconstant. The vertical bar indicates that the three marked points are mapped to the same point in κ, as specified by (2-7). Let

$$\pi_P,\pi_B:\overline{\mathcal{M}}_{1,q}\times\overline{\mathfrak{M}}_0^q(\kappa,d)\to\overline{\mathcal{M}}_{1,q},\overline{\mathfrak{M}}_0^q(\kappa,d)$$

be the projection maps.

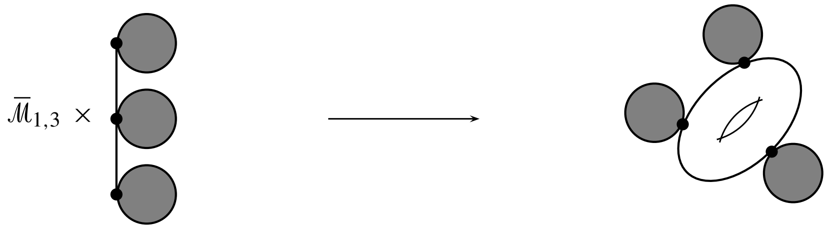


Figure 2. The node-identifying immersion (2-9) for $\overline{\mathfrak{M}}_1^3(\kappa, d)$.

For each $\kappa \in \mathcal{S}_0(Y; J)$, we denote by $N_Y\kappa$ the normal bundle of κ in Y. If $q \in \mathbb{Z}^+$ and $[b] = ([b_i])_{i\in[q]}$ is an element of $\overline{\mathfrak{M}}_0^q(\kappa, d)$, let

$$\Gamma(b; TY) = \Big\{\xi = (\xi_i)_{i\in[q]} \in \bigoplus_{i\in[q]} \Gamma(b_i; TY) : \xi_i(y_0(b_i)) = \xi_j(y_0(b_j)) \text{ for } i, j \in [q]\Big\},$$

where $y_0(b_i)$ is the marked point of the component map b_i. Since $J_Y \in \mathcal{J}^0_{\mathrm{rig}}(Y)$, by the Index Theorem the cokernel $H^1_J(b; TY)$ of the operator

$$\Gamma(b; TY) \to \Gamma^{0,1}(b; TY; J) \equiv \bigoplus_{i\in[q]} \Gamma^{0,1}(b_i; TY; J) \tag{2-10}$$

taking $D_{J,b}\big((\xi_i)_{i\in[q]}\big)$ to $(D_{J,b_i}\xi_i)_{i\in[q]}$ is a vector space of dimension $2d-2$. It is naturally isomorphic to the cokernel $H^1_J(b; N_Y\kappa)$ of the operator

$$\Gamma(b; N_Y\kappa) \to \Gamma^{0,1}(b; N_Y\kappa; J), \qquad D^{\perp}_{J,b}\big((\xi_i)_{i\in[q]}\big) = (D^{\perp}_{J,b_i}\xi_i)_{i\in[q]},$$

induced by the operator $D_{J,b}$. These cokernels induce a vector orbibundle over $\overline{\mathfrak{M}}_0^q(\kappa, d)$, which will be denoted by $\mathcal{W}^{0,q}_{\kappa,d}$. If $q = 1$, this bundle is the pullback by the forgetful map

$$\tilde{\pi} : \overline{\mathfrak{M}}_0^1(\kappa, d) \equiv \overline{\mathfrak{M}}_{0,1}(\kappa, d) \to \overline{\mathfrak{M}}_0(\kappa, d)$$

of the vector bundle defined in a similar way. We denote this last vector bundle by $\mathcal{W}^0_{\kappa,d}$. We have

$$\operatorname{rk}\mathcal{W}^{0,q}_{\kappa,d} = 2d-2 \quad \text{for all } q \in \mathbb{Z}^+ \qquad \text{and} \qquad \operatorname{rk}\mathcal{W}^{0}_{\kappa,d} = 2d-2. \tag{2-11}$$

It is straightforward to see that the cokernel bundle for the operators $D_{J,b}$ over $\overline{\mathfrak{M}}_1^q(\kappa, d)$, for $q \in \mathbb{Z}^+$, is given by

$$\iota_q^*\mathcal{W}^{1,q}_{\kappa,d} \approx \pi_P^*\mathbb{E}^* \otimes \pi_B^*\mathrm{ev}_0^*TY \oplus \pi_B^*\mathcal{W}^{0,q}_{\kappa,d}, \tag{2-12}$$

where $\mathbb{E} \to \overline{\mathcal{M}}_{1,q}$ is the Hodge line bundle and

$$\mathrm{ev}_0 : \overline{\mathfrak{M}}_0^q(\kappa, d) \to \kappa$$

is the natural evaluation map, corresponding to the marked point common to all factors. We note that

$$\operatorname{rk}\mathcal{W}^{1,q}_{\kappa,d} = 2d+1. \tag{2-13}$$

On the other hand, similarly to the genus-zero case, the cokernel $H^1_J(b;TY)$ of the operator $D_{J,b}$ for

$$b \in \mathfrak{M}^0_1(\kappa,d) \subset \overline{\mathfrak{M}}_1(\kappa,d)$$

is naturally isomorphic to the cokernel $H^1_J(b;N_Y\kappa)$ of the operator $D^{\perp}_{J,b}$ induced by $D_{J,b}$. The cokernels $H^1_J(b;N_Y\kappa)$ have the expected rank for all $b\in\overline{\mathfrak{M}}^0_1(\kappa,d)$ and thus form a vector bundle over $\overline{\mathfrak{M}}^0_1(\kappa,d)$, which we denote by $\mathcal{W}^{1,0}_{\kappa,d}$. We have

$$\operatorname{rk}\mathcal{W}^{1,0}_{\kappa,d} = 2d \qquad \text{and} \tag{2-14}$$

$$\iota_q^*\mathcal{W}^{1,0}_{\kappa,d} \approx \big(\pi_P^*\mathbb{E}^*\otimes\pi_B^*\mathrm{ev}_0^*N_Y\kappa \oplus \pi_B^*\mathcal{W}^{0,q}_{\kappa,d}\big)\big|_{\iota_q^{-1}(\overline{\mathfrak{M}}^0_1(\kappa,d))} \text{ for all } q\in\mathbb{Z}^+. \tag{2-15}$$

We are now ready to reformulate Propositions 1.3–1.5.

Proposition 2.1. *If d and $\mathfrak{L}$ are as in Theorem 1.1, $s\in H^0(\mathbb{P}^4;\mathfrak{L})$ is a transverse section, and $Y=s^{-1}(0)$, there exists $\delta(d)\in\mathbb{R}^+$ with the following property. If $J\in\mathcal{J}_{\mathrm{rig}}(s)$ and $\|J-J_0\|_{C^1}\le\delta(d)$, then*

$$N_0(d) = \sum_{\kappa\in\mathcal{S}_0(Y;J)} \big\langle e(\mathcal{W}^0_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)]\big\rangle,$$

where $\mathcal{W}^0_{\kappa,d/d_\kappa}\to\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)$ is the cokernel bundle corresponding to the almost complex structure J, as above.

Proposition 2.2. *If d, $\mathfrak{L}$, s, and Y are as in Proposition 2.1, there exists $\delta(d)\in\mathbb{R}^+$ with the following property. If $J\in\mathcal{J}_{\mathrm{reg}}(s)$ and $\|J-J_0\|_{C^1}\le\delta(d)$, then*

$$N_1(d) = \sum_{\kappa\in\mathcal{S}_1(Y;J)} \pm\big|\mathfrak{M}^0_1(\kappa,d/d_\kappa)\big| + \sum_{\kappa\in\mathcal{S}_0(Y;J)} \Big(\big\langle e(\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}^0_1(\kappa,d/d_\kappa)]\big\rangle + \frac{d/d_\kappa}{12}\big\langle e(\mathcal{W}^0_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)]\big\rangle\Big),$$

where $\mathcal{W}^0_{\kappa,d/d_\kappa}\to\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)$ and $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}\to\overline{\mathfrak{M}}^0_1(\kappa,d/d_\kappa)$ are the cokernel bundles corresponding to the almost complex structure J, as above.

Proposition 2.3. *If d, $\mathfrak{L}$, s, and Y are as in Proposition 2.1 and $\mathcal{V}^d_1\to\mathfrak{X}_1(\mathbb{P}^4,d)$ is the cone corresponding to the line bundle $\mathfrak{L}\to\mathbb{P}^4$ with its standard connection, there exists $\delta(d)\in\mathbb{R}^+$ with the following properties. If $\|J-J_0\|_{C^1}\le\delta(d)$, then the moduli space $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d;J)$ carries a rational fundamental class of dimension $10d$, the Euler class of the cone*

$$\mathcal{V}^d_1\to\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d;J)$$

is a well-defined element of $H^{10d}(\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J);\mathbb{Q})$, *and*

$$\langle e(\mathcal{V}_1^d),[\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J)]\rangle=\langle e(\mathcal{V}_1^d),[\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d)]\rangle.$$

If in addition $J\in\mathcal{J}_{\mathrm{reg}}(s)$,

$$\begin{aligned}\langle e(\mathcal{V}_1^d),&[\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d;J)]\rangle\\ &=\sum_{\kappa\in\mathcal{S}_1(Y;J)}{}^{\pm}\big|\mathfrak{M}_1^0(\kappa,d/d_\kappa)\big|+\sum_{\kappa\in\mathcal{S}_0(Y;J)}\Big(\langle e(\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}),[\overline{\mathfrak{M}}_1^0(\kappa,d/d_\kappa)]\rangle\\ &\qquad\qquad+\frac{d/d_\kappa-1}{12}\langle e(\mathcal{W}^{0}_{\kappa,d/d_\kappa}),[\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)]\rangle\Big),\end{aligned}$$

with $\mathcal{W}^0_{\kappa,d/d_\kappa}\to\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)$ *and* $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}\to\overline{\mathfrak{M}}_1^0(\kappa,d/d_\kappa)$ *as in Proposition 2.2.*

In the last two propositions, the moduli space $\mathfrak{M}_1^0(\kappa,d/d_\kappa)$, for $\kappa\in\mathcal{S}_1(Y;J)$, contains $m(d/d_\kappa)$ elements: the $m(d/d_\kappa)$ equivalence classes of the degree-d/d_κ covers of the elliptic curve κ by an elliptic curve. Since the order of the automorphism group of such a cover is d/d_κ,

$$^{\pm}\big|\mathfrak{M}_1^0(\kappa,d/d_\kappa)\big|=\pm\frac{m(d/d_\kappa)}{d/d_\kappa}.$$

The sign is determined by viewing the zero-dimensional suborbifold $\mathfrak{M}_1^0(\kappa,d/d_\kappa)$ of $\mathfrak{X}_1(Y,d)$ as a transverse zero of the section $\bar\partial_J$. This sign is the same as the sign of κ as an element of the set $\mathcal{S}_1^{d_\kappa}(Y;J)$. In particular,

$$\sum_{\kappa\in\mathcal{S}_1(Y;J)}{}^{\pm}\big|\mathfrak{M}_1^0(\kappa,d/d_\kappa)\big|=\sum_{\sigma|d}m(\sigma)\frac{n_1(d/\sigma)}{\sigma},\tag{2-16}$$

where $n_1(\cdot)$ is as in Section 1A.

If $\kappa\in\mathcal{S}_0(Y;J)$, the orientations of the vector bundles

$$\mathcal{W}^0_{\kappa,d/d_\kappa}\to\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)\quad\text{and}\quad\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}\to\overline{\mathfrak{M}}_1^0(\kappa,d/d_\kappa)$$

are determined by the linearizations of the sections $\bar\partial_J$ over $\mathfrak{X}_0(Y,d)$ and $\mathfrak{X}_1(Y,d)$. According to E. Ionel and T. Parker (seminar talk), a spectral-flow argument can be used to show that

$$\begin{aligned}\langle e(\mathcal{W}^0_{\kappa,\sigma}),[\overline{\mathfrak{M}}_0(\kappa,\sigma)]\rangle&=\pm\langle e\big(R^1\pi^\sigma_{0*}\mathrm{ev}_0^{\sigma*}(\mathcal{O}_\kappa(-1)\oplus\mathcal{O}_\kappa(-1))\big),[\overline{\mathfrak{M}}_0(\kappa,\sigma)]\rangle,\\ \langle e(\mathcal{W}^{1,0}_{\kappa,\sigma}),[\overline{\mathfrak{M}}_1^0(\kappa,\sigma)]\rangle&=\pm\langle e\big(R^1\pi^\sigma_{1*}\mathrm{ev}_1^{\sigma*}(\mathcal{O}_\kappa(-1)\oplus\mathcal{O}_\kappa(-1))\big),[\overline{\mathfrak{M}}_1^0(\kappa,\sigma)]\rangle,\end{aligned}\tag{2-17}$$

where $\sigma=d/d_\kappa$ and the sign agrees with the sign of κ as an element of $\mathcal{S}_0^{d_\kappa}(Y;J)$. By localization,

$$\langle e\big(R^1\pi^\sigma_{0*}\mathrm{ev}_0^{\sigma*}(\mathcal{O}_\kappa(-1)\oplus\mathcal{O}_\kappa(-1))\big),[\overline{\mathfrak{M}}_0(\kappa,\sigma)]\rangle=\frac{1}{\sigma^3};\tag{2-18}$$

see Section 27.5 of [Hori et al. 2003]. Using the desingularization of $\overline{\mathfrak{M}}_1^0(\kappa,\sigma)$ constructed in [Vakil and Zinger 2006], one can show that

$$\big\langle e\big(R^1\pi^{\sigma}_{1*}\mathrm{ev}_1^{\sigma*}(\mathbb{O}_\kappa(-1)\oplus\mathbb{O}_\kappa(-1))\big),[\overline{\mathfrak{M}}_1^0(\kappa,\sigma)]\big\rangle=\frac{1}{12}\,\frac{\sigma-1}{\sigma^2}. \tag{2-19}$$

Propositions 1.3–1.5 follow from Propositions 2.1–2.3 via (2-16)–(2-19).

Since Theorem 1.1 follows immediately from Propositions 2.1–2.3, we do not need to deduce Propositions 1.3–1.5 from Propositions 2.1–2.3. We prove Propositions 2.2 and 2.3 in Sections 3 and 4, respectively; see also Propositions 2.5 and 2.6. The proof of Proposition 2.1 is very similar to the proof of Proposition 2.2, but simpler, and we omit it.

2C. ***Summary of the proof of Proposition 2.2.*** A key notion in our argument, which is also used in the proof of Proposition 2.3, is Definition 2.4 below. For its purposes, we will call either of the two tuples $\mathcal{V}_g^d(\bar\partial;J)$ and $\mathcal{V}_1^d(s;J)$, defined in Section 2A, a *generalized vector bundle*. The first tuple involves an infinite-rank bundle over an infinite-dimensional space; the second one involves finite-dimensional objects, albeit nonsmooth ones. Nevertheless, both are generalizations of a rank-n vector bundle $\mathfrak{F}$ over an n-dimensional complex compact manifold $\mathfrak{X}$, with a choice of a section φ and of an appropriate subset $\mathcal{A}(\varphi)$ of $\Gamma(\mathfrak{X};\mathfrak{F})$ of second category. Such a collection of data can also be considered to be a generalized vector bundle.

Definition 2.4. Suppose $\mathcal{V}=\big(\mathfrak{X},\mathfrak{F},\pi;\varphi,\mathcal{A}(\varphi)\big)$ is a generalized vector bundle. Subset $\mathcal{Z}$ of $\varphi^{-1}(0)$ is a *regular set for* $\mathcal{V}$ if there exists $\mathcal{C}_{\mathcal{Z}}(\mathcal{V})\in\mathbb{Q}$ and a dense open subset $\mathcal{A}_{\mathcal{Z}}(\varphi)$ of $\mathcal{A}(\varphi)$ with the following properties. For every $\nu\in\mathcal{A}_{\mathcal{Z}}(\varphi)$,

(a) there exists $\epsilon_\nu\in\mathbb{R}^+$ such that $t\nu\in\mathcal{A}(\varphi)$ for all $t\in(0,\epsilon_\nu)$, and

(b) there exist a compact subset $K_\nu\subset\mathcal{Z}$, open neighborhood $U_\nu(K)$ of K in $\mathfrak{X}$ for each compact subset $K\subset\mathcal{Z}$, and $\epsilon_\nu(U)\in(0,\epsilon_\nu)$ for each open subset U of $\mathfrak{X}$ such that

$$^{\pm}\big|\{\varphi+t\nu\}^{-1}\cap U\big|=\mathcal{C}_{\mathcal{Z}}(\mathcal{V})\quad \text{if } t\in\big(0,\epsilon_\nu(U)\big) \text{ and } K_\nu\subset K\subset U\subset U_\nu(K). \tag{2-20}$$

Every connected component of $\varphi^{-1}(0)$ is regular. However, a regular subset of $\varphi^{-1}(0)$ need not be closed. For example, if φ is a holomorphic section of a rank-k algebraic vector bundle $\mathfrak{F}$ over a k-dimensional compact algebraic variety $\mathfrak{X}$, every Zariski open subset of $\varphi^{-1}(0)$ is regular. The sections $\bar\partial_J$ and s_1^d that play a central role in this paper are in a sense *generalized holomorphic sections*.

If $\mathcal{Z}$ is a regular set for the generalized vector bundle $\mathcal{V}$, we will call the number $\mathcal{C}_{\mathcal{Z}}(\mathcal{V})$ the *φ-contribution of $\mathcal{Z}$ to the Euler class of $\mathcal{V}$*. Note that if $\varphi^{-1}(0)=$

$\bigsqcup_{i\in I}\mathscr{Z}_i$ is a partition of $\varphi^{-1}(0)$ into regular sets, the Euler class of $\mathscr{V}$, or its Poincaré dual, is the sum of φ-contributions:

$$e(\mathscr{V})=\sum_{i\in I}\mathscr{C}_{\mathscr{Z}_i}(\mathscr{V}). \tag{2-21}$$

We prove Theorem 1.1 by expressing each of the three terms appearing in (1-4) in the form (2-21) and show that we end up with the same terms on the two sides of (1-4).

If d, s, and Y are as in the previous subsection and $J\in\mathcal{J}_{\mathrm{rig}}(s)$,

$$\overline{\mathfrak{M}}_1(Y,d;J)=\bigsqcup_{\kappa\in\mathcal{S}_0(Y;J)}\overline{\mathfrak{M}}_1(\kappa,d/d_\kappa)\;\sqcup\bigsqcup_{\kappa\in\mathcal{S}_1(Y;J)}\mathfrak{M}_1^0(\kappa,d/d_\kappa). \tag{2-22}$$

For any $\kappa\in\mathcal{S}_0(Y;J)$, $\sigma\in\mathbb{Z}^+$, and subset ϱ of $\bar{\mathbb{Z}}^+\equiv\mathbb{Z}^+\cup\{0\}$, let

$$\mathfrak{M}_1^{\varrho}(\kappa,\sigma)=\bigcap_{q\in\varrho}\overline{\mathfrak{M}}_1^q(\kappa,\sigma)\;-\bigcup_{q\in\bar{\mathbb{Z}}^+-\varrho}\overline{\mathfrak{M}}_1^q(\kappa,\sigma).$$

Proposition 2.5. *If d, $\mathfrak{L}$, s, and Y are as in Proposition 2.1, $J\in\mathcal{J}_{\mathrm{rig}}(s)$ is taken sufficiently close to J_0, and $\kappa\in\mathcal{S}_1(Y;J)$, then*

$$\mathscr{C}_{\mathfrak{M}_1^0(\kappa,d/d_\kappa)}\big(\mathscr{V}_1^d(\bar\partial;J)\big)=\pm\big|\mathfrak{M}_1^0(\kappa,d/d_\kappa)\big|.$$

If $\kappa\in\mathcal{S}_0(Y;J)$,

$$\mathscr{C}_{\mathfrak{M}_1^{\{0\}}(\kappa,d/d_\kappa)}\big(\mathscr{V}_1^d(\bar\partial;J)\big)=\big\langle e(\mathscr{W}^{1,0}_{\kappa,d/d_\kappa}),[\overline{\mathfrak{M}}_1^0(\kappa,d/d_\kappa)]\big\rangle.$$

Proposition 2.6. *If d, $\mathfrak{L}$, s, Y, and J are as in Proposition 2.5, $\kappa\in\mathcal{S}_0(Y;J)$, and ϱ is a subset $\bar{\mathbb{Z}}^+$ different from $\{0\}$, then*

$$\mathscr{C}_{\mathfrak{M}_1^{\varrho}(\kappa,d/d_\kappa)}\big(\mathscr{V}_1^d(\bar\partial;J)\big)=\begin{cases}\dfrac{d/d_\kappa}{12}\big\langle e(\mathscr{W}^0_{\kappa,d/d_\kappa}),[\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)]\big\rangle & \text{if } \varrho=\{1\},\\ 0 & \text{if } \varrho\neq\{1\}.\end{cases}$$

One consequence of Propositions 2.5 and 2.6 is that most boundary strata of the moduli space $\overline{\mathfrak{M}}_1(Y,d;J)$ do not contribute to the number $N_1(d)$. In fact, we will show that only the strata $\mathfrak{M}_1^0(Y,d;J)$ and $\mathfrak{M}_1^1(Y,d;J)$ contribute to the number $N_1(d)$.

We now outline the proofs of Propositions 2.5 and 2.6. Let

$$\nu\in\Gamma\big(\mathfrak{X}_1(\mathbb{P}^4,d);\Gamma_1^{0,1}(\mathbb{P}^4,d;J)\big)$$

be a small generic multisection such that

$$\nu\in\Gamma\big(\mathfrak{X}_s;\Gamma_1^{0,1}(Y,d;J)\big)$$

for a small neighborhood $\mathfrak{X}_s$ of $\overline{\mathfrak{M}}_1(Y,d;J)$ in $\mathfrak{X}_1(\mathbb{P}^4,d)$ and vanishes outside of $\mathcal{U}_s$. By definition, $N_1(d)$ is the number of elements $\exp_u\xi\in\mathfrak{X}_1(\mathbb{P}^4,d)$ such that (u,ξ) solves the system

$$\text{(2-23)}\qquad \begin{cases}\bar\partial_J\exp_u\xi+\nu(\exp_u\xi)=0,\\ s\circ\exp_u\xi=0,\end{cases}\qquad u\in\overline{\mathfrak{M}}_1(\mathbb{P}^4,d;J),\ \ \xi\in T_u\mathfrak{X}_1(\mathbb{P}^4,d).$$

Note that

$$\bar\partial_{\nabla,\exp_u\xi}s_1^d(\exp_u\xi)=0$$

if (u,ξ) solves the first equation, due to our assumptions on ν. If $u\in\mathfrak{M}_1^0(\mathbb{P}^4,d;J)$ and ν is sufficiently small, the first equation has a unique small solution $\xi_\nu(u)$ in $\Gamma_+(u)$, the orthogonal complement of $T_u\overline{\mathfrak{M}}_1(\mathbb{P}^4,d;J)$ in $T_u\mathfrak{X}_1(\mathbb{P}^4,d)$. Plugging this solution into the second equation, we obtain

$$\text{(2-24)}\qquad 0=s\circ\exp_u\xi_\nu=s_1^d(u)+\pi_{TY}^{\perp}\xi_\nu(u)\in\mathcal{V}_1^d,$$

where $\pi_{TY}^{\perp}$ is the projection map $T\mathbb{P}^4\to T\mathbb{P}^4/TY$, defined on a neighborhood of Y in $\mathbb{P}^4$. Since all solutions of the system (2-23) are transverse, so are the solutions of (2-24). Thus, the zeros of a generic perturbation ν of the section $\bar\partial_J$ that lie close to $\mathfrak{M}_1^0(Y,d;J)$ correspond to the zeros of a perturbation of the section s_1^d that lie close to $\mathfrak{M}_1^0(Y,d;J)$. In Section 3B, we show that the number of these zeros that lie near each component $\mathfrak{M}_1^0(\kappa,d/d_\kappa)$ of $\mathfrak{M}_1^0(Y,d;J)$ is the Euler class of the bundle $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$ over $\overline{\mathfrak{M}}_1^0(\kappa,d/d_\kappa)$.

We next look for solutions near $\mathfrak{M}_1^{\{1\}}(Y,d;J)$; i.e., we assume $u\in\mathfrak{M}_1^{\{1\}}(Y,d;J)$. Note that

$$\text{(2-25)}\qquad \overline{\mathfrak{M}}_1^1(Y,d;J)\approx\overline{\mathcal{M}}_{1,1}\times\overline{\mathfrak{M}}_{0,1}(Y,d;J).$$

We denote by π_B and π_P, respectively, the projection maps onto $\overline{\mathfrak{M}}_{0,1}(Y,d;J)$ and $\overline{\mathcal{M}}_{1,1}$. Let

$$\tilde\pi_B:\overline{\mathfrak{M}}_1^1(Y,d;J)\to\overline{\mathfrak{M}}_0(Y,d;J)$$

be the composition of π_B with the forgetful map $\overline{\mathfrak{M}}_{0,1}(Y,d;J)\to\overline{\mathfrak{M}}_0(Y,d;J)$. The bundle $T\mathfrak{X}_1(Y,d)$ contains the line subbundle $\mathcal{L}\equiv\pi_P^*L_{P,1}\otimes\pi_B^*L_0$, where

$$L_{P,1}\to\overline{\mathcal{M}}_{1,1}\quad\text{and}\quad L_0\to\overline{\mathfrak{M}}_{0,1}(Y,d;J)$$

are the universal tangent line bundles at the marked points. If $u\in\overline{\mathfrak{M}}_1^1(Y,d;J)$ and $\upsilon\in\mathcal{L}_u$ is small, we denote by u_υ the element $\exp_u\upsilon$ of $\mathfrak{X}_1(Y,d)$. Let

$$\mathrm{ev}_P:\overline{\mathfrak{M}}_1^1(Y,d;J)\to Y$$

be the composition of π_B with the evaluation map at the marked point. This map sends an element $[\mathcal{C},u]$ of $\overline{\mathfrak{M}}_1^1(Y,d;J)$ to the value of u on the principal component of $\mathcal{C}$.

In this case, we work with the analogue of (2-23) intrinsic to Y, i.e. we look for solutions of the equation

$$\bar\partial_J \exp_{u_\upsilon}\xi + \nu(\exp_{u_\upsilon}\xi) = 0, \qquad \text{where } u \in \overline{\mathfrak{M}}_1^{\{1\}}(Y, d; J),\ \xi \in \Gamma(\upsilon; TY) \equiv \Gamma(u_\upsilon^* TY). \tag{2-26}$$

This equation usually does not have a small solution in ξ for a fixed u_υ, as there is an obstruction bundle

$$\Gamma_-^{0,1}(u; TY; J) = \tilde\pi_B^* H^1(u_B^* TY) \oplus \pi_P^* \mathbb{E}^* \otimes \mathrm{ev}_P^* TY \subset \Gamma_1^{0,1}(Y, d; J),$$

where u_B is the restriction of u to the bubble components. Taking the projections $(\pi_{-,B}^{0,1} \oplus \pi_{-,P}^{0,1})$ and $\pi_+^{0,1}$ of (2-26) onto $\Gamma_-^{0,1}(u; TY; J)$ and its complement $\Gamma_+^{0,1}(u; TY; J)$ in $\Gamma_1^{0,1}(Y, d; J)$, respectively, we obtain

$$\begin{cases} \pi_+^{0,1}\bar\partial \exp_{u_\upsilon}\xi + \pi_+^{0,1}\nu(\exp_{u_\upsilon}\xi) = 0 \in \Gamma_+^{0,1}(u; TY; J), \\ \pi_{-,B}^{0,1}\bar\partial \exp_{u_\upsilon}\xi + \pi_{-,1}^{0,1}\nu(\exp_{u_\upsilon}\xi) = 0 \in \tilde\pi_B^* H^1(u_B^* TY), \\ \pi_{-,P}^{0,1}\bar\partial \exp_{u_\upsilon}\xi + \pi_{-,0}^{0,1}\nu(\exp_{u_\upsilon}\xi) = 0 \in \pi_P^*\mathbb{E}^* \otimes \mathrm{ev}_P^* TY. \end{cases} \tag{2-27}$$

If ν and υ are sufficiently small, the first equation has a unique small solution $\xi_\nu(u, \upsilon)$ in $\Gamma_+(u; TY)$. With appropriate choices of neighborhood charts and of the perturbation ν, the value of the left-hand side of the middle equation in (2-27) at $\xi = \xi_\nu(u, \upsilon)$ depends only on u_B, and the system (2-27) is equivalent to

$$\pi_{-,P}^{0,1}\bar\partial \exp_{u_\upsilon}\xi_\nu(u,\upsilon) + \pi_{-,P}^{0,1}\nu(\exp_{u_\upsilon}\xi_\nu(u,\upsilon)) = 0 \in \pi_P^*\mathbb{E}^* \otimes \mathrm{ev}_P^* TY, \qquad u_B \in \mathscr{Z}_0,\ \upsilon \in \mathscr{L}, \tag{2-28}$$

where $\mathscr{Z}_0$ is the zero set of a section of the first component of the bundle $\Gamma_-^{0,1}(\cdot, TY)$ over $\overline{\mathfrak{M}}_0(Y, d; J)$. In particular, ${}^{\pm}|\mathscr{Z}_0| = N_0(d)$.

