POWERS OF THETA FUNCTIONS

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Dedicated to Michael Hirschhorn on the occasion of his sixtieth birthday.

The Ramanujan–Mordell Theorem for sums of an even number of squares is extended to other quadratic forms. A number of explicit examples is given. As an application, the value of the convolution sum

\[ \sum_{1 \leq m < n/23} \sigma(m)\sigma(n - 23m) \]

is determined, where \( \sigma(m) \) denotes the sum of the divisors of \( m \).

1. Introduction

Throughout this work let \( \tau \) be a complex number with positive imaginary part, and let \( q = e^{2\pi i \tau} \). Dedekind’s eta-function is defined by

(1) \[ \eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j). \]

Let

\[ z = z(\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2 + n^2} \text{ and } \Lambda(\tau) = \frac{\eta(2\tau)^{12}}{z^6}. \]

The following result was stated by S. Ramanujan \[1916; 2000, \text{ p. 159, eq. (14)} \] and first proved by L. Mordell in \[1917\].

**Theorem 1.1** (Ramanujan–Mordell). Suppose \( k \) is a positive integer. Then

\[ z^k = F_k(\tau) + z^k \sum_{1 \leq j \leq (k-1)/4} c_{j,k} \Lambda^j, \]

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where $c_{j,k}$ are constants that depend on $j$ and $k$, and $F_k(\tau)$ is an Eisenstein series given by:

$$F_1(\tau) = 1 + 4 \sum_{j=1}^{\infty} \frac{q^j}{1 + q^{2j}} = 1 + 4 \sum_{j=1}^{\infty} \frac{(-1)^{j+1} q^{2j-1}}{1 - q^{2j-1}},$$

and for $k \geq 1$,

$$F_{2k}(\tau) = 1 - \frac{4k(-1)^k}{(2^k - 1)B_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1} q^j}{1 - (-1)^k q^j}, \quad \text{and} \quad F_{2k+1}(\tau) = 1 + \frac{4(-1)^k}{E_{2k}} \sum_{j=1}^{\infty} \left( \frac{(2j)^{2k} q^j}{1 + q^{2j}} - \frac{(-1)^{k+j} (2j - 1)^{2k} q^{2j-1}}{1 - q^{2j-1}} \right).$$

Here $B_k$ and $E_k$ are the Bernoulli numbers and Euler numbers, respectively, defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \quad \text{and} \quad \text{sech } x = \sum_{k=0}^{\infty} \frac{E_k}{k!} x^k.$$

For the values $k = 1, 2, 3$ and $4$, the condition $1 \leq j \leq (k - 1)/4$ is empty, and therefore Theorem 1.1 gives a representation of $z, z^2, z^3$ and $z^4$ solely in terms of an Eisenstein series. These are the familiar Lambert series for sums of $2, 4, 6$ and $8$ squares, originally due to C. G. J. Jacobi [1969]. The result for $k = 5$ was known in part to G. Eisenstein (without proof) [1988, p. 501], and stated in full by J. Liouville (without proof) in [1866]. The result for $k = 6$ was known in part to Liouville (without proof) in [1860; 1864]. The results for $1 \leq k \leq 9$ were proved by J. W. L. Glaisher in a series of papers culminating in [1907]. The general statement of Theorem 1.1 is due to Ramanujan (without proof) [2000, Eqs. (145)–(147)], and the first proof is due to Mordell in [1917]. Other proofs of Theorem 1.1 have been given by R. A. Rankin in [1977, pp. 241–244] and S. Cooper in [2001].

