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**CENTRAL EXTENSIONS, SYMBOLS AND RECIPROCITY LAWS ON GL($n$, $\mathcal{F}$)**


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# CENTRAL EXTENSIONS, SYMBOLS AND RECIPROCITY LAWS ON GL($n$, $\mathcal{F}$)

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**For a discrete valuation field $\mathcal{F}$, using commensurability on valuation rings, we construct arithmetic symbols on the linear group GL($n$, $\mathcal{F}$) that generalize classical symbols such as the tame symbol and the Hilbert norm residue symbol on an algebraic curve. We also offer reciprocity laws for these symbols on GL($n$, $\Sigma_C$).**

## 1. Introduction

J. Milnor [1971] defined the tame symbol $d_v$ associated with a discrete valuation $v$ on a field $\mathcal{F}$. Explicitly, if $\mathcal{O}_v$ is the valuation ring, $\mathfrak{m}_v$ is the unique maximal ideal and $\mathcal{K}(v) = \mathcal{O}_v/\mathfrak{m}_v$ is the residue class field, Milnor defined $d_v \colon \mathcal{F}^* \times \mathcal{F}^* \to \mathcal{K}(v)^*$ by

$$d_v(x, y) = (-1)^{v(x)\cdot v(y)} \frac{x^{v(y)}}{y^{v(x)}} (\text{mod } \mathfrak{m}_v).$$

If $\mathcal{F}^*$ is a topological group with the $v$-topology and $\mathcal{K}(v)^*$ is a discrete topological group, the tame symbol is a continuous Steinberg symbol (bimultiplicative and satisfying $d_v(f, 1-f) = 1$ for all $f \neq 1$). The tame symbol is used in algebraic K-theory to study the group $K_2\mathcal{F}$.

During the last thirty years, characterizations of algebraic symbols (in particular the tame symbol) have been obtained from the properties of infinite-dimensional vector spaces in order to provide new interpretations for these symbols and to deduce standard theorems from the new definitions in an easy way [Arbarello et al. 1989; Pablos Romo 2004; 2002].

Given a discrete valuation field $\mathcal{F}$, the aim of this work is to provide a method for constructing arithmetic symbols on the linear group GL($n$, $\mathcal{F}$) that generalize classical symbols, and to offer reciprocity laws for some of these symbols. To do

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so, we use a new definition of commensurability on valuation rings, and a general definition of the tame symbol over the linear group $\mathrm{GL}(n, \mathscr{F})$.

In [Muñoz Porras and Pablos Romo $\geq$ 2008] we offered a reciprocity law for the tame symbol on $\mathrm{GL}(n, \Sigma_C)$, $\Sigma_C$ being the function field of a complete, irreducible and nonsingular curve over a perfect field, and we have explored the explicit reciprocity laws for the case $\mathrm{GL}(2, \Sigma_C)$. The present work generalizes the statements of that paper in a dual way: the explicit expressions of symbols are valid for each positive integer $n$ and for an arbitrary discrete valuation field $\mathscr{F}$. The proof of the reciprocity law for the tame symbol on $\mathrm{GL}(n, \Sigma_C)$ offered here is similar to the Tate's proof [1968] of the residue theorem, and rather different from the one given in [Muñoz Porras and Pablos Romo $\geq$ 2008], which was been deduced from the triviality of an adelic central extension by considering rational points of a Sato Grassmannian $\mathrm{Gr}(\mathbb{A}_C, \mathbb{A}_C^+)$.

Section 2 introduces commensurability on $k$-vector spaces ($k$ being a field), together with its application to the study of the tame symbol on an algebraic curve. Section 3 contains the technical results of this work. Commensurability on valuation rings is defined, and from it a central extension is constructed and a commutator pairing is studied. Section 4 is devoted to constructing, from that commutator pairing, arithmetic symbols on the linear group $\mathrm{GL}(n, \mathscr{F})$, $\mathscr{F}$ being a discrete valuation field. Thus, we define symbols that generalize classical symbols such as the tame symbol and the Hilbert norm residue symbol on an algebraic curve. Moreover, we show that these symbols can be extended to the infinite general linear group $\mathrm{GL}(\mathscr{F}) = \varinjlim \mathrm{GL}(n, \mathscr{F})$.

In Section 5 we offer reciprocity laws for the symbols defined previously. If $\Sigma_C$ is the function field of an algebraic curve over a perfect field, we prove reciprocity laws for symbols obtained from the tame symbol on $\mathrm{GL}(n, \Sigma_C)$, in particular for the generalizations of the tame symbol and the Hilbert norm residue symbol. When $k$ is an algebraically closed field and $A = k$, the reciprocity law of the generalized tame symbol coincides with the law for the Contou–Carrère symbol on $\mathrm{GL}(n, \Sigma_C \otimes_k A)$ that was proved in [Pablos Romo 2006] (where we defined a Contou–Carrère symbol on $\mathrm{GL}\big(n, A((t))\big)$ for an artinian local ring $A$).

We remark that the statements in Sections 3 and 4 are valid for an arbitrary discrete valuation field $\mathscr{F}$, so it should be possible to study other properties of the linear group $\mathrm{GL}(n, \mathscr{F})$ using the results presented here, and perhaps to apply them in the context of the geometric Langlands program. As far as we know, except for a reference in [Beilinson et al. 2002] and the above-mentioned papers [Muñoz Porras and Pablos Romo $\geq$ 2008] and [Pablos Romo 2006], arithmetic symbols on the linear group $\mathrm{GL}(n, \mathscr{F})$ have not been stated explicitly in the literature. The present work, together with [Muñoz Porras and Pablos Romo $\geq$ 2008] and [Pablos Romo 2006], develop the main properties of this theory.

## 2. Preliminaries

This section contains a review of the notion of commensurability on $k$-vector spaces ($k$ being a field), together with its application to the study of the tame symbol on an algebraic curve.

Let $V$ be a vector space (in general infinite-dimensional) over a field $k$.

**Definition 2.1** [Tate 1968]. Two vector subspaces $A$ and $B$ of $V$ are said to be commensurable if $\dim_k(A+B/A\cap B)<\infty$. We shall use the symbol $A\sim B$ to denote commensurable vector subspaces.

Let $V_+\subset V$ be a fixed vector subspace, and assume $k$ is algebraically closed. Setting

$$\mathrm{GL}(V, V_+)=\{f\in \mathrm{Aut}_k(V) \text{ such that } f(V_+)\sim V_+\},$$

E. Arbarello, C. de Concini and V.-G. Kac constructed in [Arbarello et al. 1989] a central extension of groups

$$1\to k^*\to \mathrm{GL}\widetilde{(V, V_+)}\to \mathrm{GL}(V, V_+)\to 1,$$

and used it to study the tame symbol on an algebraic curve.

Given an element $f\in \mathrm{GL}(V, V_+)$, the index of $f$ over $V_+$ is the integer

$$i_k(f, V_+)=\dim_k(V_+/V_+\cap fV_+)-\dim_k(fV_+/V_+\cap fV_+).$$

This construction was generalized in [Pablos Romo 2002] to define a tame symbol for an algebraic curve over a perfect field. Now let $C$ be a nonsingular and irreducible curve over a perfect field, and let $\Sigma_C$ again be its function field. We keep the notation from that reference. If $x\in C$ is a closed point and we denote by $A_x=\widehat{\mathcal{O}}_x$ the completion of the local ring $\mathcal{O}_x$, and by $K_x=(\widehat{\mathcal{O}}_x)_0$ the field of fractions of $\widehat{\mathcal{O}}_x$ (which coincides with the completion of $\Sigma_C$ with respect to the valuation ring $\mathcal{O}_x$), it follows from [Pablos Romo 2002, Section 5] that, as in [Arbarello et al. 1989], we have a central extension of groups

$$1\to k^*\to \widetilde{\mathrm{GL}}(K_x, A_x)\to \mathrm{GL}(K_x, A_x)\to 1,$$

which, since $\Sigma_C^*\subseteq \mathrm{GL}(K_x, A_x)$, induces by restriction another central extension

$$1\to k^*\to \widetilde{\Sigma_C^*}\to \Sigma_C^*\to 1\,,$$

whose commutator, for all $f, g\in \Sigma_C^*$, is given by

$$\{f, g\}_{A_x}^{K_x}=N_{k(x)/k}\biggl(\frac{f^{v_x(g)}}{g^{v_x(f)}}(p)\biggr)\in k^*\,,$$

where $k(x)$ is the residue class field of the closed point $x$ and $N_{k(x)/k}$ is the norm of the extension $k\hookrightarrow k(x)$.

Moreover, if $\deg(x) = \dim_k k(x)$, it is known that

$$i_k(f, A_x) = \dim_k(A_x/A_x \cap f A_x) - \dim_k(f A_x/A_x \cap f A_x) = \deg(x) \cdot v_x(f).$$

**Definition 2.2.** Let $C$ be a nonsingular and irreducible curve over a perfect field, and let $\Sigma_C$ be its function field. The *tame symbol* associated with a closed point $x \in C$ is the map

$$(\ ,\ )_x \colon \Sigma_C^* \times \Sigma_C^* \to k^*$$

defined by

$$(f, g)_x = (-1)^{\deg(x) \cdot v_x(f) \cdot v_x(g)}\, N_{k(x)/k}\left(\frac{f^{v_x(g)}}{g^{v_x(f)}}(p)\right) \quad \text{for } f, g \in \Sigma_C^*.$$

When $x$ is a rational point of $C$, this definition coincides with the multiplicative local symbol of [Serre 1959]. Using the same method as Tate's proof [1968] of the residue theorem, starting from the properties of the commutator $\{\ ,\ \}_{A_x}^{K_x}$ and the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$, we prove the reciprocity law

$$\prod_{x \in C} (f, g)_x = 1 \quad \text{for } f, g \in \Sigma_C^*.$$

## 3. Commensurability and central extensions on valuation rings

Let $\mathcal{F}$ be a discrete valuation field and $\mathcal{K}(v)$ its residue class field. The valuation ring associated with $v$ is again denoted by $\mathcal{O}_v$, and $\mathfrak{m}_v$ is its maximal ideal.

***Commensurability on valuation rings.*** Let $\mathcal{V}$ be an $\mathcal{O}_v$-module and let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{O}_v$-submodules of $\mathcal{V}$. Note that $\mathcal{O}_v$ is a local principal ideal domain.

**Definition 3.1.** $\mathcal{M}$ and $\mathcal{N}$ are said to be *commensurable* if $\mathcal{M}+\mathcal{N}/\mathcal{M}\cap\mathcal{N}$ is a finitely generated torsion $\mathcal{O}_v$-module. We use the symbol $\mathcal{M} \sim \mathcal{N}$ to denote commensurable submodules.

