CONVEXITY IN LOCALLY CONFORMALLY FLAT MANIFOLDS WITH BOUNDARY

MARCOS PETRÚCIO DE A. CAVALCANTE
Given a closed subset $\Lambda$ of the open unit ball $B_1 \subset \mathbb{R}^n$ for $n \geq 3$, we consider a complete Riemannian metric $g$ on $\overline{B}_1 \setminus \Lambda$ of constant scalar curvature equal to $n(n-1)$ and conformally related to the Euclidean metric. We prove that every closed Euclidean ball $B \subset B_1 \setminus \Lambda$ is convex with respect to the metric $g$, assuming the mean curvature of the boundary $\partial B_1$ is nonnegative with respect to the inward normal.

1. Introduction

Let $B_1$ denote the open unit ball of $\mathbb{R}^n$ for $n \geq 3$. Given a closed subset $\Lambda \subset B_1$, we will consider a complete Riemannian metric $g$ on $\overline{B}_1 \setminus \Lambda$ of constant positive scalar curvature $R(g) = n(n-1)$ and conformally related to the Euclidean metric $\delta$. We will also assume that $g$ has nonnegative boundary mean curvature. Here and throughout, second fundamental forms will be computed with respect to the inward unit normal vector.

In this paper we prove

**Theorem 1.1.** If $B \subset B_1 \setminus \Lambda$ is a standard Euclidean ball, then $\partial B$ is convex with respect to the metric $g$.

Here, we say that $\partial B$ is convex if its second fundamental form is positive definite. Since $\partial B$ is umbilical in the Euclidean metric and the notion of an umbilical point is conformally invariant, we know that $\partial B$ is also umbilic in the metric $g$. In that case, $\partial B$ is convex if its mean curvature $h$ is positive everywhere.

This theorem is motivated by an analogous one on the sphere due to R. Schoen [1991]. He shows that if $\Lambda \subset S^n$ for $n \geq 3$ is closed and nonempty and $g$ is a complete Riemannian metric on $S^n \setminus \Lambda$ that is conformal to the standard round metric $g_0$ and has constant positive scalar curvature $n(n-1)$, then every standard ball $B \subset S^n \setminus \Lambda$ is convex with respect to the metric $g$. Schoen used this geometrical result to prove the compactness of the set of solutions to the Yamabe problem in the locally conformally flat case. D. Pollack [1993] also used Schoen’s theorem to
prove a compactness result for the singular Yamabe problem on the sphere where the singular set is a finite collection of points \( \Lambda = \{ p_1, \ldots, p_k \} \subset S^n \) for \( n \geq 3 \).

In this context Theorem 1.1 can be viewed as the first step in the direction of proving compactness for the singular Yamabe problem with boundary conditions. As we will see, the problem of finding a metric satisfying the hypotheses of Theorem 1.1 is equivalent to finding a positive solution to an elliptic PDE with critical Sobolev exponent. This problem is invariant by conformal transformations. So, by applying a convenient inversion on the Euclidean space, we may consider the same problem on an unbounded subset of \( \mathbb{R}^n \). The idea of the proof is to show that if \( \partial B \) is not convex, we can find a smaller ball \( \tilde{B} \subset B \) with a nonconvex boundary as well. To do this we will use the hypothesis on the mean curvature of \( \partial B \) and get geometrical information from that equation by applying the moving planes method as in [Gidas et al. 1979]. The contradiction follows by the construction of these balls.

2. Preliminaries

Here we will introduce some notation and recall some results that will be used in the proof of Theorem 1.1. We will also describe a useful example.

Let \((M^n, g_0)\) for \( n \geq 3 \) be a smooth orientable Riemannian manifold, possibly with boundary. Let us denote by \( R(g_0) \) its scalar curvature and by \( h(g_0) \) its boundary mean curvature. Let \( g = u^{4/(n-2)}g_0 \) be a metric conformal to \( g_0 \). Then the positive function \( u \) satisfies the following nonlinear elliptic partial differential equation with critical Sobolev exponent:

\[
\Delta_{g_0} u - \frac{n-2}{4(n-1)} R(g_0) u + \frac{n-2}{4(n-1)} R(g) u^{(n+2)/(n-2)} = 0 \quad \text{in} \ M,
\]

\[
\frac{\partial u}{\partial \nu} - \frac{n-2}{2} h(g_0) u + \frac{n-2}{2} h(g) u^{n/(n-2)} = 0 \quad \text{on} \ \partial M,
\]

where \( \nu \) is the inward unit normal vector field to \( \partial M \).

