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TWO REMARKS ON A THEOREM OF DIPENDRA PRASAD

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We show two results on local theta correspondence and restrictions of irreducible admissible representations of GL(2) over p-adic fields. Let F be a nonarchimedean local field of characteristic 0, and let L be a quadratic extension of F. Let $\epsilon_{L/F}$ is the character of $F^\times$ corresponding to the extension L/F, and let $\mathrm{GL}_2(F)^+$ be the subgroup of $\mathrm{GL}_2(F)$ consisting of elements with $\epsilon_{L/F}(\det g) = 1$. The first result is that the theorem of Moen–Rogawski on the theta correspondence for the dual pair $(U(1), U(1))$ is equivalent to a result by D. Prasad on the restriction to $\mathrm{GL}_2(F)^+$ of the principal series representation of $\mathrm{GL}_2(F)$ associated with 1, $\epsilon_{L/F}$. As the second result, we show that we can deduce from this a theorem of D. Prasad on the restrictions to $\mathrm{GL}_2(F)^+$ of irreducible supercuspidal representations of $\mathrm{GL}_2(F)$ associated to characters of $L^\times$.

1. Introduction

The purpose of this paper is to give two remarks on the comment in the last Remark in Section 3 of [Prasad 2007] and Theorem 1.2 in [Prasad 1994].

Let F be a nonarchimedean local field of characteristic 0, and let L be an quadratic extension of F. We denote by $\epsilon_{L/F}$ the quadratic character of $F^\times$ corresponding to the extension L/F.

Let $\mathrm{Ps}(1, \epsilon_{L/F})$ be the normalized principal series representation of $\mathrm{GL}_2(F)$ associated to the characters 1 and $\epsilon_{L/F}$. We fix an embedding of $L^\times$ into $\mathrm{GL}_2(F)$. The restriction of $\mathrm{Ps}(1, \epsilon_{L/F})$ to $L^\times$ is a multiplicity-free direct sum. Let $\mathrm{GL}_2(F)^+$ be the subgroup of $\mathrm{GL}_2(F)$ consisting of elements with determinant belonging to $N_{L/F}(L^\times)$. Then $L^\times$ is contained in $\mathrm{GL}_2(F)^+$, and the restriction of $\mathrm{Ps}(1, \epsilon_{L/F})$ to $\mathrm{GL}_2(F)^+$ decomposes into two irreducible subspaces $\mathrm{Ps}^\pm(1, \epsilon_{L/F})$. In this situation, Lemma 4 in [Prasad 2007] states that a character ϕ of $L^\times$, whose restriction to $F^\times$ is $\epsilon_{L/F}$, appears in $\mathrm{Ps}^+(1, \epsilon_{L/F})$ (resp. $\mathrm{Ps}^-(1, \epsilon_{L/F})$) if and only if $\varepsilon(\phi, \psi_0) = 1$ (resp. -1). Here ψ_0 is a character of L, the precise definition of which will be given in Section 3. On the other hand, we fix a character χ of $L^\times$ whose restriction to

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$F^\times$ is $\epsilon_{L/F}$, and consider the theta correspondence for the dual pair $(U(1), U(1))$ with respect to χ. Then the theorem of Moen–Rogawski states that a character η of L^1 appears in this theta correspondence if and only if $\varepsilon(\chi\eta_L^{-1}, \psi_0) = 1$ (see [Moen 1987; Rogawski 1992]). Here η_L is the character of $L^\times$ given by

$$\eta_L(x) = \eta(x/\bar{x})$$

for $x \in L^\times$. Now the correspondence $\eta \mapsto \chi\eta_L^{-1}$ yields a one to one correspondence between characters of L^1 and characters of $L^\times$ whose restriction to $F^\times$ is $\epsilon_{L/F}$. Thus the factor $\varepsilon(\phi, \psi_0)$ appears in formulas expressing characters of linear and nonlinear groups. The Remark in Section 3 of [Prasad 2007] raises the question whether there is a natural explanation for this phenomenon. Our first remark is an answer to this question. Our result is that Lemma 4 in Prasad's article is equivalent to the theorem of Moen–Rogawski. We show this in Sections 3 and 4 using seesaw diagrams after some preparations on seesaw diagrams in Section 2. We note that both the theorem of Moen–Rogawski and Prasad's Lemma 4 were originally proved by local methods for F with odd residual characteristic, and the general cases were proved by these local results and global methods (see [Moen 1987], Proposition 3.4 of [Rogawski 1992], and Lemma 4 of [Prasad 2007]). Later a purely local proof for the theorem of Moen–Rogawski was given by Harris, Kudla and Sweet (see Corollaries 8.5 and A.9 of [Harris et al. 1996]), and that of Lemma 4 of [Prasad 2007] was given by the author (see Appendix of [Prasad 2007]).

The second remark is concerned with Theorem 1.2 in [Prasad 1994]. Let π be the irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ associated to a character λ of $L^\times$ by theta correspondence. Then $\pi|_{L^\times}$ is multiplicity-free, and $\pi|_{\mathrm{GL}_2(F)^+}$ decomposes into two irreducible subspaces π^+ and π^-. In the article in question, D. Prasad proved that ϕ with $\lambda\phi^{-1}|_{F^\times} = \epsilon_{L/F}$ appears in $\pi^\pm$ if and only if $\varepsilon(\lambda\phi^{-1}, \psi_0) = \varepsilon(\bar{\lambda}\phi^{-1}, \psi_0) = \pm 1$. In Section 3 we deduce an analogue of this theorem for unitary groups of degree 2 (Theorem 3.5) from the theorem of Moen–Rogawski using a seesaw diagram. In Section 4 we show the above theorem of D. Prasad from this again using a seesaw diagram, which is found in [Harris 1993]. This is the first half of Theorem 1.2 in [Prasad 1994]. In Section 5, we treat a similar problem for representations of multiplicative group of the division quaternion algebra. This is the second half of Theorem 1.2 in [Prasad 1994].

2. Seesaw diagrams

In this section, we introduce notation and recall some seesaw diagrams which will be used in later sections. Let F, L and $\epsilon_{L/F}$ be as before, and fix a nontrivial additive character ψ of F. For $\alpha \in L$, we denote by $\bar{\alpha}$ its conjugate over F. We fix $\delta \in L^\times$ such that $\bar{\delta} = -\delta$ and $n_0 \in F^\times$ not contained in $N_{L/F}(L^\times)$.