**Example 3.2.** Let $\widehat{\mathcal{O}}_v = \varprojlim_n \mathcal{O}_v/\mathfrak{m}_v{}^n\mathcal{O}_v$ be the $\mathfrak{m}_v$-adic completion of $\mathcal{O}_v$ and let $(\widehat{\mathcal{O}}_v)_0$ be its field of fractions. It is clear that there exists a commutative diagram of morphisms of $\mathcal{O}_v$-modules

$$\begin{tikzcd} \mathcal{O}_v \arrow[r, hook] \arrow[d, hook] & \mathcal{F} \arrow[d, hook] \\ \widehat{\mathcal{O}}_v \arrow[r, hook] & (\widehat{\mathcal{O}}_v)_0 \end{tikzcd}$$

Hence, if $f \in \mathcal{F}^*$ and $v(f) = \beta$, when $\beta > 0$ we have the isomorphisms of $\mathcal{O}_v$-modules

$$\widehat{\mathcal{O}}_v + f \cdot \widehat{\mathcal{O}}_v/\widehat{\mathcal{O}}_v \cap f \cdot \widehat{\mathcal{O}}_v \simeq \widehat{\mathcal{O}}_v/\mathfrak{m}_v{}^\beta\widehat{\mathcal{O}}_v \simeq \mathcal{O}_v/\mathfrak{m}_v{}^\beta\mathcal{O}_v$$

and when $\beta < 0$ we have

$$\widehat{\mathcal{O}}_v + f \cdot \widehat{\mathcal{O}}_v / \widehat{\mathcal{O}}_v \cap f \cdot \widehat{\mathcal{O}}_v \simeq \mathfrak{m}_v{}^{\beta} \widehat{\mathcal{O}}_v / \widehat{\mathcal{O}}_v \simeq \mathfrak{m}_v{}^{\beta} \mathcal{O}_v / \mathcal{O}_v .$$

In both cases $\widehat{\mathcal{O}}_v + f \cdot \widehat{\mathcal{O}}_v / \widehat{\mathcal{O}}_v \cap f \cdot \widehat{\mathcal{O}}_v$ is a torsion finitely generated $\mathcal{O}_v$-module and $\widehat{\mathcal{O}}_v \sim f \cdot \widehat{\mathcal{O}}_v$.

**Remark 3.3.** If $k$ is a field such that $\mathcal{K}(v)$ is a finite $k$-algebra, $\mathcal{O}_v$ is a $k$-module, and $\mathcal{V}$ is an $\mathcal{O}_v$-module, it is clear that two subspaces $\mathcal{M}, \mathcal{N} \subset \mathcal{V}$ are commensurable according to Definition 3.1 if and only if

$$\dim_k (\mathcal{M} + \mathcal{N} / \mathcal{M} \cap \mathcal{N}) < \infty .$$

Thus, this definition coincides with Tate's definition (Definition 2.1) when $\mathcal{V}$ is a vector space over $k$ and $\mathcal{M}$ and $\mathcal{N}$ are $k$-vector subspaces (and is slightly different from the definition of commensurability with respect to an ideal $\alpha$ offered in [Pablos Romo 2004]).

A finitely generated $\mathcal{O}_v$-module $\mathcal{W}$ is isomorphic to $\mathcal{W} \simeq \mathcal{O}_v^{\alpha} \oplus T(\mathcal{W})$, where $\alpha = \dim_{\mathcal{F}} (\mathcal{W} \otimes_{\mathcal{O}_v} \mathcal{F})$ and $T(\mathcal{W})$ is the torsion submodule of $\mathcal{W}$. Hence, from an exact sequence of $\mathcal{O}_v$-modules

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0 ,$$

one concludes that $\mathcal{M}_2$ is a torsion finitely generated $\mathcal{O}_v$-module if and only if $\mathcal{M}_1$ and $\mathcal{M}_3$ are torsion finitely generated $\mathcal{O}_v$-modules. Thus, if $\mathcal{M}, \mathcal{N}$ are $\mathcal{O}_v$-submodules of $\mathcal{V}$, the commensurability $\mathcal{M} \sim \mathcal{N}$ is equivalent to each of the following properties:

- $\mathcal{M} / \mathcal{M} \cap \mathcal{N}$ and $\mathcal{N} / \mathcal{N} \cap \mathcal{M}$ are finitely generated torsion $\mathcal{O}_v$-modules.
- $\mathcal{M} + \mathcal{N} / \mathcal{M}$ and $\mathcal{M} + \mathcal{N} / \mathcal{N}$ are finitely generated torsion $\mathcal{O}_v$-modules.

As in [Arbarello et al. 1989], one has:

**Lemma 3.4.** (1) *If $\mathcal{M} \sim \mathcal{N}$ and $\mathcal{N} \sim \mathcal{P}$, then $\mathcal{M} + \mathcal{N} + \mathcal{P} / \mathcal{M} \cap \mathcal{N} \cap \mathcal{P}$ is a torsion finitely generated $\mathcal{O}_v$-module.*

(2) *Commensurability is an equivalence relation.*

(3) *Let $\mathcal{M}, \mathcal{N}, \mathcal{M}', \mathcal{N}'$ be submodules of $\mathcal{V}$ and assume that $\mathcal{M} \sim \mathcal{M}'$ and $\mathcal{N} \sim \mathcal{N}'$. Then, $\mathcal{M} + \mathcal{N} \sim \mathcal{M}' + \mathcal{N}'$ and $\mathcal{M} \cap \mathcal{N} \sim \mathcal{M}' \cap \mathcal{N}'$.*

Moreover, if $\mathcal{W}$ is a $\mathcal{O}_v$-module, we denote by $\mathcal{G}_{\mathfrak{m}_v}(\mathcal{W})$ its graded $\mathcal{K}(v)$-module induced by the $\mathfrak{m}_v$-filtration $\{\mathfrak{m}_v^n \mathcal{W}\}_{n \geq 0}$. Given commensurable submodules $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{V}$, we set

$$[\mathcal{M} | \mathcal{N}]_{\mathfrak{m}_v} = \dim_{\mathcal{K}(v)} \mathcal{G}_{\mathfrak{m}_v} (\mathcal{M} / \mathcal{M} \cap \mathcal{N}) - \dim_{\mathcal{K}(v)} \mathcal{G}_{\mathfrak{m}_v} (\mathcal{N} / \mathcal{M} \cap \mathcal{N}).$$

**Lemma 3.5.** *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ be pairwise commensurable submodules of $\mathcal{V}$. Then*

$$[\mathcal{M}|\mathcal{N}]_{\mathfrak{m}_v} + [\mathcal{N}|\mathcal{P}]_{\mathfrak{m}_v} = [\mathcal{M}|\mathcal{P}]_{\mathfrak{m}_v}.$$

*Proof.* From the exact sequence of $\mathcal{O}_v$-modules

$$0 \to \mathcal{M}\cap\mathcal{N}/\mathcal{M}\cap\mathcal{N}\cap\mathcal{P} \to \mathcal{M}/\mathcal{M}\cap\mathcal{N}\cap\mathcal{P} \to \mathcal{M}/\mathcal{M}\cap\mathcal{N} \to 0,$$

one has

$$\begin{aligned}\dim_{\mathcal{K}(v)}&\mathcal{G}_{\mathfrak{m}_v}(\mathcal{M}/\mathcal{M}\cap\mathcal{N})\\ &= \dim_{\mathcal{K}(v)}\mathcal{G}_{\mathfrak{m}_v}(\mathcal{M}/\mathcal{M}\cap\mathcal{N}\cap\mathcal{P}) - \dim_{\mathcal{K}(v)}\mathcal{G}_{\mathfrak{m}_v}(\mathcal{M}\cap\mathcal{N}/\mathcal{M}\cap\mathcal{N}\cap\mathcal{P}),\end{aligned}$$

and the claim is deduced. □

**Remark 3.6.** With the notations of Example 3.2, one has

$$[\widehat{\mathcal{O}}_v|f\cdot\widehat{\mathcal{O}}_v]_{\mathfrak{m}_v} = v(f).$$

***The central extension $\widetilde{G^{\mathcal{V}}_{\mathcal{V}_+}}$.*** Let $\mathcal{V}$ be again an $\mathcal{O}_v$-module and let $\mathcal{V}_+$ be a fixed $\mathcal{O}_v$-submodule of $\mathcal{V}$.

**Definition 3.7.** Define the group $G^{\mathcal{V}}_{\mathcal{V}_+} \subseteq \mathrm{Aut}_{\mathcal{O}_v}(\mathcal{V})$ by

$$G^{\mathcal{V}}_{\mathcal{V}_+} = \{\tau \in \mathrm{Aut}_{\mathcal{O}_v}(\mathcal{V}) \text{ such that } \tau(\mathcal{V}_+)\sim\mathcal{V}_+ \text{ and } \tau^{-1}(\mathcal{V}_+)\sim\mathcal{V}_+\}.$$

As in [Pablos Romo 2002], if $\tau \in G^{\mathcal{V}}_{\mathcal{V}_+}$ and $\Lambda$ is the maximal exterior power, we set

$$\mathrm{Det}\,\mathcal{C}^{\bullet}_{\tau\mathcal{V}_+} = \Lambda\mathcal{G}_{\mathfrak{m}_v}\big(\mathcal{V}_+/\mathcal{V}_+\cap\tau\mathcal{V}_+\big) \underset{\mathcal{K}(v)}{\otimes} \Lambda\big[\mathcal{G}_{\mathfrak{m}_v}\big(\tau\mathcal{V}_+/\mathcal{V}_+\cap\tau\mathcal{V}_+\big)\big]^*,$$

which is a $\mathcal{K}(v)$-vector space of dimension one.

If $\tau\mathcal{V}_+ = \mathcal{V}_+$, then $\mathrm{Det}\,\mathcal{C}^{\bullet}_{\tau\mathcal{V}_+} = \mathcal{K}(v)$.

From the computations made in [Arbarello et al. 1989; Pablos Romo 2004; 2002] one deduces the existence of a map

$$\mathrm{Det}\,\mathcal{C}^{\bullet}_{\tau\mathcal{V}_+} \underset{\mathcal{K}(v)}{\otimes} \mathrm{Det}\,\mathcal{C}^{\bullet}_{\sigma\mathcal{V}_+} \xrightarrow{\varphi^{\sigma}_{\tau}} \mathrm{Det}\,\mathcal{C}^{\bullet}_{\tau\sigma\mathcal{V}_+},$$

which we shall write as $\varphi^{\sigma}_{\tau}(s_1\otimes s_2) = s_1\cdot\bar{\tau}(s_2)$, where $\bar{\tau}(s_2)$ is an element of

$$\Lambda\mathcal{G}_{\mathfrak{m}_v}\big(\tau\mathcal{V}_+/\tau\mathcal{V}_+\cap\tau\sigma\mathcal{V}_+\big) \underset{\mathcal{K}(v)}{\otimes} \Lambda\big[\mathcal{G}_{\mathfrak{m}_v}\big(\tau\sigma\mathcal{V}_+/\tau\mathcal{V}_+\cap\tau\sigma\mathcal{V}_+\big)\big]^*.$$

Now consider the set

$$\widetilde{G^{\mathcal{V}}_{\mathcal{V}_+}} = \{(\tau, s) \text{ with } \tau\in G^{\mathcal{V}}_{\mathcal{V}_+},\ 0\neq s\in \mathrm{Det}\,\mathcal{C}^{\bullet}_{\tau\mathcal{V}_+}\},$$

together with the operation

$$(\tau, s_1)\cdot(\sigma, s_2) = (\tau\sigma, s_1\cdot\bar{\tau}(s_2)).$$

It follows from the same three references that $\widetilde{G^{\mathcal{V}}_{\mathcal{V}_+}}$, with this operation, forms a group and that there exists a central extension of groups

$$1 \to \mathcal{K}(v)^* \xrightarrow{i} \widetilde{G^{\mathcal{V}}_{\mathcal{V}_+}} \xrightarrow{\pi} G^{\mathcal{V}}_{\mathcal{V}_+} \to 1, \tag{3-1}$$

where $i(\lambda) = (\mathrm{Id}, \lambda)$ and $\pi(\tau, s) = \tau$.