The goal of this work is to prove the analogue of the Ramanujan–Mordell Theorem for which the quadratic form $m^2 + n^2$ in the definition of $z$ is replaced with $m^2 + mn + n^2, m^2 + mn + 2n^2, m^2 + mn + 3n^2, m^2 + mn + 6n^2$, or $2m^2 + mn + 3n^2$. Before stating the result we make some definitions. For $k \geq 1$, define the normalized Eisenstein series by

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1} q^j}{1 - q^j},$$

(2)
where $B_{2k}$ denotes the Bernoulli numbers. Let $p$ be an odd prime. The generalized Bernoulli numbers $B_{k,p}$ are defined by

$$ x \frac{e^{px} - 1}{e^x - 1} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) e^{jx} = \sum_{k=0}^{\infty} B_{k,p} \frac{x^k}{k!}, $$

where $\left( \frac{j}{p} \right)$ is the Legendre symbol. Let $k$ be a positive integer which satisfies

$$ k \equiv \frac{p - 1}{2} \quad (\text{mod } 2). $$

The generalized Eisenstein series $E_0^k(\tau; \chi_p)$ and $E_\infty^k(\tau; \chi_p)$ are defined by

$$ E_0^k(\tau; \chi_p) = \delta_{k,1} - \frac{2k}{B_{k,p}} \sum_{j=1}^{\infty} j^{k-1} \sum_{\ell=1}^{p-1} \left( \frac{\ell}{p} \right) q^{j\ell}, \quad \text{and} $$

$$ E_\infty^k(\tau; \chi_p) = 1 - \frac{2k}{B_{k,p}} \sum_{j=1}^{\infty} \left( \frac{j}{p} \right) j^{k-1} q^j \frac{1}{1-q^j}, $$

where $\delta_{m,n}$ is the Kronecker delta function, defined by

$$ \delta_{m,n} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} $$

If $p$ is a prime of the form $p \equiv 3 \quad (\text{mod } 4)$, let

$$ F_1(\tau; p) = E_1^\infty(\tau; \chi_p), $$

and for $k \geq 1$, let

$$ F_{2k}(\tau; p) = \frac{E_{2k}(\tau) + (-p)^k E_{2k}(p\tau)}{1 + (-p)^k}, $$

$$ F_{2k+1}(\tau; p) = E_{2k+1}^\infty(\tau; \chi_p) + (-p)^k E_{2k+1}^0(\tau; \chi_p). $$

For $p = 3, 7, 11$ or 23, let

$$ z_p = z_p(\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2 + mn + (p+1)n^2/4} $$

and

$$ \Lambda_p = \Lambda_p(\tau) = \left( \frac{\eta(\tau) \eta(p\tau)}{z_p} \right)^{24/(p+1)}. $$
Furthermore, let
\begin{equation}
 z_{23} = z_{23}(\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{2m^2+mn+n^2}
\end{equation}
and
\begin{equation}
 \Lambda_{23} = \Lambda_{23}(\tau) = \frac{\eta(\tau)\eta(23\tau)}{z_{23}}.
\end{equation}

The analogue of the Ramanujan–Mordell Theorem, and the main result of this work, is:

**Theorem 1.2.** Suppose \( p = 3, 7, 11 \) or 23 and let \( k \) be a positive integer. Let \( F_k(\tau; p), z_p \) and \( \Lambda_p \) be defined by (4)–(8). Then
\begin{equation}
 z^k_p = F_k(\tau; p) + z_p \sum_{1 \leq j \leq (p+1)k/24} c_{p,k,j} \Lambda_p^j,
\end{equation}
where \( c_{p,k,j} \) are numerical constants that depend only on \( p, k \) and \( j \).

A similar result holds for \( z'_{23} \) and \( \Lambda'_{23} \) defined by (9) and (10), namely
\begin{equation}
 z_{23}^k = F_k(\tau; 23) + z_{23} \sum_{1 \leq j \leq k} a_{k,j} \Lambda_{23}^j,
\end{equation}
where \( a_{k,j} \) are numerical constants that depend only on \( k \) and \( j \).

A proof of Theorem 1.2 will be given in Section 2. In the remainder of this section we describe some special cases of Theorem 1.2.