For a finite-dimensional L-space W equipped with hermitian or antihermitian form, we denote by $U(W)$ its unitary group and by $\mathrm{GU}(W)$ its unitary similitude group. For a vector space $\mathbb{W}$ with symplectic form, we denote by $\mathrm{Sp}(\mathbb{W})$ its symplectic group and by $\mathrm{GSp}(\mathbb{W})$ its symplectic similitude group. We denote by $\mathrm{Mp}(\mathbb{W})$ the metaplectic group of $\mathbb{W}$. Let V' be a finite-dimensional right F-space with symmetric bilinear form $\langle v, v'\rangle_F$ for $v, v' \in V'$. We denote by $\mathrm{SO}(V')$, $O(V')$, and $\mathrm{GO}(V')$ the special orthogonal group, the orthogonal group, and the orthogonal similitude group of V' respectively. We denote by $\mathrm{GO}^+(V')$ the group of proper similitudes of V'.

Let V be a finite-dimensional right L-space with hermitian form satisfying

$$\langle v_1\alpha, v_2\beta\rangle = \bar{\alpha}\langle v_1, v_2\rangle\beta, \quad v_1, v_2 \in V$$

and let W be a left L-space with antihermitian form satisfying

$$\langle \alpha w_1, \beta w_2\rangle = \alpha\langle w_1, w_2\rangle\bar{\beta}, \quad w_1, w_2 \in W$$

for $\alpha, \beta \in L$. Then on $\mathbb{W} = V \otimes_L W$, we can define a symplectic form by

$$\langle\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle\rangle = \tfrac{1}{2}\,\mathrm{tr}_{L/F}\big(\langle v_1, v_2\rangle\overline{\langle w_1, w_2\rangle}\big).$$

For W, V, we have a dual reductive pair $(U(W), U(V))$ in $\mathrm{Sp}(\mathbb{W})$. We denote the natural embeddings by

$$\iota_V : U(W) \to \mathrm{Sp}(\mathbb{W}),$$
$$\iota_W : U(V) \to \mathrm{Sp}(\mathbb{W}).$$

Assume W is a direct sum of two antihermitian spaces W_1, W_2 for L/F, and set $\mathbb{W}_i = V \otimes W_i$ for $i = 1, 2$. Similarly as above, we have dual pairs $(U(W_1), U(V))$ in $\mathrm{Sp}(\mathbb{W}_1)$ and $(U(W_2), U(V))$ in $\mathrm{Sp}(\mathbb{W}_2)$, and the embeddings

$$\iota_{V,1} : U(W_1) \to \mathrm{Sp}(\mathbb{W}_1),$$
$$\iota_{W_1} : U(V) \to \mathrm{Sp}(\mathbb{W}_1),$$
$$\iota_{V,2} : U(W_2) \to \mathrm{Sp}(\mathbb{W}_2),$$
$$\iota_{W_2} : U(V) \to \mathrm{Sp}(\mathbb{W}_2).$$

These dual pairs yield the seesaw diagram

$$\begin{array}{ccc} U(W) & & U(V) \times \mathrm{Sp}(\mathbb{W}_2) \\ \Big| & \times & \Big| \\ U(W_1) \times \{1\} & & U(V) \end{array} \tag{2-1}$$

The right vertical line is the map

$$\iota_{W_1} \times \iota_{W_2} : U(V) \to U(V) \times \mathrm{Sp}(\mathbb{W}_2) \subset \mathrm{Sp}(\mathbb{W}_1) \times \mathrm{Sp}(\mathbb{W}_2).$$

We recall one more seesaw diagram from [Harris 1993]. Let W' be a finite-dimensional left F-space with symplectic form $\langle\ ,\ \rangle_F$. We can define an antihermitian form on $W'_L = L \otimes_F W'$ by

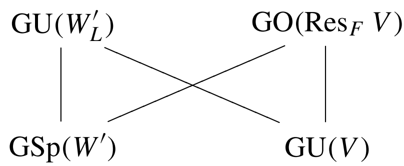
$$\left\langle \sum_i \alpha_i \otimes v_i, \sum_j \beta_j \otimes v'_j \right\rangle = \sum_{i,j} \alpha_i \bar{\beta}_j \langle v_i, v'_j \rangle_F$$

for α_i, $\beta_j \in L$, and $v_i.\ v_j \in V'$. Conversely, let V be a right L-space with hermitian form $\langle\ ,\ \rangle$. Then composing the hermitian form with $\mathrm{tr}_{L/F}$, we can define a symmetric bilinear form

$$\tfrac{1}{2}\,\mathrm{tr}_{L/F}(\langle v, v' \rangle)$$

on $\mathrm{Res}_F\, V$, the space V considered as an F-space. In this notation we have, from [Harris 1993, (3.5.1.1)],

(2-2)

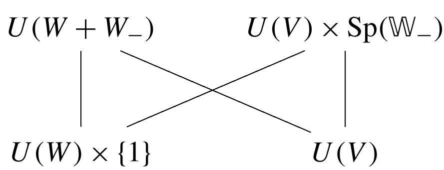


3. Application of the theorem of Moen–Rogawski

In this section, using the diagram (2-1) and the theorem of Moen–Rogawski, we deduce an analogue of Theorem 1.2 in [Prasad 1994] for unitary groups of degree 2.

For $\alpha \in L^\times$ with $\bar{\alpha} = -\alpha$, we denote by $W(\alpha)$ the 1 dimensional left L-space L with antihermitian form $\langle x, y \rangle = \alpha x \bar{y}$ for $x, y \in L$. For $\alpha,\ \beta \in L^\times$, we set $W(\alpha, \beta) = W(\alpha) \oplus W(\beta)$. For $a \in F^\times$, we denote by $V(a)$ the 1 dimensional right L-space L with hermitian form $\langle x, y \rangle = a\bar{x}y$.