***The commutator pairing* $\{\ ,\ \}^{\mathcal{V}}_{\mathcal{V}_+}$.** If $\tau, \sigma \in G^{\mathcal{V}}_{\mathcal{V}_+}$ commute and $\widetilde{\tau}, \widetilde{\sigma} \in \widetilde{G^{\mathcal{V}}_{\mathcal{V}_+}}$ are elements such that $\pi(\widetilde{\tau}) = \tau$ and $\pi(\widetilde{\sigma}) = \sigma$, there is a commutator pairing

$$\{\tau, \sigma\}^{\mathcal{V}}_{\mathcal{V}_+} = \widetilde{\tau} \cdot \widetilde{\sigma} \cdot \widetilde{\tau}^{-1} \cdot \widetilde{\sigma}^{-1} \in \mathcal{K}(v)^*.$$

Fix elements $\sigma, \sigma', \tau, \tau' \in G^{\mathcal{V}}_{\mathcal{V}_+}$ such that each of $\sigma, \sigma'$ commutes with each of $\tau, \tau'$. From the definition of the commutator pairing or the construction of $\widetilde{G^{\mathcal{V}}_{\mathcal{V}_+}}$, we have:

- $\{\sigma, \sigma\}^{\mathcal{V}}_{\mathcal{V}_+} = 1$.
- $\{\sigma, \tau\}^{\mathcal{V}}_{\mathcal{V}_+} = \left(\{\tau, \sigma\}^{\mathcal{V}}_{\mathcal{V}_+}\right)^{-1}$.
- $\{\sigma\sigma', \tau\}^{\mathcal{V}}_{\mathcal{V}_+} = \{\sigma, \tau\}^{\mathcal{V}}_{\mathcal{V}_+}\{\sigma', \tau\}^{\mathcal{V}}_{\mathcal{V}_+}$.
- $\{\sigma, \tau\tau'\}^{\mathcal{V}}_{\mathcal{V}_+} = \{\sigma, \tau\}^{\mathcal{V}}_{\mathcal{V}_+}\{\sigma, \tau'\}^{\mathcal{V}}_{\mathcal{V}_+}$.
- If $\sigma\mathcal{V}_+ = \mathcal{V}_+ = \tau\mathcal{V}_+ \Rightarrow \{\sigma, \tau\}^{\mathcal{V}}_{\mathcal{V}_+} = 1$.
- If $\mathcal{V}_+ = \{0\}$ or $\mathcal{V}_+ = \mathcal{V}$, then $\{\sigma, \tau\}^{\mathcal{V}}_{\mathcal{V}_+} = 1$.
- $\{\sigma, \tau\}^{\mathcal{V}}_{\mathcal{V}_+}$ depends only on the commensurability class of $\mathcal{V}_+$.

**Lemma 3.8.** *Assume $\mathcal{V}$ is equipped with a direct sum decomposition $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$. Put $\mathcal{V}_{i+} := \mathcal{V}_i \cap \mathcal{V}_+$ for $i = 0, 1$ and assume that $\mathcal{V}_+ = \mathcal{V}_{0+} \oplus \mathcal{V}_{1+}$. Let commuting elements $\sigma_0, \sigma_1 \in G^{\mathcal{V}}_{\mathcal{V}_+}$ be given such that*

$$\sigma_i|_{\mathcal{V}_0} \in G^{\mathcal{V}_0}_{\mathcal{V}_{0+}}, \quad \sigma_i|_{\mathcal{V}_1} = 1$$

*for $i = 0, 1$. Then*

$$\left\{\sigma_0|_{\mathcal{V}_0}, \sigma_1|_{\mathcal{V}_0}\right\}^{\mathcal{V}_0}_{\mathcal{V}_{0+}} = \{\sigma_0, \sigma_1\}^{\mathcal{V}}_{\mathcal{V}_+}.$$

*Proof.* Since $\sigma_i\mathcal{V}_+ = [\sigma_i|_{\mathcal{V}_0}\mathcal{V}_{0+}] \oplus \mathcal{V}_{1+}$, then

$$\mathcal{V}_+ \cap \sigma_i\mathcal{V}_+ = [\mathcal{V}_{0+} \cap \sigma_i|_{\mathcal{V}_0}\mathcal{V}_{0+}] \oplus \mathcal{V}_{1+}.$$

Hence

$$\begin{aligned}
\mathrm{Det}\,\mathcal{C}^{\bullet}_{\sigma_i\mathcal{V}_+} &= \Lambda\mathcal{G}_{\mathfrak{m}_v}\left(\mathcal{V}_+/\mathcal{V}_+\cap\sigma_i\mathcal{V}_+\right) \underset{\mathcal{K}(v)}{\otimes} \Lambda\left[\mathcal{G}_{\mathfrak{m}_v}\left(\sigma_i\mathcal{V}_+/\mathcal{V}_+\cap\sigma_i\mathcal{V}_+\right)\right]^* \\
&= \Lambda\mathcal{G}_{\mathfrak{m}_v}\left(\mathcal{V}_{0+}/\mathcal{V}_{0+}\cap\sigma_i|_{\mathcal{V}_0}\mathcal{V}_{0+}\right) \underset{\mathcal{K}(v)}{\otimes} \Lambda\left[\mathcal{G}_{\mathfrak{m}_v}\left(\sigma_i|_{\mathcal{V}_0}\mathcal{V}_{0+}/\mathcal{V}_{0+}\cap\sigma_i|_{\mathcal{V}_0}\mathcal{V}_{0+}\right)\right]^* \\
&= \mathrm{Det}\,\mathcal{C}^{\bullet}_{\sigma_i|_{\mathcal{V}_0}\mathcal{V}_{0+}},
\end{aligned}$$

and there exists a commutative diagram

$$\begin{tikzcd} \operatorname{Det}\mathscr{C}^{\bullet}_{\sigma_i\mathscr{V}_+} \underset{\mathscr{K}(v)}{\otimes} \operatorname{Det}\mathscr{C}^{\bullet}_{\sigma_{1-i}\mathscr{V}_+} \arrow[r, "\varphi^{\sigma_{1-i}}_{\sigma_i}"] \arrow[d, equal] & \operatorname{Det}\mathscr{C}^{\bullet}_{\sigma_i\sigma_{1-i}\mathscr{V}_+} \arrow[d, equal] \\ \operatorname{Det}\mathscr{C}^{\bullet}_{\sigma_i|_{\mathscr{V}_0}\mathscr{V}_{0+}} \underset{\mathscr{K}(v)}{\otimes} \operatorname{Det}\mathscr{C}^{\bullet}_{\sigma_{1-i}|_{\mathscr{V}_0}\mathscr{V}_{0+}} \arrow[r, "\varphi^{\sigma_{1-i}|_{\mathscr{V}_0}}_{\sigma_i|_{\mathscr{V}_0}}"] & \operatorname{Det}\mathscr{C}^{\bullet}_{(\sigma_i\sigma_{1-i})|_{\mathscr{V}_0}\mathscr{V}_{0+}}, \end{tikzcd}$$

whence the lemma can be deduced. □

**Definition 3.9.** Given an element $\tau \in G^{\mathscr{V}}_{\mathscr{V}_+}$, we shall call the integer number

$$i_{\mathscr{K}(v)}(\tau, \mathscr{V}_+) = [\mathscr{V}_+|\tau\mathscr{V}_+]_{\mathfrak{m}_v}$$

the index of $\tau$ over $\mathscr{V}_+$.

**Remark 3.10.** Note that

$$i_{\mathscr{K}(v)}(\tau, \mathscr{V}_+) = \dim_{\mathscr{K}(v)}\mathscr{G}_{\mathfrak{m}_v}(\mathscr{V}_+/\tau\mathscr{V}_+ \cap \mathscr{V}_+) - \dim_{\mathscr{K}(v)}\mathscr{G}_{\mathfrak{m}_v}(\mathscr{V}_+/\tau^{-1}\mathscr{V}_+ \cap \mathscr{V}_+).$$

Moreover, if $\mathscr{M} \sim \mathscr{V}_+$, then $i_{\mathscr{K}(v)}(\tau, \mathscr{M}) = i_{\mathscr{K}(v)}(\tau, \mathscr{V}_+)$.

**Lemma 3.11.** *Again, assume $\mathscr{V}$ is equipped with a direct sum decomposition $\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_1$, put $\mathscr{V}_{i+} := \mathscr{V}_i \cap \mathscr{V}_+$ for $i = 0, 1$ and assume that $\mathscr{V}_+ = \mathscr{V}_{0+} \oplus \mathscr{V}_{1+}$. Let $\sigma_0, \sigma_1 \in G^{\mathscr{V}}_{\mathscr{V}_+}$ be given such that*

$$\sigma_i|_{\mathscr{V}_i} \in G^{\mathscr{V}_i}_{\mathscr{V}_{i+}}, \qquad \sigma_i|_{\mathscr{V}_{1-i}} = 1$$

*for $i = 0, 1$. (Necessarily, $\sigma_0$ and $\sigma_1$ commute.) Then*

$$\{\sigma_0, \sigma_1\}^{\mathscr{V}}_{\mathscr{V}_+} = (-1)^{\alpha_0\alpha_1},$$

*where*

$$\alpha_i := i_{\mathscr{K}(v)}(\sigma_i|_{\mathscr{V}_i}, \mathscr{V}_{i+}) = i_{\mathscr{K}(v)}(\sigma_i, \mathscr{V}_+) \quad \textit{for } i = 0, 1.$$

*Proof.* Consider nonzero elements

$$\begin{aligned} a &\in \Lambda\mathscr{G}_{\mathfrak{m}_v}(\mathscr{V}_+/\sigma_0\mathscr{V}_+ \cap \mathscr{V}_+) &&\simeq \Lambda\mathscr{G}_{\mathfrak{m}_v}(\mathscr{V}_{0+}/\sigma_0|_{\mathscr{V}_0}\mathscr{V}_{0+} \cap \mathscr{V}_{0+}), \\ b &\in \Lambda\mathscr{G}_{\mathfrak{m}_v}(\sigma_0\mathscr{V}_+/\sigma_0\mathscr{V}_+ \cap \mathscr{V}_+)^* &&\simeq \Lambda\mathscr{G}_{\mathfrak{m}_v}(\sigma_0|_{\mathscr{V}_0}\mathscr{V}_{0+}/\sigma_0|_{\mathscr{V}_0}\mathscr{V}_{0+} \cap \mathscr{V}_{0+})^*, \\ c &\in \Lambda\mathscr{G}_{\mathfrak{m}_v}(\mathscr{V}_+/\sigma_1\mathscr{V}_+ \cap \mathscr{V}_+) &&\simeq \Lambda\mathscr{G}_{\mathfrak{m}_v}(\mathscr{V}_{1+}/\sigma_1|_{\mathscr{V}_1}\mathscr{V}_{1+} \cap \mathscr{V}_{1+}), \\ d &\in \Lambda\mathscr{G}_{\mathfrak{m}_v}(\sigma_1\mathscr{V}_+/\sigma_1\mathscr{V}_+ \cap \mathscr{V}_+)^* &&\simeq \Lambda\mathscr{G}_{\mathfrak{m}_v}(\sigma_1|_{\mathscr{V}_1}\mathscr{V}_{1+}/\sigma_1|_{\mathscr{V}_1}\mathscr{V}_{1+} \cap \mathscr{V}_{1+})^*, \end{aligned}$$

where $\Lambda$ is the maximal exterior power.