Equation (2-28) is equivalent to

$$\mathfrak{D}_u \upsilon + \pi_{-,P}^{0,1}\nu(u) = 0 \in \pi_P^*\mathbb{E}^* \otimes T_{\mathrm{ev}_P(u)}Y, \quad u_B \in \mathscr{Z}_0,\ \upsilon \in \mathscr{L}, \tag{2-29}$$

where $\mathfrak{D}_u \in \mathrm{Hom}(L_0, T_{\mathrm{ev}_0(u_B)}Y)$. The image of $\mathfrak{D}_u$ in $T_{\mathrm{ev}_0(u_B)}Y$ is precisely the tangent line at $\mathrm{ev}_0(u_B)$ to the rational curve $\mathrm{Im}\, u_B$, as long as the differential of the map u_B does not vanish at the marked point. Thus, for each $u_B \in \mathscr{Z}_0$, the number of solutions of (2-29) is the number of times $\pi_{-,P}^{0,1}\nu(u)$ lies in $\mathbb{E}^* \otimes T_{\mathrm{ev}_P(u)}\,\mathrm{Im}\, u_B$. We conclude that

$$\begin{aligned} &\mathscr{C}_{\mathfrak{M}_1^{\{1\}}(Y,d;J)}\big(\mathscr{V}_1^d(\bar\partial; J)\big) \\ &\qquad = \sum_{u_B \in \mathscr{Z}_0} \big\langle c(\pi_P^*\mathbb{E}^* \otimes \mathrm{ev}_P^* TY)\, c(\pi_P^*\mathbb{E}^* \otimes \mathrm{ev}_P^* T\,\mathrm{Im}\, u_B)^{-1}, [\tilde\pi_B^{-1}(u_B)]\big\rangle \end{aligned} \tag{2-30}$$

$$= \sum_{u_B \in \mathcal{Z}_0} \big\langle c_1(\mathbb{E}^*)(c_1(TY) - c_1(T\operatorname{Im} u_B)), [\overline{\mathcal{M}}_{1,1}] \times [\mathbb{P}^1] \big\rangle$$
$$= -\frac{1}{24}(0-2) \sum_{\kappa \in \mathcal{S}_0(Y;J)} (d/d_\kappa) \cdot {}^{\pm}\big|\mathcal{Z}_0 \cap \overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)\big|$$
$$= \sum_{\kappa \in \mathcal{S}_0(Y;J)} \frac{d/d_\kappa}{12} \big\langle e(\mathcal{W}^0_{\kappa, d/d_\kappa}), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \big\rangle,$$

as claimed in Proposition 2.6.

We analyze the contribution to the number $N_1(d)$ from the complement of $\mathfrak{M}_1^{\{0\}}(\mathbb{P}^4, d; J)$ and $\mathfrak{M}_1^{\{1\}}(\mathbb{P}^4, d; J)$ in $\overline{\mathfrak{M}}_1(Y, d; J)$ in a similar way, but we encounter one of two key differences. If $\varrho = \{0, 1\}$ and $u \in \mathfrak{M}_1^{\varrho}(Y, d; J)$, $\mathcal{D}_u = 0$. Equation (2-29) has no solutions near $\mathfrak{M}_1^{\varrho}(Y, d; J)$ if ν is generic. On the other hand, if ϱ is any other subset of $\bar{\mathbb{Z}}^+$ containing 0 and at least one other element, the analogue of the set $\mathcal{Z}_0$ is empty for dimensional reasons. Thus,

$$\mathcal{C}_{\mathfrak{M}_1^{\varrho}(Y,d;J)}\big(\mathcal{V}_1^d(\bar{\partial}; J)\big) = 0$$

if $\{0\} \subsetneq \varrho \subset \bar{\mathbb{Z}}^+$, as claimed.

The computation of the contribution from $\mathfrak{M}_1^0(Y, d; J)$ to the number $N_1(d)$ can also be carried out in Y, instead of $\mathbb{P}^4$. However, the version of the computation presented here is meant to indicate why the cone $\mathcal{V}_1^d$ should enter into the Gromov–Witten theory of Y.

We supply more details of the proof of Propositions 2.5 and 2.6 in Section 3. In particular, in order to use the gluing and obstruction-bundle setup described in [Zinger 2004a], we stratify the moduli spaces that appear in the statements of Propositions 2.5 and 2.6 according to the bubble type, or the dual graph, of stable maps. The notion of contribution to the Euler class used in this paper is a direct adaptation, to the orbifold and multisection setting of [Fukaya and Ono 1999] and [Li and Tian 1998], of the analogous notion used in [Zinger 2003] and [Zinger 2005a]. However, in the present case, we can get by with far less detailed understanding of the behavior of the bundle sections involved.

2D. ***Notation: genus-zero maps.*** We now summarize our notation for bubble maps from genus-zero Riemann surfaces, with one marked point, and for related objects. For more details on the notation described below, the reader is referred to Section 2 in [Zinger 2004a].

In general, moduli spaces of stable maps can be stratified by the dual graph. However, in the present situation, it is more convenient to use *linearly ordered sets*:

Definition 2.7. (1) A finite nonempty partially ordered set I is a *linearly ordered set* if for all $i_1, i_2, h \in I$ such that $i_1, i_2 < h$, either $i_1 \leq i_2$ or $i_2 \leq i_1$.

(2) A linearly ordered set I is a *rooted tree* if I has a unique minimal element, i.e., there exists $\hat{0} \in I$ such that $\hat{0} \le i$ for all $i \in I$.

We use rooted trees to stratify the moduli space $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^4, d; J)$ of degree-d J-holomorphic maps from genus-zero Riemann surfaces with one marked point to $\mathbb{P}^4$.

If I is a linearly ordered set, let $\hat{I}$ be the subset of the nonminimal elements of I. For every $h \in \hat{I}$, denote by $\iota_h \in I$ the largest element of I smaller than h:

$$\iota_h = \max\big\{i \in I : i < h\big\}.$$

A *genus-zero* $\mathbb{P}^4$*-valued bubble map* is a tuple $b = (I; x, u)$, where I is a rooted tree, and

$$x : \hat{I} \to \mathbb{C} = S^2 - \{\infty\} \quad \text{and} \quad u : I \to C^\infty(S^2; \mathbb{P}^4)$$

are maps such that $u_h(\infty) = u_{\iota_h}(x_h)$ for all $h \in \hat{I}$. Such a tuple describes a Riemann surface Σ_b and a continuous map $u_b : \Sigma_b \to \mathbb{P}^4$. The irreducible components $\Sigma_{b,i}$ of Σ_b are indexed by the set I and $u_b|_{\Sigma_{b,i}} = u_i$. The Riemann surface Σ_b carries a marked point, i.e. the point $(\hat{0}, \infty) \in \Sigma_{b,\hat{0}}$, if $\hat{0}$ is the minimal element of I. The general structure of genus-zero bubble maps is described by tuples $\mathcal{T} = (I; \underline{d})$, where $\underline{d} : I \to \mathbb{Z}$ is a map specifying the degree of $u_b|_{\Sigma_{b,i}}$, if b is a bubble map of type $\mathcal{T}$. We call such tuples *bubble types*.

If $\mathcal{T}$ is a bubble type, let $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ be the subset of $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^4, d; J)$ consisting of stable maps $[\mathcal{C}, y_1, u]$ such that

$$[\mathcal{C}, y_1, u] = [(\Sigma_b, (\hat{0}, \infty)), u_b],$$

for some bubble map b of type $\mathcal{T}$. Section 2.5 of [Zinger 2004a] describes a space $\mathcal{U}^{(0)}_{\mathcal{T}}(X; J)$ of *balanced* stable maps, not of equivalence classes of such maps, such that

$$\mathcal{U}_{\mathcal{T}}(X; J) = \mathcal{U}^{(0)}_{\mathcal{T}}(X; J)\big/\mathrm{Aut}(\mathcal{T}) \propto (S^1)^I,$$

for a natural action of $\mathrm{Aut}(\mathcal{T})$ on $(S^1)^I$. This space is convenient to use in gluing constructions.

2E. ***Notation: genus-one maps.*** We next set up analogous notation for genus-one stable maps; see Section 2.2 in [Zinger 2004b] for more details. In this case, we also need to specify the structure of the principal component. Thus, we index the strata of $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ by *enhanced linearly ordered sets*:

Definition 2.8. An *enhanced linearly ordered set* is a pair $(I, \aleph)$, where I is a linearly ordered set, $\aleph$ is a subset of $I_0 \times I_0$, and I_0 is the subset of minimal elements of I, such that if $|I_0| > 1$,

$$\aleph = \big\{(i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n), (i_n, i_1)\big\}$$

for some bijection $i : \{1, \ldots, n\} \to I_0$.

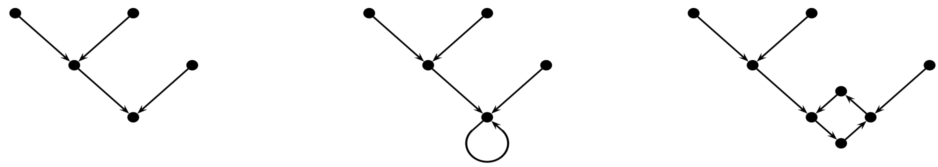


Figure 3. Some enhanced linearly ordered sets.

An enhanced linearly ordered set can be represented by an oriented connected graph. In Figure 3, the dots denote the elements of I. The arrows outside the loop, if there are any, specify the partial ordering of the linearly ordered set I. In fact, every directed edge outside of the loop connects a nonminimal element h of I with ι_h. Inside of the loop, there is a directed edge from i_1 to i_2 if and only if $(i_1, i_2) \in \aleph$.

The subset $\aleph$ of $I_0 \times I_0$ will be used to describe the structure of the principal curve of the domain of stable maps in a stratum of the moduli space $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$. If $\aleph = \varnothing$, and thus $|I_0| = 1$, the corresponding principal curve $\Sigma_\aleph$ is a smooth torus, with some complex structure. If $\aleph \neq \varnothing$, the principal components form a circle of spheres:

$$\Sigma_\aleph = \Big(\bigsqcup_{i \in I_0} \{i\} \times S^2 \Big) \Big/ \sim , \quad \text{where } (i_1, \infty) \sim (i_2, 0) \text{ if } (i_1, i_2) \in \aleph.$$

A *genus-one* $\mathbb{P}^4$*-valued bubble map* is a tuple $b = \big(I, \aleph; S, x, u\big)$, where S is a smooth Riemann surface of genus one if $\aleph = \varnothing$ and the circle of spheres $\Sigma_\aleph$ otherwise. The objects x, u, and (Σ_b, u_b) are as in the genus-zero case, except the sphere $\Sigma_{b,\hat{0}}$ is replaced by the genus-one curve $\Sigma_{b,\aleph} \equiv S$. Furthermore, if $\aleph = \varnothing$, and thus $I_0 = \{\hat{0}\}$ is a single-element set, $u_{\hat{0}} \in C^\infty(S; \mathbb{P}^4)$. In the genus-one case, the general structure of bubble maps is encoded by the tuples of the form $\mathcal{T} = (I, \aleph; \underline{d})$. Similarly to the genus-zero case, we denote by $\mathcal{U}_\mathcal{T}(\mathbb{P}^4; J)$ the subset of $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ consisting of stable maps $[\mathcal{C}, u]$ such that $[\mathcal{C}, u] = [\Sigma_b, u_b]$, for some bubble map b of type $\mathcal{T}$ as above.

If $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type as above, set

$$I_1 = \big\{h \in \hat{I} : \iota_h \in I_0\big\}, \quad \mathcal{T}_0 = \big(I_1, I_0, \aleph; \iota|_{I_1}, \underline{d}|_{I_0}\big),$$

and

$$\mathrm{Aut}^*(\mathcal{T}) = \mathrm{Aut}(\mathcal{T}) / \{g \in \mathrm{Aut}(\mathcal{T}) : g \cdot h = h \text{ for all } h \in I_1\},$$

where I_0 is the subset of minimal elements of I. For each $h \in I_1$, we put

$$I_h = \big\{i \in I : h \le i\big\} \quad \text{and} \quad \mathcal{T}_h = \big(I_h; \underline{d}|_{I_h}\big).$$

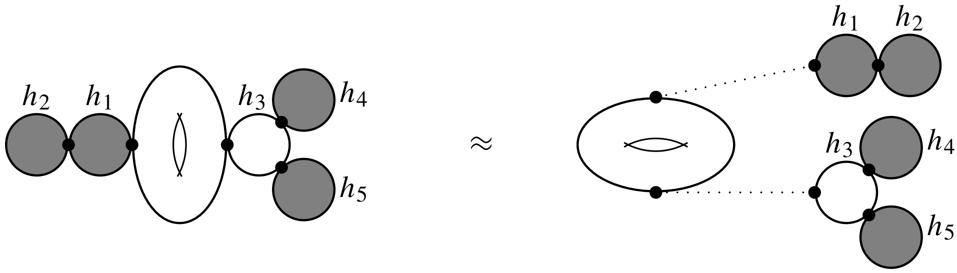


Figure 4. An example of the decomposition (2-31).

The tuple $\mathcal{T}_0$ describes bubble maps from genus-one Riemann surfaces with the marked points indexed by the set I_1; see Section 2.2 in [Zinger 2004b]. We have a natural isomorphism

$$\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \approx \Big\{ \big(b_0, (b_h)_{h\in I_1}\big) \in \mathcal{U}_{\mathcal{T}_0}(\mathbb{P}^4; J) \times \prod_{h\in I_1} \mathcal{U}_{\mathcal{T}_h}(\mathbb{P}^4; J) : \\ \mathrm{ev}_0(b_h) = \mathrm{ev}_{\iota_h}(b_0) \text{ for all } h \in I_1 \Big\} \big/ \mathrm{Aut}^*(\mathcal{T}). \tag{2-31}$$

This decomposition is illustrated in Figure 4, which represents an entire stratum of bubble maps by the domain of the stable maps in that stratum. The right-hand side of the figure represents the subset of the cartesian product of the three spaces of bubble maps, corresponding to the three drawings, on which the appropriate evaluation maps agree pairwise, as indicated by the dotted lines and defined in (2-31).

Let $\mathcal{F}T \to \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ be the bundle of gluing parameters, or of smoothings at the nodes. This orbibundle has the form

$$\mathcal{F}T = \Big(\bigoplus_{(h,i)\in\aleph} L_{h,0} \otimes L_{i,1} \oplus \bigoplus_{h\in\hat{I}} L_{h,0} \otimes L_{h,1} \Big) \big/ \mathrm{Aut}(\mathcal{T}),$$

for certain line orbibundles $L_{h,0}$ and $L_{h,1}$. Similarly to the genus-zero case,

$$\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) = \mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J) \big/ \mathrm{Aut}(\mathcal{T}) \propto (S^1)^{\hat{I}}, \tag{2-32}$$

where

$$\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J) = \Big\{ \big(b_0, (b_h)_{h\in I_1}\big) \in \mathcal{U}_{\mathcal{T}_0}(\mathbb{P}^4; J) \times \prod_{h\in I_1} \mathcal{U}^{(0)}_{\mathcal{T}_h}(\mathbb{P}^4; J) : \\ \mathrm{ev}_0(b_h) = \mathrm{ev}_{\iota_h}(b_0) \text{ for all } h \in I_1 \Big\}. \tag{2-33}$$

The line bundles $L_{h,0}$ and $L_{h,1}$ arise from the quotient (2-32), and

$$\mathcal{F}T = \tilde{\mathcal{F}}\mathcal{T} \big/ \mathrm{Aut}(\mathcal{T}) \propto (S^1)^{\hat{I}}, \qquad \text{where} \qquad \tilde{\mathcal{F}}\mathcal{T} = \tilde{\mathcal{F}}_{\aleph}\mathcal{T} \oplus \bigoplus_{h\in\hat{I}} \tilde{\mathcal{F}}_h\mathcal{T},$$

$\tilde{\mathcal{F}}_{\aleph}\mathcal{T} \to \mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J)$ is the bundle of smoothings for the $\aleph$ nodes of the circle of spheres $\Sigma_{\aleph}$ and $\tilde{\mathcal{F}}_h\mathcal{T} \to \mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J)$ is the line bundle of smoothings of the attaching node of the bubble indexed by h.

Suppose $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $d_i = 0$ for all $i \in I_0$, i.e. every element in $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ is constant on the principal components. In this case, the decomposition (2-31) is equivalent to

$$\begin{aligned}\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) &\approx \big(\mathcal{U}_{\mathcal{T}_0}(pt) \times \mathcal{U}_{\bar{\mathcal{T}}}(\mathbb{P}^4; J)\big)\big/\mathrm{Aut}^*(\mathcal{T})\\ &\subset \big(\overline{\mathcal{M}}_{1,k} \times \mathcal{U}_{\bar{\mathcal{T}}}(\mathbb{P}^4; J)\big)\big/\mathrm{Aut}^*(\mathcal{T}),\end{aligned} \tag{2-34}$$

where $k = |I_1|$ and

$$\mathcal{U}_{\bar{\mathcal{T}}}(\mathbb{P}^4; J) = \Big\{(b_h)_{h\in I_1} \in \prod_{h\in I_1} \mathcal{U}_{\mathcal{T}_h}(\mathbb{P}^4; J) : \mathrm{ev}_0(b_{h_1}) = \mathrm{ev}_0(b_{h_2}) \text{ for all } h_1, h_2 \in I_1\Big\}.$$

Similarly, (2-33) is equivalent to

$$\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J) \approx \mathcal{U}_{\mathcal{T}_0}(pt) \times \mathcal{U}^{(0)}_{\bar{\mathcal{T}}}(\mathbb{P}^4; J) \subset \overline{\mathcal{M}}_{1,k} \times \mathcal{U}^{(0)}_{\bar{\mathcal{T}}}(\mathbb{P}^4; J), \tag{2-35}$$

where

$$\begin{aligned}&\mathcal{U}^{(0)}_{\bar{\mathcal{T}}}(\mathbb{P}^4; J)\\ &\quad= \Big\{(b_h)_{h\in I_1} \in \prod_{h\in I_1} \mathcal{U}^{(0)}_{\mathcal{T}_h}(\mathbb{P}^4; J) : \mathrm{ev}_0(b_{h_1}) = \mathrm{ev}_0(b_{h_2}) \text{ for all } h_1, h_2 \in I_1\Big\}.\end{aligned} \tag{2-36}$$

We denote by

$$\pi_P : \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \to \overline{\mathcal{M}}_{1,k}/\mathrm{Aut}^*(\mathcal{T}) \quad \text{and} \quad \pi_P : \mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J) \to \overline{\mathcal{M}}_{1,k}$$

the projections onto the first component in the decompositions (2-34) and (2-35). Let

$$\mathrm{ev}_P : \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J), \mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J) \to \mathbb{P}^4$$

be the map sending each stable map (Σ, u) to its value on the principal component Σ_P of Σ, i.e., the point $u(\Sigma_P)$.

If $\mathcal{T} = (I, \aleph; \underline{d})$ is as in the previous paragraph, let

$$\chi(\mathcal{T}) = \big\{i \in \hat{I} : d_i \neq 0;\ d_h = 0 \text{ for all } h < i\big\},$$

$$\tilde{\mathfrak{F}}\mathcal{T} = \bigoplus_{i\in\chi(\mathcal{T})} \tilde{\mathcal{F}}_{h(i)}\mathcal{T} \to \mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J), \quad \text{where } h(i) = \min\{h \in \hat{I} : h \leq i\} \in I_1.$$

The subset $\chi(\mathcal{T})$ of I indexes the first-level effective bubbles of every element of $\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J)$. For each element $b = (\Sigma_b, u_b)$ of $\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J)$ and $i \in \chi(\mathcal{T})$, let

$$\mathcal{D}_i b = \big\{du_b|_{\Sigma_{b,i}}\big\}\big|_{\infty} e_\infty \in T_{\mathrm{ev}_P(b)}\mathbb{P}^4, \quad \text{where } e_\infty = (1, 0, 0) \in T_\infty S^2.$$

The complex span of $\mathcal{D}_i b$ in $T_{\mathrm{ev}_P(b)}\mathbb{P}^4$ is the tangent line to the rational component $\Sigma_{b,i}$ at the node of $\Sigma_{b,i}$ closest to a principal component of Σ_b. If the branch corresponding to $\Sigma_{b,i}$ has a cusp at this node, then $\mathcal{D}_i b = 0$.

Let $\mathbb{E} \to \overline{\mathcal{M}}_{1,k}$ denote the Hodge line bundle, that is, the line bundle of holomorphic differentials. For each $i \in \chi(\mathcal{T})$, we define the bundle map

$$\mathcal{D}_{J,i} : \tilde{\mathcal{F}}_{h(i)}\mathcal{T} \to \pi_P^*\mathbb{E}^* \otimes_J \mathrm{ev}_P^* T\mathbb{P}^4,$$

over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ by

$$\big\{\mathcal{D}_{J,i}(\tilde{\upsilon})\big\}(\psi) = \psi_{x_{h(i)}(b)}(\tilde{v}) \cdot_J \mathcal{D}_i b \in T_{\mathrm{ev}_P(b)}\mathbb{P}^4$$

if $\psi \in \pi_P^*\mathbb{E}$, $\tilde{\upsilon} = (b, \tilde{v}) \in \tilde{\mathcal{F}}_{h(i)}\mathcal{T}$, $b \in \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$, and $x_{h(i)}(b) \in \Sigma_{b,\aleph}$ is the node joining the bubble $\Sigma_{b,h(i)}$ of b to the principal component $\Sigma_{b,\aleph}$ of Σ_b. For each $\upsilon \in \tilde{\mathcal{F}}\mathcal{T}$, we put

$$\rho(\upsilon) = \big(b; \rho_i(\upsilon)\big)_{i\in\chi(\mathcal{T})} \in \tilde{\mathfrak{F}}\mathcal{T}, \quad \text{where } \rho_i(\upsilon) = \prod_{\substack{h\in\hat{I}\\ h\le i}} v_h \in \tilde{\mathcal{F}}_{h(i)}\mathcal{T},$$

if $\upsilon = \big(b; v_\aleph, (v_i)_{i\in\hat{I}}\big)$, $b \in \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$, $(b, v_\aleph) \in \tilde{\mathcal{F}}_\aleph\mathcal{T}$, $(b, v_h) \in \tilde{\mathcal{F}}_h\mathcal{T}$ if $h \in I_1$, and $v_i \in \mathbb{C}$ if $i \in \hat{I} - I_1$.

These definitions are illustrated in Figure 5 on page 461. While the bundle maps $D_{J,i}$ and ρ do not necessarily descend to the vector bundle $\mathcal{F}T$ over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, the map

$$\mathcal{D}_{\mathcal{T}} : \mathcal{F}T \to \pi_P^*\mathbb{E}^* \otimes \mathrm{ev}_P^* T\mathbb{P}^4 \big/ \mathrm{Aut}^*(\mathcal{T}), \quad \mathcal{D}_{\mathcal{T}}(\upsilon) = \sum_{i\in\chi(\mathcal{T})} \mathcal{D}_{J,i}\rho_i(\upsilon),$$

is well-defined.

Let $\tilde{\mathcal{V}}_1^d \to \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ be the vector bundle such that the fiber of $\tilde{\mathcal{V}}_1^d$ over a point $b = (\Sigma_b, u_b)$ in $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ is $\ker \bar{\partial}_{\nabla,b}$, where ∇ is the standard connection in line bundle $\mathfrak{L} = \gamma^{*\otimes 5}$ over $\mathbb{P}^4$; see Section 2A, as well as Section 3.3 in [Zinger 2007]. If $b = (\Sigma_b, u_b) \in \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$, $\xi = (\xi_h)_{h\in I} \in \Gamma(b; \mathfrak{L})$, and $i \in \chi(\mathcal{T})$, let

$$\tilde{\mathfrak{D}}_{\mathcal{T},i}\xi = \nabla_{e_\infty}\xi_i \in \mathfrak{L}_{\mathrm{ev}_0(b)},$$

as in Section 2.2 in [Zinger 2007]. We next define the bundle map

$$\mathfrak{D}_{\mathcal{T}} : \tilde{\mathcal{V}}_1^d \otimes \tilde{\mathfrak{F}}\mathcal{T} \to \pi_P^*\mathbb{E}^* \otimes \mathrm{ev}_P^*\mathfrak{L}$$

over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ by

$$\big\{\tilde{\mathfrak{D}}_{\mathcal{T}}(\xi \otimes \tilde{\upsilon})\big\}(\psi) = \sum_{i\in\chi(\mathcal{T})} \psi_{x_{h(i)}(b)}(\tilde{v}_i) \cdot \tilde{\mathfrak{D}}_{\mathcal{T},i}\xi \in \mathfrak{L}_{\mathrm{ev}_P(b)}$$

if $\xi \in \tilde{\mathcal{V}}_1^d|_b \subset \Gamma(b; \mathfrak{L})$, $\tilde{\upsilon} = (b; \tilde{v}_i)_{i\in\chi(\mathcal{T})} \in \tilde{\mathfrak{F}}\mathcal{T}|_b$, and $\psi \in \mathbb{E}_{\pi_P(b)}$.

The bundle map $\tilde{\mathfrak{D}}_{\mathcal{T}}$ induces a linear bundle map over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$:

$$\mathfrak{D}_{\mathcal{T}} : \mathcal{V}_1^d \otimes \mathfrak{F}\mathcal{T} \to \pi_P^*\mathbb{E}^* \otimes \mathrm{ev}_P^*\mathfrak{L}\big/\mathrm{Aut}^*(\mathcal{T}),$$

where

$$\mathfrak{F}\mathcal{T} = \Big(\bigoplus_{i\in\chi(\mathcal{T})} \pi_P^* L_{P,h(i)} \otimes \pi_i^* L_0\Big)\Big/\mathrm{Aut}^*(\mathcal{T}),$$

$L_{P,h} \to \overline{\mathfrak{M}}_{1,k}$ is the universal tangent line bundle at the marked point x_h, $L_0 \to \mathcal{U}_{\mathcal{T}'}(\mathbb{P}^4; J)$ is the universal tangent line bundle at the special marked point (i,∞) for any bubble type $\mathcal{T}'$ of rational stable maps, and

$$\pi_i : \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \to \mathcal{U}_{\mathcal{T}_i'}(\mathbb{P}^4; J)$$

is the projection map sending each bubble map $b = (\Sigma_b, u_b)$ to its restriction to the component $\Sigma_{b,i}$.

Finally, if $\mathcal{T}$ is any bubble type, for genus-zero or genus-one maps, and K is a subset of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, we denote by $K^{(0)}$ the preimage of K under the quotient projection map $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) \to \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$. All vector orbibundles we encounter will be assumed to be normed. Some will come with natural norms; for others, we implicitly choose a norm once and for all. If $\pi_{\mathfrak{F}} : \mathfrak{F} \to \mathfrak{X}$ is a normed vector bundle and $\delta : \mathfrak{X} \to \mathbb{R}$ is any function, possibly constant, let

$$\mathfrak{F}_\delta = \big\{ \upsilon \in \mathfrak{F} : |\upsilon| < \delta(\pi_{\mathfrak{F}}(\upsilon)) \big\}.$$

If Ω is any subset of $\mathfrak{F}$, we take $\Omega_\delta = \Omega \cap \mathfrak{F}_\delta$.