We set $W = W(\delta)$, $W_- = W(-\delta)$, and $V = V(1)$, or $W = W(n_0\delta)$, $W_- = W(-n_0\delta)$, and $V = V(1)$. Set $\mathbb{W} = V \otimes_L W$, and $\mathbb{W}_- = V \otimes_L W_-$. Then we have a seesaw diagram of type (2-1):



We recall the splittings of the above unitary groups into metaplectic groups, following Section 1 of [Harris et al. 1996]. We fix a character χ of $L^\times$ whose restriction to $F^\times$ is $\epsilon_{L/F}$. Let X be the graph of minus the identity from W to W_-, and let Y be the graph of the identity. Then $V \otimes_L X$ and $V \otimes_L Y$ are maximal isotropic subspace of $\mathbb{W}$, and $\mathbb{W} = V \otimes_L X + V \otimes_L Y$ yields a complete polarization of $\mathbb{W}$. This determines an isomorphism

$$\mathrm{Mp}(\mathbb{W}+\mathbb{W}_-) \simeq \mathrm{Sp}(\mathbb{W}+\mathbb{W}_-) \rtimes \mathbb{C}^1,$$

where the product in $\mathrm{Sp}(\mathbb{W}+\mathbb{W}_-) \rtimes \mathbb{C}^1$ is given by the Rao cocycle [1993]. The inverse image in $\mathrm{Mp}(\mathbb{W}+\mathbb{W}_-)$ of $\mathrm{Sp}(\mathbb{W}) \times \{1\}$ or $\{1\} \times \mathrm{Sp}(\mathbb{W})$ is isomorphic to $\mathrm{Mp}(\mathbb{W})$. By (1.21) of [Harris et al. 1996], we have splittings $\tilde{\iota}_{V,\chi}$, $\tilde{\iota}_{V,\chi} \times \tilde{\iota}_{V,\chi,-}$ satisfying

$$\begin{array}{ccc} U(W+W_-) & \xrightarrow{\tilde{\iota}_{V,\chi}} & \mathrm{Mp}(\mathbb{W}+\mathbb{W}_-) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \tilde{i}} \\ U(W)\times U(W) & \xrightarrow{\tilde{\iota}_{V,\chi}\times\tilde{\iota}_{V,\chi,-}} & \mathrm{Mp}(\mathbb{W})\times \mathrm{Mp}(\mathbb{W}). \end{array}$$

Here we note that $U(W_-) = U(W)$, $\mathrm{Mp}(\mathbb{W}) = \mathrm{Mp}(\mathbb{W}_-)$ and the splitting

$$\tilde{i}:\ \mathrm{Mp}(\mathbb{W}) \times \mathrm{Mp}(\mathbb{W}) \to \mathrm{Mp}(\mathbb{W}+\mathbb{W}_-)$$

of the embedding

$$i:\ \mathrm{Sp}(\mathbb{W}) \times \mathrm{Sp}(\mathbb{W}) \to \mathrm{Sp}(\mathbb{W}+\mathbb{W}_-)$$

is specified so that the restriction to central $\mathbb{C}^1$ is given by

$$\mathbb{C}^1 \times \mathbb{C}^1 \to \mathbb{C}^1, \quad (c_1, c_2) \to c_1\bar{c}_2.$$

Then, by [Harris et al. 1996, Lemma 1.1],

$$\tilde{\iota}_{V,\chi,-} = \chi^{-1}\tilde{\iota}_{V,\chi}. \tag{3-1}$$

In this case, $U(V)$ is the center of $U(W+W_-)$, and the splitting of $U(V)$ as the center of $U(W+W_-)$ by χ coincides with the splitting ι_{W+W_-,χ^2} (Corollary A.8 of the same reference).

Let $(\omega_\psi, \mathcal{S}(V \otimes_L X))$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}+\mathbb{W}_-)$ realized on the space of Schwartz–Bruhat functions on $V \otimes_F X$ as the Schrödinger model associated to the complete polarization $\mathbb{W} = V \otimes_L X + V \otimes_L Y$. For a character λ^1 of $U(V)$, let $\theta_\chi(\lambda^1, W+W_-)$ be the theta correspondence of λ^1 to $U(W+W_-)$. Namely, let $S_{V,W,\chi}(\lambda^1)$ be the maximal quotient of $\mathcal{S}(V \otimes_L X)$ on which $U(V)$ acts as multiple of λ^1. Then

$$S_{V,W,\chi}(\lambda^1) \simeq \theta_\chi(\lambda^1, W+W_-) \boxtimes \lambda^1,$$

as $U(W+W_-) \times U(V)$-spaces with an $U(W+W_-)$-module $\theta_\chi(\lambda^1, W+W_-)$.

Let $\omega_{\psi.\mathbb{W}}$ be the Weil representation of $\mathrm{Mp}(\mathbb{W})$. Let ψ_0 be the additive character of L given by $\psi_0(x) = \psi(\frac{1}{2}\operatorname{tr}_{L/F}(-\delta x))$ for $x \in L$. For a character η of L^1, we denote by η_L the character of $L^\times$ given by $\eta_L(x) = \eta(x/\bar{x})$.

Theorem 3.1 (Moen and Rogawski). *Let*

$$\epsilon = \begin{cases} 1 & \text{if } W = W(\delta), \\ -1 & \text{if } W = W(n_0\delta). \end{cases}$$

Then

$$\omega_{\psi,\mathbb{W}} \circ \tilde{\iota}_{V,\chi}|_{U(W)} = \bigoplus_{\varepsilon(\chi\eta_L^{-1},\psi_0)=\epsilon} \mathbb{C}\eta.$$

Remark 3.2. Here we use the character ψ_0 instead of $\psi \circ \mathrm{tr}_{L/F}$. This simplifies some expressions (see Remark in Introduction of [Prasad 1994]).

For a character η of $U(W)$, we denote by $\theta_\chi(\eta, V)$ the theta correspondence of η in $\mathrm{Mp}(\mathbb{W})$ to $U(V)$. Then $\theta_\chi(\eta, V) = \eta^{-1}$ if η appears in the theta correspondence. We note that $U(V) \simeq U(W) \simeq L^1$, and the embedding ι_V and ι_W are chosen so that the actions of $U(V)$ and $U(W)$ on $\mathbb{W}$ are the inverse of each other.

By the isomorphism $U(V) \simeq L^1$, we consider the restriction of χ to L^1 as a character of $U(V)$ and denote it also by χ.