If $s_0 = a \otimes b$ and $s_1 = c \otimes d$, then $(\sigma_0, s_0), (\sigma_1, s_1) \in \widetilde{G^{\mathscr{V}}_{\mathscr{V}_+}}$ and

$$\{\sigma_0, \sigma_1\}^{\mathscr{V}}_{\mathscr{V}_+} = (\sigma_0, s_0) \cdot (\sigma_1, s_1) \cdot (\sigma_0, s_0)^{-1} \cdot (\sigma_1, s_1)^{-1}.$$

Bearing in mind the multiplication of $\widetilde{G^{\mathcal{V}}_{\mathcal{V}_+}}$, an easy computation shows that

$$s_0 \cdot \bar{\sigma}_0(s_1) = (-1)^{i_{\mathcal{K}(v)}(\sigma_0,\mathcal{V}_+)\cdot i_{\mathcal{K}(v)}(\sigma_1,\mathcal{V}_+)} \cdot [s_1 \cdot \bar{\sigma}_1(s_0)],$$

and the claim is deduced. □

**Proposition 3.12.** *Let $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ and $\mathcal{V}_+ = \mathcal{V}_{0+} \oplus \mathcal{V}_{1+}$, where $\mathcal{V}_0$ and $\mathcal{V}_1$ are invariant by the action of two commuting elements $\tau, \sigma \in G^{\mathcal{V}}_{\mathcal{V}_+}$ and $\tau|_{\mathcal{V}_i}, \sigma|_{\mathcal{V}_i} \in G^{\mathcal{V}_i}_{\mathcal{V}_{i+}}$. Then*

$$\{\tau,\sigma\}^{\mathcal{V}}_{\mathcal{V}_+} = (-1)^{\alpha} \cdot \{\tau|_{\mathcal{V}_0}, \sigma|_{\mathcal{V}_0}\}^{\mathcal{V}_0}_{\mathcal{V}_{0+}} \cdot \{\tau|_{\mathcal{V}_1}, \sigma|_{\mathcal{V}_1}\}^{\mathcal{V}_1}_{\mathcal{V}_{1+}}$$

*with $\alpha = i_{\mathcal{K}(v)}(\tau|_{\mathcal{V}_0}, \mathcal{V}_{0+}) \cdot i_{\mathcal{K}(v)}(\sigma|_{\mathcal{V}_1}, \mathcal{V}_{1+}) + i_{\mathcal{K}(v)}(\tau|_{\mathcal{V}_1}, \mathcal{V}_{1+}) \cdot i_{\mathcal{K}(v)}(\sigma|_{\mathcal{V}_0}, \mathcal{V}_{0+})$.*

*Proof.* The claim is a direct consequence of Lemmas 3.8 and 3.11, given the decompositions $\tau = \tau_0 \cdot \tau_1$ and $\sigma = \sigma_0 \cdot \sigma_1$, where

$$\tau_i(v_0 + v_1) = \tau|_{\mathcal{V}_i}(v_i) + v_{1-i} \quad \text{and} \quad \sigma_i(v_0 + v_1) = \sigma|_{\mathcal{V}_i}(v_i) + v_{1-i}\ ,$$

with $v_i \in \mathcal{V}_i$. □

As in [Pablos Romo 2002], one has:

**Corollary 3.13.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{O}_v$-submodules of $\mathcal{V}$. For all commuting elements $\tau, \sigma \in G^{\mathcal{V}}_{\mathcal{M}} \cap G^{\mathcal{V}}_{\mathcal{N}}$, we have $\tau, \sigma \in G^{\mathcal{V}}_{\mathcal{M}+\mathcal{N}} \cap G^{\mathcal{V}}_{\mathcal{M}\cap\mathcal{N}}$ and*

$$\{\tau,\sigma\}^{\mathcal{V}}_{\mathcal{M}} \cdot \{\tau,\sigma\}^{\mathcal{V}}_{\mathcal{N}} = (-1)^{\beta} \cdot \{\tau,\sigma\}^{\mathcal{V}}_{\mathcal{M}+\mathcal{N}} \cdot \{\tau,\sigma\}^{\mathcal{V}}_{\mathcal{M}\cap\mathcal{N}},$$

*where*

$$\begin{aligned}\beta = {}& i_{\mathcal{K}(v)}(\tau,\mathcal{M}) \cdot i_{\mathcal{K}(v)}(\sigma,\mathcal{N}) + i_{\mathcal{K}(v)}(\tau,\mathcal{N}) \cdot i_{\mathcal{K}(v)}(\sigma,\mathcal{M}) \\ & + i_{\mathcal{K}(v)}(\tau,\mathcal{M}+\mathcal{N}) \cdot i_{\mathcal{K}(v)}(\sigma,\mathcal{M}\cap\mathcal{N}) + i_{\mathcal{K}(v)}(\tau,\mathcal{M}\cap\mathcal{N}) \cdot i_{\mathcal{K}(v)}(\sigma,\mathcal{M}+\mathcal{N}).\end{aligned}$$

**Remark 3.14.** If $\mathcal{V}_+ = \widehat{\mathcal{O}}_v$ and $\mathcal{V} = (\widehat{\mathcal{O}}_v)_0$, as in Example 3.2, we have that

$$\{f,g\}^{(\widehat{\mathcal{O}}_v)_0}_{\widehat{\mathcal{O}}_v} := \{f,g\}_v = \frac{f^{v(g)}}{g^{v(f)}} \pmod{\mathfrak{m}_v} \in \mathcal{K}(v)^*,$$

for all $f, g \in \mathcal{F}^*$, analogously to [Pablos Romo 2004, pp. 340–341].

## 4. Arithmetic symbols on GL($n,\mathcal{F}$)

With the notations of Section 3, for a positive integer number $n$, we consider the $\mathcal{O}_v$-modules

$$\mathcal{V}^n = (\widehat{\mathcal{O}}_v)_0 \oplus \overset{n\text{ times}}{\cdots\cdots} \oplus (\widehat{\mathcal{O}}_v)_0, \quad \mathcal{V}^n_+ = \widehat{\mathcal{O}}_v \oplus \overset{n\text{ times}}{\cdots\cdots} \oplus \widehat{\mathcal{O}}_v \subseteq \mathcal{V}^n.$$

Clearly, the linear group GL($n,\mathcal{F}$) is contained in $G^{\mathcal{V}^n}_{\mathcal{V}^n_+}$. Thus (3-1) yields a central extension of groups

$$1 \to \mathcal{K}(v)^* \xrightarrow{i} \widetilde{\mathrm{GL}(n,\mathcal{F})} \xrightarrow{\pi} \mathrm{GL}(n,\mathcal{F}) \to 1. \tag{4-1}$$

If $\tau$ and $\sigma$ are commuting elements of $\mathrm{GL}(n,\mathscr{F})$ and $\widetilde{\tau},\widetilde{\sigma}\in\widetilde{\mathrm{GL}(n,\mathscr{F})}$ are elements such that $\pi(\widetilde{\tau})=\tau$ and $\pi(\widetilde{\sigma})=\sigma$, there is a commutator pairing

$$\{\tau,\sigma\}^{\mathscr{V}^n}_{\mathscr{V}^n_+}=\widetilde{\tau}\cdot\widetilde{\sigma}\cdot\widetilde{\tau}^{-1}\cdot\widetilde{\sigma}^{-1}\in\mathscr{K}(v)^*.$$

**Example 4.1.** Consider the commuting matrices $\tau_{f_i},\tau_{g_j}\in\mathrm{GL}(n,\mathscr{F})$ given by

$$\tau_{f_i}=\begin{pmatrix} f_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & f_n\end{pmatrix}\quad\text{and}\quad \tau_{g_j}=\begin{pmatrix} g_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & g_n\end{pmatrix},$$

with $f_i,g_j\in\mathscr{F}^*$. It follows from Lemmas 3.8, 3.11 and the explicit expression of the commutator $\{\ ,\ \}^{(\widehat{\mathscr{O}}_v)_0}_{\widehat{\mathscr{O}}_v}$ discussed in Remark 3.14 that

$$\{\tau_{f_i},\tau_{g_j}\}^{\mathscr{V}^n}_{\mathscr{V}^n_+}=(-1)^{\sum_{i\neq j}v(f_i)v(g_j)}\cdot\prod_{s=1}^{n}\{f_s,g_s\}_v\in\mathscr{K}(v)^*.$$

We shall now study the commutator $\{\ ,\ \}^{\mathscr{V}^n}_{\mathscr{V}^n_+}$ in depth.

**Lemma 4.2.** *If $\tau\in\mathrm{GL}(n,\mathscr{F})$, then $i_{\mathscr{K}(v)}(\tau,\mathscr{V}^n_+)=v(\det\tau)$.*

*Proof.* It follows from the additivity of the index that $i_{\mathscr{K}(v)}(\tau,\mathscr{V}^n_+)=i_{\mathscr{K}(v)}(J_\tau,\mathscr{V}^n_+)$, where $J_\tau$ is the Jordan matrix associated with $\tau$.

Thus, as in [Pablos Romo 2006], from a $(m,m)$-matrix

$$\tau_a=\begin{pmatrix} 0 & 0 & \cdots & 0 & a_1\\ 1 & 0 & \cdots & 0 & a_2\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0 & a_{m-1}\\ 0 & \cdots & 0 & 1 & a_m\end{pmatrix},$$

we can consider the $(m\,s,m\,s)$-matrix

$$\tau_{(a,s)}=\begin{pmatrix} \tau_a^1 & & & \\ B & \tau_a^2 & & \\ & \ddots & \ddots & \\ & & B & \tau_a^s\end{pmatrix},$$

where $\tau_a^i=\tau_a$ and

$$B=\begin{pmatrix} 0 & \cdots & 0 & 1\\ 0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 0\end{pmatrix},$$

and a direct computation shows that

$$i_{\mathcal{K}(v)}(\tau_{(a,s)}, \mathcal{V}_+^{m\cdot s}) = i_{\mathcal{K}(v)}(\varphi_{(a,s)}, \mathcal{V}_+^{s}) = i_{\mathcal{K}(v)}(a_1^s, \widehat{\mathcal{O}}_v) = s\cdot v(a_1) = v(\det \tau_{(a,s)}).$$

Accordingly, for an arbitrary $\tau \in \mathrm{GL}(n, \mathcal{F})$, it follows from the general expression of the Jordan matrix $J_\tau$, and the above results, that

$$i_{\mathcal{K}(v)}(J_\tau, \mathcal{V}_+^n) = v(\det J_\tau),$$

from which the statement can be deduced. □

Now regard $\mathcal{F}^*$ as a subgroup of $\mathrm{GL}(n, \mathcal{F})$ via the diagonal embedding

$$f \mapsto \sigma_f^n := \begin{pmatrix} f & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f \end{pmatrix}.$$

Then $\mathcal{F}^* = Z(\mathrm{GL}(n, \mathcal{F}))$, so there exists a commutator map

$$\{\ ,\ \}_{\mathcal{V}_+^n}^{\mathcal{V}^n} : \mathcal{F}^* \times \mathrm{GL}(n, \mathcal{F}) \to k(v)^*.$$

We shall compute the explicit expression of this commutator map.