**Example 1.** For \( k = 1 \) and \( p = 3, 7 \) or 11, Theorem 1.2 gives
\begin{align*}
 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} &= 1 + 6 \sum_{j=1}^{\infty} \left( \frac{j}{3} \right) \frac{q^j}{1-q^j}, \\
 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} &= 1 + 2 \sum_{j=1}^{\infty} \left( \frac{j}{7} \right) \frac{q^j}{1-q^j}, \\
 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+3n^2} &= 1 + 2 \sum_{j=1}^{\infty} \left( \frac{j}{11} \right) \frac{q^j}{1-q^j}.
\end{align*}
These are equivalent to instances of a general theorem of Dirichlet; see [Landau
1958, Theorem 204]. When \( k = 1 \) and \( p = 23 \), Theorem 1.2 gives
\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+6n^2} = 1 + \frac{3}{2} \sum_{j=1}^{\infty} \left( \frac{j}{23} \right) \frac{q^j}{1-q^j} + \frac{3}{2} q \prod_{j=1}^{\infty} (1-q^j)(1-q^{23j}),
\]
\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} = 1 + \frac{3}{2} \sum_{j=1}^{\infty} \left( \frac{j}{23} \right) \frac{q^j}{1-q^j} - \frac{3}{2} q \prod_{j=1}^{\infty} (1-q^j)(1-q^{23j}),
\]
and these were proved by F. van der Blij in [1952]. They may be rearranged to give
\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+6n^2} + 2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} = 3 + 2 \sum_{j=1}^{\infty} \left( \frac{j}{23} \right) \frac{q^j}{1-q^j},
\]
\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+6n^2} - \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} = 2q \prod_{j=1}^{\infty} (1-q^j)(1-q^{23j}).
\]
The first of these is equivalent to another instance of Dirichlet’s theorem [Landau
1958, Theorem 204], and the second formula was noted by J.-P. Serre in [1977,
p. 242].

Example 2. For the case \( p = 3 \), results for \( 1 \leq k \leq 4 \) were given (without proof)
by Ramanujan [Andrews and Berndt 2005, pp. 402–403], and results for \( 3 \leq k \leq 6 \)
were given by H. Petersson in [1982, p. 90]. For \( 2 \leq k \leq 6 \), these results are:
\[
\left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \right)^2 = 1 + 12 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} - 36 \sum_{j=1}^{\infty} \frac{jq^{2j}}{1-q^j},
\]
\[
\left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \right)^3 = 1 - 9 \sum_{j=1}^{\infty} \left( \frac{j}{3} \right) \frac{j^2q^j}{1-q^j} + 27 \sum_{j=1}^{\infty} \frac{j^2q^j}{1+q^j+q^{2j}},
\]
\[
\left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \right)^4 = 1 + 24 \sum_{j=1}^{\infty} \frac{j^3q^j}{1-q^j} + 216 \sum_{j=1}^{\infty} \frac{j^3q^{3j}}{1-q^{3j}},
\]
\[
\left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \right)^5 = 1 + 3 \sum_{j=1}^{\infty} \left( \frac{j}{3} \right) \frac{j^4q^j}{1-q^j} + 27 \sum_{j=1}^{\infty} \frac{j^4q^j}{1+q^j+q^{2j}},
\]
\[
\left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \right)^6 = 1 + \frac{252}{13} \sum_{j=1}^{\infty} \frac{j^5q^j}{1-q^j} - \frac{6804}{13} \sum_{j=1}^{\infty} \frac{j^5q^{3j}}{1-q^{3j}} + \frac{216}{13} q \prod_{j=1}^{\infty} (1-q^j)^6 (1-q^{3j})^6.
\]
Results for \( p = 3 \), \( 1 \leq k \leq 20 \), were given by G. Lomadze in [1989a; 1989b]. Lomadze’s expansions for \( 6 \leq j \leq 20 \) are different from ours. For example, Lomadze’s formula for \( k = 6 \) has
\[
\frac{1}{12} \sum_{n=1}^{\infty} \left( \sum_{x_1^2 + x_1 y_1 + y_1^2 + x_2^2 + x_2 y_2 + y_2^2 = n} 9x_1^4 - 9n x_1^2 + n^2 \right) q^n
\]
in place of
\[
q \prod_{j=1}^{\infty} (1 - q^j)^6 (1 - q^{3j})^{6},
\]
and Lomadze’s formulas become more complicated as \( k \) increases.