Lemma 3.3. *Let the notation be as above. Let $U(W) \times \{1\}$ be the subgroup of $U(W) \times U(W) (\subset U(W + W_-))$ consisting of elements with unit in the second component. Then*

$$\dim \mathrm{Hom}_{U(W)\times\{1\}}\big(\theta_\chi(\chi^{-1}\lambda^1, W + W_-), \eta \boxtimes 1\big) = \begin{cases} 1 & \text{if } \eta \text{ and } \lambda^1\eta \text{ appear in } \omega_{\psi,\mathbb{W}} \circ \tilde{\iota}_{V,\chi}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned} &\mathrm{Hom}_{(U(W)\times\{1\})\times U(V)}(\omega_\psi, (\eta \boxtimes 1) \boxtimes \chi^{-1}\lambda^1) \\ &\quad\simeq \mathrm{Hom}_{(U(W)\times\{1\})\times U(V)}(\theta_\chi(\chi^{-1}\lambda^1, W + W_-) \boxtimes \chi^{-1}\lambda^1, (\eta \boxtimes 1) \boxtimes \chi^{-1}\lambda^1) \\ &\quad\simeq \mathrm{Hom}_{U(W)\times\{1\}}(\theta_\chi(\chi^{-1}\lambda^1, W + W_-), \eta \boxtimes 1). \end{aligned}$$

We note that $U(V)$ is embedded into $U(W) \times U(W)$ diagonally in $\mathrm{Sp}(\mathbb{W} + \mathbb{W}_-)$, and the action of $\alpha \in U(V)$ for $\alpha \in L^1$ on $\mathcal{S}(V \otimes_L X)$ is given by that of $(\alpha^{-1}, \alpha^{-1}) \in U(W) \times U(W)$. By [Mœglin et al. 1987, II.1, Remarques (5), (6)] and [Harris et al. 1996, Lemma 2.1(i)], the restriction of ω_ψ to $\mathrm{Mp}(\mathbb{W}) \times \mathrm{Mp}(\mathbb{W})$ is $\omega_{\psi,\mathbb{W}} \boxtimes \omega_{\psi,\mathbb{W}}^\vee$. Here $\omega_{\psi,\mathbb{W}}^\vee$ is the contragredient of $\omega_{\psi,\mathbb{W}}$, and by (3-1) we obtain

$$\omega_{\psi,\mathbb{W}}^\vee \circ \tilde{\iota}_{V,\chi,-} = \chi(\omega_{\psi,\mathbb{W}} \circ \tilde{\iota}_{V,\chi})^\vee.$$

Hence

$$\begin{aligned} &\mathrm{Hom}_{(U(W)\times\{1\})\times U(V)}(\omega_\psi, (\eta \boxtimes 1) \boxtimes \chi^{-1}\lambda^1) \\ &\quad\simeq \mathrm{Hom}_{U(W)\times U(V)}\big(\eta \boxtimes (\theta_\chi(\eta, V) \otimes \omega_{\psi,\mathbb{W}}^\vee \circ \tilde{\iota}_{V,\chi,-}), \eta \boxtimes \chi^{-1}\lambda^1\big) \\ &\quad\simeq \mathrm{Hom}_{U(V)}(\theta_\chi(\eta, V) \otimes (\chi(\omega_{\psi,\mathbb{W}} \circ \tilde{\iota}_{V,\chi,})^\vee), \chi^{-1}\lambda^1\big). \end{aligned}$$

Our assertion follows from this. □

Taking λ^1 to be the trivial character of L^1, by Lemma 3.3 and Theorem 3.1, we obtain:

Theorem 3.4. *Let ϵ be as above. Then*

$$\theta_\chi(\chi^{-1}, W+W_-)|_{U(W)\times\{1\}} = \bigoplus_{\varepsilon(\chi\eta_L^{-1},\psi_0)=\epsilon} \mathbb{C}\eta\boxtimes 1.$$

Theorem 3.5. *Let λ^1 be a nontrivial character of L^1, and let ϵ be as above. Then*

$$\theta_\chi(\chi^{-1}\lambda^1, W+W_-)|_{U(W)\times\{1\}} = \bigoplus_{\substack{\varepsilon(\chi(\lambda_L^1\eta_L)^{-1},\psi_0)=\\ \varepsilon(\chi\eta_L^{-1},\psi_0)=\epsilon}} \mathbb{C}\eta\boxtimes 1.$$

4. Prasad's Theorem

We rewrite the results in the previous section in terms of $\mathrm{GU}(2)$ and a torus T_L in $\mathrm{GU}(2)$ isomorphic to $L^\times$, and by restricting it to a subgroup of index 2 of $\mathrm{GL}_2(F)$, we deduce the theorem of D. Prasad using a seesaw diagram of type (2-2).

Let $W'=F^2$ be the two-dimensional left F-space with symplectic form

$$\langle v_1, v_2\rangle = x_1y_2 - y_1x_2$$

for $v_1=(x_1,y_1)$, $v_2=(x_2,y_2)\in W'$, and let $W'_L=L^2$ be the two-dimensional left L-space with antihermitian from

$$\langle \tilde v_1, \tilde v_2\rangle_H = x_1\bar y_2 - y_1\bar x_2$$

for $\tilde v_1=(x_1,y_1)$, $\tilde v_2=(x_2,y_2)\in W'_L$. Then we see $W(\delta,-\delta)\simeq W'_L$, as spaces with antihermitian forms. More explicitly, let

$$h=\begin{pmatrix}\delta & 1/2\\ -\delta & 1/2\end{pmatrix}.$$

Then

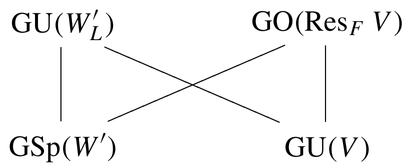
$$h\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}{}^t\bar h = \begin{pmatrix}\delta & 0\\ 0 & -\delta\end{pmatrix}.$$

If we take $n_0\langle v_1,v_2\rangle$ instead of $\langle v_1,v_2\rangle$, we get $W(n_0\delta,-n_0\delta)\simeq W'_L$. Similarly, we have

$$h\begin{pmatrix}0 & n_0\\ -n_0 & 0\end{pmatrix}{}^t\bar h = \begin{pmatrix}n_0\delta & 0\\ 0 & -n_0\delta\end{pmatrix}.$$

Let $\mathrm{Res}_F V$ be the two-dimensional right F-space with symmetric bilinear form associated with $V(1)$. For these spaces, we have the following diagram of type

(2-2):



Note that W' and V satisfy

$$\mathrm{GSp}(W')=\mathrm{GL}(W'),\quad \mathrm{Sp}(W')=\mathrm{SL}(W'),\quad U(W'_L)\supset SU(W'_L)=\mathrm{SL}(W'),$$
$$\mathrm{SO}(\mathrm{Res}_F\,V)=U(V),\quad \mathrm{GO}^+(\mathrm{Res}_F\,V)=\mathrm{GU}(V).$$

Let $\nu_W(g)$ be the similitude of $g\in U(W'_L)$. Let

$$\mathrm{GU}(W'_L)^+=\{g\in \mathrm{GU}(W'_L)\mid \epsilon_{L/F}(\nu_W(g))=1\},$$
$$\mathrm{GL}(W')^+=\{g\in \mathrm{GL}(W')\mid \epsilon_{L/F}(\det g)=1\},$$

and identify $L^\times$ with the center of $\mathrm{GU}(W_L)$. Then

$$\mathrm{GU}(W'_L)\supset \mathrm{GU}(W'_L)^+=L^\times U(W'_L)=L^\times\,\mathrm{GL}(W')^+,$$

since $N_{L/F}(L^\times)L^{\times 2}=L^1L^{\times 2}$.