The problem of existence of solutions to (1) when \( R(g) \) and \( h(g) \) are constants is referred to as the Yamabe problem. It was completely solved when \( \partial M = \emptyset \) in a sequence of works, beginning with H. Yamabe himself [1960], followed by N. Trudinger [1968] and T. Aubin [1976], and finally by R. Schoen [1984]. In the case of nonempty boundary, J. Escobar solved almost all the cases [1992a; 1992b], followed by Z. Han and Y. Li [1999], F. Marques [2005], and others.

Here, however, we wish to study solutions of (1) with \( R(g) \) constant; these become singular on a closed subset \( \Lambda \subset M \). This is the so called singular Yamabe problem. This singular behavior is equivalent, at least in the case that \( g_0 \) is conformally flat, to requiring \( g \) to be complete on \( M \setminus \Lambda \). The existence problem (with \( \partial M = \emptyset \)) displays a relationship between the size of \( \Lambda \) and the sign of \( R(g) \). It
is known that for a solution with $R(g) < 0$ to exist, it is necessary and sufficient that $\dim(\Lambda) > (n - 2)/2$ (see [Aviles and McOwen 1988; McOwen 1993; Finn and McOwen 1993]), while if a solution exists with $R(g) \geq 0$, then $\dim(\Lambda) \leq (n - 2)/2$. Here $\dim(\Lambda)$ stands for the Hausdorff dimension of $\Lambda$. In this paper we will treat the case of constant positive scalar curvature, which we suppose equal to $n(n - 1)$ after normalization. In this case the simplest examples are given by the Fowler solutions which we will now discuss briefly.

Let $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a positive smooth function such that

$$\Delta u + \frac{n(n-2)}{4} u^{(n+2)/(n-2)} = 0 \text{ in } \mathbb{R}^n \setminus \{0\} \text{ for } n \geq 3 \text{ and}$$

$$0 \text{ is an isolated singularity.}$$

Then $g = u^{4/(n-2)} \delta$ is a complete metric on $\mathbb{R}^n \setminus \{0\}$ of constant scalar curvature $n(n-1)$.

Using the invariance under conformal transformations we may work in different background metrics. The most convenient one here is the cylindrical metric $g_{\text{cyl}} = d\theta^2 + dt^2$ on $S^{n-1} \times \mathbb{R}$. Then $g = v^{4/(n-2)} g_{\text{cyl}}$, where $v$ is defined in the whole cylinder and satisfies

$$\frac{d^2 v}{dt^2} + \Delta_\theta v - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{(n+2)/(n-2)} = 0.$$

One easily verifies that the solutions to Equation (2) and (3) are related by

$$u(x) = |x|^{(2-n)/2} v(x/|x|, -\log |x|).$$

By a deep theorem of Caffarelli, Gidas and Spruck [1989, Theorem 8.1], we know that $v$ is rotationally symmetric, that is $v(\theta, t) = v(t)$, and therefore the PDE (3) reduces to the ODE

$$\frac{d^2 v}{dt^2} - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{(n+2)/(n-2)} = 0.$$

Setting $w = v'$ this equation is transformed into a first order Hamiltonian system