**Example 3.** For \( p = 7 \), the cases \( k = 2 \) and \( 3 \) of Theorem 1.2 give
\[
\left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} \right)^2 = 1 + 4 \sum_{j=1}^{\infty} \frac{j q^j}{1 - q^j} - 28 \sum_{j=1}^{\infty} \frac{j q^{7j}}{1 - q^{7j}}
\]
and
\[
\left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} \right)^3
\]
\[
= 1 - \frac{7}{8} \sum_{j=1}^{\infty} \left( \frac{j}{7} \right) - \frac{7}{8} \sum_{j=1}^{\infty} \frac{j^2 (q^j + q^{2j} - q^{3j} + q^{4j} - q^{5j} - q^{6j})}{1 - q^{7j}}
\]
\[
+ \frac{3}{4} q \prod_{j=1}^{\infty} (1 - q^j)^3 (1 - q^{7j})^3.
\]

The identity (11) was given by Ramanujan; see [Andrews and Berndt 2005, p. 405, Entry 18.2.15]. See [Chan and Ong 1999; Cooper and Toh 2008; Liu 2003] and [Williams 2006] for other proofs.

The identity (12) is a consequence of the formulas for \( E_3^\infty(q; \chi_7) \) and \( E_3^0(q; \chi_7) \) in [Chan and Cooper 2008]. In [Chan et al. 2008], it was shown that
\[
q \prod_{j=1}^{\infty} (1 - q^j)^3 (1 - q^{7j})^3 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( m + n \left( \frac{1 + i \sqrt{7}}{2} \right) \right)^2 q^{m^2+mn+2n^2}.
\]

Another result for \( \chi_7^3 \) can be obtained by combining two of Ramanujan’s results, [Andrews and Berndt 2005, p. 404, Entry 18.2.14] and [Berndt 1991, p. 467, Entry 5 (i)].
\[
\left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} \right)^3 = \prod_{j=1}^{\infty} \frac{(1 - q^j)^7}{(1 - q^{7j})} + 13q \prod_{j=1}^{\infty} (1 - q^j)^3 (1 - q^{7j})^3 + 49q^2 \prod_{j=1}^{\infty} \frac{(1 - q^{7j})^7}{(1 - q^j)}.
\]

Other proofs of (13) have been given by H. H. Chan and Y. L. Ong in [1999, Lemma 2.2] and Z.-G. Liu in [2003].

The remainder of this paper is organized as follows. We shall give a proof of Theorem 1.2 in Section 2. The proof depends on three transformation formulas (Lemmas 2.1–2.3) for \( \Gamma_0(p) \), as well as a result that says certain bounded functions must be constant (Lemma 2.4). A proof of the identity (13) using the same technique is also given. Some applications to convolution sums are given in Section 3.

2. Proofs

Let
\[
\Gamma = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\},
\]
\[
\Gamma_0(p) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : a, b, c, d \in \mathbb{Z}, \ ad - bc = 1, \ c \equiv 0 \pmod{p} \right\}.
\]
For \( p = 3, 7, 11 \) or 23, define
\[
(14) \quad \eta_p(\tau) = (\eta(\tau) \eta(p \tau))^{24/(p+1)}.
\]

The proof of Theorem 1.2 hinges on the following four lemmas.