3. Genus-one Gromov–Witten invariants

3A. ***Setup.*** Our next goal is to prove Propositions 2.5 and 2.6. We start by clarifying the setup described after Proposition 2.6. We also specify the open subsets of admissible perturbations of the $\bar\partial_J$-operator to be used in proving the propositions; see Definition 2.4.

Let U_s be the neighborhood of Y in $\mathbb{P}^4$ and $\tilde{T}Y$ the subbundle of $T\mathbb{P}^4|_{U_s}$ as in Section 2B. We set

$$\mathfrak{X}_s = \big\{ [\Sigma, j; u] \in \mathfrak{X}_1(\mathbb{P}^4, d) : u(\Sigma) \subset U_s \big\}.$$

Let ν be a multisection of the bundle $\Gamma_1^{0,1}(\mathbb{P}^4, d)$ such that

(ν1) for every open neighborhood $\mathcal{U}$ of $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ in $\mathfrak{X}_1(\mathbb{P}^4, d)$, there exists $\epsilon_\nu(\mathcal{U}) > 0$ such that $\{\bar\partial_J + t\nu\}^{-1}(0)$ is contained in $\mathcal{U}$ for all $t \in (0, \epsilon_\nu(\mathcal{U}))$;

(ν2) $\nu(b) \in \Gamma(\Sigma; \Lambda_{J,j}^{0,1} T^*\Sigma \otimes u^*\tilde{T}Y)/\mathrm{Aut}(b)$ if $b = [\Sigma, j, u] \in \mathfrak{X}_s$, and $\nu(b) = 0$ if $b \notin \mathfrak{X}_s$;

($\nu 3$) for some $\epsilon_\nu > 0$ and for all $t \in (0, \epsilon_\nu)$, the multisection $\bar\partial_J + t\nu$ does not vanish on $\mathfrak{X}_1(Y, d) - \mathfrak{X}_1^0(Y, d)$ and is transversal to the zero set in $\Gamma_1^{0,1}(Y, d; J)$ along $\mathfrak{X}_1^0(Y, d)$.

The middle condition implies that $\bar\partial_{\nabla,u}\{s \circ u\} = 0$ if $[\Sigma, j, u] \in \{\bar\partial_J + t\nu\}^{-1}(0)$. It can be shown, by slightly modifying the proof of Corollary 3.11, that the finite-dimensional conditions (ν3a)–(ν3c) stated below imply (ν3).

If ν is a section of the bundle $\Gamma_1^{0,1}(\mathbb{P}^4; d)$ over $\mathfrak{X}_1(\mathbb{P}^4, d)$ as in (ν1) and (ν2), for every $\kappa \in \mathcal{S}_0(Y; J)$, we define a section of the bundle $\mathcal{W}^{1,1}_{\kappa,d/d_\kappa} \to \overline{\mathfrak{M}}^1_1(\kappa, d/d_\kappa)$ (where $\mathcal{W}^{1,1}_{\kappa,d/d_\kappa}$ is as in Section 2B) by setting

$$\pi^1_{\nu,\kappa}(b) = [\nu(b)],$$

where $[\nu(b)]$ is the (0, 1)-cohomology class of $\nu(b)$. For each $q \in \mathbb{Z}^+$, we define a section of the bundle

$$\iota_q^*\widetilde{\mathcal{W}}^{1,q}_{\kappa,d/d_\kappa} \equiv \pi_P^*\mathbb{E}^* \otimes \pi_B^*\mathrm{ev}_0^* N_Y\kappa \oplus \pi_B^*\mathcal{W}^{0,q}_{\kappa,d/d_\kappa} \to \overline{\mathcal{M}}_{1,q} \times \overline{\mathfrak{M}}^q_0(\kappa, d/d_\kappa)$$

by

$$\tilde\pi^q_{\nu,\kappa}(b) = \pi_\kappa^\perp[\nu(\iota_q(b))], \tag{3-1}$$

where

$$\pi_\kappa^\perp : \mathcal{W}^{1,q}_{\kappa,d/d_\kappa} \to \widetilde{\mathcal{W}}^{1,q}_{\kappa,d/d_\kappa}$$

is the projection map corresponding to the quotient of $\iota_q^*\mathcal{W}^{1,q}_{\kappa,d/d_\kappa}$ by $\pi_P^*\mathbb{E}^*\otimes\pi_B^*\mathrm{ev}_0^*T\kappa$; see (2-12). Finally, we define a section of the bundle $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa} \to \overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa)$ by setting

$$\pi^0_{\nu,\kappa}(b) = \begin{cases} \tilde\pi^q_{\nu,\kappa}(\iota_q^{-1}(b)) & \text{if } b \in \overline{\mathfrak{M}}^q_1(\kappa, d/d_\kappa),\ q \in \mathbb{Z}^+, \\ [\nu(b)] & \text{otherwise;} \end{cases}$$

see (2-15). This section is well-defined on $\overline{\mathfrak{M}}^{q_1}_1(\kappa, d/d_\kappa) \cap \overline{\mathfrak{M}}^{q_2}_1(\kappa, d/d_\kappa)$.

We denote by $\tilde{\mathcal{A}}^d_1(\bar\partial, J)$ the space of multisections ν as in (ν1) and (ν2) such that, for all $\kappa \in \mathcal{S}_0(Y; J)$,

(ν3a) the section $\pi^0_{\nu,\kappa}$ does not vanish on $\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa) - \mathfrak{M}^0_1(\kappa, d/d_\kappa)$ and is transversal to the zero set on $\mathfrak{M}^0_1(\kappa, d/d_\kappa)$;

(ν3b) the section $\pi^1_{\nu,\kappa}$ does not vanish on $\overline{\mathfrak{M}}^1_1(\kappa, d/d_\kappa)$;

(ν3c) the section $\tilde\pi^1_{\nu,\kappa}$ does not vanish on $\overline{\mathfrak{M}}^1_1(\kappa, d/d_\kappa) - \mathfrak{M}^1_1(\kappa, d/d_\kappa)$ and is transversal to the zero set on $\mathfrak{M}^1_1(\kappa, d/d_\kappa)$.

By (2-6), (2-8), (2-13), and Lemmas 4.1 and 4.2, these conditions are satisfied by a dense open path-connected subset of sections ν.

3B. ***Proof of Proposition 2.5.*** We will focus on the last case of Proposition 2.5, which follows from Proposition 3.1. The claim in the first case is clear, since the finite set $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ consists of transverse zeros of the section $\bar\partial_J$ over $\mathfrak{X}_1(Y, d)$. The proof of Proposition 3.1 applies to this case as well, except there is no gluing to be done.

Let s and Y be as in Proposition 2.5. For every bubble type $\mathcal{T}$ and every rational J-holomorphic curve κ in Y, we put

$$\mathcal{U}_{\mathcal{T};\kappa} = \big\{[\mathcal{C}, u] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) : u(\mathcal{C}) = \kappa\big\}.$$

Proposition 3.1. *Suppose d, Y, and J are as in Proposition 2.5, $\nu \in \tilde{\mathcal{A}}_1^d(\bar\partial; J)$ is a generic perturbation of the $\bar\partial_J$-operator on $\mathfrak{X}_1(\mathbb{P}^4, d)$, $\kappa \in \mathcal{S}_0(Y; J)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i\in I} d_i = d$ and $d_i \neq 0$ for some minimal element i of I. If $|I| > 1$ or $\aleph \neq \varnothing$, for every compact subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\mathfrak{X}_1(Y, d)$ such that*

$$\{\bar\partial_J + t\nu\}^{-1}(0) \cap U(K) = \varnothing \qquad \textit{for all } t \in (0, \epsilon_\nu(K)).$$

If $|I| = 1$ and $\aleph = \varnothing$, for every compact subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\mathfrak{X}_1(Y, d)$ with the following properties:

(a) *the section $\bar\partial_J + t\nu$ is transverse to the zero set in $\Gamma_1^{0,1}(Y, d; J)$ over $U(K)$ for all $t \in (0, \epsilon_\nu(K))$;*

(b) *for every open subset U of $\mathfrak{X}_1(Y, d)$, there exists $\epsilon(U) \in (0, \epsilon_\nu(K))$ such that*

$$^{\pm}\big|\{\bar\partial_J + t\nu\}^{-1} \cap U\big| = \big\langle e(\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)]\big\rangle$$

if $\pi^{0\ -1}_{\nu,\kappa}(0) \subset K \subset U \subset U(K)$ and $t \in (0, \epsilon(U))$.

In other words, the contribution from the main stratum $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ to the number $N_1(d)$, as computed via the section $\bar\partial_J$, is the Euler class of the vector bundle $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$ over $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. None of the boundary strata of $\mathfrak{M}_1^{\{0\}}(\kappa, d/d_\kappa)$ contributes to $N_1(d)$.

We fix a J-compatible metric $g_{\mathbb{P}^4}$ on $\mathbb{P}^4$ and proceed as in Section 4.1 of [Zinger 2004b]. For each sufficiently small element $\upsilon = (b, v)$ of $\tilde{\mathcal{F}}\mathcal{T}^{\varnothing}$, let

$$b(\upsilon) = \big(\Sigma_\upsilon, j_\upsilon; u_\upsilon\big), \quad \text{where } u_\upsilon = u_b \circ q_\upsilon,$$

be the corresponding approximately holomorphic stable map. Here

$$q_\upsilon : \Sigma_\upsilon \to \Sigma_b$$

is the basic gluing map constructed in Section 4.1 of [Zinger 2004b]. Since $d_i \neq 0$ for some minimal element i of I, i.e. the stable map b is nonconstant on the principal curve of the domain Σ_b of b, the linearization $D_{J,b}$ of the $\bar\partial_J$-operator at b is surjective, since $\|J - J_0\|_{C^1} \le \delta(b)$. Thus, if υ is sufficiently small, the linearization

$$D_{J,\upsilon} : \Gamma(\upsilon; T\mathbb{P}^4) \equiv L^p_1(\Sigma_\upsilon; u^*_\upsilon T\mathbb{P}^4) \longrightarrow \Gamma^{0,1}(\upsilon, T\mathbb{P}^4; J) \equiv L^p(\Sigma_\upsilon; \Lambda^{0,1}_{J,j} T^*\Sigma_\upsilon \otimes u^*_\upsilon T\mathbb{P}^4)$$

of the $\bar\partial_J$-operator at $b(\upsilon)$, defined via the J-compatible connection ∇^J in $T\mathbb{P}^4$ corresponding to the Levi-Civita connection of the metric $g_{\mathbb{P}^4}$, is also surjective. In particular, we can obtain an orthogonal decomposition

$$\Gamma(\upsilon; T\mathbb{P}^4) = \Gamma_-(\upsilon; T\mathbb{P}^4) \oplus \Gamma_+(\upsilon; T\mathbb{P}^4) \tag{3-2}$$

such that the linear operator

$$D_{J,\upsilon} : \Gamma_+(\upsilon; T\mathbb{P}^4) \to \Gamma^{0,1}(\upsilon; T\mathbb{P}^4; J)$$

is an isomorphism, while

$$\Gamma_-(\upsilon; T\mathbb{P}^4) = \{\xi \circ q_\upsilon : \xi \in \Gamma_-(b; \mathbb{P}^4)\}, \quad \text{where } \Gamma_-(b; T\mathbb{P}^4) = \ker D_{J,b}.$$

The L^2-inner product on $\Gamma(\upsilon; T\mathbb{P}^4)$ used in the orthogonal decomposition is defined via the metric $g_{\mathbb{P}^4}$ on $\mathbb{P}^4$ and the metric g_υ on Σ_υ induced by the pregluing construction. The Banach spaces $\Gamma(\upsilon; T\mathbb{P}^4)$ and $\Gamma^{0,1}(\upsilon; T\mathbb{P}^4; J)$ carry the norms $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$, respectively, which are also defined by the pregluing construction. These norms are equivalent to the ones used in [Li and Tian 1998]. In particular, the norms of $D_{J,\upsilon}$ and of the inverse of its restriction to $\Gamma_+(\upsilon; T\mathbb{P}^4)$ have fiberwise uniform upper bounds, i.e. dependent only on $[b] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, and not on $\upsilon \in \tilde{\mathcal{F}}\mathcal{T}^\varnothing$.

Lemma 3.2. *If $\mathcal{T}$ is a bubble type and ν is an admissible perturbation of the $\bar\partial_J$-operator on $\mathfrak{X}_1(\mathbb{P}^4, d)$ as in Proposition 3.1, for every precompact open subset K of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, there exist $\delta_K, \epsilon_K, C_K \in \mathbb{R}^+$ and an open neighborhood U_K of K in $\mathfrak{X}_1(\mathbb{P}^4, d)$ with the following properties*:

(1) *for all $\upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}^\varnothing|_{K^{(0)}}$,*

$$\|D_{J,\upsilon}\xi\|_{\upsilon,p} \le C_K |\upsilon|^{1/p} \|\xi\|_{\upsilon,p,1} \quad \textit{for all } \xi \in \Gamma_-(\upsilon; T\mathbb{P}^4) \qquad \textit{and}$$
$$C_K^{-1}\|\xi\|_{\upsilon,p,1} \le \|D_{J,\upsilon}\xi\|_{\upsilon,p} \le C_K\|\xi\|_{\upsilon,p,1} \quad \textit{for all } \xi \in \Gamma_+(\upsilon; T\mathbb{P}^4);$$

(2) *for all $\upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}^\varnothing|_{K^{(0)}}$ and $t \in [0, \delta_K]$, the equation*

$$\bar\partial_J \exp_{u_\upsilon}\xi + t\nu(\exp_{u_\upsilon}\xi) = 0, \qquad \xi \in \Gamma_+(\upsilon; T\mathbb{P}^4),\ \|\xi\|_{\upsilon,p,1} \le \epsilon_K,$$

has a unique solution $\xi_{t\nu}(\upsilon)$, and $\|\xi_{t\nu}(\upsilon)\|_{C^0} \le C_K(t + |\upsilon|^{1/p})$;

(3) *there exist a smooth bundle map* $\zeta_\nu : \tilde{\mathcal{F}}\mathcal{T}^\varnothing \to \tilde{\Gamma}(T\mathbb{P}^4, d)$ *over* $\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4; J)$ *and a continuous function* $\varepsilon_\nu : \mathcal{F}T^\varnothing \to \mathbb{R}$ *such that for all* $\upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}^\varnothing|_{K^{(0)}}$ *and* $t \in [0, \delta_K]$,

$$\big\|\xi_{t\nu}(\upsilon) - \xi_0(\upsilon) - tq_\upsilon^*\zeta_\nu(b)\big\|_{C^0} \le C_K\big(t + \varepsilon_\nu(\upsilon)\big)t \quad \text{and} \quad \lim_{\upsilon\to b} \varepsilon_\nu(\upsilon) = 0;$$

(4) *the map*

$$\phi_{\mathcal{T},t\nu} : \mathcal{F}T^\varnothing_{\delta_K}\big|_K \to \mathfrak{X}^0_1(\mathbb{P}^4, d), \quad [\upsilon] \to [\tilde{b}_{t\nu}(\upsilon)],$$

where $\tilde{b}_{t\nu}(\upsilon) = \big(\Sigma_\upsilon, j_\upsilon; \exp_{u_\upsilon} \xi_{t\nu}(\upsilon)\big)$, *is an orientation-preserving diffeomorphism onto* $\big\{\bar\partial_J + t\nu\big\}^{-1}(0) \cap \mathfrak{X}^0_1(\mathbb{P}^4, d) \cap U_K$.

The first claim of the lemma is a special case of Lemma 4.1 in [Zinger 2004b]. The second statement is obtained by expanding the equation at u_υ and applying the Contraction Principle; see Section 3.6 in [Zinger 2004a]. The uniqueness part means that there is a unique solution for each branch of the multisection ν. In (3), $\tilde{\Gamma}(T\mathbb{P}^4, d)$ denotes the Banach bundle over the space $\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4, d; J)$ such that

$$\tilde{\Gamma}(T\mathbb{P}^4, d)\big|_{(\Sigma_b, u_b)} = \Gamma(\Sigma_b; u_b^*T\mathbb{P}^4).$$

Let P_υ and P_b, respectively, denote the inverses of $D_{J,\upsilon}$ on $\Gamma_+(\upsilon; T\mathbb{P}^4)$ and of $D_{J,b}$ on $\Gamma_+(b; T\mathbb{P}^4)$. The Banach space $\Gamma_+(b; T\mathbb{P}^4)$ is the orthogonal complement of $\Gamma_-(b; T\mathbb{P}^4)$ in

$$\Gamma(b; T\mathbb{P}^4) \equiv L^p_1(\Sigma_b; u_b^*T\mathbb{P}^4);$$

see Section 3.1 in [Zinger 2004a]. Taking the difference of the expansions for the equations in (2) describing $\xi_{t\nu}(\upsilon)$ and $\xi_0(\upsilon)$ and applying P_υ, one finds that

$$\big\|\xi_{t\nu}(\upsilon) - \xi_0(\upsilon) - tP_\upsilon\nu(u_\upsilon)\big\|_{C^0} \le C_K(t + |\upsilon|^{1/p})t.$$

On the other hand, a direct computation shows that

$$\begin{aligned}\|P_\upsilon\nu(u_\upsilon) - q_\upsilon^*P_b\nu(u_b)\big\|_{C^0} &\le C(b)\|\nu(u_\upsilon) - D_{J,\upsilon}q_\upsilon^*P_b\nu(u_b)\big\|_{\upsilon,p} + \tilde\varepsilon_\nu(\upsilon)\\ &\le C(b)\|\nu(u_\upsilon) - q_\upsilon^*\nu(u_b)\big\|_{\upsilon,p} + \tilde\varepsilon'_\nu(\upsilon) \le \varepsilon_\nu(\upsilon);\end{aligned}$$

see Section 4.1 in [Zinger 2004a] for a similar computation. These two bounds imply (3) of Lemma 3.2, with $\zeta_\nu(b) = P_b\nu(u_b)$. Finally, the proof of (4) is similar to Sections 3.8 and 4.3-4.5 of [Zinger 2004a].

Lemma 3.3. *Suppose* $\mathcal{T}$ *and* ν *are as in Lemma 3.2. For every precompact open subset* K *of* $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, *there exist* $\delta_K, \epsilon_K, C_K \in \mathbb{R}^+$, *an open neighborhood* U_K *of* K *in* $\mathfrak{X}_1(\mathbb{P}^4, d)$, *and injective vector-bundle homomorphisms*

$$\tilde\phi_{\mathcal{T},t\nu} : \pi^*_{\mathcal{F}T}\mathcal{V}^d_1\big|_{\mathcal{F}^\varnothing\mathcal{T}_{\delta_K}} \to \Gamma(\mathfrak{L}; d),$$

covering the maps $\phi_{\mathcal{T},t\nu}$ *of Lemma 3.2, with the following properties*:

(1) *requirements* (1)-(4) *of Lemma 3.2 are satisfied*;

(2) $\lim_{(\upsilon,w)\to(b,w^*)}\tilde{\phi}_{\mathcal{T},t\nu}(\upsilon;w)=w^*$ *for all* $b\in K$ *and* $w^*\in\mathcal{V}_1^d$;

(3) $s_1^d(\phi_{\mathcal{T},t\nu}(\upsilon))\equiv[s\circ\exp_{u_\upsilon}\xi_{t\nu}]\in\operatorname{Im}\tilde{\phi}_{\mathcal{T},t\nu}$, *and for all* $[\upsilon]=[b,v]\in\mathcal{F}T^{\varnothing}_{\delta_K}\big|_K$

$$\left|\tilde{\phi}^{-1}_{\mathcal{T},t\nu}s_1^d\big(\phi_{\mathcal{T},t\nu}(\upsilon)\big)-\tilde{\phi}^{-1}_{\mathcal{T},0}s_1^d\big(\phi_{\mathcal{T},0}(\upsilon)\big)-t\{\nabla s\}\zeta_\nu(b)\right|\le C_K\big(t+\varepsilon_\nu(\upsilon)\big)t,$$

where $\varepsilon_\nu:\mathcal{F}T^{\varnothing}\to\mathbb{R}$ *is a continuous function such that* $\lim_{\upsilon\to b}\varepsilon_\nu(\upsilon)=0$ *for all* $b\in\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J)$.

Proof. (1) We need to construct a lift $\tilde{\phi}_{\mathcal{T},t\nu}$ that has the desired properties. For each element $\upsilon=(b,v)$ of $\tilde{\mathcal{F}}\mathcal{T}^{\varnothing}_{\delta_K}\big|_{K^{(0)}}$, $t\in[0,\delta_K)$, and $\xi\in\Gamma(b;\mathfrak{L})$, define

$$R_\upsilon\xi\in\Gamma(\upsilon;\mathfrak{L})\equiv L^p_1\big(\Sigma_\upsilon;u^*_\upsilon\mathfrak{L}\big)\quad\text{by}\quad\big\{R_\upsilon\xi\big\}(z)=\xi\big(q_\upsilon(z)\big)\quad\text{for all }\ z\in\Sigma_\upsilon$$

and

$$R_{\upsilon,t\nu}\xi\in\Gamma(\tilde{b}_{t\nu}(\upsilon);\mathfrak{L})\equiv L^p_1\big(\Sigma_\upsilon;\{\exp_{u_\upsilon}\xi_{t\nu}(\upsilon)\}^*\mathfrak{L}\big)$$

by

$$\big\{R_{\upsilon,t\nu}\xi\big\}(z)=\Pi_{\{\xi_{t\nu}(\upsilon)\}(z)}\big\{R_\upsilon\xi\big\}(z)\quad\text{for all }\ z\in\Sigma_\upsilon,$$

where $\Pi_{\{\xi_{t\nu}(\upsilon)\}(z)}\big\{R_\upsilon\xi\big\}(z)$ is the ∇-parallel transport of $\big\{R_\upsilon\xi\big\}(z)$ along the ∇^J-geodesic

$$\gamma_{\{\xi_{t\nu}(\upsilon)\}(z)}:[0,1]\to\mathbb{P}^4,\qquad\tau\to\exp_{u_\upsilon(z)}\tau\{\xi_{t\nu}(\upsilon)\}(z).$$

We denote the image of

$$\Gamma_-(b;\mathfrak{L})\equiv\ker\bar{\partial}_{\nabla,b}$$

under the linear map $R_{\upsilon,t\nu}$ by $\tilde{\Gamma}_-(\tilde{b}_{t\nu}(\upsilon);\mathfrak{L})$. If δ_K is sufficiently small, the L^2-orthogonal projection

$$\tilde{\pi}_{\upsilon,t\nu}:\Gamma(\tilde{b}_{t\nu}(\upsilon);\mathfrak{L})\to\tilde{\Gamma}_-(\tilde{b}_{t\nu}(\upsilon);\mathfrak{L}),$$

defined with respect to the metric g_υ on Σ_υ, restricts to an isomorphism on

$$\Gamma_-\big(\tilde{b}_{t\nu}(\upsilon);\mathfrak{L}\big)\equiv\ker\bar{\partial}_{\nabla,\tilde{b}_{t\nu}(\upsilon)};$$

see Section 3.2 in [Zinger 2007]. Let $\tilde{\pi}^{-1}_{\upsilon,t\nu}$ be the inverse of this isomorphism. We set

$$\tilde{\phi}_{\mathcal{T},t\nu}([\upsilon;\xi])=[\tilde{\pi}^{-1}_{\upsilon,t\nu}R_{\upsilon,t\nu}\xi]\qquad\text{for all }\xi\in\Gamma_-(b;\mathfrak{L}).$$

(2) By our assumptions on ν,

$$\bar{\partial}_{\nabla,\tilde{b}_{t\nu}(\upsilon)}\big(s\circ\exp_{u_\upsilon}\xi_{t\nu}(\upsilon)\big)=0\qquad\Longrightarrow\qquad s_1^d\big(\phi_{\mathcal{T},t\nu}(\upsilon)\big)\in\operatorname{Im}\tilde{\phi}_{\mathcal{T},t\nu}.$$

It remains to prove the estimate in part (3) of the lemma. If ϵ_K is sufficiently small, $\upsilon\in\tilde{\mathcal{F}}\mathcal{T}^{\varnothing}_{\delta_K}|_{K^{(0)}}$, $\xi\in\Gamma(\upsilon;T\mathbb{P}^4)$, and $\|\xi_\upsilon\|_{\upsilon,p,1}<\epsilon_K$, we define $N^s_\upsilon\xi\in\Gamma\big(\upsilon;\mathfrak{L}\big)$ by

$$\Pi^{-1}_{\xi(z)}s\big(\exp_{u_\upsilon(z)}\xi(z)\big)=s(u_\upsilon(z))+\nabla s\big|_{u_\upsilon(z)}\xi(z)+\big\{N^s_\upsilon\xi\big\}(z)\quad\text{for all }z\in\Sigma_\upsilon.$$

The quadratic term N^s_υ varies smoothly with υ; moreover $N^s_\upsilon 0=0$ and

$$\big\| N^s_\upsilon\xi_1 - N^s_\upsilon\xi_2\big\|_{C^0} \le C_s\big(\|\xi_1\|_{C^0}+\|\xi_2\|_{C^0}\big)\|\xi_1-\xi_2\|_{C^0} \tag{3-3}$$

for some $C_s\in\mathbb{R}^+$ and for all $\xi_1,\xi_2\in\Gamma(\upsilon)$ such that $\|\xi_1\|_{\upsilon,p,1},\ \|\xi_2\|_{\upsilon,p,1}<\epsilon_{\mathcal{T},\nu}(K)$. If $\xi\in\Gamma_-(b;\mathfrak{L})$,

$$\big\langle\!\big\langle s_1^d\big(\phi_{\mathcal{T},t\nu}(\upsilon)\big),\, R_{\upsilon,t\nu}\xi\big\rangle\!\big\rangle = \big\langle\!\big\langle \Pi^{-1}_{\xi_{t\nu}(\upsilon)} s_1^d(\phi_{\mathcal{T},t\nu}(\upsilon)),\, \xi\circ q_\upsilon\big\rangle\!\big\rangle.$$

Thus, the estimate in (3) of Lemma 3.3 follows from (3-3) and the estimate in (3) of Lemma 3.2. □

Corollary 3.4. *Suppose $\mathcal{T}$ and ν are as in Lemma 3.2. For every precompact open subset K of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J)$, there exist $\delta_K,\epsilon_K,C_K\in\mathbb{R}^+$, an open neighborhood U_K of K in $\mathfrak{X}_1(\mathbb{P}^4,d)$, and for each $t\in(0,\epsilon(K))$ a sign-preserving bijection*

$$\{\bar\partial_J+t\nu\}^{-1}(0)\cap\mathfrak{X}_1(Y,d)\cap U_K \to \big\{u\in\mathfrak{M}_1^0(\mathbb{P}^4,d;J)\cap U_K : \{s_1^d+t\vartheta_t\}(u)=0\big\},$$

where $\vartheta_t\in\Gamma(\mathfrak{M}_1^0(\mathbb{P}^4,d;J)\cap U_K;\mathcal{V}_1^d)$ is a family of smooth sections such that

$$\lim_{\upsilon\to b,t\to 0}\vartheta_t(\upsilon)=[\{\nabla s\}P_b\nu(b)] \qquad \text{for all } b\in K \qquad \text{and}$$

$$\big|\nabla_X\tilde\phi^{-1}_{\mathcal{T},0}s_1^d\phi_{\mathcal{T},0}(\upsilon)\big|\le C_K|X| \qquad \text{for all } b\in K,\ \upsilon\in\mathcal{F}T^{\varnothing}_{\delta_K}\big|_b,\ X\in\ker D_{J,b},$$

where $\phi_{\mathcal{T},0}$ and $\tilde\phi_{\mathcal{T},0}$ are as in Lemma 3.3.