$$\frac{dv}{dt} = w,$$

$$\frac{dw}{dt} = \frac{(n-2)^2}{4} v - \frac{n(n-2)}{4} v^{(n+2)/(n-2)},$$

whose Hamiltonian energy is

$$H(v, w) = w^2 - \frac{(n-2)^2}{4} v^2 + \frac{(n-2)^2}{4} v^{2n/(n-2)}.$$
ourselves to the half-plane \( \{ v > 0 \} \) where \( g = v^{4/(n-2)} g_{cyl} \) has geometrical meaning. On the other hand, we are looking for complete metrics. Those will be generated by the Fowler solutions, that is, the periodic solutions around the equilibrium point \((v_0, 0)\). They are symmetric with respect to \( v \)-axis and can be parametrized by the minimum value \( \varepsilon \) attained by \( v \) for \( \varepsilon \in (0, v_0) \) (and a translation parameter \( T \)). We will denote them by \( v_\varepsilon \). We point out that \( v_0 \) corresponds to the scaling of \( g_{cyl} \) that makes the cylinder \( S^{n-1} \times \mathbb{R} \) have scalar curvature \( n(n - 1) \). One obtains the Fowler solutions \( u_\varepsilon \) by using the relation (4).

We can now construct metrics satisfying the hypotheses of Theorem 1.1 (with \( \Lambda = \{0\} \)) from the Fowler solutions. To do this, we just take a Fowler solution \( v \) defined for \( t \geq t_0 \), where \( t_0 \) is such that \( w = dv/dt \leq 0 \) or equivalently

\[
h(g) = -\frac{2}{n-2} v^{-n/(n-2)} \frac{dv}{dt} \geq 0.
\]

We point out that, by another result of Caffarelli, Gidas, and Spruck [1989, Theorem 1.2], it is known that, given a positive solution \( u \) to

\[
\Delta u + \frac{n(n-2)}{4} u^{(n+2)/(n-2)} = 0
\]

that is defined in the punctured ball \( B_1 \setminus \{0\} \) and that is singular at the origin, there exists a unique Fowler solution \( u_\varepsilon \) such that

\[
u(x) = (1 + o(1))u_\varepsilon(|x|) \quad \text{as} \quad |x| \to 0.
\]

Therefore, from Equation (4) or also [Korevaar et al. 1999], either \( u \) extends as a smooth solution to the ball, or there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 |x|^{(2-n)/2} \leq u(x) \leq C_2 |x|^{(2-n)/2}.
\]

3. Proof of Theorem 1.1

The proof will be by contradiction. If \( \partial B \) is not convex then, since it is umbilical, there exists a point \( q \in \partial B \) such that the mean curvature of \( \partial B \) at \( q \) (with respect to the inward unit normal vector) is \( H(q) \leq 0 \). If we write \( g = u^{4/(n-2)} \delta \), then \( u \) is a positive smooth function on \( \bar{B}_1 \setminus \Lambda \) satisfying

\[
\Delta u + \frac{n(n-2)}{4} u^{(n+2)/(n-2)} = 0 \quad \text{in} \quad B_1 \setminus \Lambda,
\]

\[
\frac{\partial u}{\partial v} - \frac{n-2}{2} u + \frac{n-2}{2} hu^{n/(n-2)} = 0 \quad \text{on} \quad \partial B_1.
\]

Now, we will choose a point \( p \in \partial B \) with \( p \neq q \) and consider the inversion

\[
I : \mathbb{R}^n \setminus \{p\} \to \mathbb{R}^n \setminus \{p\}.
\]
This map takes \( \overline{B}_1 \setminus \{ p \} \cup \Lambda \) on \( \mathbb{R}^n \setminus (B(a, r) \cup \Lambda) \), where \( B(a, r) \) is an open ball of center \( a \in \mathbb{R}^n \) and radius \( r > 0 \) and \( \Lambda \) still denotes the singular set. Let us denote by \( \Sigma \) the boundary of \( B(a, r) \), that is, \( \Sigma = \partial B_1 \).

The image of \( \partial B \setminus \{ p \} \) is a hyperplane \( \Pi \), and by a coordinate choice we may assume \( \Pi = \Pi_0 := \{ x \in \mathbb{R}^n : x^n = 0 \} \). We may suppose that the ball \( B(a, r) \) lies below \( \Pi_0 \). In this case \( \Lambda \) also lies below \( \Pi_0 \).

Since \( I \) is a conformal map we have \( I^* g = v^{4/(n-2)} \delta \), where \( v \) is the Kelvin transform of \( u \) on \( \mathbb{R}^n \setminus (B(a, r) \cup \Lambda) \).