**Lemma 2.1.** Let \( p = 3, 7, 11 \) or 23 and let \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(p) \). Then, for \( \eta_p(\tau) \) defined by (14), we have
\[
\eta_p \left( \frac{a \tau + b}{c \tau + d} \right) = \left( \frac{d}{p} \right)^{24/(p+1)} (c \tau + d)^{24/(p+1)} \eta_p(\tau)
\]
and
\[
\eta_p \left( \frac{-1}{\tau \sqrt{p}} \right) = (-i \tau)^{24/(p+1)} \eta_p \left( \frac{\tau}{\sqrt{p}} \right).
\]

**Proof:** These follow from the transformation formula for the Dedekind eta-function [Apostol 1990, p. 52, Theorem 3.4].

**Lemma 2.2.** Let \( p = 3, 7, 11 \) or 23 and let \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(p) \). Then, for \( z_p(\tau) \) defined by (7), we have
\[
z_p \left( \frac{a \tau + b}{c \tau + d} \right) = \left( \frac{d}{p} \right) (c \tau + d) z_p(\tau)
\]
and
\[ z_p \left( \frac{-1}{\tau \sqrt{p}} \right) = -i \tau z_p \left( \frac{\tau}{\sqrt{p}} \right). \]

The same transformation formulas hold when \( z_{23} \) is replaced with \( z'_{23} \).

**Proof.** The first result follows from [Schoeneberg 1974, p. 217, Theorem 4] by taking \( r = 1, A = \left( \frac{2}{1_{(p+1)/2}} \right), h = (0, 0), k = 0 \) and \( P_k = 1 \). The corresponding result for \( z'_{23} \) follows by taking \( A = \left( \frac{4}{1_6} \right) \), with the other parameters being the same as for the case \( p = 23 \).

The second result is a direct consequence of [Schoeneberg 1974, p. 205, (5)]. □

**Lemma 2.3.** Let \( p \equiv 3 \pmod{4} \) be prime and let \( n \) be a positive integer. Let \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(p) \). Then for \( F_k(\tau; p) \) defined by (4)–(6), we have
\[ F_k \left( \frac{a \tau + b}{c \tau + d}; p \right) = \left( \frac{d}{p} \right)^k (c \tau + d)^k F_k(\tau; p) \]
and
\[ F_k \left( \frac{-1}{\tau \sqrt{p}}; p \right) = (-i \tau)^k F_k \left( \frac{\tau}{\sqrt{p}}; p \right). \]

**Proof.** For odd values of \( k \), these follow from [Cooper 2008, Theorem 6.1] or [Kolberg 1968, (1.8)–(1.12)]. For even values of \( k \) with \( k \geq 4 \), these follow from the well-known transformation formulas for \( E_{2k}(\tau) \), for example, see [Serre 1973, pp. 83, 92, 95–96]. For \( k = 2 \), the results are most easily proved by appealing to the transformation formulas for the function \( \left( \frac{\psi(p \tau)}{\eta(\tau)} \right)^{24} \) in [Apostol 1990, pp. 84–85, Theorems 4.7 and 4.8], and then applying logarithmic differentiation. □

**Lemma 2.4.** Let \( f(\tau) \) be analytic and bounded in the upper half plane \( \text{Im}(\tau) > 0 \), and suppose it satisfies the transformation property
\[ f \left( \frac{a \tau + b}{c \tau + d} \right) = f(\tau) \quad \text{for all} \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(p). \]

Then \( f \) is constant.