Let T_L be the torus in $\mathrm{GL}(W')$ isomorphic to $L^\times$ given by

$$\left\{\begin{pmatrix} a & 2^{-1}b\\ 2\delta^2 b & a\end{pmatrix}\;\middle|\; {}^t(a,b)\in F^2\setminus {}^t(0,0)\right\},$$

and let

$$\alpha=a+b\delta\in L,\qquad \mu=\alpha/\bar\alpha.$$

We fix the isomorphism

$$\alpha\mapsto \begin{pmatrix} a & 2^{-1}b\\ 2\delta^2 b & a\end{pmatrix}$$

and identify T_L with $L^\times$. We have

$$\text{(4-1)}\qquad \begin{pmatrix} a & 2^{-1}b\\ 2\delta^2 b & a\end{pmatrix}=\begin{pmatrix} \bar\alpha & 0\\ 0 & \bar\alpha\end{pmatrix}\frac{1}{2}\begin{pmatrix} \mu+1 & (2\delta)^{-1}(\mu-1)\\ (2\delta)(\mu-1) & \mu+1\end{pmatrix}.$$

We note

$$\frac{1}{2}\begin{pmatrix} \mu+1 & (2\delta)^{-1}(\mu-1)\\ 2\delta(\mu-1) & \mu+1\end{pmatrix}=h^{-1}\begin{pmatrix} \mu & 0\\ 0 & 1\end{pmatrix}h.$$

We recall the action of some elements on $\mathcal{S}(V\otimes_L X)$. We write them for the pair $(U(W'_L),U(V))$. Then $X=\{(x,0)\mid x\in L\}$, $Y=\{(0,y)\mid y\in L\}$, and

$$\begin{pmatrix} \alpha & 0\\ 0 & \bar\alpha^{-1}\end{pmatrix}\in U(W'_L)$$

acts on X by α and on Y by $\bar{\alpha}^{-1}$. Hence

$$\beta_V\begin{pmatrix}\alpha & 0\\ 0 & \bar{\alpha}^{-1}\end{pmatrix} = \chi(\bar{\alpha}^{-1}) = \chi(\alpha)$$

in the notation of Theorem 3.1 of [Kudla 1994]. By the same theorem we have, for $\alpha \in L^\times$,

$$\omega_\psi\left(\tilde{\iota}_{V,\chi}\begin{pmatrix}\alpha & 0\\ 0 & \bar{\alpha}^{-1}\end{pmatrix}\right) f(x) = \chi(\alpha)|\alpha|_L^{1/2} f(\alpha x).$$

In particular, for $\alpha \in L^1$,

$$\omega_\psi\left(\tilde{\iota}_{V,\chi}\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right) f(x) = \chi(\alpha) f(\alpha x). \tag{4-2}$$

For the dual pair $(\mathrm{SL}(W'), \mathrm{SO}(\mathrm{Res}_F V))$, let $S(\lambda^1)$ be the maximal quotient of $\mathcal{S}(V \otimes_L X) = \mathcal{S}(\mathrm{Res}_F V \otimes_F X')$, $X' = \{(x, 0) \mid x \in F\}$, on which $\mathrm{SO}(\mathrm{Res}_F V)$ acts as multiple of λ^1. Here the action of $\alpha \in \mathrm{SO}(\mathrm{Res}_F V)$ with $\alpha \in L^1$ is given by $f(x) \mapsto f(\alpha^{-1}x)$. Then the above formula implies that

$$S_{V,W,\chi}(\chi^{-1}\lambda^1) = S(\lambda^1).$$

Hence the restriction of the action of $U(W'_L)$ on the space $\theta_\chi(\chi^{-1}\lambda^1, W + W_-)$ to $\mathrm{SL}(W')$ is the theta correspondence of λ^1 to $\mathrm{SL}(W')$. We denote it by $\theta(\lambda^1, W')$.

We extend the theta correspondence θ_χ of $U(V)$ to $U(W'_L)$ to that of $\mathrm{GU}(V)$ to $\mathrm{GU}(W'_L)^+$ following [Harris 1993, 3.2]. The similitude ν_V of $\mathrm{GU}(V)$ satisfies $\nu_V(\mathrm{GU}(V)) = N_{L/F}L^\times$. Let

$$R(V, W) = \{(g, h) \in \mathrm{GU}(W'_L) \times \mathrm{GU}(V) \mid \nu_V(h) = \nu_W(g)\}.$$

Then by corresponding (g, h) to the map

$$v \otimes w \mapsto h^{-1}v \otimes wg,\ v \in V,\ w \in W'_L,$$

we can takes $R(V, W)$ into $\mathrm{Sp}((V \otimes_L W'_L)$. We consider a semidirect product $U(W'_L) \ltimes \mathrm{GU}(V)$ defined by

$$hg = {}^h g h$$

with

$${}^h g = \begin{pmatrix}1 & 0\\ 0 & \nu_V(h)\end{pmatrix} g \begin{pmatrix}1 & 0\\ 0 & \nu_V(h)\end{pmatrix}^{-1}.$$

Then we have an isomorphism

$$R(V, W) \simeq U(W'_L) \ltimes \mathrm{GU}(V)$$

given by

$$(g,h)\to\left(g\begin{pmatrix}1&0\\0&\nu_V(h)\end{pmatrix}^{-1},h\right).$$

We let $\mathrm{GU}(V)$ act on $\mathcal{S}(V\otimes_L X)$ by

$$L(h)f(x)=\chi(\alpha^{-1})|\alpha|_L^{-1/2}f(\alpha^{-1}x).$$

Then $L(h)$ defines a unitary operator on $\mathcal{S}(V\otimes_L X)$, and this action with $\omega_\psi\circ\tilde{\iota}_{V,\chi}$ defines an action of $R(V,W)$ on $\mathcal{S}(V\otimes_L X)$ and a splitting of $R(V,W)$ into $\mathrm{Mp}(V\otimes_L W'_L)$.