Thus this metric has constant positive scalar curvature \( n(n-1) \) in \( \mathbb{R}^n \setminus (B(a, r) \cup \Lambda) \) and nonnegative mean curvature \( h \) on \( \Sigma \).

As before, \( v \) is a solution of the problem

\[
\Delta v + \frac{n(n-2)}{4} v^{(n+2)/(n-2)} = 0 \quad \text{in} \mathbb{R}^n \setminus (B(a, r) \cup \Lambda),
\]

\[
\frac{\partial v}{\partial v} + \frac{n-2}{2r} v + \frac{n-2}{2} h v^{(n)/(n-2)} = 0 \quad \text{on} \Sigma.
\]

Also, by hypotheses of contradiction, the mean curvature of the hyperplane \( \Pi_0 \) at \( I(q) \) (with respect to \( \partial/\partial x^n \)) is \( H \leq 0 \). By applying the boundary equation of the system (1) to \( \Pi_0 \), we obtain

\[
\frac{\partial v}{\partial x^n} + \frac{n-2}{2} H v^{n/(n-2)} = 0
\]
on \( \Pi_0 \). Thus we conclude that \( \partial v/\partial x^n(I(q)) \geq 0 \).

Now we start with the moving planes method. Given \( \lambda \geq 0 \) we will denote by \( x_\lambda \) the reflection of \( x \) with respect to the hyperplane \( \Pi_\lambda := \{ x \in \mathbb{R}^n : x^n = \lambda \} \) and set \( \Omega_\lambda = \{ x \in \mathbb{R}^n \setminus (B(a, r) \cup \Lambda) : x^n \leq \lambda \} \). We define

\[
w_\lambda(x) = v(x) - v_\lambda(x) \quad \text{for} \quad x \in \Omega_\lambda, \quad \text{where} \quad v_\lambda(x) := v(x_\lambda).
\]

Since infinity is a regular point of \( I^* g \), we have

\[
v(x) = |x|^{-n} (a + \sum b_i x^i |x|^{-2}) + O(|x|^{-n})
\]
in a neighborhood of infinity. It follows from [Caffarelli et al. 1989, Lemma 2.3] that there exist \( R > 0 \) and \( \lambda > 0 \) such that \( w_\lambda > 0 \) in interior of \( \Omega_\lambda \setminus B(0, R) \) if \( \lambda \geq \lambda \). Without loss of generality, we can choose \( R > 0 \) such that \( B(a, r) \cup \Lambda \subset B(0, R) \).

Now we note that \( v \) has a positive infimum, say \( v_0 > 0 \), in \( B(0, R) \setminus (B(a, r) \cup \Lambda) \). It follows from the fact that \( v \) is a classical solution to (5) in \( B(0, R) \setminus (B(a, r) \cup \Lambda) \).

So, since \( v \) decays in a neighborhood of infinity, we may choose \( \lambda > 0 \) large enough so that \( v_\lambda(x) < v_0/2 \) for \( x \in B(0, R) \) and for \( \lambda \geq \lambda \). Thus for sufficiently large \( \lambda \), we get \( w_\lambda > 0 \) in \( \text{int}(\Omega_\lambda) \).

We also write

\[
\Delta w_\lambda + c_\lambda(x) w_\lambda = 0 \quad \text{in} \text{int}(\Omega_\lambda),
\]

(7)
where
\[ c_\lambda(x) = \frac{n(n-2)}{4} \frac{v(x)^{(n+2)/(n-2)} - v_\lambda(x)^{(n+2)/(n-2)}}{v(x) - v_\lambda(x)}. \]

By definition, \( w_\lambda \) always vanishes on \( \Pi_\lambda \). In particular, setting \( \lambda_0 = \inf\{\lambda > 0 : w_\lambda > 0 \text{ on } \Omega_\lambda\} \) for all \( \lambda \geq \lambda_0 \), we obtain by continuity that \( w_{\lambda_0} \) satisfies (7), \( w_{\lambda_0} \geq 0 \) in \( \Omega_{\lambda_0} \), and \( w_{\lambda_0} = 0 \) on \( \Pi_{\lambda_0} \). Hence, by applying the strong maximum principle, we conclude that either \( w_{\lambda_0} > 0 \) in \( \Omega_{\lambda_0} \) or \( w_{\lambda_0} = v - v_{\lambda_0} \) vanishes identically. We point out that the second case occurs only if \( \Lambda = \emptyset \).