Proof: The section ϑ_t is given by

$$t\vartheta_t\big(\phi_{\mathcal{T},0}(\upsilon)\big)=\tilde\phi_{\mathcal{T},0}\tilde\phi^{-1}_{\mathcal{T},t\nu}\big(s_1^d(\phi_{\mathcal{T},t\nu}(\upsilon))\big)-s_1^d(\phi_{\mathcal{T},0}(\upsilon)) \qquad \text{for all } [\upsilon]\in\mathcal{F}T^{\varnothing}_{\delta_K}|_{K^{(0)}}.$$

This corollary is immediate from Lemma 3.3, with the exception of the last estimate. This estimate follows from the behavior of the various terms involved in defining ϑ_t; see Sections 3.4 and 4.2 in [Zinger 2004a].

For each $\kappa\in\mathcal{S}_0(J;Y)$, $\mathcal{U}_{\mathcal{T};\kappa}$ is a smooth suborbifold of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J)$. We denote its normal bundle by $\mathcal{N}^\kappa\mathcal{T}$. Its fiber at $[b]\in\mathcal{U}_{\mathcal{T};\kappa}$ is the quotient

$$\Gamma_-(b;T\mathbb{P}^4)\,/\,\Gamma_-(b;TY),$$

where

$$\Gamma_-(b;TY)\equiv\Gamma_-(b;T\mathbb{P}^4)\cap\Gamma(b;TY)=\Gamma_-(b;T\kappa),$$

by the assumption (J_Y2) on J_Y. We identify $\mathcal{N}^\kappa\mathcal{T}$ with the L^2-orthogonal complement of $\Gamma_-(b;TY)$ in $\Gamma_-(b;T\mathbb{P}^4)$. Let

$$\varphi_{\mathcal{T};\kappa}:\mathcal{N}^\kappa\mathcal{T}_{\delta_\kappa}\to\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J)$$

be an orientation-preserving identification of neighborhoods of $\mathcal{U}_{\mathcal{T};\kappa}$ in $\mathcal{N}^\kappa\mathcal{T}$ and in $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J)$ and let

$$\tilde\varphi_{\mathcal{T};\kappa}:\pi^*_{\mathcal{N}^\kappa\mathcal{T}}\mathcal{F}T\big|_{\mathcal{N}^\kappa\mathcal{T}_{\delta_\kappa}}\to\mathcal{F}T \quad\text{and}\quad \tilde\varphi_{\mathcal{T};\kappa}:\pi^*_{\mathcal{N}^\kappa\mathcal{T}}\mathcal{V}_1^d\big|_{\mathcal{N}^\kappa\mathcal{T}_{\delta_\kappa}}\to\mathcal{V}_1^d$$

be lifts of $\varphi_{\mathcal{T};\kappa}$ to vector-bundle isomorphisms restricting to the identity over $\mathcal{U}_{\mathcal{T};\kappa}$.

The section s_1^d is smooth on $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ and its differential along $\mathcal{U}_{\mathcal{T};\kappa}$, i.e. the homomorphism j_0 in the long exact sequence

$$0 \to \Gamma_-(b; TY) \xrightarrow{i_0} \Gamma_-(b; T\mathbb{P}^4) \xrightarrow{j_0} \Gamma_-(b; \mathfrak{L}) \xrightarrow{\partial_0} H^1_J(b; TY) \to 0, \tag{3-4}$$

is injective on $\mathcal{N}^\kappa\mathcal{T}$. We denote the image bundle of j_0 by $\mathcal{V}_+ \subset \mathcal{V}_1^d$ and its L^2-orthogonal complement in $\mathcal{V}_1^d$ by $\mathcal{V}_-$. Let π_+ and π_- be the corresponding projection maps.

Lemma 3.5. *Suppose $\mathcal{T}$ is a bubble type as in Lemma 3.2 and $\kappa \in \mathcal{S}_0(Y; J)$. For every precompact open subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist*

(a) *$\delta_K, \delta'_K \in \mathbb{R}^+$ and an open neighborhood U_K of K in $\mathfrak{X}_1(\mathbb{P}^4, d)$,*

(b) *an orientation-preserving diffeomorphism*

$$\phi_{\mathcal{T},\kappa} : \mathcal{N}^\kappa\mathcal{T}_{\delta'_K} \times_K \mathcal{F}T^{\varnothing}_{\delta_K} \to \mathfrak{M}^0_1(\mathbb{P}^4, d; J) \cap U_K,$$

(c) *and a lift $\tilde{\phi}_{\mathcal{T},\kappa} : \pi^*_{\mathcal{N}^\kappa\mathcal{T}\oplus\mathcal{F}T}\mathcal{V}_1^d \to \mathcal{V}_1^d$ of $\phi_{\mathcal{T},\kappa}$ to a vector-bundle isomorphism,*

such that the following property is satisfied. If $\vartheta_t \in \Gamma(\mathfrak{M}^0_1(\mathbb{P}^4, d; J) \cap U_K; \mathcal{V}_1^d)$ is a family of smooth sections such that

$$|\vartheta_t(u)| \le C \quad \text{and} \quad \left|\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\vartheta_t\phi_{\mathcal{T},\kappa}(X, \upsilon) - \tilde{\phi}^{-1}_{\mathcal{T},\kappa}\vartheta_t\phi_{\mathcal{T},\kappa}(X', \upsilon)\right| \le C|X - X'|$$

for some constant $C \in \mathbb{R}^+$ and for all $X, X' \in \mathcal{N}^\kappa\mathcal{T}_{\delta'_K}$ and $\upsilon \in \mathcal{F}T^{\varnothing}_{\delta_K}\big|_K$, then there exists $\epsilon \in \mathbb{R}^+$ such that for all $t \in [0, \epsilon)$, $b \in K$, and $\upsilon \in \mathcal{F}T_{\delta_K}|_b$, the equation

$$\pi_+\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(\{s_1^d + t\vartheta_t\}\phi_{\mathcal{T},\kappa}(b; X, \upsilon)\big) = 0$$

has a unique solution $X = X_t(\upsilon) \in \mathcal{N}^\kappa\mathcal{T}_{\delta'_K}|_b$. Furthermore,

$$\lim_{\substack{t\to 0\\ \upsilon\to b}} t^{-1}\pi_-\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(s_1^d\phi_{\mathcal{T},\kappa}(b; X_t(\upsilon), \upsilon)\big) = 0 \quad \textit{for all } b \in K.$$

Proof. (1) The desired maps $\phi_{\mathcal{T},\kappa}$ and $\tilde{\phi}_{\mathcal{T},\kappa}$ are just the compositions $\phi_{\mathcal{T},0} \circ \varphi_{\mathcal{T},\kappa}$ and $\tilde{\phi}_{\mathcal{T},0} \circ \tilde{\varphi}_{\mathcal{T},\kappa}$, respectively. For each $b \in K$, $X \in \mathcal{N}^\kappa\mathcal{T}|_b$, and $\upsilon \in \mathcal{F}T^{\varnothing}|_b$ sufficiently small, we define $\tilde{N}_s(X)$ and $\tilde{N}'_s(X, \upsilon)$ in $\mathcal{V}_1^d|_b$ by

$$\tilde{\varphi}^{-1}_{\mathcal{T},\kappa}\big(s_1^d\varphi_{\mathcal{T},\kappa}(b; X)\big) = s_1^d(b) + j_0X + \tilde{N}_s(X) = j_0X + \tilde{N}_s(X), \tag{3-5}$$

$$\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(s_1^d\phi_{\mathcal{T},\kappa}(b; X, \upsilon)\big) = \tilde{\varphi}^{-1}_{\mathcal{T},\kappa}s_1^d\varphi_{\mathcal{T},\kappa}(b; X) + \tilde{N}'_s(X, \upsilon). \tag{3-6}$$

Since j_0 is the derivative of s_1^d on $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, there exists $C_K \in \mathbb{R}^+$ such that

$$\begin{aligned} &\tilde{N}_s(0) = 0, \\ &\big|\tilde{N}_s(X) - \tilde{N}_s(X')\big| < C_K\big(|X| + |X'|\big)|X - X'| \quad \text{for all } X, X' \in \mathcal{N}^\kappa\mathcal{T}_{\delta'_K}|_K. \end{aligned} \tag{3-7}$$

For $\tilde{N}'_s(\cdot\,,\cdot)$, we similarly have

$$\text{(3-8)}\quad \big|\tilde{N}'_s(X,\upsilon)\big| \le C_K|\upsilon|^{1/p}, \quad \big|\tilde{N}'_s(X,\upsilon)-\tilde{N}'_s(X',\upsilon)\big| < C_K|\upsilon|^{1/p}|X-X'|$$

for all $X, X' \in \mathcal{N}^\kappa\mathcal{T}_{\delta'_K}|_K$ and $\upsilon \in \mathcal{F}T^{\varnothing}_{\delta_K}|_K$. The first of these estimates is clear from Lemma 3.2(2). The second bound follows from the analogous bound on the behavior of the vector field $\xi_0(\upsilon)$ of Lemma 3.2(2); see Section 4.2 in [Zinger 2004a].

(2) If ϑ_t is a family of smooth sections as in the statement of the lemma, by (3-5) and (3-6),

$$\text{(3-9)}\quad \pi_+\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(\{s_1^d+t\vartheta_t\}\phi_{\mathcal{T},\kappa}(b;X,\upsilon)\big) = j_0X+\pi_+\tilde{N}_s(X)+\pi_+\tilde{N}'_s(X,\upsilon)+t\pi_+\tilde{\vartheta}_t(X,\upsilon),$$

where

$$\tilde{\vartheta}_t(X,\upsilon)=\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(\vartheta_t\phi_{\mathcal{T},\kappa}(b;X,\upsilon)\big).$$

By (3-7)–(3-9) and the Contraction Principle, there exist $\delta,\delta'\in\mathbb{R}^+$, dependent on j_0 and C_K, and $\epsilon, C'\in\mathbb{R}^+$, dependent on j_0, C_K, and C, such that for all $t\in[0,\epsilon)$, $b\in K$, and $\upsilon\in\mathcal{F}T^{\varnothing}_{\delta}|_b$, the equation

$$\pi_+\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(\{s_1^d+t\vartheta_t\}\phi_{\mathcal{T},\kappa}(b;X,\upsilon)\big)=0$$

has a unique solution $X=X_t(\upsilon)\in\mathcal{N}^\kappa\mathcal{T}_{\delta'}|_b$. Furthermore,

$$\text{(3-10)}\qquad \big|X_0(\upsilon)\big|\le C'|\upsilon|^{1/p} \quad\text{and}\quad \big|X_t(\upsilon)-X_0(\upsilon)\big|\le C't.$$

(3) By (3-5)–(3-8) and (3-10),

(3-11)

$$\big|\pi_-\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(s_1^d\phi_{\mathcal{T},\kappa}(b;X_t(\upsilon),\upsilon)\big)-\pi_-\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(s_1^d\phi_{\mathcal{T},\kappa}(b;X_0(\upsilon),\upsilon)\big)\le C''\big(t+|\upsilon|^{1/p}\big)t.$$

On the other hand, as can be seen from Lemma 3.6 below,

$$\text{(3-12)}\qquad \pi_-\tilde{\phi}^{-1}_{\mathcal{T},\kappa}\big(s_1^d\phi_{\mathcal{T},\kappa}(b;X_0(\upsilon),\upsilon)\big)=0$$

for all $\upsilon\in\mathcal{F}T^{\varnothing}|_K$ sufficiently small. The last claim of the lemma follows from (3-11) and (3-12). □

Proof of Proposition 3.1. (1) By Corollary 3.4, if $t\in\mathbb{R}^+$ and $U(K)$ are sufficiently small, there is a one-to-one correspondence between $\{\bar\partial_J+t\nu\}^{-1}(0)\cap U(K)$ and the set

$$\big\{u\in\mathfrak{M}_1^0(\mathbb{P}^4,d;J)\cap U(K):\{s_1^d+t\vartheta_t\}(u)=0\big\},$$

where $\vartheta_t\in\Gamma(\mathfrak{M}_1^0(\mathbb{P}^4,d;J)\cap U_K;\mathcal{V}_1^d)$ is a family of smooth sections as in Lemma 3.5. In addition,

$$\lim_{\substack{\upsilon\to b\\ t\to 0}}\vartheta_t(\upsilon)=[\{\nabla s\}P_b\nu(b)] \quad\text{for all } b\in K.$$

The homomorphism $\mathfrak{d}_0$ in the long exact sequence (3-4) restricts to an isomorphism on $\mathcal{V}_-$ and vanishes on $\mathcal{V}_+$. By definition of $\mathfrak{d}_0$ and P_b,

$$\mathfrak{d}_0([\{\nabla s\}P_b\nu(b)]) = \pi^0_{\kappa,\nu}(b) \quad \text{for all } b \in \mathcal{U}_{\mathcal{T};\kappa}.$$

Thus, by Lemma 3.5,

$$\{\bar\partial_J + t\nu\}^{-1}(0) \cap U(K) = \varnothing \quad \text{if } \pi^{0\;-1}_{\kappa,\nu}(0) \cap K = \varnothing.$$

The case $|I| > 1$ or $\aleph \neq \varnothing$ of Proposition 3.1 now follows from the assumption (ν3a).

(2) If $|I| = 1$ and $\aleph = \varnothing$, by the assumption (ν3a) and Lemma 3.5, the section $\bar\partial_J + t\nu$ is transverse to the zero set on $U(K)$ and

$$^{\pm}\big|\{\bar\partial_J + t\nu\}^{-1} \cap U(K)\big| = {}^{\pm}\big|\pi^{0\;-1}_{\kappa,\nu}(0) \cap K\big|.$$

Since $\pi^{0\;-1}_{\kappa,\nu}(0) \subset \mathcal{U}_{\mathcal{T};\kappa} = \mathfrak{M}^0_1(\kappa, d/d_\kappa)$,

$$^{\pm}\big|\{\bar\partial_J + t\nu\}^{-1} \cap U(K)\big| = {}^{\pm}\big|\pi^{0\;-1}_{\kappa,\nu}(0)\big| = \big\langle e(\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa)]\big\rangle,$$

provided $\pi^{0\;-1}_{\kappa,\nu}(0) \subset K$ and t and $U(K)$ are sufficiently small. □

We conclude this subsection with Lemma 3.6, which was used in Lemma 3.5.

Lemma 3.6. *Suppose $\mathcal{T}$ is a bubble type as in Lemma 3.2 and $\kappa \in \mathcal{S}_0(Y, J)$. For every precompact open subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\delta \in \mathbb{R}^+$, an open neighborhood U of K in $\mathfrak{X}_1(\mathbb{P}^4, d)$, and an orientation-preserving diffeomorphism*

$$\phi'_{\mathcal{T},\kappa} : \mathcal{F}T^{\varnothing}_\delta|_K \to \mathfrak{M}^0_1(\kappa, d/d_\kappa) \cap U \subset \mathfrak{M}^0_1(\mathbb{P}^4, d; J).$$

Proof. If $\mathcal{T} = (I, \aleph; \underline{d})$, we have

$$\mathcal{U}_{\mathcal{T};\kappa} = \mathcal{U}_{\mathcal{T}'}(\kappa; J_0) \approx \mathcal{U}_{\mathcal{T}'}(\mathbb{P}^1; J_0) \quad \text{and} \quad \mathcal{F}T|_{\mathcal{U}_{\mathcal{T};\kappa}} = \mathcal{F}T' \to \mathcal{U}_{\mathcal{T}'}(\mathbb{P}^1; J_0),$$

where $\mathcal{T}' = (I, \aleph; \underline{d}')$ and $d'_i = d_i/d_\kappa$. Thus, Lemma 3.6 is the $\mathbb{P}^1$-analogue of the $t = 0$ case of Lemma 3.2(4). □

3C. *Proof of Proposition 2.6.* This proposition follows directly from the next:

Proposition 3.7. *Suppose d, Y, and J are as in Proposition 2.5, $\nu \in \tilde{\mathcal{A}}^d_1(\bar\partial; J)$ is a generic perturbation of the $\bar\partial_J$-operator on $\mathfrak{X}_1(\mathbb{P}^4, d)$, $\kappa \in \mathcal{S}_0(Y; J)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i\in I} d_i = d$ and $d_i = 0$ for all minimal elements i of I. If $|\hat I| > 1$ or $\aleph \neq \varnothing$, for every compact subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U_\nu(K)$ of K in $\mathfrak{X}_1(Y, d)$ such that*

$$\{\bar\partial_J + t\nu\}^{-1}(0) \cap U_\nu(K) = \varnothing \qquad \textit{for all } t \in (0, \epsilon_\nu(K)).$$

If $|\hat I| = 1$ and $\aleph = \varnothing$, for every compact subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\mathfrak{X}_1(Y, d)$ with the following properties:

(a) *the section* $\bar\partial_J + t\nu$ *is transverse to the zero set in* $\Gamma^{0,1}_1(Y,d;J)$ *over* $U(K)$ *for all* $t\in(0,\epsilon_\nu(K))$;

(b) *for every open subset* U *of* $\mathfrak{X}_1(Y,d)$, *there exists* $\epsilon(U)\in(0,\epsilon_\nu(K))$ *such that*

$$^{\pm}\big|\{\bar\partial_J+t\nu\}^{-1}\cap U\big| = \frac{d/d_\kappa}{12}\big\langle e(\mathcal{W}^0_{\kappa,d/d_\kappa}),[\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)]\big\rangle$$

if $\tilde\pi^{1\ -1}_{\nu,\kappa}(0)\subset K\subset U\subset U(K)$ *and* $t\in(0,\epsilon(U))$.

In simpler words, none of the strata of $\overline{\mathfrak{M}}^q_1(\kappa,d/d_\kappa)$ with $q\ge 2$ contributes to the number $N_1(d)$. Neither does any of the boundary strata of $\overline{\mathfrak{M}}^1_1(\kappa,d/d_\kappa)$. On the other hand, $\mathfrak{M}^1_1(\kappa,d/d_\kappa)$ contributes the Euler class of the bundle $\widetilde{\mathcal{W}}^{1,1}_{\kappa,d/d_\kappa}$; see Section 3A.

We will proceed similarly to Section 3B, but run the gluing construction in Y, instead of $\mathbb{P}^4$, and make use of the assumption (J_Y2) from the start. We will also use the family of metrics on $\mathbb{P}^1$ provided by Lemma 2.1 in [Zinger 2003], which we now restate:

Lemma 3.8. *There exist* $r_{\mathbb{P}^1}>0$ *and a smooth family of Kähler metrics*

$$\{g_{\mathbb{P}^1,q}: q\in\mathbb{P}^1\}$$

on $\mathbb{P}^1$ *with the following property. If* $B_q(q',r)\subset\mathbb{P}^1$ *denotes the* $g_{\mathbb{P}^1,q}$*-geodesic ball about* q', *the triple* $(B_q(q,r_{\mathbb{P}^1}),J_0,g_{\mathbb{P}^1,q})$ *is isomorphic to a ball in* $\mathbb{C}^1$ *for all* $q\in\mathbb{P}^1$.

In this case, the operators $D_{J,b}|_{\Gamma(\Sigma_b;u_b^*TY)}$ are not surjective for $b\in\mathcal{U}^{(0)}_{\mathcal{T};\kappa}$, where $\mathcal{U}^{(0)}_{\mathcal{T};\kappa}$ is the preimage of $\mathcal{U}_{\mathcal{T};\kappa}$ under the quotient projection map

$$\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^4;J)\to\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J).$$

Thus, in contrast to the case of Lemma 3.2, we encounter an obstruction bundle in trying to solve the $\bar\partial_J$-equation near $\mathcal{U}_{\mathcal{T};\kappa}$, as in Sections 3.3-3.5 of [Zinger 2004a]. Sections 3.3-3.5 in [Zinger 2003] describe a special case of an analogous construction in circumstances similar to the present situation.

First, we describe a convenient "exponential" map for Y defined on a neighborhood of each smooth curve $\kappa\in\mathcal{S}_0(Y;J)$. We identify the rational curve κ with $\mathbb{P}^1$. For each $b\in\mathcal{U}_{\mathcal{T};\kappa}$, let $g_{Y,b}$ be a J-compatible extension of the metric

$$g_{\kappa,b}\equiv g_{\mathbb{P}^1,\mathrm{ev}_P(b)}$$

on κ provided by Lemma 3.8 to a Riemannian metric on a neighborhood of κ in Y. We identify the normal bundle $N_Y\kappa$ of κ in Y with the $g_{Y,b}$-orthogonal complement of $T\kappa$ in $TY|_\kappa$. Let

$$\exp_b: T\kappa\to\kappa \quad\text{and}\quad \widetilde{\exp}_b:\pi^*_{T\kappa}N_Y\kappa\to N_Y\kappa$$

be the exponential map with respect to the metric $g_{\kappa,b}$ and a lift of $\exp_b$ to a vector bundle homomorphism restricting to the identity over κ. For example, $\widetilde{\exp}_b$ can be taken to be the $g_{Y,b}$-parallel transport along the $g_{\kappa,b}$-geodesics. For each $q \in \kappa$ and $\xi \in T_qY$ sufficiently small, let

$$\exp_b \xi = \exp_{g_{Y,b},\exp_{b,q}\xi_-}\big(\widetilde{\exp}_{b,\xi_-}\xi_+\big) \quad \text{if } \xi = \xi_- + \xi_+ \in T\kappa \oplus N_Y\kappa = TY,$$

where $\exp_{g_{Y,b}}$ is the exponential map for the metric $g_{Y,b}$. One useful property of this "exponential" map is that $\exp_b \xi \in \kappa$ if $\xi \in T\kappa \subset TY$.

For each element $b = (\Sigma_b, u_b)$ of $\mathcal{U}^{(0)}_{\mathcal{T};\kappa}$, we identify the cokernel $H^1_J(b;\, TY)$ of the operator

$$D_{J,b} : \Gamma(b;\, TY) \to \Gamma^{0,1}(b;\, TY;\, J)$$

with the space $\Gamma^{0,1}_-(b;TY)$ of (J, j)-antilinear u_b^*TY-valued harmonic forms on Σ_b. The elements of $\Gamma^{0,1}_-(b;\, TY)$ may have simple poles at the nodes of Σ_b with the residues adding up to zero at each node. If $\mathcal{H}_{b,P}$ denotes the one-dimensional vector space of harmonic antilinear differentials on the principal component(s) $\Sigma_{b,P}$ of Σ_b,

$$\Gamma^{0,1}_-(b;\, TY) = \Gamma^{0,1}_-(b;\, T\kappa) \oplus \Gamma^{0,1}_-(b;\, N_Y\kappa) = \mathcal{H}_{b,P} \otimes T_{\mathrm{ev}_P(b)}\kappa \oplus \Gamma^{0,1}_-(b;\, N_Y\kappa).$$

This decomposition is L^2-orthogonal. Furthermore, $\Gamma^{0,1}_-(b;\, N_Y\kappa)$ is isomorphic to the cokernel $H^1_J(b;\, N_Y\kappa)$ of the operator

$$D^\perp_{J,b} : \Gamma(b;\, N_Y\kappa) \to \Gamma^{0,1}(b;\, N_Y\kappa;\, J)$$

induced by the operator $D_{J,b}$ via the quotient projection map

$$\pi^\perp_\kappa : TY|_\kappa \to N_Y\kappa = TY|_\kappa \big/ T\kappa.$$

We note that if $\aleph = \varnothing$ and $|\hat{I}| = 1$, $\Gamma^{0,1}_-(b;\, TY)$ is a subspace of $\Gamma^{0,1}(b;\, TY;\, J)$.

We are now ready to proceed with the pregluing construction. For each small enough element $\upsilon = (b, v)$ of $\tilde{\mathcal{F}}\mathcal{T}^{\varnothing}$, let

$$b(\upsilon) = \big(\Sigma_\upsilon,\, j_\upsilon;\, u_\upsilon\big)$$

be the corresponding approximately holomorphic stable map, as on page 444. In the present case, the linearization $D_{J,b}$ of the $\bar\partial_J$-operator at b is not surjective. Thus, the linearization $D_{J,\upsilon}$ of the $\bar\partial_J$-operator at $b(\upsilon)$, defined via the Levi-Civita connection of the metric $\tilde{g}_{Y,b}$, is not uniformly surjective. An approximate cokernel of $D_{J,b}$ is given by

$$\Gamma^{0,1}_-(\upsilon;\, TY) = \Gamma^{0,1}_-(\upsilon;\, T\kappa) \oplus \Gamma^{0,1}_-(\upsilon;\, N_Y\kappa), \tag{3-13}$$

with the vector spaces $\Gamma^{0,1}_-(\upsilon;\, T\kappa)$ and $\Gamma^{0,1}_-(\upsilon;\, N_Y\kappa)$ explicitly describable from $\Gamma^{0,1}_-(b;\, T\kappa)$ and $\Gamma^{0,1}_-(b;\, N_Y\kappa)$, respectively, via the basic gluing map $q_\upsilon : \Sigma_\upsilon \to \Sigma_b$.

In fact, we can simply take

$$\Gamma_-^{0,1}(\upsilon; N_Y\kappa) = \{q_\upsilon^*\eta : \eta \in \Gamma_-^{0,1}(b; N_Y\kappa)\}. \tag{3-14}$$

While we can define the space $\Gamma_-^{0,1}(\upsilon; T\kappa)$ in the same way from $\Gamma_-^{0,1}(b; T\kappa)$, in the $\aleph = \varnothing$, $|\hat{I}| = 1$ case it is more convenient to take

$$\Gamma_-^{0,1}(\upsilon; T\kappa) = \{R_\upsilon\eta : \eta \in \Gamma_-^{0,1}(b; T\kappa)\},$$

where $R_\upsilon\eta$ is a smooth extension of η such that $R_\upsilon\eta$ is harmonic on the neck attaching the only bubble $\Sigma_{b,h}$ of Σ_b and below a small collar of the neck and vanishes past a slighter larger collar. For an explicit description of $R_\upsilon\eta$, see the construction at the beginning of Section 2.2 in [Zinger 2003]. We observe that

$$\langle\langle\eta_\kappa, \tilde{\eta}\rangle\rangle_{\upsilon,2} = 0, \quad \langle\langle\bar{\partial}_J u_\upsilon, \tilde{\eta}\rangle\rangle_{\upsilon,2} = 0, \quad \langle\langle D_{J,\upsilon}\xi, \tilde{\eta}\rangle\rangle_{\upsilon,2} = 0, \tag{3-15}$$

for all $\xi \in \Gamma(\upsilon; T\kappa)$, $\eta_\kappa \in \Gamma_-^{0,1}(\upsilon; T\kappa)$ and $\tilde{\eta} \in \Gamma_-^{0,1}(\upsilon; N_Y\kappa)$, where $\langle\langle\cdot,\cdot\rangle\rangle_{\upsilon,2}$ is the L^2-inner product of the metric $g_{Y,b}$ on Y. This inner product is independent of the choice of a metric on Σ_υ compatible with the complex structure j_υ on Σ_υ, though we will always view Σ_υ as carrying the metric g_υ induced by the pregluing construction. If $\aleph = \varnothing$ and $|\hat{I}| = 1$, $\Gamma_-^{0,1}(\upsilon; TY)$ is a subspace of $\Gamma^{0,1}(\upsilon; TY; J)$, and we denote its L^2-orthogonal complement by $\Gamma_+^{0,1}(\upsilon; TY)$. Let

$$\pi_{\upsilon;\kappa}^{0,1}, \tilde{\pi}_\upsilon^{0,1}, \pi_{\upsilon;+}^{0,1} : \Gamma^{0,1}(\upsilon; TY; J) \to \Gamma_-^{0,1}(\upsilon; T\kappa), \Gamma_-^{0,1}(\upsilon; N_Y\kappa), \Gamma_+^{0,1}(\upsilon; TY)$$

be the L^2-projection maps.