**Proof.** This is Theorem 4.4 in [Apostol 1990, p. 79]. □

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( p = 3, 7, 11 \) or 23, and let \( k \) be a positive integer. Let \( \ell \) be the smallest integer that satisfies \( \frac{24\ell}{p+1} \geq k \). Consider the functions
\[ \varphi(\tau) = \varphi_{p,k}(\tau) = \frac{F_k(\tau; p)}{(z_p(\tau))^k \left( \frac{\eta(\tau) \eta(p \tau)}{\eta(\tau) \eta(p \tau)} \right)^{24/(p+1)}} \]
and
\[ \psi(\tau) = \psi_{p}(\tau) = \left( \frac{z_p(\tau)}{\eta(\tau) \eta(p \tau)} \right)^{24/(p+1)}. \]
By Lemmas 2.1–2.3, $\varphi(\tau)$ and $\psi(\tau)$ satisfy the transformation property (15). Furthermore, $\varphi$ and $\psi$ are both analytic in the upper half plane $0 < \text{Im}(\tau) < \infty$, as $\eta(\tau)$ does not vanish in this region. Let us analyze the behavior at $\tau = i\infty$. From the $q$-expansions, we find that

$$
\varphi(\tau) = \frac{(1 + O(q))}{(1 + O(q))^{k}} \left( \frac{1 + O(q)}{q + O(q^{2})} \right)^{\ell} = q^{-\ell} + O(q^{-\ell+1}) \quad \text{as} \quad \tau \to i\infty.
$$

Therefore $\varphi(\tau)$ has a pole of order $\ell$ at $i\infty$. Similarly, we find that $\psi(\tau)$ has a pole of order 1 at $i\infty$. It follows that there exist constants $b_{1}, \ldots, b_{\ell}$, such that the function

$$
\lambda(\tau) := \varphi(\tau) - \sum_{j=1}^{\ell} b_{j} (\psi(\tau))^{j}
$$

has no pole at $i\infty$. That is to say,

$$
\lambda(\tau) = b_{0} + O(q) \quad \text{as} \quad \tau \to i\infty
$$

for some constant $b_{0}$. Let us consider the behavior of $\lambda(\tau)$ at $\tau = 0$. By the second result in each of Lemmas 2.1–2.3, we find that

$$
\varphi\left(\frac{-1}{\sqrt{\tau}}\right) = \varphi\left(\frac{\tau}{\sqrt{p}}\right) \quad \text{and} \quad \psi\left(\frac{-1}{\sqrt{\tau}}\right) = \psi\left(\frac{\tau}{\sqrt{p}}\right).
$$

Therefore

$$
\lambda(\tau) = \lambda\left(\frac{-1}{p\tau}\right) \to b_{0} \quad \text{as} \quad \tau \to 0.
$$

It follows from the description of the fundamental region for $\Gamma_{0}(p)$ given in [Apostol 1990, p. 76, Theorem 4.2] that $\lambda(\tau)$ is bounded in the upper half plane. Hence by Lemma 2.4, $\lambda(\tau)$ is constant, that is, $\lambda(\tau) \equiv b_{0}$. Therefore, from (16) we have

$$
\varphi(\tau) = \sum_{j=0}^{\ell} b_{j} (\psi(\tau))^{j}.
$$

Using the fact that $\psi(\tau) = 1/\Lambda_{p}(\tau)$, this is equivalent to

$$
F_{k}(\tau; p) = z_{p}^{k} \sum_{j=0}^{\ell} b_{j} \Lambda_{p}^{\ell-j} = z_{p}^{k} \sum_{0 \leq j \leq (p+1)k/24} c_{j} \Lambda_{p}^{j},
$$

where $c_{j} = b_{\ell-j}$. Letting $q = 0$ on both sides we deduce that $c_{0} = 1$.

If we replace $z_{23}$ and $\Lambda_{23}$ by $z'_{23}$ and $\Lambda'_{23}$, respectively, at every step in the proof, we establish the result for $z'_{23}$ and $\Lambda'_{23}$.