If \( w_{\lambda_0} \equiv 0 \), then \( \Pi_{\lambda_0} \) is a hyperplane of symmetry of \( v \) and therefore \( v \) extends to a global positive solution of (5) on the entire \( \mathbb{R}^n \). Using [Caffarelli et al. 1989], we conclude that \( (B_1, g) \) is a convex spherical cap and the result is obvious.

If \( w_{\lambda_0} > 0 \) in \( \Omega_{\lambda_0} \), we apply the E. Hopf maximum principle to conclude
\[ \frac{\partial w_{\lambda_0}}{\partial x^n} = 2 \frac{\partial v}{\partial x^n} < 0 \quad \text{in } \Pi_{\lambda_0}, \]
and since \( \partial v/\partial x^n(I(q)) \geq 0 \), we have \( \lambda_0 > 0 \). In this case, by definition of \( \lambda_0 \), we can choose sequences \( \lambda_k \uparrow \lambda_0 \) and \( x_k \in \Omega_{\lambda_k} \) such that \( w_{\lambda_k}(x_k) < 0 \).

It follows from the work in [Korevaar et al. 1999] that \( w_\lambda \) achieves its infimum. Then we lose no generality in assuming \( x_k \) is a minimum of \( w_{\lambda_k} \) in \( \Omega_{\lambda_k} \).

We have \( x_k \notin \Pi_k \) because \( w_{\lambda_k} \) always vanishes on \( \Pi_{\lambda_k} \). So, either \( x_k \) is in \( \Sigma \) or it is an interior point. Even when \( x_k \) is an interior point we claim that the \( x_k \) form a bounded sequence. More precisely:

**Claim 3.1** [Chen and Lin 1998, Section 2]. There exists \( R_0 > 0 \), independent of \( \lambda \), such that if \( w_\lambda \) solves (7) and is negative somewhere in \( \text{int}(\Omega) \) and if \( x_0 \in \text{int}(\Omega) \) is a minimum point of \( w_\lambda \), then \( |x_0| < R_0 \).

For completeness we present a proof in the Appendix.

So, we can take a convergent subsequence \( x_k \rightarrow \bar{x} \in \Omega_{\lambda_{\lambda_0}} \). Since \( w_{\lambda_k}(x_k) < 0 \) and \( w_{\lambda_0} \geq 0 \) in \( \Omega_{\lambda_0} \), we necessarily have \( w_{\lambda_0}(\bar{x}) = 0 \) and therefore \( \bar{x} \in \partial \Omega_{\lambda_0} = \Pi_{\lambda_0} \cup \Sigma \).

If \( x \in \Pi_{\lambda_0} \) then \( x \) is an interior minimum point to \( w_{\lambda_0} \), and hence \( \nabla w_{\lambda_0}(\bar{x}) = 0 \), which cannot occur by inequality (8). Thus we have \( \bar{x} \in \Sigma \) and, by the E. Hopf maximum principle again,
\[ \frac{\partial w_{\lambda_0}}{\partial \eta}(\bar{x}) = \frac{\partial v}{\partial \eta}(\bar{x}) - \frac{\partial v}{\partial \eta}(\bar{x}_{\lambda_0}) < 0, \]
where \( \eta := -v \) is the inward unit normal vector to \( \Sigma \).

Now, we recall that
\[ \frac{\partial v}{\partial w} + \frac{n-2}{2r}v + \frac{n-2}{2}hv^{(n+2)/(n-2)} = 0 \quad \text{on } \Sigma. \]

Thus, since \( v(\bar{x}) = v(\bar{x}_{\lambda_0}) \) we have from (9) and (10) that the mean curvature of \( \Sigma_{\lambda_0} \) at \( \bar{x}_{\lambda_0} \) (with respect to the inward unit normal vector) is strictly less than \(-h\).
Since $h$ is nonnegative, $\varphi_{\lambda_0}$ is a nonconvex point in the reflected sphere $\Sigma_{\lambda_0}$. Considering the problem back to $B_1$, we denote by $K_1$ the ball corresponding to the one whose boundary is $\Sigma_{\lambda_0}$ and by $P_1$ the ball corresponding to $\Pi_{\lambda_0}$. Thus, we have obtained a strictly smaller ball $K_1 \subset B$ with a nonconvex boundary which is the reflection of $\partial B_1$ with respect to $\partial P_1$.