**Lemma 4.3.** *If $f \in \mathcal{F}^*$, then $i_{\mathcal{K}(v)}(\sigma_f^n, \mathcal{V}_+^n) = n \cdot v(f)$.*

*Proof.* The claim is a direct consequence of Lemma 4.2. □

**Proposition 4.4.** *For all $f \in \mathcal{F}^*$ and $\tau \in \mathrm{GL}(n, \mathcal{F})$,*

$$\begin{aligned}\{\sigma_f^n, \tau\}_{\mathcal{V}_+^n}^{\mathcal{V}^n} &= (-1)^{(n-1)\cdot v(f)\cdot v(\det\tau)} \cdot \{f, \det\tau\}_v \\ &= (-1)^{(n-1)\cdot v(f)\cdot v(\det\tau)} \cdot \left(\frac{f^{v(\det\tau)}}{[\det\tau]^{v(f)}} (\textit{mod}\ \mathfrak{m}_v)\right).\end{aligned}$$

*Proof.* Write $\mathrm{Sl}(n, R) = \{g \in \mathrm{GL}(n, R) \text{ such that } \det g = 1\}$ for a commutative ring $R$ (the special linear group of $R$).

We have the decomposition $\tau = \tau_0 \cdot \tau_{\det}$, where $\tau_0 \in \mathrm{Sl}(n, \mathcal{F})$ and

$$\tau_{\det} := \begin{pmatrix} \det\tau & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

By [Klingenberg 1961, p. 139, Korollar 1], $\mathrm{Sl}(n, R)$ is always a commutator subgroup of $\mathrm{GL}(n, R)$ when $R$ is a commutative local ring, so

$$\{\sigma_f^n, \tau\}_{\mathcal{V}_+^n}^{\mathcal{V}^n} = \{\sigma_f^n, \tau_{\det}\}_{\mathcal{V}_+^n}^{\mathcal{V}^n}.$$

Now the claim is easily deduced from the properties of the commutator $\{\ ,\ \}_{\mathcal{V}_+^n}^{\mathcal{V}^n}$.

□

**Remark 4.5.** When $n = 1$, the above expression is the commutator $\{\ ,\ \}_v$ that determines the tame symbol on an arbitrary discrete valuation field $\mathcal{F}$ [Pablos Romo 2004], and, when $n=2$, if $X$ is an algebraic curve over an algebraically closed field $k$, $x \in X$ is a closed point, $\mathcal{F} = \Sigma_X$ (the function field of $X$), and $v = v_x$ (the discrete valuation induced by the point $x$), the statement of Proposition 4.4 coincides with the characterization of the commutator map $\{\ ,\ \}^{K_x^2}_{A_x^2}\colon \Sigma_X^* \times \mathrm{GL}(2, \Sigma_X) \to k^*$ obtained in [Muñoz Porras and Pablos Romo $\geq$ 2008].

***The tame symbol on*** $\mathbf{GL}(n, \mathcal{F})$. We shall now define a generalization of the tame symbol from the commutator $\{\ ,\ \}^{\mathcal{V}^n}_{\mathcal{V}^n_+}$.

**Definition 4.6.** The tame symbol on $\mathrm{GL}(n, \mathcal{F})$ associated with $v$ is the assignment

$$(\tau, \varphi)^n_v = (-1)^{v(\det\tau)\cdot v(\det\varphi)} \cdot \{\tau, \sigma\}^{\mathcal{V}^n}_{\mathcal{V}^n_+} \in \mathcal{K}(v)^*,$$

where $\tau, \varphi$ are commuting matrices of $\mathrm{GL}(n, \mathcal{F})$.

**Remark 4.7.** When $n = 1$, the map $(\ ,\ )^1_v\colon \mathcal{F}^* \times \mathcal{F}^* \to \mathcal{K}(v)^*$ is the usual tame symbol associated with the discrete valuation field $\mathcal{F}$.

Now fix elements $\sigma, \sigma', \tau, \tau' \in \mathrm{GL}(n, \mathcal{F})$ such that $\sigma, \sigma'$ commute with $\tau, \tau'$. The following relations are easily deduced from the definitions:

- $(\sigma, \sigma)^n_v = 1$.
- $(\sigma, \tau)^n_v = [(\tau, \sigma)^n_v]^{-1}$.
- $(\sigma\sigma', \tau)^n_v = (\sigma, \tau)^n_v \cdot (\sigma', \tau)^n_v$.
- $(\sigma, \tau\tau')^n_v = (\sigma, \tau)^n_v \cdot (\sigma, \tau')^n_v$.
- $(\tau, -\tau)^n_v = 1$.

**Remark 4.8.** Given a diagonal matrix $\tau_{f_i}$ with $f_i \neq 1$, it is clear that

$$(\tau_{f_i}, 1 - \tau_{f_i})^n_v = 1.$$

A remaining problem is to determine whether this property of Steinberg symbols on $\mathcal{F}$ holds for arbitrary matrices $\tau, 1 - \tau \in \mathrm{GL}(n, \mathcal{F})$.

**Example 4.9.** Consider again the commuting matrices $\tau_{f_i}, \tau_{g_j} \in \mathrm{GL}(n, \mathcal{F})$ of Example 4.1. Since

$$\{\tau_{f_i}, \tau_{g_j}\}^{\mathcal{V}^n}_{\mathcal{V}^n_+} = (-1)^{\sum_{i\neq j} v(f_i)v(g_j)} \cdot \prod_{s=1}^{n} \{f_s, g_s\}_v \in \mathcal{K}(v)^*\,,$$

one has

$$(\tau_{f_i}, \tau_{g_j})^n_v = \prod_{s=1}^{n} (f_s, g_s)_v,$$

where $(\ ,\ )_v$ is the tame symbol.

**Example 4.10.** It follows from Proposition 4.4 that the explicit expression of the tame symbol $(\ ,\ )^n_v \colon \mathcal{F}^* \times \mathrm{GL}(n, \mathcal{F}) \to \mathcal{K}(v)^*$ is

$$(\sigma^n_f, \tau)^n_v = (-1)^{v(f)\cdot v(\det\tau)} \cdot \left(\frac{f^{v(\det\tau)}}{[\det\tau]^{v(f)}} (\mathrm{mod}\ \mathfrak{m}_v)\right).$$

In particular, if $\widetilde{\mathcal{F}}$ is a number field (i.e., it is a finite extension of the field of rational numbers) with a discrete valuation $\widetilde{v}$, in the theory of modular forms the subgroup $\mathrm{GL}^+(n, \widetilde{\mathcal{F}}) \subset \mathrm{GL}(n, \widetilde{\mathcal{F}})$ appears, such that $\tau \in \mathrm{GL}^+(n, \widetilde{\mathcal{F}})$ implies $\widetilde{v}(\det\tau) = 0$. Hence, the restricted symbol $(\ ,\ )^n_{\widetilde{v}} \colon \widetilde{\mathcal{F}}^* \times \mathrm{GL}^+(n, \widetilde{\mathcal{F}}) \to \mathcal{K}(\widetilde{v})^*$ is trivial.

**Example 4.11.** Consider a nonsingular and irreducible curve $C$ over an algebraically closed field $k$ and a closed point $x \in C$. If $\Sigma_C$ is the function field of $C$ and $v_x$ is the discrete valuation on $\Sigma_C$ associated with $x$, by using the above method we can define the tame symbol on $\mathrm{GL}(n, \Sigma_C)$ associated with $x$, whose explicit expression is

$$(\tau, \sigma)^n_x = (-1)^{v_x(\det\tau)\cdot v_x(\det\sigma)} \cdot \{\tau, \sigma\}^{K^n_x}_{A^n_x} \in k^*,$$

where $\tau, \sigma \in \mathrm{GL}(n, \Sigma_C)$ are commuting matrices, $A^n_x = \widehat{\mathcal{O}}_x \oplus \overset{n\,\text{times}}{\cdots\cdots} \oplus \widehat{\mathcal{O}}_x$ and $K^n_x = (\widehat{\mathcal{O}}_x)_0 \oplus \overset{n\,\text{times}}{\cdots\cdots} \oplus (\widehat{\mathcal{O}}_x)_0$.

**Example 4.12** (Generalized Parshin symbol on a surface). Let $C$ be an irreducible and nonsingular algebraic curve on a smooth, proper, geometrically irreducible surface $S$ over an algebraically closed field $k$. If $\Sigma_S$ is the function field of $S$, the curve $C$ defines a discrete valuation

$$v_C \colon \Sigma^*_S \to \mathbb{Z},$$

whose residue class field is $\Sigma_C$ (the function field of $C$).

Thus, it follows from the central extension (4-1) that there exists a central extension of groups

$$1 \to \Sigma^*_C \xrightarrow{i} \widetilde{\mathrm{GL}(n, \Sigma_S)} \xrightarrow{\pi} \mathrm{GL}(n, \Sigma_S) \to 1\,, \tag{4-2}$$

and its commutator will be denoted by $\{\ ,\ \}^n_{v_C}$.

Now setting an element $z \in \Sigma^*_S$ with $v_C(z) = 1$, given a closed point $x \in C$, we denote

$$v^{n,z}_x(\tau) = v_x(\{\tau, \sigma^n_z\}^n_{v_C}) \in \mathbb{Z}$$

for a matrix $\tau \in \mathrm{GL}(n, \Sigma_S)$.

If $\{\ ,\ \}_{A_x}^{K_x}$ is again the commutator referred to in Section 2, we define the map

$$\{\ ,\ ,\ \}_{x,C}^{n,z}\colon \operatorname{GL}(n,\Sigma_S)\times\operatorname{GL}(n,\Sigma_S)\times\operatorname{GL}(n,\Sigma_S)\to k^*$$

by the expression

$$\begin{aligned}\{\tau_1,\tau_2,\tau_3\}_{x,C}^{n,z} &= \bigl(\{\{\tau_1,\sigma_z^n\}_{v_C}^n,\{\tau_2,\sigma_z^n\}_{v_C}^n\}_{A_x}^{K_x}\bigr)^{-v_C(\det\tau_3)}\\ &= (-1)^{(n-1)[v_C(\det\tau_1)+v_C(\det\tau_2)]\,v_C(\det\tau_3)}\left(\frac{(\det\tau_1)^{-v_C(\det\tau_3)\,v_x^{n,z}(\tau_2)}}{(\det\tau_2)^{-v_C(\det\tau_3)\,v_x^{n,z}(\tau_1)}}(p)\right),\end{aligned}$$

for all matrices $\tau_1,\tau_2,\tau_3\in\operatorname{GL}(n,\Sigma_S)$.