As in Section 3B, if υ is sufficiently small, we can also obtain a decomposition

$$\Gamma(\upsilon; TY) = \Gamma_-(\upsilon; TY) \oplus \tilde{\Gamma}_+(\upsilon; TY) \tag{3-16}$$

such that the linear operator

$$D_{J,\upsilon} : \Gamma_+(\upsilon; TY) \to \Gamma^{0,1}(\upsilon; TY; J)$$

is injective, while

$$\Gamma_-(\upsilon; TY) = \{\xi \circ q_\upsilon : \xi \in \Gamma_-(b; TY)\}.$$

In this case, $D_{J,\upsilon}$ denotes the linearization of the $\bar{\partial}_J$-operator at $b(\upsilon)$ with the respect to the "exponential" map chosen above. In (3-16), we can take the space $\tilde{\Gamma}_+(\upsilon; TY)$ to be the L^2-orthogonal complement of $\Gamma_-(\upsilon; TY)$, and we do so unless $\aleph = \varnothing$ and $|\hat{I}| = 1$. If $\aleph = \varnothing$ and $|\hat{I}| = 1$, we can choose $\tilde{\Gamma}_+(\upsilon; TY)$ in such a way that

$$\langle\langle D_{J,\upsilon}\xi, \eta\rangle\rangle_{\upsilon,2} = 0 \ \text{ for all } \xi \in \tilde{\Gamma}_+(\upsilon; TY) \cap \Gamma(\upsilon; T\kappa),\ \eta \in \Gamma_-^{0,1}(\upsilon; TY), \tag{3-17}$$

the operator

$$D_{J,\upsilon} : \tilde{\Gamma}_+(\upsilon; T\kappa) \equiv \tilde{\Gamma}_+(\upsilon; TY) \cap \Gamma(\upsilon; T\kappa) \\ \longrightarrow \Gamma_+^{0,1}(\upsilon; T\kappa) \equiv \Gamma_+^{0,1}(\upsilon; TY) \cap \Gamma^{0,1}(\upsilon; T\kappa)$$

is an isomorphism, and the intersection of $\tilde{\Gamma}_+(\upsilon; TY)$ with the L^2-orthogonal complement of $\Gamma_-(\upsilon; TY)$ has codimension one in both spaces. The subspace $\tilde{\Gamma}_+(\upsilon; TY)$ of $\Gamma(\upsilon; TY)$ is constructed by restricting the procedure described in Section 2.3 of [Zinger 2003] to the line $\mathscr{H}_{b,\aleph} \otimes T_{\mathrm{ev}_P(b)}\kappa$.

Similarly, let $\Gamma_+(\upsilon; N_Y\kappa)$ be the L^2-orthogonal complement of

$$\Gamma_-(\upsilon; N_Y\kappa) = \big\{\xi \circ q_\upsilon : \xi \in \Gamma_-(b; N_\kappa Y)\big\}$$

in $\Gamma(\upsilon; N_Y\kappa) \equiv L_1^p(\Sigma_\upsilon; u_\upsilon^* N_Y\kappa)$. If υ is sufficiently small, the linear operator

$$D_{J,\upsilon}^\perp : \Gamma_+(\upsilon; N_Y\kappa) \to \Gamma^{0,1}(\upsilon; N_Y\kappa; J)$$

is injective. The key properties of this setup are described in Lemma 3.9:

Lemma 3.9. *If $\mathcal{T}$, ν, and κ are as in Proposition 3.7, for every precompact open subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\delta_K, C_K \in \mathbb{R}^+$ and an open neighborhood U_K of K in $\mathfrak{X}_1(Y, d)$ with the following properties*:

(1) *for every $[\tilde{b}] \in \mathfrak{X}_1^0(Y, d) \cap U_K$, there exist $\upsilon \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^\varnothing|_{K^{(0)}}$ and $\zeta \in \tilde{\Gamma}_+(\upsilon; TY)$ such that $\|\zeta\|_{\upsilon,p,1} < \delta_K$ and $[\exp_{b(\upsilon)} \zeta] = [\tilde{b}]$, and the pair (b, ζ) is unique up to the action of the group $Aut(\mathcal{T}) \propto (S^1)^{\hat{I}}$*;

(2) *for all $\upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^\varnothing|_{K^{(0)}}$,*

$$\|\bar{\partial}_J u_\upsilon\|_{\upsilon,p} \le C_K|\upsilon|^{1/p};$$
$$C_K^{-1}\|\zeta\|_{\upsilon,p,1} \le \|D_{J,\upsilon}\zeta\|_{\upsilon,p} \le C_K\|\zeta\|_{\upsilon,p,1} \quad \textit{for all } \zeta \in \tilde{\Gamma}_+(\upsilon; TY);$$
$$C_K^{-1}\|\xi\|_{\upsilon,p,1} \le \|D_{J,\upsilon}^\perp\xi\|_{\upsilon,p} \le C_K\|\xi\|_{\upsilon,p,1} \quad \textit{for all } \xi \in \Gamma_+(\upsilon; N_\kappa Y);$$

(3) *for all $\upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^\varnothing|_{K^{(0)}}$, $\xi \in \Gamma(\upsilon; N_\kappa Y)$, and $\eta \in \Gamma_-^{0,1}(\upsilon; N_\kappa Y)$,*

$$\big|\langle\langle D_{J,\upsilon}^\perp\xi, \eta\rangle\rangle_{\upsilon,2}\big| \le C_K|\upsilon|^{1/p}\|\xi\|_{\upsilon,p,1}\|\eta\|_{\upsilon,1}.$$

In the first claim of Lemma 3.9,

$$\exp_{b(\upsilon)} \zeta = \big(\Sigma_\upsilon, j_\upsilon; \exp_{b,u_\upsilon} \zeta\big).$$

This statement is a variation on (2) of Lemma 4.4 in [Zinger 2004b] and holds for the same reasons. The first estimate in (2) and (3) of Lemma 3.9 can be obtained by direct computations. The two remaining estimates are proved analogously to the corresponding estimates of Lemma 3.2.

Corollary 3.10. *Suppose ν, $\mathcal{T}$, and κ are as in Proposition 3.7. If $q\in\mathbb{Z}^+$ and K is a compact subset of $\mathcal{U}_{\mathcal{T};\kappa}\subset\overline{\mathfrak{M}}_1^q(\kappa,d/d_\kappa)$ such that*

$$\tilde{\pi}^{q}_{\kappa,\nu}{}^{-1}(0)\cap K=\varnothing,$$

then there exist $\epsilon_\nu(K)\in\mathbb{R}^+$ and an open neighborhood $U_\nu(K)$ of K in $\mathfrak{X}_1(Y,d)$ such that

$$\{\bar\partial_J+t\nu\}^{-1}(0)\cap U_\nu(K)=\varnothing\quad\text{for all } t\in(0,\epsilon_\nu(K)).$$

Proof. (1) As usually, for all $\zeta\in\Gamma(\upsilon;TY)$ sufficiently small,

$$\Pi_\zeta^{-1}\{\bar\partial_J+t\nu\}\exp_{b(\upsilon)}\zeta=\bar\partial_J u_\upsilon+D_{J,\upsilon}\zeta+N_\upsilon\zeta+tN_{\nu,\upsilon}\zeta+t\nu|_{u_\upsilon},$$

where Π_ζ denotes the parallel transport with respect to the Levi-Civita connection of the metric $g_{Y,b}$ along the geodesics of the map $\exp_b$. The nonlinear terms satisfy

$$\begin{aligned}\|N_\upsilon\zeta-N_\upsilon\zeta'\|_{\upsilon,p}&\le C_K\big(\|\zeta\|_{\upsilon,p,1}+\|\zeta'\|_{\upsilon,p,1}\big)\|\zeta-\zeta'\|_{\upsilon,p,1}\\ \|N_{\nu,\upsilon}\zeta-N_{\nu,\upsilon}\zeta'\|_{\upsilon,p}&\le C_K\|\zeta-\zeta'\|_{\upsilon,p,1}\end{aligned}\tag{3-18}$$

for all $\zeta,\zeta'\in\Gamma(\upsilon;TY)$; see Section 3.6 in [Zinger 2004a] for example. Our choice of the map $\exp_b$ also implies that

$$N_\upsilon\zeta\in\Gamma^{0,1}(\upsilon;T\kappa)\quad\text{for all }\zeta\in\Gamma(\upsilon;T\kappa)\subset\Gamma(\upsilon;TY).\tag{3-19}$$

(2) Suppose $\upsilon=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}^{\varnothing}|_{K^{(0)}}$, $\zeta\in\Gamma_+(\upsilon;TY)$, and

$$\{\bar\partial_J+t\nu\}\exp_{b(\upsilon)}\zeta=0\implies\bar\partial_J u_\upsilon+D_{J,\upsilon}\zeta+N_\upsilon\zeta+tN_{\nu,\upsilon}\zeta+t\nu|_{u_\upsilon}=0.\tag{3-20}$$

From (2) of Lemma 3.9 and (3-18), we then obtain

$$\|\zeta\|_{\upsilon,p,1}\le C_K\big(|\upsilon|^{1/p}+t\big).\tag{3-21}$$

On the other hand, applying the projection map $\pi_\kappa^\perp$ to both sides of (3-20), we get

$$D^\perp_{J,\upsilon}\zeta^\perp+N^\perp_\upsilon\zeta+tN^\perp_{\nu,\upsilon}\zeta+t\nu^\perp|_{u_\upsilon}=0\in\Gamma^{0,1}(\upsilon;N_Y\kappa;J),\tag{3-22}$$

if $\zeta=\zeta^t+\zeta^\perp\in\Gamma(\upsilon;T\kappa)\oplus\Gamma(\upsilon;N_Y\kappa)$, $N^\perp_\upsilon\zeta=\pi^\perp_\kappa N_\upsilon\zeta$, $N^\perp_{\nu,\upsilon}\zeta=\pi^\perp_\kappa N_{\nu,\upsilon}\zeta$, $\nu^\perp=\pi^\perp_\kappa\nu$. By (3-18), (3-19), and (3-21),

$$\big\|N^\perp_\upsilon\zeta\big\|_{\upsilon,p}=\big\|\pi^\perp_\kappa\big(N_\upsilon(\zeta^t+\zeta^\perp)-N_\upsilon\zeta^t\big)\big\|_{\upsilon,p}\le C_K\big(|\upsilon|^{1/p}+t\big)\,\|\zeta^\perp\|_{\upsilon,p,1}.$$

Thus, by (2) of Lemma 3.9, (3-18), and (3-22),

$$\|\zeta^\perp\|_{\upsilon,p,1}\le C_Kt,\tag{3-23}$$

provided δ_K is sufficiently small. Combining (3) of Lemma 3.9, (3-18), (3-22), and (3-23), we obtain

$$\big|\langle\!\langle\nu|_{u_\upsilon},\eta\rangle\!\rangle_{\upsilon,2}\big|\le C_K\big(|\upsilon|^{1/p}+t\big)\|\eta\|_{\upsilon,1}\quad\text{for all }\eta\in\Gamma^{0,1}_-(\upsilon;N_Y\kappa).$$

Since the section $\tilde{\pi}^q_{\kappa,\nu}$ of the bundle $\iota_q^*\widetilde{\mathcal{W}}^{1,q}_{\kappa,d/d_\kappa}$ does not vanish over the compact set K, it follows that

$$\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap U_\nu(K) = \varnothing$$

if t and $U_\nu(K)$ are sufficiently small. □

By Lemma 4.2, the spaces $\overline{\mathfrak{M}}^q_1(\kappa, d/d_\kappa)$ with $q \geq 2$ are contained in $\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa)$. In particular, if $\mathcal{T}$ is a bubble type as in the first claim of Proposition 3.7,

$$\mathcal{U}_{\mathcal{T};\kappa} \subset \big(\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa) - \mathfrak{M}^0_1(\kappa, d/d_\kappa)\big) \cup \big(\overline{\mathfrak{M}}^1_1(\kappa, d/d_\kappa) - \mathfrak{M}^1_1(\kappa, d/d_\kappa)\big).$$

Thus, Corollary 3.10, along with the regularity assumptions (ν3a) and (ν3c), implies the first claim of Proposition 3.7.

Corollary 3.11. *Suppose ν, $\mathcal{T}$, and κ are as in Proposition 3.7. If $\aleph = \varnothing$ and $\hat{I}$ has one element, there exist, for every compact subset K of $\mathcal{U}_{\mathcal{T};\kappa}$ containing $\tilde{\pi}^{1\ -1}_{\nu,\kappa}(0)$, some $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\mathfrak{X}_1(Y, d)$ with the following properties*:

(a) *the section $\bar{\partial}_J + t\nu$ is transverse to the zero set in $\Gamma^{0,1}_1(Y, d; J)$ over $U(K)$ for all $t \in (0, \epsilon_\nu(K))$;*

(b) *for every open subset U of $\mathfrak{X}_1(Y, d)$, there exists $\epsilon(U) \in (0, \epsilon_\nu(K))$ such that*

$$^{\pm}\big|\{\bar{\partial}_J + t\nu\}^{-1} \cap U\big| = \frac{d/d_\kappa}{12}\big\langle e(\mathcal{W}^0_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)]\big\rangle$$

if $K \subset U \subset U(K)$, $t \in (0, \epsilon(U))$.

Proof. (1) By Corollary 3.10 and the assumption (ν3a) on ν, it can be assumed that the compact set K is disjoint from $\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa)$. Thus, if h is the unique element of $\hat{I}$,

$$\big|\mathfrak{D}^{(1)}_{J,h}\upsilon\big| \geq C_K|\upsilon| \qquad \text{for all } [\upsilon] = [b, v] \in \mathcal{F}T|_K; \tag{3-24}$$

see Lemma 4.2. By Lemma 3.9 and the proof of Corollary 3.10, we need to determine the number of solutions $[\upsilon, \zeta]$ of the equation

$$\bar{\partial}_J u_\upsilon + D_{J,\upsilon}\zeta + N_\upsilon\zeta + tN_{\nu,\upsilon}\zeta + t\nu = 0, \tag{3-25}$$

with $\upsilon \in \tilde{\mathcal{F}}\mathcal{T}^{\varnothing}_{\delta_K}|_{K^{(0)}}$, $\zeta \in \tilde{\Gamma}_+(\upsilon; TY)$, $\|\zeta\|_{\upsilon,p,1} \leq \epsilon_K$. In this case, $\Gamma^{0,1}_-(\upsilon; TY)$ is a subspace of $\Gamma^{0,1}(\upsilon; TY; J)$, and the middle estimate in (2) of Lemma 3.9 implies that

$$C_K^{-1}\|\zeta\|_{\upsilon,p,1} \leq \|\pi^{0,1}_{\upsilon;+}D_{J,\upsilon}\zeta\|_{\upsilon,p} \leq C_K\|\zeta\|_{\upsilon,p,1} \quad \text{for all } \zeta \in \tilde{\Gamma}_+(\upsilon; TY).$$

Thus, the linear operator

$$\pi^{0,1}_{\upsilon;+}D_{J,\upsilon} : \tilde{\Gamma}_+(\upsilon; TY) \to \Gamma^{0,1}_+(\upsilon; TY)$$

is an isomorphism. It then follows from the Contraction Principle, the first estimate in (2) of Lemma 3.9, and (3-18) that the equation

$$\pi^{0,1}_{\upsilon;+}\big(\bar\partial_J u_\upsilon + D_{J,\upsilon}\zeta + N_\upsilon\zeta + tN_{\nu,\upsilon}\zeta + t\nu\big)=0, \qquad \zeta\in\tilde\Gamma_+(\upsilon;TY),\ \|\zeta\|_{\upsilon,p,1}\le\epsilon_K,$$

has a unique solution $\zeta_\upsilon\in\tilde\Gamma_+(\upsilon;TY)$, provided $\upsilon\in\tilde{\mathcal{F}}\mathcal{T}^{\varnothing}_{\delta_K}|_{K^{(0)}}$ is sufficiently small. Furthermore,

$$\|\zeta_\upsilon\|_{\upsilon,p,1}\le C_K\big(|\upsilon|^{1/p}+t\big). \tag{3-26}$$

Thus, the number of solutions $[\upsilon,\zeta]$ of (3-25) is the same as the number of solutions of

$$\Psi_{t\nu}(\upsilon)\equiv t^{-1}\cdot\pi^{0,1}_{\upsilon;-}\big(\bar\partial_J u_\upsilon + D_{J,\upsilon}\zeta_\upsilon + N_\upsilon\zeta_\upsilon + tN_{\nu,\upsilon}\zeta_\upsilon + t\nu\big)=0, \qquad [\upsilon]\in\mathcal{F}T^{\varnothing}_{\delta_K}|_{K^{(0)}}, \tag{3-27}$$

where

$$\pi^{0,1}_{\upsilon;-}=\pi^{0,1}_{\upsilon;\kappa}\oplus\tilde\pi^{0,1}_{\upsilon}:\Gamma^{0,1}(\upsilon;TY;J)\to\Gamma^{0,1}_{-}(\upsilon;T\kappa)\oplus\Gamma^{0,1}_{-}(\upsilon;N_Y\kappa)$$

is the L^2-projection map.

(2) With our choice of the space $\Gamma^{0,1}_{-}(\upsilon;T\kappa)$,

$$\pi^{0,1}_{\upsilon;\kappa}\bar\partial_J u_\upsilon = R_\upsilon\mathfrak{D}_{J,h}\upsilon\in\Gamma^{0,1}_{-}(\upsilon;T\kappa); \tag{3-28}$$

see Section 4.1 in [Zinger 2003]. Furthermore,

$$\pi^{0,1}_{\upsilon;\kappa}N_\upsilon\zeta=0 \qquad \text{for all } \zeta\in\Gamma(\upsilon;T\kappa), \tag{3-29}$$

since the supports of all elements of $\eta\in\Gamma^{0,1}_{-}(\upsilon;T\kappa)$ are disjoint from the support of $N_\upsilon\zeta$, for $\zeta\in\Gamma(\upsilon;T\kappa)$, due to our choice of the "exponential" map. By (3-18), (3-26), (3-28), (3-29), and the same argument as in the proof of Corollary 3.10,

$$\begin{aligned}\big\|\tilde\pi^{0,1}_\upsilon\Psi_{t\nu}(\upsilon)-R_\upsilon\tilde\pi^{1}_{\kappa,\nu}(b)\big\|_{\upsilon,2}&\le C_K\big(|\upsilon|^{1/p}+t\big)+\|\nu(u_\upsilon)-R_\upsilon\nu(b)\|_{\upsilon,2},\\ \big\|\pi^{0,1}_{\upsilon;\kappa}\Psi_{t\nu}(\upsilon)-R_\upsilon\big(\pi^1_{\kappa,\nu;\kappa}(b)+t^{-1}\mathfrak{D}_{J,h}\upsilon\big)\big\|_{\upsilon,2}&\\ \le C_K\big(|\upsilon|^{1/p}+t\big)+\|\nu(u_\upsilon)-R_\upsilon\nu(b)\|_{\upsilon,2},&\end{aligned} \tag{3-30}$$

where $R_\upsilon\eta=q_\upsilon^*\eta$ for $\eta\in\Gamma^{0,1}_{-}(b;N_Y\kappa)$ and

$$\pi^1_{\kappa,\nu;\kappa}(b)=\pi_\kappa\pi^1_{\kappa,\nu}(b)\equiv\pi^1_{\kappa,\nu}(b)-\pi^\perp_\kappa\pi^1_{\kappa,\nu}(b)\in\Gamma^{0,1}_{-}(b;T\kappa).$$

Since

$$\lim_{t\to0,|\upsilon|\to0}\big(|\upsilon|^{1/p}+t\big)+\|\nu(u_\upsilon)-R_\upsilon\nu(b)\|_{\upsilon,2}=0,$$

by (3-24), (3-30), and the same cobordism argument as in Section 3.1 of [Zinger 2003], the number of solutions of (3-27) is the same as the number of solutions of the system

$$\begin{cases} \tilde{\pi}^1_{\kappa,\upsilon}(b) = 0 \in \widetilde{\mathcal{W}}^{1,1}_{\kappa,d/d_\kappa}, \\ \pi^1_{\kappa,\upsilon;\kappa}(b) + \mathfrak{D}_{J,h}\upsilon = 0 \in \pi_P^*\mathbb{E}^* \otimes \pi_B^*\mathrm{ev}_0^*T\kappa, \end{cases}$$

with $[\upsilon] = [b, v] \in \mathcal{F}T \to \overline{\mathfrak{M}}^1_1(k, d/d_\kappa)$, if the interior of the compact set K contains the finite set $\tilde{\pi}^{1\;-1}_{\upsilon,\kappa}(0)$. Since $D_{J,h}$ does not vanish on $\tilde{\pi}^{1\;-1}_{\kappa,\upsilon}(0)$, the number of solution of this system is

$$\begin{aligned} {}^\pm\big|\tilde{\pi}^{1\;-1}_{\upsilon,\kappa}(0)\big| &= \big\langle e(\widetilde{\mathcal{W}}^{1,1}_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}^1_1(\kappa, d/d_\kappa)]\big\rangle \\ &= \big\langle e\big(\pi_P^*\mathbb{E}^* \otimes \pi_B^*\mathrm{ev}_0^*N_Y\kappa\big) e(\tilde{\pi}_B^*\mathcal{W}^0_{\kappa,d/d_\kappa}), [\overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,1}(\kappa, d/d_\kappa)]\big\rangle \\ &= -\tfrac{1}{24}\big(-2(d/d_\kappa)\big)\big\langle e(\mathcal{W}^0_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)]\big\rangle, \end{aligned}$$

as claimed in Proposition 3.7. □

4. On the Euler class of the cone $\mathcal{V}^d_1 \to \overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d; J)$

4A. ***The structure of the moduli spaces*** $\overline{\mathfrak{M}}^0_1(\mathbb{P}^n, d; J)$**.** In this section, we prove Proposition 2.3 by constructing a perturbation ϑ of the section s^d_1 of the cone $\mathcal{V}^d_1$ over $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d; J)$ and counting the number of zeros of the multisection $s^d_1 + t\vartheta$ for a small $t \in \mathbb{R}^+$ that lie near each stratum of

$$\text{(4-1)}\quad \begin{aligned} s^{d\;-1}_1(0) \cap \overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d; J) &= \overline{\mathfrak{M}}^0_1(Y, d; J) \\ &= \bigsqcup_{\kappa \in \mathcal{S}_0(Y;J)} \overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa) \sqcup \bigsqcup_{\kappa \in \mathcal{S}_1(Y;J)} \mathfrak{M}^0_1(\kappa, d/d_\kappa). \end{aligned}$$

Since the finite set $\mathfrak{M}^0_1(\kappa, d/d_\kappa)$ consists of transverse zeros of s^d_1, for $\kappa \in \mathcal{S}_1(Y; J)$,

$$\text{(4-2)}\qquad \mathcal{C}_{\mathfrak{M}^0_1(\kappa,d/d_\kappa)}\big(s^d_1\big) = {}^\pm\big|\{s^d_1 + t\vartheta\}^{-1}(0) \cap U_\kappa\big| = {}^\pm\big|\mathfrak{M}^0_1(\kappa, d/d_\kappa)\big|,$$

if U_κ is a small neighborhood of $\mathfrak{M}^0_1(\kappa, d/d_\kappa)$ in $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d; J)$. The second equality in (4-2) holds for every multisection ϑ of $\mathcal{V}^d_1$ and every $t \in \mathbb{R}$ sufficiently small. Thus, the key to proving Proposition 2.3 is computing the s^d_1-contribution from each stratum of the moduli space $\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa)$. This is achieved by Proposition 4.5 and Corollary 4.7.

In this subsection, we describe the structure of the moduli space $\overline{\mathfrak{M}}^0_1(\mathbb{P}^n, d; J)$, with J sufficiently close to J_0. Lemmas 4.1 and 4.2 are special cases of Lemmas 2.3 and 2.4, respectively, in [Zinger 2007]. In turn, the latter two lemmas follow immediately from Theorems 1.6 and 2.3 in [Zinger 2004b].

Lemma 4.1. *If $n, d \in \mathbb{Z}^+$, there exists $\delta_n(d) \in \mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$, such that $\|J - J_0\|_{C^1} < \delta_n(d)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i \neq 0$ for some minimal element i of I, then $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ is a smooth orbifold,*

$$\dim \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) = 2\big(d(n+1) - |\aleph| - |\hat{I}|\big) \quad \textit{and} \quad \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) \subset \overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J).$$

Furthermore, there exist $\delta \in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J); \mathbb{R}^+)$, an open neighborhood $U_{\mathcal{T}}$ of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ in $\mathfrak{X}_1(\mathbb{P}^n, d)$, and an orientation-preserving homeomorphism

$$\phi_{\mathcal{T}} : \mathcal{F}T_\delta \to \overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J) \cap U_{\mathcal{T}},$$

which restricts to a diffeomorphism $\mathcal{F}T_\delta^{\varnothing} \to \mathfrak{M}_1^0(\mathbb{P}^n, d; J) \cap U_{\mathcal{T}}$.

Lemma 4.2. *If $n, d \in \mathbb{Z}^+$, there exists $\delta_n(d) \in \mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$, such that $\|J - J_0\|_{C^1} < \delta_n(d)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i = 0$ for all minimal elements i of I, then $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ is a smooth orbifold,*

$$\dim \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) = 2\big(d(n+1) - |\aleph| - |\hat{I}| + n\big),$$

and

$$\overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J) \cap \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) = \mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n; J),$$

where

$$\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n; J) = \big\{[b] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) : \dim_{\mathbb{C}} \mathrm{Span}_{(\mathbb{C},J)}\{\mathcal{D}_i b : i \in \chi(\mathcal{T})\} < |\chi(\mathcal{T})|\big\}.$$

The space $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n; J)$ has a stratification by smooth suborbifolds of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$:

$$\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n; J) = \bigsqcup_{m=\max(|\chi(\mathcal{T})|-n,1)}^{m=|\chi(\mathcal{T})|} \mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n; J),$$

where

$$\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n; J) = \big\{[b] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) : \dim_{\mathbb{C}} \mathrm{Span}_{(\mathbb{C},J)}\{\mathcal{D}_i b : i \in \chi(\mathcal{T})\} = |\chi(\mathcal{T})| - m\big\},$$

with dimension

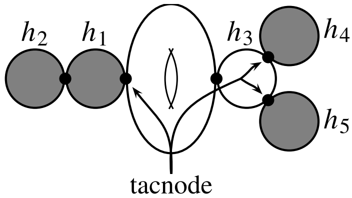
$$2\big(d(n+1) - |\aleph| - |\hat{I}| + n + \big(|\chi(\mathcal{T})| - n - m\big)m\big) \le \dim \mathfrak{M}_1^0(\mathbb{P}^n, d; J) - 2.$$

The space

$$\mathcal{F}^1\mathcal{T}^{\varnothing} \equiv \big\{[b, \upsilon] \in \mathcal{F}T^{\varnothing} : \mathcal{D}_{\mathcal{T}}(\upsilon) = 0\big\}$$

is a smooth oriented suborbifold of $\mathcal{F}T$. Finally, there exist $\delta \in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J); \mathbb{R}^+)$, an open neighborhood $U_{\mathcal{T}}$ of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ in $\mathfrak{X}_1(\mathbb{P}^n, d)$, and an orientation-preserving diffeomorphism

$$\phi_{\mathcal{T}} : \mathcal{F}^1\mathcal{T}_\delta^{\varnothing} \to \mathfrak{M}_1^0(\mathbb{P}^n, d; J) \cap U_{\mathcal{T}},$$



$$\chi(\mathscr{T})=\{h_1,h_4,h_5\},\ \rho(\upsilon)=(\upsilon_{h_1},\upsilon_{h_3}\upsilon_{h_4},\upsilon_{h_3}\upsilon_{h_5})$$

$$\mathscr{F}^1\mathscr{T}^{\varnothing}=\big\{[b;v_1,v_2,v_3,v_4,v_5]:v_h\in\mathbb{C}^*\\ \mathfrak{D}_{J,h_1}v_{h_1}+\mathfrak{D}_{J,h_4}v_{h_3}v_{h_4}+\mathfrak{D}_{J,h_5}v_{h_3}v_{h_5}=0\big\}$$

Figure 5. An illustration of Lemma 4.2.

which extends to a homeomorphism

$$\phi_{\mathscr{T}}:\mathscr{F}^1\mathscr{T}_\delta\to\overline{\mathfrak{M}}_1^0(\mathbb{P}^n,d;J)\cap U_{\mathscr{T}},$$

where $\mathscr{F}^1\mathscr{T}$ is the closure of $\mathscr{F}^1\mathscr{T}^{\varnothing}$ in $\mathscr{F}T$.