We can repeat this argument to obtain a sequence of balls with nonconvex points on the boundaries, that is, $B \supset K_1 \supset \cdots \supset K_j \supset \cdots$.

This sequence cannot converge to a point, since small balls are always convex. On the other hand, if $K_j \to K_{\infty}$ where $K_{\infty}$ is not a point, then $K_{\infty} \subset B$ is a ball in $B_1 \setminus \Lambda$ whose boundary is the reflection of $\partial B_1$ with respect to itself. This is a contradiction.

**Appendix. Proof of Claim 3.1**

First write Equation (7), setting $c_\lambda(x) = 0$ when $w_\lambda(x) = 0$. Fix $0 < \mu < n - 2$, and define $g(x) = |x|^{-\mu}$ and $\phi(x) = w_\lambda(x)/g(x)$. Then, using (7),

$$\Delta \phi + \frac{2}{g}(\nabla g, \nabla \phi) + \left(c_\lambda(x) + \frac{\Delta g}{g}\right)\phi = 0.$$ 

By a computation we get $\Delta g = -\mu(n - 2 - \mu)|x|^{-\mu - 2}$, that is, 

$$\frac{\Delta g}{g} = -\mu(n - 2 - \mu)|x|^{-2}.$$

On the other hand, the expansion of $v$ in a neighborhood of infinity implies that $w_\lambda(x) = O(|x|^{2-n})$ and consequently $c_\lambda(x) = O(|x|^{-n-2+2n}) = O(|x|^{-4})$. Hence we obtain 

$$c_\lambda(x) + \frac{\Delta g}{g} \leq C(|x|^{-4} - \mu(n - 2 - \mu)|x|^{-2}).$$

In particular $c(x) + \Delta g/g < 0$ for large $|x|$. Choose $R_0$ with $B(a, r) \cup \Lambda \subset B(0, R_0)$ such that

\begin{equation}
C(|x|^{-4} - \mu(n - 2 - \mu)|x|^{-2}) < 0 \quad \text{for} \ |x| \geq R_0.
\end{equation}

Now let $x_0 \in \text{int}(\Omega_\lambda)$ so that $w_\lambda(x_0) = \inf_{\text{int}(\Omega_\lambda)} w_\lambda < 0$.

Since $\lim_{|x| \to +\infty} \phi(x) = 0$ and $\phi(x) \geq 0$ on $\partial \Omega_\lambda$, there exists $\bar{x}_0$ such that $\phi$ has its minimum at $\bar{x}_0$. By applying the maximum principle for $\phi$ at $\bar{x}_0$ we get $c_\lambda(\bar{x}_0) + \Delta g(\bar{x}_0)/g \geq 0$ and by (11), we get $|\bar{x}_0| < R_0$. Now we have 

$$\frac{w_\lambda(x_0)}{g(\bar{x}_0)} \leq \frac{w_\lambda(\bar{x}_0)}{g(\bar{x}_0)} = \phi(\bar{x}_0) \leq \phi(x_0) = \frac{w_\lambda(x_0)}{g(x_0)}.$$ 

This implies $|x_0| \leq |\bar{x}_0| \leq R_0$ and proves the claim.
Acknowledgements

The content of this paper is part of the author’s doctoral thesis [Cavalcante 2006]. The author would like to express his gratitude to Professor Manfredo do Carmo for the encouragement and to Professor Fernando Coda Marques for many useful discussions during this work. While the author was at IMPA in Rio de Janeiro, he was fully supported by CNPq-Brazil.

References


MARCOS PETRÚCIO DE A. CAVALCANTE
INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DE ALAGOA
CAMPU AMILCAR SIMÕES, KM 97
57072-970 MACEÍA, AL,
BRASIL
marcos.petrucio@gmail.com