This completes the proof of Theorem 1.2.
Remarks. For \( p = 3, 7, 11 \) or 23, the genus of the normalizer of \( \Gamma_0(p) \) in \( SL_2(\mathbb{R}) \) (denoted by \( \Gamma_0(p)+ \)) is 0. It turns out that for each \( p \), the field of functions invariant under \( \Gamma_0(p)+ \) is generated by \( \psi_p(\tau) \), which has a simple pole at \( \tau = i\infty \). Since \( \varphi_{p,k}(\tau) \) has a pole of order \( \ell \) at \( \tau = i\infty \) and \( \varphi_{p,k}(\tau) \) is a function on \( \Gamma_0(p)+ \), it follows that \( \varphi_{p,k}(\tau) \) is a polynomial in \( \psi_p(\tau) \) with degree exactly \( \ell \). This explains the existence of relation (16).

The identity (13) may be proved similarly.

Proof of (13). Let

\[
F(\tau) = \frac{z^3}{\eta^3(\tau)\eta^3(7\tau)} \quad \text{and} \quad G(\tau) = \frac{\eta^4(\tau)}{\eta^4(7\tau)}.
\]

Lemmas 2.1 and 2.2 imply \( F(\tau) \) satisfies the transformation formula (15). Furthermore, [Apostol 1990, p. 87, Theorem 4.9] implies that \( G(\tau) \) also satisfies the transformation formula (15). The \( q \)-expansions are

\[
F(\tau) = \frac{1}{q} + O(1) \quad \text{and} \quad G(\tau) = \frac{1}{q} + O(1) \quad \text{as} \quad \tau \to i\infty.
\]

Hence \( F(\tau) \) and \( G(\tau) \) both have a pole of order 1 at \( \tau = i\infty \).

By the second parts of Lemmas 2.1 and 2.2, and by the transformation formula for the Dedekind eta-function [Apostol 1990, p. 52, Theorem 3.4], we have

\[
F\left(\frac{-1}{\tau}\right) = F(\tau) \quad \text{and} \quad G\left(\frac{-1}{\tau}\right) = \frac{49}{G(\tau)}.
\]

Therefore at the point \( \tau = 0 \), \( F(\tau) \) has a pole of order 1 and \( G(\tau) \) has a zero of order 1.

Let

\[
H(\tau) := F(\tau) - aG(\tau) - \frac{b}{G(\tau)},
\]

where \( a \) and \( b \) are constants that will be chosen so that \( H(\tau) \) has no pole at 0 or \( i\infty \). In order for there to be no pole at \( \tau = i\infty \), (17) implies \( a = 1 \). In order for there to be no pole at \( \tau = 0 \), (17) and (18) imply \( b = 49 \). It follows that the function \( H(\tau) \) with these values of \( a \) and \( b \) is bounded in the upper half plane, and Lemma 2.4 implies that it is constant. That is,

\[
\frac{z^3}{\eta^3(\tau)\eta^3(7\tau)} = c + \frac{\eta^4(\tau)}{\eta^4(7\tau)} + 49\frac{\eta^4(7\tau)}{\eta^4(\tau)},
\]

for some constant \( c \). If we multiply by \( \eta^3(\tau)\eta^3(7\tau) \) and compare coefficients of \( q \) on both sides, we find that \( c = 13 \). This completes the proof of (13).
3. Application to convolution sums

Let $\sigma_j(n)$ denote the sum of the $j$-th powers of the divisors of $n$, and let $\sigma(n) = \sigma_1(n)$. The convolution sum

$$W_k(n) = \sum_{1 \leq m < n/k} \sigma(m)\sigma(n-km)$$

has been evaluated for $1 \leq k \leq 14$ and $k = 16, 18$ and 24. See [Alaca et al. 2007] and [Royer 2007] for references. In this section, we show how Theorem 1.2 leads to an evaluation of $W_k(n)$ for the cases $k = 3, 7, 11$ and 23. The case $k = 23$ is new. Let

$$P(q) = E_2(\tau) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j},$$
$$Q(q) = E_4(\tau) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3q^j}{1-q^j},$$
$$S(q) = -q \frac{d}{24 dq} P(q) = \sum_{j=1}^{\infty} \frac{j^2q^j}{(1-q^j)^2}.$$