Thus, analogously to [Pablos Romo 2004, Section 3], we can consider the map $\{\ ,\ ,\ \}_{x,C}^{n}$ given by

$$\begin{aligned}&\{\tau_1,\tau_2,\tau_3\}_{x,C}^{n}\\ &=\{\tau_1,\tau_2,\tau_3\}_{x,C}^{n,z}\cdot\{\tau_2,\tau_3,\tau_1\}_{x,C}^{n,z}\cdot\{\tau_3,\tau_1,\tau_2\}_{x,C}^{n,z}\\ &=\bigl((\det\tau_1)^{v_C(\det\tau_2)\cdot v_x^{n,z}(\tau_3)-v_C(\det\tau_3)\cdot v_x^{n,z}(\tau_2)}\cdot(\det\tau_2)^{v_C(\det\tau_3)\cdot v_x^{n,z}(\tau_1)-v_C(\det\tau_1)\cdot v_x^{n,z}(\tau_3)}\\ &\qquad\cdot(\det\tau_3)^{v_C(\det\tau_1)\cdot v_x^{n,z}(\tau_2)-v_C(\det\tau_2)\cdot v_x^{n,z}(\tau_1)}\bigr)(p)\\ &=\{\det\tau_1,\det\tau_2,\det\tau_3\}_{x,C},\end{aligned}$$

for all matrices $\tau_1,\tau_2,\tau_3\in\operatorname{GL}(n,\Sigma_S)$. This map is independent of the choice of the parameter $z$.

If we consider the symbol

$$\langle\ ,\ ,\ \rangle_{x,C}^{n}\colon \operatorname{GL}(n,\Sigma_S)\times\operatorname{GL}(n,\Sigma_S)\times\operatorname{GL}(n,\Sigma_S)\to k^*$$

defined by

$$\langle\tau_1,\tau_2,\tau_3\rangle_{x,C}^{n}=(-1)^{\alpha(\tau_1,\tau_2,\tau_3)}\cdot\{\tau_1,\tau_2,\tau_3\}_{x,C}^{n},$$

with

$$\begin{aligned}\alpha(\tau_1,\tau_2,\tau_3)v_C(\det\tau_1)\cdot v_C(\det\tau_2)\cdot v_x^{n,z}(\tau_3)+v_C(\det\tau_1)\cdot v_C(\det\tau_3)\cdot v_x^{n,z}(\tau_2)&\\ +v_C(\det\tau_2)\cdot v_C(\det\tau_3)\cdot v_x^{n,z}(\tau_1)+v_C(\det\tau_1)\cdot v_x^{n,z}(\tau_2)\cdot v_x^{n,z}(\tau_3)&\\ +v_C(\det\tau_2)\cdot v_x^{n,z}(\tau_1)\cdot v_x^{n,z}(\tau_3)+v_C(\det\tau_3)\cdot v_x^{n,z}(\tau_1)\cdot v_x^{n,z}(\tau_2)&,\end{aligned}$$

bearing in mind the statements of [Pablos Romo 2004, Section 3], we see that this symbol is independent of the choice of the parameter $z$ and generalizes the symbol on $\Sigma_S$ offered by A. N. Parshin [1984].

**Remark 4.13** (Tame symbol on the infinite general linear group $\operatorname{GL}(\mathscr{F})$). Similar to [Milnor 1971], let $\operatorname{GL}(\mathscr{F})$ denote the direct limit of the sequence

$$\operatorname{GL}(1,\mathscr{F})\subset\operatorname{GL}(2,\mathscr{F})\subset\operatorname{GL}(3,\mathscr{F})\subset\cdots,$$

where each GL($n, \mathcal{F}$) is injected into GL($n+1, \mathcal{F}$) by the correspondence

$$\tau \mapsto \phi_n^{n+1}(\tau) = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}.$$

The commutator subgroup of GL($\mathcal{F}$), $E(\mathcal{F})$, consists of all elementary matrices; that is, $\sigma \in E(\mathcal{F})$ if and only if $\sigma$ coincides with the identity matrix except for a single off-diagonal entry. The abelian quotient group GL($\mathcal{F}$)$/E(\mathcal{F})$ is the Whitehead group $K_1(\mathcal{F})$, and it is known that $K_1(\mathcal{F}) \simeq \mathcal{F}^*$ by the assignment: $[\tau] \mapsto \det \tau$.

Since

$$(\sigma_f^n, \tau)_v^n = (\sigma_f^{n+1}, \phi_n^{n+1}(\tau))_v^{n+1}$$

for all $\tau \in \mathrm{GL}(n, \mathcal{F})$, the symbols $(\ ,\ )_v^n$ induce a map

$$(\ ,\ )_v^\infty : \mathcal{F}^* \times \mathrm{GL}(\mathcal{F}) \to \mathcal{K}(v)^*,$$

such that, for all $n$, there exist commutative diagrams

$$\begin{tikzcd} \mathcal{F}^* \times \mathrm{GL}(\mathcal{F}) \arrow[r, "(\ ,\ )_v^\infty"] & \mathcal{K}(v)^* \\ \mathcal{F}^* \times \mathrm{GL}(n, \mathcal{F}) \arrow[u, hook] \arrow[ur, "(\ ,\ )_v^n"'] & \end{tikzcd} ,$$

Moreover, bearing in mind that the restriction of $(\ ,\ )_v^\infty$ to $\mathcal{F}^* \times E(\mathcal{F})$ is trivial, because each elementary matrix is included in a special linear group Sl($n, \mathcal{F}$), one has a map

$$\overline{(\ ,\ )_v}^\infty : \mathcal{F}^* \times K_1(\mathcal{F}) \to \mathcal{K}(v)^*,$$

$K_1(\mathcal{F})$ being the Whitehead group of $\mathcal{F}$, and such that $(\ ,\ )_v^\infty$ factorizes through $\overline{(\ ,\ )_v}^\infty$ and the natural projection $\mathcal{F}^* \times \mathrm{GL}(\mathcal{F}) \to \mathcal{F}^* \times K_1(\mathcal{F})$.

It follows from the isomorphism $K_1(\mathcal{F}) \simeq \mathcal{F}^*$ that

$$\overline{(f, [\tau])_v}^\infty = (f, \det \tau)_v^1 .$$

Thus, we have a double relation between these symbols: $(\ ,\ )_v^\infty$ is an extension of the classical tame symbol $(\ ,\ )_v^1$ and, also, $(\ ,\ )_v^1$ is a "quotient" of $(\ ,\ )_v^\infty$.

Note that with this method it is not possible to define a map over GL($\mathcal{F}$) from the commutator $\{\ ,\ \}_{\mathcal{V}_+^n}^{\mathcal{V}^n}$ because

$$\{\sigma_f^n, \tau\}_{\mathcal{V}_+^n}^{\mathcal{V}^n} \neq \{\sigma_f^{n+1}, \phi_n^{n+1}(\tau)\}_{\mathcal{V}_+^{n+1}}^{\mathcal{V}^{n+1}} .$$

***Symbols on* $\mathrm{GL}(n,\mathscr{F})$ *associated with a morphism* $\mathscr{K}(v)^* \to G$.** Let $G$ be an abelian group and let us consider a morphism of groups

$$\psi\colon \mathscr{K}(v)^* \to G.$$

The central extension (4-1) induces another central extension of groups

$$1 \to G \xrightarrow{i} \widetilde{\mathrm{GL}(n,\mathscr{F})}_{\psi} \xrightarrow{\pi} \mathrm{GL}(n,\mathscr{F}) \to 1, \tag{4-3}$$

whose commutator is

$$\{\tau,\sigma\}^{\mathscr{V}^n}_{\mathscr{V}^n_+,\psi} = \widetilde{\tau}\cdot\widetilde{\sigma}\cdot\widetilde{\tau}^{-1}\cdot\widetilde{\sigma}^{-1},$$

$\tau,\sigma \in \mathrm{GL}(n,\mathscr{F})$ being two commuting matrices, $\widetilde{\tau},\widetilde{\sigma} \in \widetilde{\mathrm{GL}(n,\mathscr{F})}_{\psi}$, $\pi(\widetilde{\tau}) = \tau$ and $\pi(\widetilde{\sigma}) = \sigma$. It is clear that $\{\ ,\ \}^{\mathscr{V}^n}_{\mathscr{V}^n_+,\psi} = \psi \circ \{\ ,\ \}^{\mathscr{V}^n}_{\mathscr{V}^n_+}$.

**Example 4.14.** Let $C$ be a nonsingular and irreducible curve over a perfect field $k$ and let $x \in C$ be a closed point. If $\Sigma_C$ is again the function field of $C$, $v_x$ is the discrete valuation on $\Sigma_C$ associated with $x$, $A_x = \widehat{\mathcal{O}}_x$, $K_x = (\widehat{\mathcal{O}}_x)_0$, and $k(x)$ is the residue class field of $x$, from the norm $N_{k(x)/k}$ of the extension $k \hookrightarrow k(x)$, we have a commutator $\{\ ,\ \}^{K^n_x}_{A^n_x,N_{k(x)/k}}$ defined by

$$\{\tau,\sigma\}^{K^n_x}_{A^n_x,N_{k(x)/k}} = N_{k(x)/k}\big[\{\tau,\sigma\}^{K^n_x}_{A^n_x}\big] \in k^*,$$

for all commuting matrices $\tau,\sigma \in \mathrm{GL}(n,\Sigma_C)$.

**Example 4.15.** Let $C$ be a nonsingular and irreducible curve over a finite perfect field $\mathbb{F}_q$ that contains the $m$-th roots of unity. If $\#\mathbb{F}_q = q$, one has the morphism of groups

$$\psi_m\colon \mathbb{F}_q^* \to \mu_m, \quad a \mapsto a^{(q-1)/m},$$

and, with the notation of Example 4.14, we can consider the morphism of groups:

$$\phi_m := \psi_m \circ N_{k(x)/k}\colon k(x)^* \to \mu_m,$$

and the induced commutator is

$$\{\tau,\sigma\}^{K^n_x}_{A^n_x,\phi_m} = N_{k(x)/k}\big[\{\tau,\sigma\}^{K^n_x}_{A^n_x}\big]^{(q-1)/m} \in \mu_m,$$

for all commuting matrices $\tau,\sigma \in \mathrm{GL}(n,\Sigma_C)$. Note that we can also obtain this expression by considering the commensurability of $k$-modules instead of $k(x)$-modules (see Section 2 for the case $n=1$).

**Definition 4.16.** Given an abelian group $G$ and a morphism of groups $\psi\colon \mathscr{K}(v)^* \to G$, we define the tame symbol on $\mathrm{GL}(n,\mathscr{F})$ associated with $v$ and the morphism $\psi$ as the assignment

$$(\tau,\varphi)^n_{v,\psi} = \psi\big[(-1)^{v(\det\tau)\cdot v(\det\varphi)}\big]\cdot\{\tau,\sigma\}^{\mathscr{V}^n}_{\mathscr{V}^n_+,\psi} \in G,$$

where $\tau, \varphi$ are two commuting matrices of $\mathrm{GL}(n, \mathscr{F})$.