We illustrate Lemma 4.2 in Figure 5. As before, the shaded discs represent the components of the domain on which every stable map $[b]$ in $\mathscr{U}_{\mathscr{T}}(\mathbb{P}^n;J)$ is nonconstant. The element $[\Sigma_b,u_b]$ of $\mathscr{U}_{\mathscr{T}}(\mathbb{P}^n;J)$ is in the stable-map closure of $\mathfrak{M}_1^0(\mathbb{P}^n,d;J)$ if and only if the branches of $u_b(\Sigma_b)$ corresponding to the attaching nodes on the first-level effective bubbles of $[\Sigma_b,u_b]$ form a generalized tacnode. In the case of Figure 5, this means that either

(a) for some $h\in\{h_1,h_4,h_5\}$, the branch of $u_b|_{\Sigma_{b,h}}$ at the node ∞ has a cusp, or

(b) for all $h\in\{h_1,h_4,h_5\}$, the branch of $u_b|_{\Sigma_{b,h}}$ at the node ∞ is smooth, but the dimension of the span of the three lines tangent to these branches is less than three.

If $\kappa\in\mathscr{S}_0(Y;J)$, we put

$$\mathscr{U}_{\mathscr{T};\kappa;1}=\mathscr{U}_{\mathscr{T};\kappa}\cap\mathscr{U}_{\mathscr{T};1}(\mathbb{P}^4;J)\subset\overline{\mathfrak{M}}_1^0(\kappa,d/d_\kappa),$$
$$\mathscr{U}^m_{\mathscr{T};\kappa;1}=\mathscr{U}_{\mathscr{T};\kappa}\cap\mathscr{U}^m_{\mathscr{T};1}(\mathbb{P}^4;J)\subset\mathscr{U}_{\mathscr{T};\kappa;1}.$$

By the $n=1$ case of Lemma 4.2,

$$\mathscr{U}_{\mathscr{T};\kappa;1}=\mathscr{U}_{\mathscr{T};\kappa}=\mathscr{U}^{|\chi(\mathscr{T})|-1}_{\mathscr{T};\kappa;1}\cup\mathscr{U}^{|\chi(\mathscr{T})|}_{\mathscr{T};\kappa;1}\quad\text{if } |\chi(\mathscr{T})|\ge 2.$$

The last space may be empty. In particular, $\overline{\mathfrak{M}}_1^q(\kappa,d/d_\kappa)\subset\overline{\mathfrak{M}}_1^0(\kappa,d/d_\kappa)$ if $q\ge 2$.

Let

$$\mathfrak{M}^0_{0,1;1}(\mathbb{P}^4,d;J)=\big\{[\mathbb{P}^1,u]\in\mathfrak{M}^0_{0,1}(\mathbb{P}^4,d;J):du|_\infty=0\big\}.$$

In other words, $\mathfrak{M}^0_{0,1;1}(\mathbb{P}^4,d;J)$ is the subset of $\mathfrak{M}^0_{0,1}(\mathbb{P}^4,d;J)$ consisting of the elements $[\mathbb{P}^1,u]$ such that the differential of u vanishes at the marked point of $\mathbb{P}^1$, which we always take to be ∞. The image of a generic element in $\mathfrak{M}^0_{0,1;1}(\mathbb{P}^4,d;J)$ is a rational curve J-holomorphic curve in $\mathbb{P}^4$ with a cusp at the image of the marked point. We denote by $\overline{\mathfrak{M}}_{0,1;1}(\mathbb{P}^4, d; J)$ the closure of $\mathfrak{M}^0_{0,1;1}(\mathbb{P}^4, d; J)$ in $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^4, d; J)$. If $\kappa \in \mathcal{S}_0(Y; J)$, we put

$$\begin{aligned}
\mathfrak{M}^0_{0,1;1}(\kappa, d/d_\kappa) &= \mathfrak{M}^0_{0,1;1}(\mathbb{P}^4, d; J) \cap \overline{\mathfrak{M}}_{0,1}(\kappa, d/d_\kappa),\\
\overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_\kappa) &= \overline{\mathfrak{M}}_{0,1;1}(\mathbb{P}^4, d; J) \cap \overline{\mathfrak{M}}_{0,1}(\kappa, d/d_\kappa),\\
\mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa) &= \overline{\mathcal{M}}_{1,1} \times \mathfrak{M}^0_{0,1;1}(\kappa, d/d_\kappa),\\
\overline{\mathfrak{M}}^1_{1;1}(\kappa, d/d_\kappa) &= \overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_\kappa).
\end{aligned}$$

By Lemma 4.2,

$$\overline{\mathfrak{M}}^1_1(\kappa, d) \cap \overline{\mathfrak{M}}^0_1(\kappa, d) = \overline{\mathfrak{M}}^1_{1;1}(\kappa, d) \qquad \text{for all } d \in \mathbb{Z}^+.$$

We note that

$$\dim_{\mathbb{C}} \overline{\mathfrak{M}}_{0,1;1}(\kappa, d) = 2d - 2 \quad \text{and} \quad \dim_{\mathbb{C}} \overline{\mathfrak{M}}^1_{1;1}(\kappa, d) = 2d - 1. \tag{4-3}$$

4B. ***The structure of the cone*** $\mathcal{V}^d_1 \to \overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d; J)$**.** We next describe the structure of the cone $\mathcal{V}^d_1$ near each stratum $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ and $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^4; J)$ of $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d; J)$. We then state several regularity conditions that we will require the perturbation ϑ of s^d_1 to satisfy. The first lemma stated is a special case of Lemma 3.2 in [Zinger 2007].

Lemma 4.3. *If d, $\mathfrak{L}$, and $\mathcal{V}^d_1$ are as in Proposition 2.3, there exists $\delta(d) \in \mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^4$, such that $\|J - J_0\|_{C^1} < \delta_n(d)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i\in I} d_i = d$ and $d_i \neq 0$ for some minimal element i of I, then the requirements of Lemma 4.1 are satisfied. Furthermore, the restriction $\mathcal{V}^d_1 \to \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ is a smooth complex vector orbibundle of rank $5d$. Finally, there exists a smooth vector-bundle isomorphism*

$$\tilde{\phi}_{\mathcal{T}} : \pi^*_{\mathcal{F}T^\varnothing_\delta}\big(\mathcal{V}^d_1|_{\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J)}\big) \to \mathcal{V}^d_1|_{\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d;J)\cap U_{\mathcal{T}}},$$

covering the homeomorphism $\phi_{\mathcal{T}}$ of Lemma 4.1, such that $\tilde{\phi}_{\mathcal{T}}$ is the identity over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ and is smooth over $\mathcal{F}T^\varnothing_\delta$.

For every $\kappa \in \mathcal{S}_0(Y; J)$, the family of boundary operators $\mathfrak{d}_0$ in the long exact sequence (3-4), with $b \in \mathcal{U}^{(0)}_{\mathcal{T};\kappa}$ and $\mathcal{T}$ as in Lemma 4.3, induces a surjective bundle homomorphism

$$\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa} : \mathcal{V}^d_1 \to \mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$$

over $\mathfrak{M}^{\{0\}}_1(\kappa, d/d_\kappa)$. The first two regularity conditions on a perturbation ϑ of the section s^d_1 over $\overline{\mathfrak{M}}^0_1(\mathbb{P}^4, d; J)$ are that, for every $\kappa \in \mathcal{S}_0(Y; J)$,

(ϑ1a) the section $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta|_{\mathfrak{M}^0_1(\kappa,d/d_\kappa)}$ is transverse to the zero set in $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$;

(ϑ1b) the section $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta$ does not vanish on $\mathfrak{M}^{\{0\}}_1(\kappa, d/d_\kappa) - \mathfrak{M}^0_1(\kappa, d/d_\kappa)$.

By Lemma 4.3, the $n = 1$ case of Lemma 4.1, and (2-15), the collection of multisections ϑ of $\mathcal{V}_1^d$ that satisfy ($\vartheta 1a$) and ($\vartheta 1b$) is open and dense in the space of all multisections of $\mathcal{V}_1^d$.

The next lemma, which is the analogue of Lemma 4.3 for the strata $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^4; J)$ of Lemma 4.2, is a special case of Proposition 3.3 and Lemma 3.4 in [Zinger 2007]. For any $b \in \mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^4; J)$, we put

$$\mathfrak{F}^1\mathcal{T}_b = \Big\{\tilde{\upsilon} = (\tilde{\upsilon}_i)_{i\in\chi(\mathcal{T})} \in \mathfrak{F}\mathcal{T}_b : \sum_{i\in\chi(\mathcal{T})} \mathcal{D}_{J,i}\tilde{\upsilon}_i = 0\Big\}.$$

Lemma 4.4. *If d, $\mathfrak{L}$, and $\mathcal{V}_1^d$ are as in Proposition 2.3, there exists $\delta(d) \in \mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^4$ such that $\|J - J_0\|_{C^1} < \delta(d)$, then the requirements of Lemmas 4.2 and 4.3 are satisfied for all appropriate bubble types. Further, if $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i\in I} d_i = d$ and $d_i = 0$ for all minimal elements i of I, the restriction $\mathcal{V}_1^d \to \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ is a smooth complex vector orbibundle of rank $5d + 1$. In addition, for every integer*

$$m \in \big(\max(|\chi(\mathcal{T})| - 4, 1), |\chi(\mathcal{T}|\big),$$

there exist a neighborhood $U^m_{\mathcal{T}}$ of $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^4; J)$ in $\mathfrak{X}_1(\mathbb{P}^4, d)$ and a topological vector orbibundle

$$\mathcal{V}^{d;m}_{1;\mathcal{T}} \to \overline{\mathfrak{M}}{}^0_1(\mathbb{P}^4, d; J) \cap U^m_{\mathcal{T}}$$

such that $\mathcal{V}^{d;m}_{1;\mathcal{T}} \to \mathfrak{M}^0_1(\mathbb{P}^4, d; J) \cap U^m_{\mathcal{T}}$ is a smooth complex vector orbibundle contained in $\mathcal{V}_1^d$ and

$$\mathcal{V}^{d;m}_{1;\mathcal{T}}|_{\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^4;J)} = \big\{\xi \in \mathcal{V}_1^d|_b : b \in \mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^4; J);\ \mathfrak{D}_{\mathcal{T}}(\xi \otimes \tilde{\upsilon}) = 0 \text{ for all } \tilde{\upsilon} \in \mathfrak{F}^1\mathcal{T}_b\big\}.$$

There also exists a continuous vector-bundle isomorphism

$$\tilde{\phi}^m_{\mathcal{T}} : \pi^*_{\mathcal{F}^1\mathcal{T}_\delta}\big(\mathcal{V}^{d;m}_{1;\mathcal{T}}|_{\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4;J)\cap U^m_{\mathcal{T}}}\big) \to \mathcal{V}^{d;m}_{1;\mathcal{T}}|_{\overline{\mathfrak{M}}{}^0_1(\mathbb{P}^4,d;J)\cap U^m_{\mathcal{T}}},$$

covering the homeomorphism $\phi_{\mathcal{T}}$ of Lemma 4.1, such that $\tilde{\phi}^m_{\mathcal{T}}$ is the identity over $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^4; J)$. Finally, if $\mathcal{T}$ and $\mathcal{T}'$ are two bubble types as above and $m, m' \in \mathbb{Z}^+$, then

$$\mathcal{V}^{d;m'}_{1;\mathcal{T}'}\big|_{\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^4;J)\cap U^{m'}_{\mathcal{T}'}} \subset \mathcal{V}^{d;m}_{1;\mathcal{T}}\big|_{\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^4;J)\cap U^{m'}_{\mathcal{T}'}} \quad \textit{if } m' \ge m.$$

If $[b] \in \mathcal{U}^m_{\mathcal{T};\kappa;1}$, we put

$$\Gamma_-(b; \mathfrak{L}; 0) = \big\{\xi \in \Gamma_-(b; \mathfrak{L}) : \tilde{\mathfrak{D}}_{\mathcal{T},i}\xi = 0 \text{ for all } i \in \chi(\mathcal{T})\big\},$$
$$\tilde{\mathcal{V}}^d_{1;\mathcal{T}}|_b = \big\{[\xi] \in \mathcal{V}^d_{1;\mathcal{T}}|_b : \mathfrak{D}_{\mathcal{T},i}\xi = 0 \text{ for all } i \in \chi(\mathcal{T})\big\} \subset \mathcal{V}^{d;m}_{1;\mathcal{T}}.$$

In this case, the standard analogue for b of the long exact sequence (3-4) has six terms. However, replacing the fourth term by the kernel of the outgoing map at the fourth term, we get

$$0 \to \Gamma_-(b; TY) \xrightarrow{i_0} \Gamma_-(b; T\mathbb{P}^4) \xrightarrow{j_0} \Gamma_-(b; \mathfrak{L}) \xrightarrow{\mathfrak{d}_0} H^1_J(\pi_B(b); TY) \to 0. \tag{4-4}$$

By Theorem 1.6 in [Zinger 2004b], the linear operator

$$\tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},i} : \{\zeta \in \Gamma_-(b; T\mathbb{P}^4) : \zeta|_{\Sigma_{b,\aleph}} = 0\} \to T_{\mathrm{ev}_P(b)}\mathbb{P}^4, \quad \zeta \to \nabla^J_{e_\infty}(\zeta|_{\Sigma_{b,i}}),$$

is surjective for every $[b] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, with $\mathcal{T}$ as in Lemma 4.2 and $i \in \chi(\mathcal{T})$, if J is sufficiently close to J_0. It follows that the homomorphism

$$\tilde{j}_0 : \Gamma_-(b; T\mathbb{P}^4) \to \Gamma_-(b; \mathfrak{L}) \big/ \Gamma_-(b; \mathfrak{L}; 0),$$

induced by the map j_0 in (4-4) is surjective for every $[b] \in \mathcal{U}_{\mathcal{T};\kappa}$. Thus, the family of boundary operators $\mathfrak{d}_0$ in (4-4) with $[b] \in \mathcal{U}_{\mathcal{T};\kappa}$ induces a surjective bundle homomorphism

$$\mathfrak{d}^{1,q}_{\kappa,d/d_\kappa} : \tilde{\mathcal{V}}^d_{1;\mathcal{T}} \to \pi^*_B \mathcal{W}^{0,q}_{\kappa,d/d_\kappa} \subset \mathcal{W}^{1,0}_{\kappa,d/d_\kappa} \tag{4-5}$$

over $\mathcal{U}_{\mathcal{T};\kappa;1}$, if $\mathcal{U}_{\mathcal{T};\kappa} \subset \overline{\mathfrak{M}}^q_1(\kappa, d/d_\kappa)$. Furthermore,

$$\mathfrak{d}^{1,q}_{\kappa,d/d_\kappa} = \pi^*_B \mathfrak{d}^{0,q}_{\kappa,d/d_\kappa}, \tag{4-6}$$

where $\mathfrak{d}^{0,q}_{\kappa,d/d_\kappa}$ is the surjective bundle homomorphism over $\mathcal{U}_{\bar{\mathcal{T}};\kappa} \subset \overline{\mathfrak{M}}^q_0(\kappa, d/d_\kappa)$ defined similarly to $\mathfrak{d}^{1,q}_{\kappa,d/d_\kappa}$.

We now state additional regularity conditions on a perturbation ϑ of s^d_1. We will require that for every $\kappa \in \mathcal{S}_0(Y; J)$:

(ϑ2a) the sections $\mathfrak{d}^{1,1}_{\kappa,d/d_\kappa}\vartheta|_{\mathcal{M}_{1,1}\times\mathfrak{M}^0_{0,1;1}(\kappa,d/d_\kappa)}$ and $\mathfrak{d}^{1,1}_{\kappa,d/d_\kappa}\vartheta|_{\partial\bar{\mathcal{M}}_{1,1}\times\mathfrak{M}^0_{0,1;1}(\kappa,d/d_\kappa)}$ are transverse to the zero set in $\pi^*_B\mathcal{W}^{0,1}_{\kappa,d/d_\kappa}$;

(ϑ2b) the section $\mathfrak{d}^{1,1}_{\kappa,d/d_\kappa}\vartheta$ does not vanish on $\overline{\mathfrak{M}}^1_{1;1}(\kappa, d/d_\kappa) - \mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa)$;

(ϑ2c) for $q \geq 2$, the section $\mathfrak{d}^{1,q}_{\kappa,d/d_\kappa}\vartheta$ does not vanish on $\overline{\mathfrak{M}}^q_1(\kappa, d/d_\kappa)$.

By Lemma 4.4, (2-6), and (2-11), the collection of multisections ϑ of $\mathcal{V}^d_1$ that satisfy (ϑ2a) and (ϑ2c) with $q \geq 4$ is open and dense in the space of all multisections of $\mathcal{V}^d_1$. By (2-8), (2-11), (4-3), and (4-6), the collection of multisections ϑ of $\mathcal{V}^d_1$ that satisfy one of the three remaining conditions, i.e. (ϑ2b), (ϑ2c) with $q = 2$, or (ϑ2c) with $q = 3$, is nonempty and open in the space of all multisections of $\mathcal{V}^d_1$, but not dense. Nevertheless, by considering the decompositions of the intersections of the corresponding subspaces of $\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa)$ analogous to (2-9), it is straightforward to see that the intersection of these three open sets is still nonempty. Alternatively, note that

$$\dim_{\mathbb{C}}\big(\overline{\mathfrak{M}}^1_{1;1}(\kappa, d/d_\kappa) - \mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa)\big) = 2(d/d_\kappa) - 2 = \operatorname{rk}\mathcal{W}^{0,1}_{\kappa,d/d_\kappa};$$
$$\dim_{\mathbb{C}} \overline{\mathfrak{M}}^3_1(\kappa, d/d_\kappa) = 2(d/d_\kappa) - 2 = \operatorname{rk}\mathcal{W}^{0,3}_{\kappa,d/d_\kappa};$$
$$\dim_{\mathbb{C}} \overline{\mathfrak{M}}^2_1(\kappa, d/d_\kappa) \cap \overline{\mathfrak{M}}^3_1(\kappa, d/d_\kappa) \le 2(d/d_\kappa) - 3.$$

Furthermore, the space $\overline{\mathfrak{M}}^1_{1;1}(\kappa, d/d_\kappa) - \mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa)$ has two irreducible components. One is contained in $\overline{\mathfrak{M}}^2_1(\kappa, d/d_\kappa)$, while the other intersects $\overline{\mathfrak{M}}^2_1(\kappa, d/d_\kappa)$ in subvariety of complex dimension $2(d/d_\kappa) - 3$. Thus, if ϑ is a generic multisection that satisfies (ϑ2c) with $q = 2$, its restrictions to

$$\overline{\mathfrak{M}}^1_{1;1}(\kappa, d/d_\kappa) - \mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa) \qquad \text{and} \qquad \overline{\mathfrak{M}}^3_1(\kappa, d/d_\kappa) \tag{4-7}$$

have finite zero sets, divided equally between positive and negative zeros. These zeros can be removed in pairs by modifying ϑ outside of the boundary strata of (4-7).

It remains to state one more regularity assumption on ϑ. If $\hat{I} = \chi(\mathcal{T}) = \{h\}$ is a single-element set, for every $[b] \in \mathcal{U}_{\mathcal{T};\kappa;1}$, we put

$$\Gamma_-(b; T\mathbb{P}^4; TY) = \big\{\zeta \in \Gamma_-(b; T\mathbb{P}^4) : \tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},h}\zeta \in T_{\mathrm{ev}_P(b)}Y\big\}$$
$$\Gamma_-(b; T\mathbb{P}^4; T\kappa) = \big\{\zeta \in \Gamma_-(b; T\mathbb{P}^4) : \tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},h}\zeta \in T_{\mathrm{ev}_P(b)}\kappa\big\} \subset \Gamma_-(b; T\mathbb{P}^4; TY).$$

Since $d\{u_b|_{\Sigma_{b,h}}\}|_\infty = 0$ by Lemma 4.2, the subspaces

$$\Gamma_-(b; T\mathbb{P}^4; TY), \Gamma_-(b; T\mathbb{P}^4; T\kappa) \subset \Gamma_-(b; T\mathbb{P}^4)$$

are in fact independent of the choice of connection ∇^J in $T\mathbb{P}^4$. Furthermore, by Theorem 1.6 in [Zinger 2004b],

$$\Gamma_-(b; T\mathbb{P}^4; TY)\,/\,\Gamma_-(b; T\mathbb{P}^4; T\kappa) \approx N_Y\kappa|_{\mathrm{ev}_P(b)} \quad \text{via} \quad \zeta \to [\tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},h}\zeta]. \tag{4-8}$$

By the paragraph following Lemma 4.4 and condition (J_Y2) of Definition 1.2, we have

$$\begin{aligned} \operatorname{Im} j_0|_{\Gamma_-(b;T\mathbb{P}^4;TY)} &= \ker \mathfrak{d}_0 \cap \Gamma_-(b; \mathfrak{L}; 0),\\ \ker j_0 &\subset \Gamma_-(b; T\mathbb{P}^4; T\kappa), \end{aligned} \tag{4-9}$$

where j_0 and $\mathfrak{d}_0$ are as in (4-4). Let

$$\tilde{H}^1_J(\pi_B(b); TY) = \Gamma_-(b; \mathfrak{L}; 0)\big/\operatorname{Im} j_0|_{\Gamma_-(b;T\mathbb{P}^4;T\kappa)}.$$

The vector spaces $\tilde{H}^1_J(\pi_B(b); TY)$ and the quotient projection maps induce a vector bundle over $\mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa)$, which we denote by $Q^{1,1}_{\kappa,d/d_\kappa}$, and a surjective bundle homomorphism

$$\tilde{\mathfrak{d}}^{1,1}_{\kappa,d/d_\kappa} : \tilde{\mathcal{V}}^d_{\mathcal{T};1} = \mathcal{V}^{d;1}_{\mathcal{T};1} \to Q^{1,1}_{\kappa,d/d_\kappa}.$$

On the other hand, the boundary operators $\mathfrak{d}_0$ in (4-4) induce a surjective bundle homomorphism

$$\pi_\kappa^+ : Q^{1,1}_{\kappa,d/d_\kappa} \to \pi_B^* \mathcal{W}^{0,1}_{\kappa,d/d_\kappa}$$

over $\mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa)$. By (4-8) and (4-9),

$$\ker \pi_\kappa^+ \approx \pi_B^*(L_0^* \otimes \mathrm{ev}_0^* N_Y \kappa).$$

We also have a surjective bundle homomorphism

$$\pi_\kappa^- : Q^{1,1}_{\kappa,d/d_\kappa} \to \pi_B^*(L_0^* \otimes \mathrm{ev}_0^* N_Y \kappa).$$

It is induced by the map

$$j_0 \zeta \to (-2\pi J)\big[\nabla^J_{e_\infty} \zeta|_{\Sigma_{b,h}}\big] \in N_Y \kappa, \tag{4-10}$$

where $\zeta \in \Gamma(b; T\mathbb{P}^4)$, $j_0\zeta \in \Gamma_-(b; \mathfrak{L}; 0)$. Thus, we obtain a splitting of $Q^{1,1}_{\kappa,d/d_\kappa}$ over $\mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa)$:

$$\pi_\kappa^- \oplus \pi_\kappa^+ : Q^{1,1}_{\kappa,d/d_\kappa} \to \pi_B^*(L_0^* \otimes \mathrm{ev}_0^* N_Y \kappa) \oplus \pi_B^* \mathcal{W}^{0,1}_{\kappa,d/d_\kappa}. \tag{4-11}$$

We note that

$$\mathrm{rk}\, Q^{1,1}_{\kappa,d} = 2d \tag{4-12}$$

for all $d \in \mathbb{Z}^+$.

Our final regularity condition on ϑ is that, for every $\kappa \in \mathcal{S}_0(Y; J)$,

(ϑ3) the section $\tilde{\mathfrak{d}}^{1,1}_{\kappa,d/d\kappa} \vartheta$ does not vanish over $\mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa)$.

By Lemma 4.4, (4-3), and (4-12), the collection of multisections ϑ of $\mathcal{V}_1^d$ satisfying (ϑ3) is open and dense in the space of all multisections of $\mathcal{V}_1^d$. Denote by $\tilde{\mathcal{A}}_1^d(s; J)$ the collection of multivalued perturbations of the section s_1^d of $\mathcal{V}_1^d$ over $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ that satisfy the regularity conditions (ϑ1)–(ϑ3). By the above, this is a nonempty open, but not dense, subset of the space of all multisections of $\mathcal{V}_1^d$.

It is possible to use a dense open collection of perturbations in the statement of Proposition 4.5 below. However, using such a collection would needlessly complicate its proof by enlarging the zero set of the sections $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa} \vartheta$ by homologically trivial subspaces of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. This would also require stating the analogue of (ϑ3) for the $q = 2, 3$ cases of (ϑ2c).

4C. ***Proof of Proposition 2.3.*** In this section, we prove Proposition 2.3; it follows immediately from (4-1) and (4-2) plus Proposition 4.5 and Corollary 4.7 below.

Proposition 4.5. *Suppose J, d, $\mathfrak{L}$, and $\mathcal{V}_1^d$ are as in Proposition 2.3, $\vartheta \in \tilde{\mathcal{A}}_1^d(s; J)$ is a regular perturbation of the section s_1^d of $\mathcal{V}_1^d$ on $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$, $\kappa \in \mathcal{S}_0(Y; J)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i\in I} d_i = d$. If $|I| > 1$ or $\aleph \neq \varnothing$, for every compact subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\epsilon_\vartheta(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ such that*

$$\{s_1^d + t\vartheta\}^{-1}(0) \cap U(K) = \varnothing \qquad \textit{for all } t \in (0, \epsilon_\vartheta(K)).$$

If $|I| = 1$ and $\aleph = \varnothing$, for every compact subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\epsilon_\vartheta(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ with the following properties:

(a) *the section $s_1^d + t\vartheta$ is transverse to the zero set in $\mathcal{V}_1^d$ over $U(K)$ for all $t \in (0, \epsilon_\vartheta(K))$;*

(b) *for every open subset U of $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$, there exists $\epsilon(U) \in (0, \epsilon_\vartheta(K))$ such that*

$$^{\pm}\big|\{s_1^d + t\vartheta\}^{-1} \cap U\big| = \big\langle e(\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)]\big\rangle - \mathcal{C}_{\mathfrak{M}^{1;0}_{1;1}(\kappa,d/d_\kappa)}\big(\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\big)$$

$$\textit{if } \big\{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\big\}^{-1}(0) - \mathfrak{M}^1_{1;1}(\kappa, d/d_\kappa) \subset K \subset U \subset U(K),\ t \in (0, \epsilon(U)).$$

In other words, the s_1^d-contribution from the main stratum $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ of the space $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ to the number

$$\big\langle e(\mathcal{V}_1^d), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)]\big\rangle,$$

as computed via a perturbation from the open collection $\tilde{\mathcal{A}}_1^d(s; J)$, is the Euler class of the vector bundle $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$ over $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ minus the $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta$-contribution to the latter Euler class from the zeros of $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta$ that lie in $\partial\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. Since $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta|_{\mathfrak{M}_1^0(\kappa,d/d_\kappa)}$ is transverse to the zero set in $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$,

$$^{\pm}\big|\big\{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\big\}^{-1}(0) \cap \mathfrak{M}_1^0(\kappa, d/d_\kappa)\big| = \big\langle e(\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)]\big\rangle - \mathcal{C}_{\partial\overline{\mathfrak{M}}_1^0(\kappa,d/d_\kappa)}\big(\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\big),$$

by Definition 2.4. None of the boundary strata of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ contributes to the Euler class of $\mathcal{V}_1^d$.