**Theorem 3.1.** For $p = 3, 7, 11$ and 23 we have

$$P(q)P(q^p) = \frac{1}{p^2+1}(Q(q)+p^2Q(q^p)) - \frac{144}{p}(S(q)+p^2S(q^p))-576z^4u_p(\Lambda_p),$$

where

$$u_3(\Lambda_3) = 0,$$
$$u_7(\Lambda_7) = \frac{1}{31} \Lambda_7,$$
$$u_{11}(\Lambda_{11}) = \frac{1}{671} (15\Lambda_{11} - 17\Lambda_{11}^2),$$
$$u_{23}(\Lambda_{23}) = \frac{1}{2438} (77\Lambda_{23} - 222\Lambda_{23}^2 + 201\Lambda_{23}^3 - 30\Lambda_{23}^4).$$

**Proof:** By Theorem 1.2 with $k = 2$ and 4, we have

$$\frac{pP(q^p) - P(q)}{p-1} = z_p^2 \left( 1 - \sum_{1 \leq j \leq (p+1)/12} c_{p,j} \Lambda_p^j \right),$$
$$\frac{p^2Q(q^p) + Q(q)}{p^2+1} = z_p^4 \left( 1 - \sum_{1 \leq j \leq (p+1)/6} d_{p,j} \Lambda_p^j \right),$$

for some constants $c_{p,j}$ and $d_{p,j}$. If we square (19) and subtract the result from (20), we obtain

$$\frac{p^2Q(q^p) + Q(q)}{p^2+1} - \left( \frac{pP(q^p) - P(q)}{p-1} \right)^2 = z_p^4 \sum_{1 \leq j \leq (p+1)/6} d_{p,j}^* \Lambda_p^j.$$
for some constants $d'_{p,j}$. This may be rewritten as

$$P(q)P(q^p) = \frac{1}{2p} \left( \frac{p^2 P^2(q) + P^2(q)}{2p^3 + 1} \right) - \frac{(p-1)^2}{2p^3} \left( P^2(Q(q) + Q(q)) \right)$$

$$+ z^4 \sum_{1 \leq j \leq (p+1)/6} d''_{p,j} \Lambda^j_p,$$

for some constants $d''_{p,j}$. Now use the result (see [Chan 2007; Glaisher 1885] or [Ramanujan 2000, p. 142, Eq. (30)])

$$P^2(q) = Q(q) - 288S(q)$$

to get

$$P(q)P(q^p) = \frac{1}{p^2 + 1} \left( Q(q) + p^2 Q(q^p) \right) - \frac{144}{p} \left( S(q) + p^2 S(q^p) \right)$$

$$+ z^4 \sum_{1 \leq j \leq (p+1)/6} d''_{p,j} \Lambda^j_p.$$

The values of the coefficients $d''_{p,j}$ may be determined by expanding in powers of $q$ and equating coefficients of $q^j$ for $1 \leq j \leq (p+1)/6$. In this way we obtain the polynomials $u_p(\Lambda_p)$ given in the statement of the theorem. This completes the proof.

\[\square\]

**Theorem 3.2.** For $p = 3, 7, 11$ and 23 we have

$$W_p(n) = \frac{5}{12(p^2 + 1)} \left( \frac{\sigma_3(n) + p^2 \sigma_3 \left( \frac{n}{p} \right)}{6} \right)$$

$$+ \left( \frac{1}{24} - \frac{n}{4p} \right) \sigma(n) + \left( \frac{1}{24} - \frac{n}{4} \right) \sigma \left( \frac{n}{p} \right) - c_p(n).$$

Here $c_p(n)$ is defined by

$$\sum_{n=1}^{\infty} c_p(n)q^n = z^4 u_p(\Lambda_p),$$

and $u_p(\Lambda_p)$ is as in Theorem 3.1.

**Proof.** Equate coefficients of $q^n$ on both sides of the identity in Theorem 3.1.  \[\square\]

**References**

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