**Example 4.17.** With the hypothesis of Example 4.14, one has the tame symbol $( \ , \ )^n_{x,N_{k(x)/k}}$ on $\mathrm{GL}(n, \Sigma_C)$ associated with $x$:

$$(\tau, \sigma)^n_{x,N_{k(x)/k}} := (\tau, \sigma)^n_x = (-1)^{\deg(x)\cdot v_x(\det\tau)\cdot v_x(\det\sigma)} \cdot N_{k(x)/k}\big[\{\tau, \sigma\}^{K^n_x}_{A^n_x}\big] \in k^*,$$

where $\tau, \sigma \in \mathrm{GL}(n, \Sigma_C)$ are commuting matrices, and $\deg(x) = \dim_k k(x)$ is the degree of the closed point $x$. We keep the notation of Example 4.11, because if $x$ is a rational point of $C$ both expressions coincide. Moreover, for a morphism of groups $\varphi\colon k^* \to G$, we denote by $( \quad , \quad )^n_{x,\varphi}$ the tame symbol on $\mathrm{GL}(n, \Sigma_C)$ associated with $x$ and the morphism $\varphi$.

**Example 4.18.** With the hypothesis of Example 4.15, a particular case of Example 4.17 is the Hilbert norm residue symbol $( \ , \ )^n_{x,\phi_m}$ on $\mathrm{GL}(n, \Sigma_C)$ associated with $x$:

$$(\tau, \sigma)^n_{x,\phi_m} = (-1)^{\frac{q-1}{m}\cdot\deg(x)\cdot v_x(\det\tau)\cdot v_x(\det\sigma)} \cdot N_{k(x)/k}\big[\{\tau, \sigma\}^{K^n_x}_{A^n_x}\big]^{\frac{q-1}{m}} \in \mu_m,$$

where $\tau, \sigma \in \mathrm{GL}(n, \Sigma_C)$ are commuting matrices. This formula generalizes the Hilbert norm residue symbol [Schmid 1936] on $\Sigma_C$ associated with the closed point $x \in C$.

**Example 4.19.** Let $\mathscr{F}_p$ be a local field of characteristic $p$, $p \neq 2$, with a discrete valuation $v_p$. If $\mu_2$ is the group of the $2^{nd}$ roots of the unity, from the morphism of groups

$$\phi_p\colon (\mathbb{Z}/p)^* \to \mu_2, \quad a \mapsto a^{(p-1)/2},$$

we have a symbol $( \ , \ )^n_{p,\phi_p}$, defined by

$$(\tau, \sigma)^n_{p,\phi_p} = (-1)^{((p-1)/2)\cdot v_p(\det\tau)\cdot v_p(\det\sigma)} \cdot [(\tau, \sigma)^n_p]^{(p-1)/2} \in \mu_2,$$

for all commuting matrices $\tau, \sigma \in \mathrm{GL}(n, \mathscr{F}_p)$, $( \ , \ )^n_p$ being the tame symbol on $\mathrm{GL}(n, \mathscr{F}_p)$ associated with $v_p$.

**Remark 4.20.** With the above notation, similar to Remark 4.13, it is possible to define morphisms

$$( \ , \ )^\infty_{v,\psi}\colon \mathscr{F}^* \times \mathrm{GL}(\mathscr{F}) \to G,$$
$$\overline{( \ , \ )}^\infty_{v,\psi}\colon \mathscr{F}^* \times K_1(\mathscr{F}) \to G,$$

$K_1(\mathscr{F})$ again being the Whitehead group of $\mathscr{F}$, and such that there exist commutative diagrams

$$\begin{tikzcd} \mathscr{F}^* \times \mathrm{GL}(\mathscr{F}) \arrow[r, "( \ , \ )^\infty_{v,\psi}"] & G, \\ \mathscr{F}^* \times \mathrm{GL}(n, \mathscr{F}) \arrow[u, hook] \arrow[ur, "( \ , \ )^n_{v,\psi}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}^* \times \mathrm{GL}(\mathscr{F}) \arrow[r, "( \ , \ )^\infty_{v,\psi}"] \arrow[d] & G. \\ \mathscr{F}^* \times K_1(\mathscr{F}) \arrow[ur, "\overline{( \ , \ )}^\infty_{v,\psi}"'] & \end{tikzcd}$$

Moreover, using the natural isomorphism $\mathcal{F}^* \simeq K_1(\mathcal{F})$, we can recover the Steinberg symbol on $\mathcal{F}$ with values in $G$ whose generalization is $(\ ,\ )^{\infty}_{v,\psi}$ as a quotient of this generalized symbol.

## 5. Reciprocity laws on $\mathrm{GL}(n, \Sigma_C)$

Let $C$ again be a nonsingular and irreducible curve over a perfect field $k$. For each closed point $x \in C$, $k(x)$ is its residue class field and $N_{k(x)/k}$ is the norm of the extension $k \hookrightarrow k(x)$. We can now consider the $k$-vector spaces $A_x = \widehat{\mathcal{O}}_x$ and $K_x = (\widehat{\mathcal{O}}_x)_0$, and the ring of adeles $\mathbb{A}_C$ is

$$\mathbb{A}_C = \prod_{x\in C}{}' K_x = \{f = (f_x) \text{ such that } f_x \in K_x \text{ and } f_x \in A_x \text{ for almost all } x\}.$$

We write $\mathbb{A}^+_C = \prod_{x\in C} A_x$, and for each subset of closed points $T \subset C$, we put

$$\mathbb{A}_T = \prod_{y\in T}{}' K_y, \quad \mathbb{A}^+_T = \prod_{y\in T} A_y.$$

We will be concerned with commensurability on $k$-vector spaces. Consider a $k$-vector space $V$ and a vector subspace $V_+ \subseteq V$. According to [Arbarello et al. 1989] and [Pablos Romo 2002], if

$$G^V_{V_+} = \{f \in \mathrm{Aut}_k(V) \text{ such that } fV_+ \sim V_+\},$$

there exists a central extension of groups

$$1 \to k^* \to \widetilde{G^V_{V_+}} \to G^V_{V_+} \to 1\,,$$

whose commutator is denoted by $\{\quad,\quad\}^V_{V_+}$.

For each closed point $x \in C$, setting again $A^n_x = \widehat{\mathcal{O}}_x \oplus \overset{n \text{ times}}{\cdots\cdots} \oplus \widehat{\mathcal{O}}_x$ and $K^n_x = (\widehat{\mathcal{O}}_x)_0 \oplus \overset{n \text{ times}}{\cdots\cdots} \oplus (\widehat{\mathcal{O}}_x)_0$, since

$$\mathrm{GL}(n, \Sigma_C) \subseteq G^{K^n_x}_{A^n_x},$$

then there exists a central extension of groups

$$1 \to k^* \to \widetilde{\mathrm{GL}(n, \Sigma_C)}_x \to \mathrm{GL}(n, \Sigma_C) \to 1. \tag{5-1}$$

We write $\widetilde{\{\ ,\ \}}^{K^n_x}_{A^n_x}$ to denote the commutator of (5-1).

**Lemma 5.1.** *For every two commuting matrices $\tau, \sigma \in \mathrm{GL}(n, \Sigma_C)$ one has*

$$\widetilde{\{\tau, \sigma\}}^{K^n_x}_{A^n_x} = \{\tau, \sigma\}^{K^n_x}_{A^n_x, N_{k(x)/k}},$$

*where $\{\ ,\ \}^{K^n_x}_{A^n_x, N_{k(x)/k}}$ is the commutator referred to in Example 4.14.*

*Proof.* Bearing in mind the definitions of the commutators $\widetilde{\{\ ,\ \}}^{K_x^n}_{A_x^n}$ and $\{\ ,\ \}^{K_x^n}_{A_x^n}$ (both depend on determinants of the same finite-dimensional $k(x)$-vector spaces, considering the natural structure as $k$-modules in the first case, and the structure as $k(x)$-modules in the second one; see [Anderson and Pablos Romo 2004, p. 88], the four square identity), the lemma is a direct consequence of the following well-known property of determinants of vector spaces: *If $k \hookrightarrow K$ is a finite extension of fields, $V$ is a $K$-vector space, and $\phi\colon V \simeq V$ is an automorphism of $K$-vector spaces, then*

$$\det_k(\phi) = N_{K/k}[\det_K(\phi)]. \qquad \square$$

Moreover, if $\tau \in \mathrm{GL}(n, \Sigma_C)$,

$$\begin{aligned} i_k(\tau, A_x^n) &= \dim_k(A_x^n/A_x^n \cap \tau A_x^n) - \dim_k(\tau A_x^n/A_x^n \cap \tau A_x^n) \\ &= \deg(x) \cdot v_x(\det \tau). \end{aligned} \tag{5-2}$$

**Lemma 5.2** [Pablos Romo 2002, Theorem 4.5, p. 4357]. *If $V = V_1 \oplus V_2$ and $V_+ = V_+^1 \oplus V_+^2$ such that $V_1$ and $V_2$ are invariant by the action of $f, g \in G^V_{V_+}$, then*

$$\{f, g\}^V_{V_+} = (-1)^\alpha \cdot \{f, g\}^{V_1}_{V_+^1} \cdot \{f, g\}^{V_2}_{V_+^2}$$

*with $\alpha = i_k(f, V_+^1) \cdot i_k(g, V_+^2) + i_k(f, V_+^2) \cdot i_k(g, V_+^1)$.*

**Lemma 5.3** [Pablos Romo 2002, Theorem 4.8, p. 4359]. *If $M$ and $N$ are two arbitrary vector subspaces of $V$, such that $fM \sim M$ and $fN \sim N$ for all $f \in G^V_{V_+}$, then*

$$\{f, g\}^V_M \cdot \{f, g\}^V_N = (-1)^\beta \cdot \{f, g\}^V_{M+N} \cdot \{f, g\}^V_{M\cap N}$$

*where $\beta = i_k(f, M) \cdot i_k(g, N) + i_k(f, N) \cdot i_k(g, M) + i_k(f, M+N) \cdot i_k(g, M \cap N) + i_k(f, M \cap N) \cdot i_k(g, M+N)$.*

We now give a proof of a reciprocity law for the symbols on $\mathrm{GL}(n, \Sigma_C)$ defined in Section 4.

Using the diagonal embedding, there exist immersions

$$\mathrm{GL}(n, \Sigma_C) \hookrightarrow G^{K^n_{x_1}\oplus\cdots\oplus K^n_{x_s}}_{A^n_{x_1}\oplus\cdots\oplus B^n_{x_s}},$$

for finite sets of closed points $\{x_1, \dots, x_s\} \subset X$. Thus, we can consider the central extension of groups

$$1 \to k^* \to \widetilde{\mathrm{GL}(n, \Sigma_C)}_{\{x_1,\dots,x_s\}} \to \mathrm{GL}(n, \Sigma_C) \to 1, \tag{5-3}$$

and we write $\widetilde{\{\ ,\ \}}^{K^n_{x_1}\oplus\cdots\oplus K^n_{x_s}}_{A^n_{x_1}\oplus\cdots\oplus A^n_{x_s}}$ to denote the commutator of this central extension.