Proof. (1) If $\mathcal{T}$ is a bubble type such that $d_i \neq 0$ for some minimal element $i \in I$, the conclusion of Proposition 4.5 follows by the same argument as in the proof of Lemma 3.5 and at the end of Section 3B. The key difference in the case $|I| = 1$, $\aleph = \varnothing$ is that the section $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta$ may vanish on $\partial\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. In addition, by the regularity assumptions, (ϑ1b), (ϑ2b), and (ϑ2c), the condition

$$\big\{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\big\}^{-1}(0) - \mathfrak{M}_1^0(\kappa, d/d_\kappa) \subset \mathfrak{M}^{1;0}_{1;1}(\kappa, d/d_\kappa)$$

implies

$$\pm\big|\big\{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\big\}^{-1}(0)\cap\mathfrak{M}^0_1(\kappa,d/d_\kappa)\big| \\ =\big\langle e(\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}),[\overline{\mathfrak{M}}^0_1(\kappa,d/d_\kappa)]\big\rangle-\mathscr{C}_{\mathfrak{M}^{1;0}_{1;1}(\kappa,d/d_\kappa)}\big(\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\big).$$

(2) If $\mathcal{T}$ is a bubble type such that $d_i=0$ for all minimal elements $i\in I$ and $|\hat{I}|=1$, or more generally $|\chi(\mathcal{T})|=1$, nearly the same argument still applies. In this case, $\mathcal{F}^1\mathcal{T}=\mathcal{F}T$ and the conclusions of Lemmas 3.5 and 3.6 are still valid. The key difference is that the normal bundle of $\mathcal{U}_{\mathcal{T};\kappa;1}$ in $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4;J)$ is not given by the cokernels of the homomorphisms i_0 in the long exact sequence (3-4). Instead, up to the action of the automorphism group of b, the fiber of $\mathcal{N}^\kappa\mathcal{T}$ at $[b]\in\mathcal{U}_{\mathcal{T};\kappa;1}$ is

$$\mathcal{N}^\kappa\mathcal{T}=\Gamma_-(b;T\mathbb{P}^4;0)\big/\Gamma_-(b;TY;0),$$

where

$$\Gamma_-(b;T\mathbb{P}^4;0)=\big\{\zeta\in\Gamma_-(b;T\mathbb{P}^4):\tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},h}\zeta=0\big\},$$
$$\Gamma_-(b;TY;0)=\big\{\zeta\in\Gamma_-(b;TY):\tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},h}\zeta=0\big\}\subset\Gamma_-(b;T\kappa),$$

if h is the unique element of $\chi(\mathcal{T})$. The reason for this is that

$$\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4;J)=\big\{b\in\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J):d\{u_b|_{\Sigma_{b,h}}\}=0\big\},$$

by Lemma 4.2. Since the linear operator

$$\mathfrak{D}^\kappa_{\mathcal{T},h}:\big\{\zeta\in\Gamma_-(b;T\kappa):\zeta|_{\Sigma_{b,\aleph}}=0\big\}\to T_{\mathrm{ev}_P(b)}\kappa,\qquad \zeta\to\nabla^\kappa_{e_\infty}\big(\zeta|_{\Sigma_{b,h}}\big),$$

is surjective, the image of the homomorphism j_0 in the long exact sequence (4-4) on $\Gamma_-(b;T\mathbb{P}^4;0)$ is the same as on $\Gamma_-(b;T\mathbb{P}^4;T\kappa)$. Thus, the analogue of the bundle $\mathcal{V}_-$ of page 449 in this case is the bundle $Q^{1,1}_{\kappa,d/d_\kappa}$ defined on page 465. The section of $\mathcal{V}_-$ induced by ϑ is $\tilde{\mathfrak{d}}^{1,1}_{\kappa,d/d_\kappa}\vartheta$. Its composition with the map π^+_κ in (4-11) is $\mathfrak{d}^{1,1}_{\kappa,d/d_\kappa}\vartheta$. Thus, Proposition 4.5 in this case follows from the regularity assumptions (ϑ2b) and (ϑ3), by the same argument as in Section 3B.

(3) Finally, suppose $\mathcal{T}=(I,\aleph;\underline{d})$ is a bubble type such that $d_i=0$ for all minimal elements $i\in I$ and $|\hat{I}|\geq 2$. In this case, the dimension of the fibers of $\mathcal{F}^1\mathcal{T}$ may not be constant over $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4;J)$. Thus, we modify the setup of the second part of Section 3B by working directly with the normal bundle to the smooth submanifold $\mathcal{F}^1\mathcal{T}^\varnothing|_{\mathcal{U}_{\mathcal{T};\kappa;1}}$ in $\mathcal{F}^1\mathcal{T}^\varnothing|_{\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4;J)}$. Let $\gamma\to\mathbb{P}\mathfrak{F}\mathcal{T}$ be the tautological line bundle and

$$V=\pi^*_{\mathbb{P}\mathfrak{F}\mathcal{T}}\big(\pi^*_P\mathbb{E}^*\otimes\mathrm{ev}^*_PT\mathbb{P}^4\big)\to\mathbb{P}\mathfrak{F}\mathcal{T}.$$

We define the section $\alpha_{\mathcal{T}}$ of $\gamma^*\otimes V$ over $\mathbb{P}\mathfrak{F}\mathcal{T}$ by

$$\big\{\alpha_{\mathcal{T}}\big(b,(\tilde{v}_i)_{i\in\chi(\mathcal{T})}\big)\big\}(b,\psi)=\sum_{i\in\chi(\mathcal{T})}\mathcal{D}_{\mathcal{T},i}\big(b,\psi_{x_{h(i)}}\tilde{v}_i\big)\in T_{\mathrm{ev}_P(b)}\mathbb{P}^4$$

if $\big(b,(\tilde{\upsilon}_i)_{i\in\chi(\mathcal{T})}\big)\in\gamma$ and $(b,\psi)\in\mathbb{E}_{\pi_P(b)}$. With our assumptions on J, this section is transverse to the zero set and thus

$$\mathcal{U}_1(\mathbb{P}^4;J)\equiv\alpha_{\mathcal{T}}^{-1}(0)$$

is a smooth suborbifold of $\mathbb{P}\mathfrak{F}\mathcal{T}$. For a similar reason, so is

$$\mathcal{U}_{1;\kappa}\equiv\mathcal{U}_1\cap\mathbb{P}\mathfrak{F}\mathcal{T}\big|_{\mathcal{U}_{\mathcal{T};\kappa}}.$$

Let $\mathcal{N}^\kappa\mathcal{T}$ denote the normal bundle of $\mathcal{U}_{1;\kappa}$ in $\mathcal{U}_1(\mathbb{P}^4;J)$. Up to the action of the automorphism group of $[b,[\tilde{\upsilon}]]\in\mathcal{U}_{1;\kappa}$, we have

$$\mathcal{N}^\kappa\mathcal{T}|_{[b,[\tilde{\upsilon}]]}=\Gamma_-(b;T\mathbb{P}^4;\tilde{\upsilon})\big/\Gamma_-(b;TY;\tilde{\upsilon}),$$

where

$$\Gamma_-(b;T\mathbb{P}^4;\tilde{\upsilon})=\Big\{\zeta\in\Gamma_-(b;T\mathbb{P}^4):\sum_{i\in\chi(\mathcal{T})}(\psi_{x_{h(i)}}\tilde{\upsilon}_i)\tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},i}\zeta=0\text{ for all }\psi\in\mathbb{E}_{\pi_P(b)}\Big\},$$

$$\Gamma_-(b;TY;\tilde{\upsilon})=\Big\{\zeta\in\Gamma_-(b;TY):\sum_{i\in\chi(\mathcal{T})}(\psi_{x_{h(i)}}\tilde{\upsilon}_i)\tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},i}\zeta=0\text{ for all }\psi\in\mathbb{E}_{\pi_P(b)}\Big\}\subset\Gamma_-(b;T\kappa).$$

Thus, there is a natural surjective bundle homomorphism

$$\mathcal{V}_-\equiv\pi^*_{\mathbb{P}\mathfrak{F}\mathcal{T}}\mathcal{V}^d_1\big/j_\kappa(\mathcal{N}^\kappa\mathcal{T})\to\pi_B^*\mathcal{W}^{0,q}_{\kappa,d/d_\kappa}\quad\text{if }\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4;J)\subset\overline{\mathfrak{M}}{}^q_1(\mathbb{P}^4,d;J),$$

where $j_\kappa:\mathcal{N}^\kappa\mathcal{T}\to\pi^*_{\mathbb{P}\mathfrak{F}\mathcal{T}}\mathcal{V}^d_1$ is the injective bundle homomorphism induced by the maps j_0 in (4-4). We put

$$\mathcal{F}=\pi^*_{\mathbb{P}\mathfrak{F}\mathcal{T}}\mathcal{F}T\to\mathbb{P}\mathfrak{F}\mathcal{T},$$
$$\mathcal{F}^{1,\varnothing}=\big\{\big(b,[\tilde{\upsilon}];\upsilon\big)\in\mathcal{F}^\varnothing:\big(b,[\tilde{\upsilon}]\big)\in\mathcal{U}_1(\mathbb{P}^4;J);\ [\rho(\upsilon)]=[\tilde{\upsilon}]\big\}.$$

The smooth orbifold $\mathcal{F}^{1,\varnothing}$ is diffeomorphic to $\mathcal{F}^1\mathcal{T}^\varnothing$ by the projection map

$$\big(b,[\tilde{\upsilon}];\upsilon\big)\to(b;\upsilon).$$

Furthermore, $\mathcal{F}^{1,\varnothing}\to\mathcal{U}_1(\mathbb{P}^4;J)$ is a fiber bundle of smooth varieties. We can thus apply the same argument as in the proof of Lemma 3.5 and the end of Section 3B, along with the regularity assumption (ϑ2b), to show that

$$\{s^d_1+t\vartheta\}^{-1}(0)\cap U(K)=\varnothing\qquad\text{for all }t\in(0,\epsilon_\vartheta(K))$$

if $U(K)$ and t are sufficiently small. □

It remains to compute the $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta$-contribution to the Euler class of the bundle $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$ over $\overline{\mathfrak{M}}{}^0_1(\kappa,d/d_\kappa)$ from the set

$$\mathcal{Z}_{\kappa,\vartheta}\equiv\big\{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\big\}^{-1}(0)-\mathfrak{M}^0_1(\kappa,d/d_\kappa)=\big\{\mathfrak{d}^{1,1}_{\kappa,d/d_\kappa}\vartheta\big\}^{-1}(0)\subset\mathfrak{M}^{1,0}_{1;1}(\kappa,d/d_\kappa).$$

This contribution is computed by counting the zeros of the section $\mathfrak{D}^{1,0}_{\kappa,d/d_\kappa}\vartheta + t\nu$, for a generic section ν of $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$, that lie near $\mathfrak{X}_{\kappa,\vartheta}$. First, let

$$\pi^{\perp}_{-;\kappa} : \widetilde{\mathcal{W}}^{1,1}_{\kappa,d/d_\kappa} \to \pi_P^*\mathbb{E}^* \otimes \pi_B^*\mathrm{ev}_0^* N_Y\kappa$$

denote the (quotient) projection map; see (3-1). Our regularity assumptions on ν will be that the affine map

$$\begin{gathered}\psi_{\vartheta,\nu} : \pi_P^* L_{P,1} \otimes \pi_B^* L_0 \to \pi_P^*\mathbb{E}^* \otimes \pi_B^*\mathrm{ev}_0^* N_Y\kappa,\\ \psi_{\vartheta,\nu}(b;\upsilon) = \pi^{\perp}_{-;\kappa}\nu(b) + \{\tilde{\mathfrak{D}}^{1,1}_{\kappa,d/d_\kappa}\vartheta\}|_b\upsilon,\end{gathered} \tag{4-13}$$

over $\mathfrak{X}_{\kappa,\vartheta}$ is transverse to the zero set and all zeros of $\psi_{\vartheta,\nu}$ lie over

$$\mathfrak{X}^0_{\kappa,\vartheta} \equiv \mathfrak{X}_{\kappa,\vartheta} \cap \big(\mathcal{M}_{1,1} \times \mathfrak{M}^0_{0,1;1}(\kappa, d/d_\kappa)\big).$$

Since the set $\psi^{-1}_{\vartheta,\nu}(0)$ is finite, it follows that it lies over a compact subset $K_{\vartheta,\nu}$ of $\mathfrak{X}^0_{\kappa,\vartheta}$. By the regularity assumption (ϑ2a), these conditions are satisfied by sections ν in a dense open subset of the space of all sections of $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$. We put

$$\partial\mathfrak{X}_{\kappa,\vartheta} = \mathfrak{X}_{\kappa,\vartheta} \cap \big(\partial\overline{\mathcal{M}}_{1,1} \times \mathfrak{M}^0_{0,1;1}(\kappa, d/d_\kappa)\big).$$

Lemma 4.6. *Suppose J, d, $\mathfrak{L}$, $\mathcal{V}^d_1$, $\vartheta \in \tilde{\mathcal{A}}^d_1(s;J)$, and $\kappa \in \mathcal{S}_0(Y;J)$ are as in Proposition 4.5 and in Lemma 4.4. If $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $d_i = 0$ for all minimal elements i of I and $\hat{I} = \{h\}$ is a single-element set, then there exist $\delta \in C(\mathcal{U}_{\mathcal{T};\kappa;1}; \mathbb{R}^+)$, $U^1_{\mathcal{T}}$, and $\phi^1_{\mathcal{T}}$ as in the $n = 1$ case of Lemma 4.2, $\varepsilon \in C(\mathcal{F}T;\mathbb{R})$, and a vector bundle isomorphism*

$$\Phi_{\mathcal{T}} : \pi^*_{\mathcal{F}T_\delta}\big(\widetilde{\mathcal{W}}^{1,1}_{\kappa,d/d_\kappa}|_{\mathcal{U}_{\mathcal{T};\kappa;1}}\big) \to \mathcal{W}^{1,0}_{\kappa,d/d_\kappa}\big|_{\overline{\mathfrak{M}}^0_1(\kappa,d/d_\kappa)\cap U^1_{\mathcal{T}}},$$

covering the homeomorphism $\phi^1_{\mathcal{T}}$ and restricting to the identity over $\mathcal{U}_{\mathcal{T};\kappa;1}$, all such that $\lim_{|\upsilon|\to 0}\varepsilon(\upsilon) = 0$ *and*

$$\big|\pi^{\perp}_{-;\kappa}\Phi^{-1}_{\mathcal{T}}\big(\{\mathfrak{D}^{1,0}_{\kappa,d/d_\kappa}\vartheta\}(\phi^1_{\mathcal{T}}(\upsilon)\big) - \big\{\pi^-_\kappa\tilde{\mathfrak{D}}^{1,1}_{\kappa,d/d_\kappa}\vartheta|_b\big\}\rho(\upsilon)\big| \le \varepsilon(\upsilon)\big|\rho(\upsilon)\big|$$

for all $\upsilon = (b, v) \in \mathcal{F}T^{\varnothing}_\delta$.

In this case, $\aleph = \varnothing$ or $\aleph$ contains one element, and

$$\mathcal{U}_{\mathcal{T};\kappa;1} = \begin{cases}\mathcal{M}_{1,1} \times \mathcal{M}^0_{0,1;1}(\kappa, d/d_\kappa) & \text{if } \aleph = \varnothing,\\ \partial\overline{\mathcal{M}}_{1,1} \times \mathfrak{M}^0_{0,1;1}(\kappa, d/d_\kappa) & \text{otherwise.}\end{cases}$$

In either case, by Lemma 4.2, the normal bundle $\mathcal{F}^1\mathcal{T}$ of $\mathcal{U}_{\mathcal{T};\kappa}$ in $\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa)$ is $\mathcal{F}T$. If $\aleph = \varnothing$,

$$\mathcal{F}T = \pi_P^* L_{P,1} \otimes \pi_B^* L_0 \quad \text{and} \quad \rho(\upsilon) = \upsilon.$$

Otherwise, $\mathcal{F}T$ is the direct sum of $\pi_P^* L_{P,1} \otimes \pi_B^* L_0$ with the line of smoothings of the node of $\Sigma_{b,\aleph}$, which in this case is a sphere with two points identified. If $\upsilon \in \mathcal{F}T$, $\rho(\upsilon)$ is the $\pi_P^* L_{P,1} \otimes \pi_B^* L_0$-component of υ.

Lemma 4.6 follows fairly easily from constructions in [Zinger 2004b; 2007]. However, its proof is notationally involved, and we postpone it until the next subsection.

Corollary 4.7. *Suppose d, $\mathfrak{L}$, $\mathcal{V}_1^d$, J, $\vartheta \in \tilde{\mathcal{A}}_1^d(s; J)$, and $\kappa \in \mathcal{S}_0(Y; J)$ are as in Proposition 4.5. If ν is a generic perturbation of the section $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta$ of $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$ over $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$, there exist $\epsilon_\nu \in \mathbb{R}^+$ and an open neighborhood U of $\partial\mathfrak{X}_{\kappa,\vartheta}$ in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ such that*

$$\left\{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta + t\nu\right\}^{-1}(0) \cap U = \varnothing \qquad \text{for all } t \in (0, \epsilon_\nu).$$

Furthermore, for every compact subset K of $\mathfrak{X}^0_{\kappa,\vartheta}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ with the following properties:

(a) *the section $\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta + t\nu$ is transverse to the zero set in $\mathcal{W}^{1,0}_{\kappa,d/d_\kappa}$ over $U(K)$ for all $t \in (0, \epsilon_\nu(K))$;*

(b) *for every open subset U of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$, there exists $\epsilon(U) \in (0, \epsilon_\nu(K))$ such that*

$$^{\pm}\left|\{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta + t\nu\}^{-1} \cap U\right| = -\frac{d/d_\kappa - 1}{12}\left\langle e(\mathcal{W}^0_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)]\right\rangle$$

if $K_{\vartheta,\nu} \subset K \subset U \subset U(K)$, $t \in (0, \epsilon(U))$.

Proof. (1) Let $\mathcal{T} = (I, \aleph; \underline{d})$ be a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i = 0$ for all minimal elements i of I. By the regularity assumption (ϑ2b), if

$$\mathfrak{X}_{\vartheta,\mathcal{T}} \equiv \{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta\}^{-1}(0) \cap \mathcal{U}_{\mathcal{T};\kappa;1} \neq \varnothing,$$

then $\hat{I} = \{h\}$ is a single-element set, while $|\aleph| \in 0, 1$.

(2) We denote by $\mathcal{N}^\vartheta\mathcal{T}$ the normal bundle of $\mathfrak{X}_{\vartheta,\mathcal{T}}$ in $\mathcal{U}_{\mathcal{T};\kappa;1}$. Similarly to the proof of Lemma 3.5, using the homeomorphism $\phi^1_{\mathcal{T}}$ of Lemma 4.2 and the bundle isomorphism $\Phi_{\mathcal{T}}$ of Lemma 4.6, we can obtain an identification of neighborhoods of $\mathfrak{X}_{\vartheta,\mathcal{T}}$ in $\mathcal{N}^\vartheta\mathcal{T} \oplus \mathcal{F}T$ and in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$,

$$\phi_{\mathcal{T};\vartheta} : \mathcal{N}^\vartheta\mathcal{T}_\delta \times_{\mathfrak{X}_{\vartheta,\mathcal{T}}} \mathcal{F}T_\delta \to \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa),$$

and a lift of $\phi_{\mathcal{T};\vartheta}$ to a bundle isomorphism,

$$\Phi_{\mathcal{T};\vartheta} : \pi^*_{\mathcal{N}^\vartheta\mathcal{T} \times_{\mathfrak{X}_{\vartheta,\mathcal{T}}} \mathcal{F}T_\delta}\left(\widetilde{\mathcal{W}}^{1,1}_{\kappa,d/d_\kappa}|_{\mathfrak{X}_{\vartheta,\mathcal{T}}}\right) \to \mathcal{W}^{0,1}_{\kappa,d/d_\kappa}.$$

For $(b; X, \upsilon) \in \mathcal{N}^{\vartheta}\mathcal{T} \times_{\mathfrak{X}_{\vartheta,\mathcal{T}}} \mathcal{F}T_{\delta}$, we put

$$s(b; X, \upsilon) = \Phi_{\mathcal{T};\vartheta}^{-1}\big(\{\mathfrak{D}_{\kappa,d/d_\kappa}^{1,0}\vartheta\}\phi_{\mathcal{T}}^{1}(b; X, \upsilon)\big) \in \widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,1}\big|_b;$$
$$\tilde{\nu}(b; X, \upsilon) = \Phi_{\mathcal{T};\vartheta}^{-1}\big(\nu(\phi_{\mathcal{T}}^{1}(b; X, \upsilon))\big) \in \widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,1}\big|_b.$$

We define $N_s(X)$ and $N_s'(X, \upsilon)$ in $\widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,1}|_b$ by

$$s(b; X, 0) = s(b; 0, 0) + j_b X + N_s X = j|_b X + N_s X; \tag{4-14}$$

$$s(b; X, \upsilon) = s(b; X, 0) + N_s'(X, \upsilon), \tag{4-15}$$

where $j_b : \mathcal{N}^{\vartheta}\mathcal{T} \to \pi_B^*\mathcal{W}_{\kappa,d/d_\kappa}^{0,1}|_b \subset \widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,1}|_b$ is the derivative of s. Thus,

$$\begin{gathered} N_s(0) = 0, \\ \big|N_s(X) - N_s(X')\big| < C_K\big(|X| + |X'|\big)|X - X'| \quad \text{for all } X, X' \in \mathcal{N}^{\vartheta}\mathcal{T}_{\delta}. \end{gathered} \tag{4-16}$$

By the continuity ϑ, there exists $\varepsilon \in C(\mathcal{F}T; \mathbb{R}^+)$ such that $\lim_{|\upsilon|\to 0} \varepsilon(\upsilon) = 0$ and

$$\begin{gathered} \big|N_s'(X, \upsilon)\big| \le \varepsilon(\upsilon), \\ \big|N_s'(X, \upsilon) - N_s'(X', \upsilon)\big| \le \varepsilon(\upsilon)\big|X - X'\big| \end{gathered} \tag{4-17}$$

for all $X, X' \in \mathcal{N}^{\vartheta}\mathcal{T}_{\delta},\ \upsilon \in \mathcal{F}T_{\delta}$. We also have some $C \in C(\mathfrak{X}_{\vartheta,\mathcal{T}}; \mathbb{R})$ such that

$$\begin{gathered} \big|\tilde{\nu}(b; X, \upsilon)\big| \le C(b) \\ \big|\tilde{\nu}(b; X, \upsilon) - \tilde{\nu}(b; X', \upsilon)\big| \le C(b)|X - X'| \end{gathered} \tag{4-18}$$

for all $b \in \mathfrak{X}_{\vartheta,\mathcal{T}},\ X, X' \in \mathcal{N}^{\vartheta}\mathcal{T}_{\delta},\ \upsilon \in \mathcal{F}T_{\delta}$.

(3) Let $\pi_{-;B} : \widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,1} \to \pi_B^*\mathcal{W}_{\kappa,d/d_\kappa}^{0,1}$ be the natural projection map; see (3-1). Let K be a precompact open subset of $\mathfrak{X}_{\vartheta,\mathcal{T}}$. Since the homomorphism

$$j_b : \mathcal{N}^{\vartheta}\mathcal{T} \to \pi_B^*\mathcal{W}_{\kappa,d/d_\kappa}^{0,1}$$

is an isomorphism by the regularity assumption (ϑ2a), by (4-14)–(4-18) and the Contraction Principle, the equation

$$\pi_{-;B}\big(s(b; X, \upsilon) + t\tilde{\nu}(b; X, \nu)\big) = 0$$

has a unique small solution $X = X_t(\upsilon) \in \mathcal{N}^{\vartheta}\mathcal{T}_b$ for all $t \in [0, \delta_K)$, $\upsilon \in \mathcal{F}T_{\delta_K}|_b$, and $b \in K$. Furthermore,

$$\big|X_t(\upsilon)\big| \le C_K\big(t + \varepsilon(\upsilon)\big). \tag{4-19}$$

(4) By the above, the number of zeros of $\mathfrak{D}^{1,0}_{\kappa,d/d_\kappa}\vartheta + t\nu$, for $t \in (0, \delta_K)$, in a small neighborhood U_K of K in $\overline{\mathfrak{M}}^0_1(\kappa, d/d_\kappa)$ is the number of solutions of the equation

$$\begin{aligned}\Psi_t(b;\upsilon) &\equiv t^{-1}\pi^{\perp}_{-;\kappa}\big(s(b; X_t(\upsilon);\upsilon) + t\tilde{\nu}(b; X_t(\upsilon);\upsilon)\big)\\ &= t^{-1}\pi^{\perp}_{-;\kappa}N'_s\big(\upsilon; X_t(\upsilon)\big) + \pi^{\perp}_{-;\kappa}\tilde{\nu}\big(b; X_t(\upsilon);\upsilon\big),\end{aligned}$$

since $\mathfrak{D}^{1,0}_{\kappa,d/d_\kappa}\vartheta|_{\mathcal{U}_{\mathcal{T};\kappa;1}}$ is a section of $\pi_B^*\mathcal{W}^{0,1}_{\kappa,d/d_\kappa}$ and thus $\pi^{\perp}_{-;\kappa}N_sX = 0$ for all $X \in \mathcal{N}^{\vartheta}\mathcal{T}_\delta$. By the estimate of Lemma 4.6, (4-19), and the smoothness of ν

$$\Big|\Psi_t(b;\upsilon) - \big(\pi^{\perp}_{-;\kappa}\nu\big(b\big) + t^{-1}\{\pi^-_\kappa\tilde{\mathfrak{D}}^{1,1}_{\kappa,d/d_\kappa}\vartheta\}|_b\rho(\upsilon)\big)\Big| \le C_K\big(t+\varepsilon(\upsilon)\big)|\rho(\upsilon)|. \tag{4-20}$$

By the regularity assumption (ϑ3), the section $\pi^-_\kappa\tilde{\mathfrak{D}}^{1,1}_{\kappa,d/d_\kappa}\vartheta$ does not vanish over $\mathfrak{X}_{\vartheta,\mathcal{T}}$. Suppose $\mathcal{T}$ is a bubble type as in the $|\aleph| = 1$ case in (1) above. By our assumptions on ν, the affine map

$$\mathcal{F}T \to \pi_P^*\mathbb{E}^* \otimes \pi_B^*\mathrm{ev}_0^*N_Y\kappa, \quad \upsilon \to \pi^{\perp}_{-;\kappa}\nu\big(b\big) + \{\pi^-_\kappa\tilde{\mathfrak{D}}^{1,1}_{\kappa,d/d_\kappa}\vartheta\}|_b\rho(\upsilon), \tag{4-21}$$

which factors through $\psi_{\vartheta,\nu}$, does not vanish over the compact set $\partial\mathfrak{X}_{\kappa,\vartheta}$. Thus, $\Psi_t(b;\upsilon) \neq 0$ for all $t \in \mathbb{R}^+$ and $\upsilon \in \mathcal{F}T|_{\partial\mathfrak{X}_{\kappa,\vartheta}}$ sufficiently small. This concludes the proof of the first statement of Corollary 4.7.