Let λ be a character of $\mathrm{GU}(V)$ whose restriction to $U(V)$ is λ^1. We identify λ with a character of $L^\times$ by $\mathrm{GU}(V)\simeq L^\times$. For a character λ of $L^\times$, let $\bar{\lambda}$ be the character of $L^\times$ given by $\bar{\lambda}(\alpha)=\lambda(\bar{\alpha})$ for $\alpha\in L^\times$. By the projection to the second factor $\mathrm{GU}(V)$ of $\mathrm{GU}(W'_L)\times\mathrm{GU}(V)$, we may see $\chi\bar{\lambda}$ as a character of $R(V,W)$. Define

$$(\mathcal{S}(V\otimes_L X)\otimes\chi\bar{\lambda})_{U(V)}$$

to be the maximal quotient of $\mathcal{S}(V\otimes_L X)\otimes\chi\bar{\lambda}$ on which $U(V)$ acts trivially. Then $\mathrm{GU}(W'_L)^+$ acts on this space as follows. For $g\in\mathrm{GU}(W'_L)^+$, choose $h\in\mathrm{GU}(V)$ satisfying $\nu_W(g)=\nu_V(h)$. Define the action of g as that of $(g,h)\in R(W,V)$. Then this is independent of the choice of h. As $U(W'_L)$-modules, we have

$$(\mathcal{S}(V\otimes_L X)\otimes\chi\bar{\lambda})_{U(V)}\simeq S_{V,W,\chi}(\chi^{-1}\lambda^1),$$

and on this space, $\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)\in\mathrm{GU}(W'_L)^+$ acts by $\chi\bar{\lambda}$. We denote the restriction to $\mathrm{GL}(W')^+$ of this representation by $\theta(\lambda,\mathrm{GL}(W'))^\epsilon$. Here ϵ is as in Section 3.

Let $a=\alpha\bar{\alpha}$. Then

$$\begin{pmatrix}a&0\\0&1\end{pmatrix}=\begin{pmatrix}\bar{\alpha}&0\\0&\bar{\alpha}\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\bar{\alpha}^{-1}\end{pmatrix}$$

Hence $\left(\begin{smallmatrix}\alpha&0\\0&1\end{smallmatrix}\right)$ acts on $\tilde{f}\in S(\lambda^1)$ sending it to the class in $S(\lambda^1)$ of the function

$$\chi(\bar{\alpha})\lambda(\alpha)\chi(\alpha)|\alpha|_L^{1/2}f(\alpha x)=\lambda(\alpha)|\alpha|_L^{1/2}f(\alpha x).$$

This coincides with the extension of the action of $\mathrm{SL}(W')$ to $\mathrm{GL}(W')^+$ in [Jacquet and Langlands 1970, Proposition 1.5]. For a character λ of $L^\times$, we set

$$\theta(\lambda,\mathrm{GL}(W'))=\mathrm{Ind}_{\mathrm{GL}(W')^+}^{\mathrm{GL}(W')}\theta(\lambda,\mathrm{GL}(W')^+)^+.$$

Then as $\mathrm{GL}(W')^+$-modules, we have

$$\mathrm{Res}_{\mathrm{GL}(W')^+}^{\mathrm{GL}(W')}\theta(\lambda,\mathrm{GL}(W'))=\theta(\lambda,\mathrm{GL}(W')^+)^+\oplus\theta(\lambda,\mathrm{GL}(W')^+)^-;$$

see [Mœglin et al. 1987, II.1, Remarque (3)].

Let $\mathrm{Ps}(1,\epsilon_{L/F})$ be the principal series representation of $\mathrm{GL}_2(F)$ associated with characters $(1,\epsilon_{L/F})$. Then $\theta(1,\mathrm{GL}(W'))$ is isomorphic to $\mathrm{Ps}(1,\epsilon_{L/F})$ by [Jacquet and Langlands 1970, Theorem 4.7]. We set $\mathrm{Ps}(1, \epsilon_{L/F})^{\epsilon}$ the subspace corresponding to $\theta(\lambda, \mathrm{GL}(W'))^{\epsilon}$. By setting $\phi = \chi\eta_L^{-1}$, we see that Theorem 3.4 is equivalent to the following:

Theorem 4.1 [Prasad 2007, Lemma 4]. *For* $\epsilon = \pm 1$,

$$\mathrm{Ps}(1, \omega_{L/F})^{\epsilon}|_{T_L} = \oplus_{\varepsilon(\phi,\psi_0)=\epsilon}\mathbb{C}\phi,$$

where ϕ *runs through all characters of* $L^\times$ *whose restriction to* $F^\times$ *is equal to* $\epsilon_{L/F}$.

Remark 4.2. The map $\eta \mapsto \chi^{-1}\eta_L$ induces a one to one correspondence between the set of characters of L^1 and the set of characters of $L^\times$ whose restriction to $F^\times$ is $\epsilon_{L/F}$. Therefore the theorem of Moen–Rogawski is equivalent to the preceding theorem through Theorem 3.4.

For λ such that $\lambda|_{L^1}$ is not trivial, $\theta(\lambda, \mathrm{GL}(W'))$ is an irreducible supercuspidal representation of $\mathrm{GL}(W')$ by Theorem 4.6 of [Jacquet and Langlands 1970]. In this case, Theorem 3.5 can be stated as follows:

Theorem 4.3 [Prasad 1994, Theorem 1.2]. *Under the action of* $\mathrm{GL}_2(F)^+$, *the space* $\theta(\lambda, \mathrm{GL}(W'))$ *decomposes into two subspaces* $\theta(\lambda, \mathrm{GL}(W'))^{\pm}$, *and for* $\epsilon = \pm 1$, *one has*

$$\theta(\lambda, \mathrm{GL}(W'))^{\epsilon}|_{T_L} = \bigoplus_{\substack{\varepsilon(\lambda\phi^{-1},\psi_0)= \\ \varepsilon(\bar{\lambda}\phi^{-1},\psi_0)=\epsilon}} \mathbb{C}\phi,$$

where ϕ *runs through all characters of* $L^\times$ *which satisfy* $\lambda\phi^{-1}|_{F^\times} = \epsilon_{L/F}$.