Writing

$$\mathbb{A}^n_C \quad := \prod_{x\in C}{}' K^n_x = \{\varphi = (\varphi_x) : \varphi_x \in K^n_x \text{ and } \varphi_x \in A^n_x \text{ for almost all } x \in C\},$$
$$(\mathbb{A}^n_C)^+ := \prod_{x\in C} A^n_x,$$

the diagonal embedding also induces an immersion of linear groups

$$\mathrm{GL}(n, \Sigma_C) \hookrightarrow G^{\mathbb{A}^n_C}_{(\mathbb{A}^n_C)^+}.$$

In general, for an arbitrary $k$-subspace $H \subseteq \mathbb{A}^n_C$, such that

$$\mathrm{GL}(n, \Sigma_C) \hookrightarrow G^{\mathbb{A}^n_C}_H,$$

we can consider the corresponding central extension of groups

$$1 \to k^* \to \widetilde{\mathrm{GL}(n, \Sigma_C)}^{\mathbb{A}^n_C}_H \to \mathrm{GL}(n, \Sigma_C) \to 1. \tag{5-4}$$

and we denote by $\widetilde{\{\ ,\ \}}^{\mathbb{A}^n_C}_H$ its commutator.

From the diagonal embedding of $k$-vector spaces

$$(\Sigma_C)^n := \Sigma_C \oplus \overset{n\text{ times}}{\cdots\cdots} \oplus \Sigma_C \hookrightarrow \prod_{x\in C}{}' K^n_x,$$

according to the statements of [Tate 1968], we have

$$(\mathbb{A}^n_C)^+ \cap (\Sigma_C)^n \simeq H^0(C, \mathcal{O}_C) \oplus \overset{n\text{ times}}{\cdots\cdots} \oplus H^0(C, \mathcal{O}_C)$$
$$\mathbb{A}^n_C \big/ [(\mathbb{A}^n_C)^+ + (\Sigma_C)^n] \simeq H^1(C, \mathcal{O}_C) \oplus \overset{n\text{ times}}{\cdots\cdots} \oplus H^1(C, \mathcal{O}_C).$$

Thus, when $C$ is a complete curve, $(\mathbb{A}^n_C)^+ \cap (\Sigma_C)^n \sim (0)$ and $(\mathbb{A}^n_C)^+ + (\Sigma_C)^n \sim \mathbb{A}^n_C$. Bearing in mind the properties of the commutator $\{\ ,\ \}^V_{V_+}$, for commuting matrices $\sigma, \tau \in G^{\mathbb{A}^n_C}_{(\mathbb{A}^n_C)^+} \cap G^{\mathbb{A}^n_C}_{(\Sigma_C)^n}$, one can see that

$$\widetilde{\{\sigma, \tau\}}^{\mathbb{A}^n_C}_{[(\mathbb{A}^n_C)^+]\cap(\Sigma_C)^n} = \widetilde{\{\sigma, \tau\}}^{\mathbb{A}^n_C}_{[(\mathbb{A}^n_C)^+]+(\Sigma_C)^n} = 1\,.$$

In particular, these equalities hold when $\sigma, \tau \in \mathrm{GL}(n, \Sigma_C)$.

Furthermore, since $(\Sigma_C)^n \subset \mathbb{A}^n_C$ is invariant under the action of $\mathrm{GL}(n, \Sigma_C)$, we also have

$$\widetilde{\{\sigma, \tau\}}^{\mathbb{A}^n_C}_{(\Sigma_C)^n} = 1,$$

for all commuting matrices $\sigma, \tau \in \mathrm{GL}(n, \Sigma_C)$.

**Lemma 5.4.** *If $C$ is a nonsingular, irreducible and complete curve over a perfect field $k$, then*

$$\widetilde{\{\sigma, \tau\}}^{\mathbb{A}^n_C}_{(\mathbb{A}^n_C)^+} = 1,$$

*for all commuting matrices* $\sigma, \tau \in \mathrm{GL}(n, \Sigma_C)$.

*Proof.* Since

$$i_k(\sigma, (\Sigma_C)^n) = i_k(\sigma, (\mathbb{A}^n_C)^+ \cap (\Sigma_C)^n) = i_k(\sigma, (\mathbb{A}^n_C)^+ + (\Sigma_C)^n) = 0$$

for every matrix $\sigma \in \mathrm{GL}(n, \Sigma_C)$, the claim follows from the above equalities of commutators and Lemma 5.3. □

**Lemma 5.5.** *Let $C$ be a nonsingular and irreducible curve over a perfect field $k$ and let $T \subset C$ be a subset of closed points. If*

$$\mathbb{A}^n_T = \prod_{y\in T}{}' K^n_y, \quad (\mathbb{A}^+_T)^n = \prod_{y\in T} A^n_y,$$

*and $\tau = (\tau_{ij}), \sigma = (\sigma_{ij}) \in \mathrm{GL}(n, \Sigma_C)$ are commuting matrices such that $v_y(\tau_{ij}) = v_y(\sigma_{ij}) = v_y(\det\tau) = v_y(\det\sigma) = 0$ for all $y \in T$, from the natural immersions $\mathrm{GL}(n, \Sigma_C) \hookrightarrow \mathrm{GL}(n, \mathbb{A}_T) \subseteq G^{\mathbb{A}^n_T}_{(\mathbb{A}^+_T)^n}$ one has*

- $i_k(\tau, (\mathbb{A}^+_T)^n) = i_k(\sigma, (\mathbb{A}^+_T)^n) = 0,$
- $\{\tau, \sigma\}^{\mathbb{A}^n_T}_{(\mathbb{A}^+_T)^n} = 1.$

*Proof.* Since $\tau, \sigma \in \mathrm{GL}(n, A_y)$ for all $y \in T$, then $\tau, \sigma \in \mathrm{GL}(n, \mathbb{A}^+_T)$ and $\tau[(\mathbb{A}^+_T)^n] = \sigma[(\mathbb{A}^+_T)^n] = (\mathbb{A}^+_T)^n$, whence the claim can be deduced. □

**Lemma 5.6.** *Consider again a nonsingular and irreducible curve $C$ over a perfect field $k$, and two commuting matrices $\tau = (\tau_{ij}), \sigma = (\sigma_{ij}) \in \mathrm{GL}(n, \Sigma_C)$. If $X = \{x_1, \dots, x_s\} \subset C$ is a finite set of closed points of $C$, one has*

$$\widetilde{\{\tau, \sigma\}}^{K^n_{x_1}\oplus\cdots\oplus K^n_{x_s}}_{A^n_{x_1}\oplus\cdots\oplus A^n_{x_s}} = (-1)^{\sum_{i\neq j}\deg(x_i)\cdot\deg(x_j)\cdot v_{x_i}(\det\tau)\cdot v_{x_j}(\det\sigma)} \cdot \prod_{h=1}^{s} \{\tau, \sigma\}^{K^n_{x_h}}_{A^n_{x_h}, N_{k(x)/k}},$$

*where* $\widetilde{\{\ ,\ \}}^{K^n_{x_1}\oplus\cdots\oplus K^n_{x_s}}_{A^n_{x_1}\oplus\cdots\oplus A^n_{x_s}}$ *is the commutator of the central extension* (5-3).

*Proof.* The statement is a direct consequence of Lemma 5.1, Lemma 5.2 and the expression (5-2). □

**Proposition 5.7.** *If $C$ is a complete, nonsingular and irreducible curve over a perfect field $k$, given two commuting matrices $\tau, \sigma \in \mathrm{GL}(n, \Sigma_C)$, one has*

$$\prod_{x\in C} \{\tau, \sigma\}^{K^n_{x_h}}_{A^n_{x_h}, N_{k(x)/k}} = (-1)^{\sum_{x\in C}\deg(x)\cdot v_x(\det\tau)\cdot v_x(\det\sigma)},$$

*almost all terms of the product being* 1.

*Proof.* If we consider a finite set of closed points $X = \{x_1, \dots, x_s\} \subset C$ containing all zeros and poles of $\tau_{ij}$, $\sigma_{ij}$, $\det\tau$ and $\det\sigma$ and $T = C - X$, bearing in mind that $T$ satisfies the conditions of Lemma 5.5 and that

$$\mathbb{A}_C^n \simeq K_{x_1}^n \oplus \cdots \oplus K_{x_s}^n \oplus \mathbb{A}_T^n, \quad (\mathbb{A}_C^+)^n \simeq A_{x_1}^n \oplus \cdots \oplus A_{x_s}^n \oplus (\mathbb{A}_T^+)^n,$$

the claim can be deduced immediately from Lemmas 5.2, 5.4, 5.5, 5.6 and a well-known property of complete curves over a perfect field:

$$\sum_{x\in C} \deg(x)\cdot v_x(f) = 0 \text{ for all } f \in \Sigma_C^*. \qquad \square$$

**Theorem 5.8** (Reciprocity law). *If $C$ is a complete, nonsingular and irreducible curve over a perfect field $k$, $G$ is an abelian group and $\psi\colon k^* \to G$ is a morphism of groups, given two commuting matrices $\tau, \sigma \in \mathrm{GL}(n, \Sigma_C)$ one has*

$$\prod_{x\in C} (\tau, \sigma)_{x,\psi}^n = 1,$$

*where $(\ ,\ )_{x,\psi}^n$ is the tame symbol on $\mathrm{GL}(n, \Sigma_C)$ associated with the closed point $x$ and the morphism $\psi$.*

*Proof.* This is a direct consequence of the definition of the symbol $(\ ,\ )_{x,\psi}^n$ (see Definition 4.16 and Example 4.17) and the statement of Proposition 5.7. $\square$

**Corollary 5.9.** *With the hypothesis of Theorem 5.8, if*

$$(\ ,\ )_{v,\psi}^\infty\colon \Sigma_C^* \times \mathrm{GL}(\Sigma_C) \to G$$

*is the morphism referred to in Remark 4.20, the*

$$\prod_{x\in C} (f, \tau)_{v,\psi}^\infty = 1 \quad \textit{for all } f \in \Sigma_C^* \textit{ and } \tau \in \mathrm{GL}(\Sigma_C).$$

**Remark 5.10.** Theorem 5.8 gives reciprocity laws for the symbols constructed in Example 4.11, Example 4.17 and Example 4.18 (the Hilbert norm residue symbol on $\mathrm{GL}(n, \Sigma_C)$). There remains the task of obtaining a reciprocity law on $\mathrm{GL}(n, \mathbb{Q})$ generalizing Gauss's reciprocity law, or one on $\mathrm{GL}(n, \mathcal{F})$ generalizing the $m$-th power reciprocity theorem [Milnor 1971], $\mathcal{F}$ being a finite extension of the field of rational numbers. Note that both reciprocity laws are related to symbols that are not linear algebra objects, so it is difficult to study them using linear algebra techniques. In fact, as far as we know interpretations of the classical Gauss reciprocity law and the $m$-th power reciprocity theorem using the Tate and Arbarello–de Concini–Kac method have not previously been stated explicitly in the literature. However, the important arithmetical applications that will be the hypothetical generalizations of both reciprocity laws imply that we believe that it is worth trying to obtain an answer to this question.

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