(5) Finally, suppose $\mathcal{T}$ is a bubble type as in the $|\aleph| = 0$ case in (1). If K is a compact subset of $\mathfrak{X}^0_{\kappa,\vartheta}$ containing $K_{\vartheta,\nu}$, by (4-20), the regularity assumption (ϑ3), and the same cobordism argument as in Section 3.1 of [Zinger 2003], the number of solutions of $\Psi_t(b;\upsilon) = 0$ with $t \in \mathbb{R}^+$ and $\upsilon \in \mathcal{F}T|_K$ sufficiently small is the number of zeros of the affine map in (4-21), i.e. ${}^{\pm}|\psi^{-1}_{\vartheta,\nu}(0)|$. Since $\pi^-_\kappa\tilde{\mathfrak{D}}^{1,1}_{\kappa,d/d_\kappa}\vartheta$ does not vanish over $\mathfrak{X}_{\kappa,\vartheta}$,

$$\begin{aligned}{}^{\pm}\big|\psi^{-1}_{\vartheta,\nu}(0)\big| &= \big\langle c(\pi_P^*\mathbb{E}^*\otimes\pi_B^*\mathrm{ev}_0^*N_Y\kappa)c(\pi_P^*L_{P,1}\otimes\pi_B^*L_0)^{-1}, [\mathfrak{X}_{\kappa,\vartheta}]\big\rangle\\ &= \big\langle\pi_P^*(-2\lambda+\psi_{P,1}) + \pi_B^*(c_1(\mathrm{ev}_0^*N_Y\kappa)+\psi_0), [\mathfrak{X}_{\kappa,\vartheta}]\big\rangle,\end{aligned}$$

where λ and $\psi_{P,1}$ are the usual tautological classes on $\overline{\mathcal{M}}_{1,1}$. The space $\mathfrak{X}_{\kappa,\vartheta}$ is the zero set of the section $\mathfrak{D}^{1,1}_{\kappa,d/d_\kappa}\vartheta$ of the bundle $\pi_B^*\mathcal{W}^{0,1}_{\kappa,d/d_\kappa}$ over $\overline{\mathfrak{M}}^1_{1;1}(\kappa, d/d_\kappa)$. Since this section is transverse to the zero set by (ϑ2a),

$$\begin{aligned}{}^{\pm}\big|\psi^{-1}_{\vartheta,\nu}(0)\big| &= \big\langle\pi_P^*(-2\lambda+\psi_{P,1})\cdot\pi_B^*e(\mathcal{W}^{0,1}_{\kappa,d/d_\kappa}), [\overline{\mathcal{M}}_{1,1}\times\overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_\kappa)]\big\rangle\\ &= -\tfrac{1}{24}\big\langle e(\mathcal{W}^{0,1}_{\kappa,d/d_\kappa}), [\overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_\kappa)]\big\rangle.\end{aligned}$$

We note that a generic fiber of the forgetful map

$$\tilde{\pi} : \overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_\kappa) \to \overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)$$

consists of $2(d/d_\kappa)-2$ points, corresponding to the branch points a degree-d/d_κ cover $\mathbb{P}^1\to\kappa$. We conclude that for every compact subset K of $\mathfrak{X}_{\kappa,\vartheta}$ containing $K_{\vartheta,\nu}$

$$\begin{aligned}{}^{\pm}\big|\{\mathfrak{d}^{1,0}_{\kappa,d/d_\kappa}\vartheta+t\nu\}^{-1}\cap U\big| = {}^{\pm}\big|\psi^{-1}_{\vartheta,\nu}(0)\big| &= -\tfrac{1}{24}\big\langle\tilde{\pi}^*e(\mathcal{W}^0_{\kappa,d/d_\kappa}),[\overline{\mathfrak{M}}_{0,1;1}(\kappa,d/d_\kappa)]\big\rangle\\ &= -\tfrac{1}{24}\big(2(d/d_\kappa)-2\big)\big\langle e(\mathcal{W}^0_{\kappa,d/d_\kappa}),[\overline{\mathfrak{M}}_0(\kappa,d/d_\kappa)]\big\rangle,\end{aligned}$$

provided that U is a sufficiently small neighborhood of K in $\overline{\mathfrak{M}}^0_1(\kappa,d/d_\kappa)$ and $t\in(0,\delta(U))$. □

4D. ***A genus-one gluing procedure.*** In this subsection, we prove Lemma 4.6. We review the genus-one gluing procedure of Section 4.2 in [Zinger 2004b] and its extensions to the spaces $\Gamma_-(b;T\mathbb{P}^4)$ and $\Gamma_-(b;\mathfrak{L})$. As a result, we will be able to describe the behavior of the boundary operator $\mathfrak{d}_0$ in the long exact sequence (3-4) for $[\tilde{b}(\upsilon)]\in\mathfrak{M}^0_1(\kappa,d/d_\kappa)$ with $\upsilon\in\tilde{\mathcal{F}}\mathcal{T}^{\varnothing}$ sufficiently small.

Let $\mathcal{T}=(I,\aleph;\underline{d})$ be a bubble type as in the statement of Lemma 4.6. If $\upsilon=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}^{\varnothing}_\delta$ is small gluing parameter, let

$$b(\upsilon)=(\Sigma_\upsilon,j_\upsilon;u_\upsilon),\quad \text{where } u_\upsilon=u_b\circ\tilde{q}_\upsilon,$$

be the (second-stage) approximately J-holomorphic map. Here

$$\tilde{q}_\upsilon=\tilde{q}_{\upsilon_0;2}:\Sigma_\upsilon\to\Sigma_b$$

is the basic gluing map constructed in Section 4.2 of [Zinger 2004b]. In the present case, there is no first stage in this usually two-stage gluing construction, as there is only one level of bubbles (in fact, only one bubble) to attach. The key advantage of this gluing construction is that the map $\tilde{q}_\upsilon$ is closer to being holomorphic than in the gluing construction used in Section 3. In particular,

$$\big\|\bar{\partial}_J u_\upsilon\big\|_{\upsilon,p}\le C(b)\big|\rho(\upsilon)\big|.$$

If $b\in\mathcal{U}_{\mathcal{T};\kappa;1}$, then $du_{b,h}|_\infty=0$ and this estimate improves to

$$\big\|\bar{\partial}_J u_\upsilon\big\|_{\upsilon,p}\le C(b)\big|\rho(\upsilon)\big|^2. \tag{4-22}$$

This is immediate from the definition of the map $\tilde{q}_\upsilon$.

We extend the metric $g_{Y,b}$ introduced on page 452 to a metric $g_{\mathbb{P}^4,b}$ on the bundle $T\mathbb{P}^4$. Let ∇^J be the J-compatible connection corresponding to the Levi-Civita of the metric $g_{\mathbb{P}^4,b}$. As in Section 3C, let

$$\Gamma^{0,1}_-(b;T\mathbb{P}^4)=\mathcal{H}^{0,1}_{b,P}\otimes T_{\mathrm{ev}_P(b)}\mathbb{P}^4$$

be the space of $u_b^*T\mathbb{P}^4$-valued harmonic $(0,1)$-forms on Σ_b. If $\upsilon=(b,v)$ and $b\in\mathcal{U}_{\mathcal{T};\kappa;1}$ are as above, we put

$$\Gamma_-^{0,1}(\upsilon;T\mathbb{P}^4)=\{R_\upsilon\eta:\eta\in\Gamma_-^{0,1}(b;T\mathbb{P}^4)\}\subset\Gamma^{0,1}(\upsilon;T\mathbb{P}^4),$$

where $R_\upsilon\eta$ is a smooth extension of η such that $R_\upsilon\eta$ is nearly harmonic on the neck attaching the only bubble $\Sigma_{b,h}$ of Σ_b and below a small collar of the neck and vanishes past a slighter larger collar; see Section 4.2 in [Zinger 2004b]. Let

$$\pi_{\upsilon;-}^{0,1}:\Gamma^{0,1}(\upsilon;T\mathbb{P}^4)\to\Gamma_-^{0,1}(b;T\mathbb{P}^4)$$

be the L^2-projection map. We denote its kernel by $\Gamma_+^{0,1}(\upsilon;T\mathbb{P}^4)$. By the same argument as in Section 2.3 in [Zinger 2003], we have a decomposition

$$\Gamma(\upsilon;T\mathbb{P}^4)=\Gamma_-(\upsilon;T\mathbb{P}^4)\oplus\tilde{\Gamma}_+(\upsilon;T\mathbb{P}^4),$$

where

$$\Gamma_-(\upsilon;T\mathbb{P}^4)=\{R_\upsilon\zeta\equiv\xi\circ\tilde{q}_\upsilon:\zeta\in\Gamma_-(b;T\mathbb{P}^4)\},$$

such that

$$D_{J,\upsilon}:\tilde{\Gamma}_+(\upsilon;T\mathbb{P}^4)\to\Gamma_+^{0,1}(\upsilon;T\mathbb{P}^4)$$

is an isomorphism with fiber-uniformly bounded inverse and $\langle\!\langle D_{J,\upsilon}\zeta,\eta\rangle\!\rangle_{\upsilon,2}=0$ for all $\zeta\in\tilde{\Gamma}_+(\upsilon;T\kappa)\equiv\tilde{\Gamma}_+(\upsilon;T\mathbb{P}^4)\cap\Gamma(\upsilon;T\kappa)$, $\eta\in\Gamma_-^{0,1}(\upsilon;T\mathbb{P}^4)$. Analogously to (4-22), we also have

$$\|D_{J,\upsilon}\zeta\|_{\upsilon,p}\le C(b)|\rho(\upsilon)|\|\zeta\|_{\upsilon,p,1}\quad\text{for all }\zeta\in\Gamma_-(\upsilon;T\mathbb{P}^4).\tag{4-23}$$

Furthermore,

$$\big|\pi_{\upsilon;-}^{0,1}D_{J,\upsilon}R_\upsilon\zeta+2\pi\rho(\upsilon)J\,R_\upsilon\tilde{\mathfrak{D}}_{\mathcal{T},h}^{\mathbb{P}^4}\zeta\big|\le C(b)|\rho(\upsilon)|^2\|\zeta\|_{b,p,1}\tag{4-24}$$

for all $\zeta\in\Gamma_-(b;T\mathbb{P}^4)$; see Lemma 4.4(5) in [Zinger 2004b]. Due to the assumption that $du_{b,h}|_\infty=0$, we do not need to require that $\zeta|_{\Sigma_{b,\aleph}}=0$. We also get a slightly sharper bound, though this is not essential. The estimate (4-24) is the fundamental fact behind the estimate of Lemma 4.6.

Similarly to Section 3C, the restriction of the homeomorphism $\phi_{\mathcal{T}}^1$ of Lemma 4.2 can be taken to be of the form

$$\phi_{\mathcal{T}}^1([\upsilon])=([\tilde{b}(\upsilon)]),$$

where $\tilde{b}(\upsilon)=(\Sigma_\upsilon,j_\upsilon;\tilde{u}_\upsilon)$, $\tilde{u}_\upsilon=\exp_{b,u_\upsilon}\zeta_\upsilon$, $\zeta_\upsilon\in\tilde{\Gamma}_+(\upsilon;T\kappa)$, and

$$\|\zeta_\upsilon\|_{\upsilon,p,1}\le C(b)|\rho(\upsilon)|^2.\tag{4-25}$$

The last estimate follows from (4-22) by the usual argument.

We denote by

$$\Pi_\upsilon^J : \Gamma(\upsilon; T\mathbb{P}^4) \to \tilde{\Gamma}(\upsilon; T\mathbb{P}^4) \equiv L_1^p(\tilde{b}(\upsilon); T\mathbb{P}^4)$$

the ∇^J-parallel transport in $T\mathbb{P}^4$ and by

$$\Pi_\upsilon : \Gamma(\upsilon; \mathfrak{L}) \equiv L_1^p(b(\upsilon); \mathfrak{L}) \to \tilde{\Gamma}(\upsilon; \mathfrak{L}) \equiv L_1^p(\tilde{b}(\upsilon); \mathfrak{L})$$

the ∇-parallel transport in $\mathfrak{L}$ along the geodesics γ_{ζ_υ} in κ of the metric $g_{\kappa,b}$. By (4-23)–(4-25) and the same argument as in Section 4.3 of [Zinger 2007], there exists an isomorphism

$$\tilde{R}_\upsilon : \Gamma_-(b; T\mathbb{P}^4; 0) \to \tilde{\Gamma}_-(\upsilon; T\mathbb{P}^4) \equiv \ker D_{J,\tilde{b}(\upsilon)}$$

such that

$$\big\| \tilde{R}_\upsilon\zeta - \Pi_\upsilon^J R_\upsilon\zeta \big\|_{\upsilon,p,1} \le C(b)\big|\rho(\upsilon)\big|^2 \|\zeta\|_{b,p,1} \quad \text{for all } \zeta \in \Gamma_-(b; T\mathbb{P}^4; 0). \tag{4-26}$$

Similarly, there exists an isomorphism

$$\tilde{R}_\upsilon : \Gamma_-(b; \mathfrak{L}; 0) \to \tilde{\Gamma}_-(\upsilon; \mathfrak{L}) \equiv \ker \bar{\partial}_{\nabla,\tilde{b}(\upsilon)}$$

such that

$$\big\| \tilde{R}_\upsilon\xi - \Pi_\upsilon R_\upsilon\xi \big\|_{\upsilon,p,1} \le C(b)\big|\rho(\upsilon)\big|^2 \|\xi\|_{b,p,1} \quad \text{for all } \xi \in \Gamma_-(b; \mathfrak{L}; 0), \tag{4-27}$$

where again $R_\upsilon\xi = \xi \circ \tilde{q}_\upsilon$.

We next describe a convenient family of finite-dimensional spaces

$$\Gamma(b; T\mathbb{P}^4; \mathfrak{L}) \subset \Gamma(b; T\mathbb{P}^4),$$

parameterized by $b \in \mathcal{U}^{(0)}_{\mathcal{T};\kappa;1}$, such that the homomorphism

$$j_0 : \Gamma(b; T\mathbb{P}^4; \mathfrak{L}) \to \Gamma_-(b; \mathfrak{L}; 0)$$

is an isomorphism. For every $b \in \mathcal{U}^{(0)}_{\mathcal{T};\kappa;1}$, let

$$\Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa) \approx \mathrm{ev}_P^* N_Y\kappa \quad \text{and} \quad \Gamma_{-,+}(b; T\mathbb{P}^4; \kappa)$$

be the L^2-orthogonal complements of $\Gamma_-(b; T\mathbb{P}^4; T\kappa)$ in $\Gamma_-(b; T\mathbb{P}^4; TY)$ and of $\Gamma_-(b; T\kappa; 0)$ in $\Gamma_-(b; T\mathbb{P}^4; 0)$, respectively. The map in (4-10) induces a surjective homomorphism

$$\tilde{\pi}_\kappa^- : \Gamma_-(b; \mathfrak{L}; 0) \to \mathrm{ev}_P^* N_Y\kappa,$$

which restricts to an isomorphism on $j_0(\Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa))$ and vanishes on $j_0(\Gamma_{-,+}(b; T\mathbb{P}^4; \kappa))$, where j_0 is as in (4-4). Let $\Gamma_{-,+}(b; \mathfrak{L}; 0)$ be the L^2-orthogonal complement of $j_0(\Gamma_{-,+}(b; T\mathbb{P}^4; \kappa))$ in $\ker \tilde{\pi}_\kappa^-$. We set

$$\Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) = \big\{ \zeta \in \Gamma(b; N_{\mathbb{P}^4}Y) : \pi_Y^\perp \circ \zeta \in \Gamma_{-,+}(b; \mathfrak{L}; 0) \big\},$$

where the normal bundle $N_{\mathbb{P}^4}Y$ of Y in $\mathbb{P}^4$ is identified with the $g_{\mathbb{P}^4,b}$-orthogonal complement of TY in $T\mathbb{P}^4$ and

$$\pi_Y^\perp : T\mathbb{P}^4 \to \mathfrak{L} \approx N_{\mathbb{P}^4}Y$$

is the quotient projection. Then, the map

$$\pi_Y^\perp : \Gamma(b; T\mathbb{P}^4; \mathfrak{L}) \equiv \Gamma_{-,+}(b; T\mathbb{P}^4; \kappa) \oplus \Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa) \oplus \Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \longrightarrow \Gamma_-(b; \mathfrak{L}; 0)$$

is an isomorphism. Furthermore, since

$$\Gamma_{-,+}(b; T\mathbb{P}^4; \kappa) \oplus \Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa) \subset \Gamma_-(b; T\mathbb{P}^4),$$

the map

$$\mathfrak{d}_b : \Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \to H^1_J(\pi_B(b); N_Y\kappa), \quad \zeta \to [\pi_\kappa^\perp D_{J,b}\zeta],$$

is also an isomorphism, by the definition of the boundary operator $\mathfrak{d}_0$ in (4-4).

We now use the subspace $\Gamma(b; T\mathbb{P}^4; \mathfrak{L})$ of $\Gamma(b; T\mathbb{P}^4)$ to construct an analogous subspace $\tilde{\Gamma}(\upsilon; T\mathbb{P}^4; \mathfrak{L})$ of $\tilde{\Gamma}(\upsilon; T\mathbb{P}^4)$ for $\upsilon \in \tilde{\mathscr{F}}\mathscr{T}_\delta^\varnothing$ sufficiently small. For every

$$\zeta \in \Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa) \oplus \Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y),$$

we define $\tilde{R}_\upsilon\zeta \in \tilde{\Gamma}(\upsilon; T\mathbb{P}^4)$ by

$$\pi_Y^\perp \tilde{R}_\upsilon\zeta = \tilde{R}_\upsilon\pi_Y^\perp\zeta \in \tilde{\Gamma}_-(\upsilon; \mathfrak{L}) \quad \text{and} \quad \tilde{R}_\upsilon\zeta - \Pi_\upsilon^J R_\upsilon\zeta \in \Gamma(\Sigma_\upsilon; \tilde{u}_\upsilon^* N_{\mathbb{P}^4}Y),$$

where again $R_\upsilon\zeta = \zeta \circ \tilde{q}_\upsilon$. By (4-26) and (4-27),

$$\|\tilde{R}_\upsilon\zeta - \Pi_\upsilon R_\upsilon\zeta\|_{\upsilon,p,1} \le C(b)\big|\rho(\upsilon)\big|^2\|\zeta\|_{b,p,1} \quad \text{for all } \zeta \in \Gamma(b; \mathbb{P}^4; \mathfrak{L}). \tag{4-28}$$

Let $\tilde{\Gamma}_{-,+}(\upsilon; T\mathbb{P}^4; \kappa)$, $\tilde{\Gamma}_{-,+}(\upsilon; T\mathbb{P}^4; N_Y\kappa)$, $\tilde{\Gamma}_{+,-}(\upsilon; T\mathbb{P}^4; N_{\mathbb{P}^4}Y)$, and $\tilde{\Gamma}(\upsilon; T\mathbb{P}^4; \mathfrak{L})$ denote the images of $\Gamma_{-,+}(b; T\mathbb{P}^4; \kappa)$, $\Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa)$, $\Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y)$, and $\Gamma(b; T\mathbb{P}^4; \mathfrak{L})$ under $\tilde{R}_\upsilon$. By (4-28), the map

$$\pi_Y^\perp : \tilde{\Gamma}(\upsilon; T\mathbb{P}^4; \mathfrak{L}) \to \tilde{\Gamma}_-(\upsilon; \mathfrak{L})$$

is injective and thus an isomorphism. Furthermore, since

$$\tilde{\Gamma}_{-,+}(\upsilon; T\mathbb{P}^4; \kappa) \subset \tilde{\Gamma}_-(\upsilon; T\mathbb{P}^4),$$

by the definition of the boundary operator $\mathfrak{d}_0$ in (3-4) the map

$$\mathfrak{d}_\upsilon : \tilde{\Gamma}_{-,+}(\upsilon; T\mathbb{P}^4; N_Y\kappa) \oplus \tilde{\Gamma}_{+,-}(\upsilon; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \to H^1_J(\tilde{b}(\upsilon); N_Y\kappa)$$

given by

$$\zeta \to [\pi_\kappa^\perp D_{J,\tilde{b}(\upsilon)}\zeta]$$

is surjective and thus an isomorphism. We set

$$\begin{aligned}\tilde{\Gamma}^{0,1}_{-,+}(\upsilon; T\mathbb{P}^4; N_Y\kappa) &= \{\pi^{\perp}_{\kappa} D_{J,\tilde{b}(\upsilon)}\zeta : \zeta \in \tilde{\Gamma}_{-,+}(\upsilon; T\mathbb{P}^4; N_Y\kappa)\}\\ &\subset \tilde{\Gamma}^{0,1}(\upsilon; N_Y\kappa) \equiv L^p(\tilde{b}(\upsilon); N_Y\kappa);\\ \tilde{\Gamma}^{0,1}_{+,-}(\upsilon; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) &= \{\pi^{\perp}_{\kappa} D_{J,\tilde{b}(\upsilon)}\zeta : \zeta \in \tilde{\Gamma}_{+,-}(\upsilon; T\mathbb{P}^4; N_{\mathbb{P}^4}Y)\}\\ &\subset \tilde{\Gamma}^{0,1}(\upsilon; N_Y\kappa).\end{aligned}$$

It follows from above that the projection map

$$\tilde{\pi}^{0,1}_{\upsilon} : \tilde{\Gamma}^{0,1}_{-,+}(\upsilon; T\mathbb{P}^4; N_Y\kappa) \oplus \tilde{\Gamma}^{0,1}_{+,-}(\upsilon; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \to H^1_J(\tilde{b}(\upsilon); N_Y\kappa)$$

is an isomorphism.

The space $\Gamma^{0,1}_{-}(b; N_Y\kappa)$ of $u_b^* N_Y\kappa$-valued harmonic (0, 1)-forms on Σ_b splits as

$$\Gamma^{0,1}_{-}(b; N_Y\kappa) = \Gamma^{0,1}_{-;P}(b; N_Y\kappa) \oplus \Gamma^{0,1}_{-;B}(b; N_Y\kappa) = \mathscr{H}_{b,P} \otimes \mathrm{ev}_P^* N_Y\kappa \oplus \Gamma^{0,1}_{-;B}(b; N_Y\kappa).$$

Here $\Gamma^{0,1}_{-;P}(b; N_Y\kappa)$ and $\Gamma^{0,1}_{-;B}(b; N_Y\kappa)$ are the subspaces of $\Gamma^{0,1}_{-}(b; N_Y\kappa)$ consisting of the differentials supported on the main components $\Sigma_{b,\aleph}$ of Σ_b and on the only bubble component $\Sigma_{b,h}$ of Σ_b, respectively. Let

$$\pi^{0,1}_{b;P}, \pi^{0,1}_{b;B} : \Gamma^{0,1}_{-}(b; N_Y\kappa) \to \Gamma^{0,1}_{-;P}(b; N_Y\kappa),\ \Gamma^{0,1}_{-;B}(b; N_Y\kappa)$$

denote the projection maps. If $\eta \in \mathscr{H}_{b,P} \otimes \mathrm{ev}_P^* N_Y\kappa$, we define $R_\upsilon\eta \in \Gamma^{0,1}(\upsilon; N_Y\kappa)$ as above by identifying $N_Y\kappa$ with the $g_{Y,b}$-orthogonal complement of $T\kappa$ in $TY \subset T\mathbb{P}^4$. If $\eta \in \Gamma^{0,1}_{-;B}(b; N_Y\kappa)$, let $R_\upsilon\eta = \tilde{q}_\upsilon^*\eta$. We denote by

$$\tilde{\Gamma}^{0,1}_{-;P}(\upsilon; N_Y\kappa),\ \tilde{\Gamma}^{0,1}_{-;B}(\upsilon; N_Y\kappa) \subset \tilde{\Gamma}^{0,1}(\upsilon; N_Y\kappa)$$

the images of $\Gamma^{0,1}_{-;P}(b; N_Y\kappa)$ and $\Gamma^{0,1}_{-;B}(b; N_Y\kappa)$ under the map $\tilde{R}_\upsilon \equiv \Pi^J_\upsilon R_\upsilon$. Let

$$\tilde{\pi}^{0,1}_{\upsilon;P} : \tilde{\Gamma}^{0,1}(\upsilon; N_Y\kappa) \to \tilde{\Gamma}^{0,1}_{-;P}(\upsilon; N_Y\kappa) \text{ and } \tilde{\pi}^{0,1}_{\upsilon;B} : \tilde{\Gamma}^{0,1}(\upsilon; N_Y\kappa) \to \tilde{\Gamma}^{0,1}_{-;B}(\upsilon; N_Y\kappa)$$

be the L^2-projection maps. By (4-24), (4-25), and (4-28), we have

$$\left|\tilde{\pi}^{0,1}_{\upsilon;P} D_{J,\tilde{b}(\upsilon)} \tilde{R}_\upsilon\zeta + 2\pi\rho(\upsilon) J\, \tilde{R}_\upsilon \pi^{\perp}_{\kappa} \tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{J},h}\zeta\right| \le C(b)\left|\rho(\upsilon)\right|^2 \|\zeta\|_{b,p,1} \tag{4-29}$$

for all $\zeta \in \Gamma^{0,1}_{-,+}(\upsilon; T\mathbb{P}^4; N_Y\kappa)$. In particular, the projection map

$$\tilde{\pi}^{0,1}_{\upsilon;P} : \tilde{\Gamma}^{0,1}_{-,+}(\upsilon; T\mathbb{P}^4; N_Y\kappa) \to \tilde{\Gamma}^{0,1}_{-;P}(\upsilon; N_Y\kappa)$$

is an isomorphism. We denote its inverse by $S_{\upsilon;P}$. The projection map

$$\tilde{\pi}^{0,1}_{\upsilon;B} : \tilde{\Gamma}^{0,1}_{+,-}(\upsilon; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \to \tilde{\Gamma}^{B,1}_{-;1}(\upsilon; N_Y\kappa).$$

is also an isomorphism, since the map $\mathfrak{d}_b$ is. We denote its inverse by $S_{\upsilon;B}$.

Finally, let

$$T_\upsilon = \tilde{\pi}_\upsilon^{0,1} \circ (S_{\upsilon;P} \oplus S_{\upsilon;B}) \circ \tilde{R}_\upsilon : \Gamma_-^{0,1}(b; N_Y\kappa) \to H_J^1(\tilde{b}(\upsilon); N_Y\kappa).$$

The maps T_υ with $\upsilon \in \tilde{\mathcal{F}}\mathcal{T}_\delta^\varnothing$ induce a bundle isomorphism

$$\Phi_{\mathcal{T}} : \pi^*_{\mathcal{F}T_\delta^\varnothing}\big(\widetilde{\mathcal{W}}^{1,1}_{\kappa,d/d_\kappa}|_{\mathcal{U}_{\mathcal{T};\kappa;1}}\big) \to \mathcal{W}^{0,1}_{\kappa,d/d_\kappa}\big|_{\mathfrak{M}_1^0(\kappa,d/d_\kappa)\cap U^1_{\mathcal{T}}},$$

covering $\phi^1_{\mathcal{T}}|_{\mathcal{F}T_\delta^\varnothing}$. This isomorphism extends continuously over $\mathcal{F}T_\delta - \mathcal{F}T_\delta^\varnothing$, as can be seen directly from the definition.

If $\vartheta(\upsilon) \in \tilde{\Gamma}_-(\upsilon; \mathfrak{L})$, we can find a unique

$$\begin{aligned}\zeta_\vartheta(\upsilon) = {} & \zeta_\vartheta^-(\upsilon) \oplus \zeta_\vartheta^0(\upsilon) \oplus \zeta_\vartheta^+(\upsilon) \\ & \in \tilde{\Gamma}_{-,+}(\upsilon; T\mathbb{P}^4; \kappa) \oplus \tilde{\Gamma}_{-,+}(\upsilon; T\mathbb{P}^4; N_Y\kappa) \oplus \tilde{\Gamma}_{+,-}(\upsilon; T\mathbb{P}^4; N_{\mathbb{P}^4}Y)\end{aligned}$$

such that $\pi_Y^\perp \zeta_\vartheta(\upsilon) = \vartheta(\upsilon)$. By (4-29),

$$\big|\pi^{0,1}_{b;P} T_\upsilon^{-1} \mathfrak{d}_0 \vartheta(\upsilon) + 2\pi\rho(\upsilon) J\, \pi_\kappa^\perp \tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},h} \tilde{R}_\upsilon^{-1} \zeta^0_\vartheta(\upsilon)\big| \le C_\vartheta(b)\big|\rho(\upsilon)\big|^2. \tag{4-30}$$

On the other hand, by the definition of the map π_k^- in Section 4B,

$$\pi_k^- \tilde{\mathfrak{d}}^{1,1}_{\kappa,d/d_\kappa} \vartheta(b) = -2\pi\, J\, \pi_\kappa^\perp \tilde{\mathfrak{D}}^{\mathbb{P}^4}_{\mathcal{T},h} \zeta^0_\vartheta(b). \tag{4-31}$$

The estimate of Lemma 4.6 follows from (4-28), (4-30), (4-31), and the continuity of the section ϑ.

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