Proof. Set $\phi = \chi^{-1}\lambda\eta_L$. Since $\lambda_L^1 = \lambda\bar{\lambda}^{-1}$, we see

$$\chi(\eta_L\lambda_L^1)^{-1} = (\chi\lambda^{-1}\eta_L^{-1})\bar{\lambda} = \bar{\lambda}\phi^{-1},$$
$$\chi\eta_L^{-1} = (\chi\lambda^{-1}\eta_L^{-1})\lambda = \lambda\phi^{-1}.$$

We note $\bar{\lambda}\phi^{-1}|_{F^\times} = \lambda\phi^{-1}|_{F^\times} = \epsilon_{L/F}$. By (4-1), we can see the action of T_L by that of L^1. For $v \in \theta_\chi(\chi^{-1}\lambda^1, W + W_-)$, $U(W) \times \{1\}$ acts on v via $\eta \boxtimes 1$ if and only if T_L acts on v via $\bar{\chi}\lambda\eta_L = \chi^{-1}\lambda\eta_L$. The assertion follows from this and Theorem 3.5. □

5. Nonsplit case

We now consider the nonsplit case. Let

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ n_0\bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in L \right\}.$$

Then B is the division quaternion algebra over F. Let

$$B^+ = \{x \in B \mid \epsilon_{L/F}(N(x)) = 1\}, \quad B^1 = \{x \in B \mid N(x) = 1\}.$$

Here $N(x)$ is the reduced norm of $x \in B$. We set

$$T_L = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \middle| \alpha \in L^\times \right\}.$$

Then $T_L \simeq L^\times$. We note

$$\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \alpha/\bar{\alpha} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5-1}$$

Let $\alpha = \delta$, $\beta = -n_0\delta$, or $\alpha = n_0\delta$, $\beta = -n_0^2\delta$. Then $B^\times \subset \mathrm{GU}(W(\alpha, \beta))$, and

$$T_L \subset B^+ \subset \mathrm{GU}(W(\alpha, \beta))^+ = L^\times U(W(\alpha, \beta)) = L^\times B^+.$$

Here $\mathrm{GU}(W(\alpha, \beta))^+$ is the subgroup of $\mathrm{GU}(W(\alpha, \beta))$ consisting of elements with similitude in $N_{L/F}(L^\times)$.

We define splittings. Let $W = W(\alpha, -\beta)$. We embed $W(\alpha, \beta)$ into $W + W_-$ and consider $U(W(\alpha, \beta))$ as a subgroup of $U(W + W_-)$. Let $\mathbb{W} = V \otimes_F W$, and $\mathbb{W}_- = V \otimes_F W_-$. We may consider $\mathbb{W}(\alpha, \beta) = V \otimes_F W(\alpha, \beta)$ as a symplectic subspace of $\mathbb{W} + \mathbb{W}_-$ and $\mathrm{Sp}(\mathbb{W}(\alpha, \beta)))$ as a subgroup of $\mathrm{Sp}(\mathbb{W} + \mathbb{W}_-)$. Then we have splittings $\tilde{\iota}_{V,\chi}$, $\tilde{\iota}_{V,\chi,-}$ satisfying

$$\begin{array}{ccc} U(W + W_-) & \xrightarrow{\tilde{\iota}_{V,\chi}} & \mathrm{Mp}(\mathbb{W} + \mathbb{W}_-) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \tilde{i}} \\ U(W) \times U(W) & \xrightarrow{\tilde{\iota}_{V,\chi} \times \tilde{\iota}_{V,\chi,-}} & \mathrm{Mp}(\mathbb{W}) \times \mathrm{Mp}(\mathbb{W}). \end{array}$$

We choose the embedding of $\mathrm{Mp}(\mathbb{W}) \times \mathrm{Mp}(\mathbb{W})$ into $\mathrm{Mp}(\mathbb{W}+\mathbb{W}_-)$ so that it induces the map $(c_1, c_2) \mapsto c_1\bar{c}_2$ on the center $\mathbb{C}^1 \times \mathbb{C}^1$. Let $\mathbb{W}(\alpha) = V \otimes_L W(\alpha)$, and $\mathbb{W}(\beta) = V \otimes_L W(\beta)$. Restricting the above diagram to $\mathrm{Mp}(\mathbb{W}(\alpha, \beta))$, we obtain

$$\begin{array}{ccc} U(W(\alpha, \beta)) & \xrightarrow{\tilde{\iota}_{V,\chi}} & \mathrm{Mp}(\mathbb{W}(\alpha, \beta)) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \tilde{i}} \\ U(W(\alpha)) \times U(W(\beta)) & \xrightarrow{\tilde{\iota}_{V,\chi} \times \tilde{\iota}_{V,\chi,-}} & \mathrm{Mp}(\mathbb{W}(\alpha)) \times \mathrm{Mp}(\mathbb{W}(\beta)) \end{array}$$

Here $\mathrm{Mp}(\mathbb{W}(\alpha, \beta))$ is the inverse image of $\mathrm{Sp}(\mathbb{W}(\alpha, \beta))$ in $\mathrm{Mp}(\mathbb{W} + \mathbb{W}_-)$, and $\mathrm{Mp}(\mathbb{W}(\alpha))$ and $\mathrm{Mp}(\mathbb{W}(\beta))$ are the inverse images of $\mathrm{Sp}(\mathbb{W}(\alpha))$ and $\mathrm{Sp}(\mathbb{W}(\beta))$ in $\mathrm{Mp}(\mathbb{W})$ on the first and the second factor in the above diagram respectively. The restriction of $\tilde{\iota}_{V,\chi} : U(W(\alpha, -\beta)) \to \mathrm{Mp}(\mathbb{W})$ to $U(W(-\beta))$ induces a map

$$U(W(-\beta)) = U(W(\beta)) \xrightarrow{\tilde{\iota}_{V,\chi}} \mathrm{Mp}(\mathbb{W}(-\beta)) = \mathrm{Mp}(\mathbb{W}(\beta)).$$

Then $\tilde{\iota}_{V,\chi,-}$ and $\chi^{-1}\tilde{\iota}_{V,\chi}$ coincide as homomorphisms of $U(W(\beta))$ to $\mathrm{Mp}(\mathbb{W}(\beta))$, by [Harris et al. 1996, Lemma 1.1]. We have

$$\omega_{\psi,\mathbb{W}+\mathbb{W}} \circ \tilde{i} = \omega_{\psi,\mathbb{W}(\alpha,-\beta)} \boxtimes \omega^{\vee}_{\psi,\mathbb{W}(\alpha,-\beta)}$$

by [Mœglin et al. 1987, II.1 Remarques (5), (6)] and [Harris et al. 1996, Lemma 2.1(i)]. By restricting this to $\mathrm{Mp}(\mathbb{W}(\alpha)) \times \mathrm{Mp}(W(\beta))$, we obtain

$$\omega_{\psi,\mathbb{W}(\alpha,\beta)} \circ \tilde{i} = \omega_{\psi,\mathbb{W}(\alpha)} \boxtimes \chi\omega^{\vee}_{\psi,\mathbb{W}(-\beta)},$$

$$\begin{aligned}\omega_{\psi,\mathbb{W}(\alpha,\beta)} \circ \tilde{i} \circ (\tilde{\iota}_{V,\chi} \times \tilde{\iota}_{V,\chi,-}) &= \omega_{\psi,\mathbb{W}(\alpha)} \circ \tilde{\iota}_{V,\chi} \boxtimes \omega^{\vee}_{\psi,\mathbb{W}(-\beta)} \circ \tilde{\iota}_{V,\chi,-} \\ &= \omega_{\psi,W(\alpha)} \circ \tilde{\iota}_{V,\chi} \boxtimes \chi\omega^{\vee}_{\psi,W(-\beta)} \circ \tilde{\iota}_{V,\chi}\end{aligned}$$

As for the splitting for $U(V)$, we may take $\tilde{\iota}_{W+W_-,\chi^4}$ or that induced by $\tilde{\iota}_{V,\chi}$.

Let $\theta_\chi(\chi^{-1}\lambda^1, W(\alpha, \beta))$ be the theta correspondence of the character $\chi^{-1}\lambda^1$ of $U(V)$ to $U(W(\alpha, \beta))$ in $\mathrm{Mp}(\mathbb{W}(\alpha, \beta))$. By the same calculation as in the split case, we obtain:

Lemma 5.1. *Let $U(W(\alpha)) \times \{1\}$ be the subgroup of $U(W(\alpha)) \times U(W(\beta))$. Then*

$$\dim \mathrm{Hom}_{U(W(\alpha))\times\{1\}}\big(\theta_\chi(\chi^{-1}\lambda^1, W(\alpha, \beta)), \eta \boxtimes 1\big) = \begin{cases} 1 & \text{if } \eta \text{ appears in } \omega_{\psi,\mathbb{W}(\alpha)} \circ \tilde{\iota}_{V,\chi} \text{ and } \lambda^1\eta \text{ appears in } \omega_{\psi,\mathbb{W}(-\beta)} \circ \tilde{\iota}_{V,\chi}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\epsilon_{L/F}(-\beta/\alpha) = -1$, the trivial character does not satisfy the above condition for λ^1. In the case of a nontrivial λ^1, we have:

Theorem 5.2. *Let λ^1 be a nontrivial character of L^1, and let $\epsilon = \epsilon_{L/F}(\alpha/\delta)$. Then*

$$\theta_\chi(\chi^{-1}\lambda^1, W(\alpha, \beta))|_{U(W(\alpha))\times\{1\}} = \bigoplus_{\substack{-\varepsilon(\chi(\lambda^1_L\eta_L)^{-1}\lambda^1_L,\psi_0) = \\ \varepsilon(\chi\eta_L^{-1},\psi_0) = \epsilon}} \mathbb{C}\eta \boxtimes 1.$$

As in the split case, we can interpret this result by the dual reductive pair $(B^\times, \mathrm{GO}(V))$. In the same way as in the split case, we can define $\theta(\lambda^1, B^1)$. Let λ be a character of $L^\times$ which restriction to L^1 is λ^1. We define the action of $L^\times$, the center of $U(W(\alpha, \beta))$, on $\theta_\chi(\chi^{-1}\lambda^1, W(\alpha, \beta))$ by $\chi\bar{\lambda}$. Then this yields a well-defined smooth action of $L^\times U(W(\alpha, \beta))$ on $\theta_\chi(\chi^{-1}\lambda^1, W(\alpha, \beta))$, since $L^\times \cap U(W(\alpha, \beta)) = L^1$. By restriction, we obtain an action of B^+, since $B^+ \subset L^\times U(W(\alpha, \beta))$. We denote this representation of B^+ by $\theta(\lambda, B^+)^\epsilon$ for $\epsilon = \epsilon_{L/F}(\alpha/\delta)$. We induce it to $B^\times$ and denote it by $\theta(\lambda, B^\times)$.

By Theorem 5.2 and (5-1), we obtain:

Theorem 5.3. *Under the action of* B^+, $\theta(\lambda, B^\times)$ *decomposes into two subspaces* $\theta(\lambda, B^\times)^\epsilon$ *for* $\epsilon = \pm 1$, *and*

$$\theta(B^\times, \lambda)^\epsilon|_{T_L} = \bigoplus_{\substack{-\varepsilon(\bar{\lambda}\phi^{-1}, \psi_0)= \\ \varepsilon(\lambda\phi^{-1}, \psi_0)=\epsilon}} \mathbb{C}\phi,$$

where ϕ *runs through all characters of* $L^\times$ *that satisfy* $\lambda\phi^{-1}|_{F^\times} = \epsilon_{L/F}$.

Remark 5.4. The representations $\theta(\lambda, \mathrm{GL}(W'))$ and $\theta(\lambda, B^\times)$ are in Jacquet–Langlands correspondence with each other, and Theorem 5.3 gives the latter half of Theorem 1.2 in [Prasad 1994].

By [Mœglin et al. 1987, Chapitre 3, IV, Corollaire 9], an irreducible quotient of

$$\theta\big(\chi^{-1}\lambda^1, W(U(\alpha, \beta))\big)$$

is uniquely determined. Since $U(W(\alpha, \beta))$ is compact, $\theta(\chi^{-1}\lambda^1, U(W(\alpha, \beta)))$ is a multiple of this irreducible representation. Lemma 5.1 implies that the multiplicity is 1, and $\theta(\chi^{-1}\lambda^1, W(\alpha, \beta))$ is irreducible. Let $\pi = \theta(\lambda, B^\times)$. Since $\lambda|_{L^1}$ is not trivial, $\theta(\lambda, \mathrm{GL}(W'))$ is supercuspidal. Let π' be the representation of $B^\times$ which corresponds to $\theta(\lambda, \mathrm{GL}(W'))$ under the Jacquet–Langlands correspondence. We denote by χ_π, $\chi_{\pi'}$ the characters of π, π'. Then π and π' satisfy

$$\pi \otimes \epsilon_{L/F} \simeq \pi, \quad \pi' \otimes \epsilon_{L/F} \simeq \pi',$$

and $\chi_\pi = \chi_{\pi'}$ on $L^\times$. By Corollaries 1.7 and 1.15 of [Hijikata et al. 1993] and Theorem 4.6 (and the remark following it) in [Takahashi 1996], this implies that $\chi_\pi = \chi_{\pi'}$ on all the other elliptic torus of $B^\times$. Therefore $\pi \simeq \pi'$.

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