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Dedicated to Michael Hirschhorn on the occasion of his sixtieth birthday.

The Ramanujan–Mordell Theorem for sums of an even number of squares is extended to other quadratic forms. A number of explicit examples is given. As an application, the value of the convolution sum

$$\sum_{1\le m<n/23} \sigma(m)\sigma(n-23m)$$

is determined, where $\sigma(m)$ denotes the sum of the divisors of m.

1. Introduction

Throughout this work let τ be a complex number with positive imaginary part, and let $q = e^{2\pi i\tau}$. Dedekind's eta-function is defined by

$$\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j). \tag{1}$$

Let

$$z = z(\tau) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+n^2} \quad \text{and} \quad \Lambda = \Lambda(\tau) = \frac{\eta(2\tau)^{12}}{z^6}.$$

The following result was stated by S. Ramanujan [1916; 2000, p. 159, eq. (14)] and first proved by L. Mordell in [1917].

Theorem 1.1 (Ramanujan–Mordell). *Suppose k is a positive integer. Then*

$$z^k = F_k(\tau) + z^k \sum_{1\le j\le (k-1)/4} c_{j,k}\Lambda^j,$$

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where $c_{j,k}$ are constants that depend on j and k, and $F_k(\tau)$ is an Eisenstein series given by:

$$F_1(\tau) = 1 + 4\sum_{j=1}^{\infty} \frac{q^j}{1+q^{2j}} = 1 + 4\sum_{j=1}^{\infty} \frac{(-1)^{j+1}q^{2j-1}}{1-q^{2j-1}},$$

and for $k \geq 1$,

$$F_{2k}(\tau) = 1 - \frac{4k(-1)^k}{(2^{2k}-1)B_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1}q^j}{1-(-1)^{k+j}q^j}, \quad \textit{and}$$

$$F_{2k+1}(\tau) = 1 + \frac{4(-1)^k}{E_{2k}} \sum_{j=1}^{\infty} \Big(\frac{(2j)^{2k}q^j}{1+q^{2j}} - \frac{(-1)^{k+j}(2j-1)^{2k}q^{2j-1}}{1-q^{2j-1}} \Big).$$

Here B_k and E_k are the Bernoulli numbers and Euler numbers, *respectively*, *defined by*

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}x^k \quad \textit{and} \quad \operatorname{sech} x = \sum_{k=0}^{\infty} \frac{E_k}{k!}x^k.$$

For the values $k = 1$, 2, 3 and 4, the condition $1 \leq j \leq (k-1)/4$ is empty, and therefore Theorem 1.1 gives a representation of z, z^2, z^3 and z^4 solely in terms of an Eisenstein series. These are the familiar Lambert series for sums of 2, 4, 6 and 8 squares, originally due to C. G. J. Jacobi [1969]. The result for $k = 5$ was known in part to G. Eisenstein (without proof) [1988, p. 501], and stated in full by J. Liouville (without proof) in [1866]. The result for $k = 6$ was known in part to Liouville (without proof) in [1860; 1864]. The results for $1 \leq k \leq 9$ were proved by J. W. L. Glaisher in a series of papers culminating in [1907]. The general statement of Theorem 1.1 is due to Ramanujan (without proof) [2000, Eqs. (145)–(147)], and the first proof is due to Mordell in [1917]. Other proofs of Theorem 1.1 have been given by R. A. Rankin in [1977, pp. 241–244] and S. Cooper in [2001].

The goal of this work is to prove the analogue of the Ramanujan–Mordell Theorem for which the quadratic form m^2+n^2 in the definition of z is replaced with m^2+mn+n^2, $m^2+mn+2n^2$, $m^2+mn+3n^2$, $m^2+mn+6n^2$, or $2m^2+mn+3n^2$. Before stating the result we make some definitions. For $k \geq 1$, define the normalized Eisenstein series by

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1}q^j}{1-q^j}, \tag{2}$$

where B_{2k} denotes the Bernoulli numbers. Let p be an odd prime. The generalized Bernoulli numbers $B_{k,p}$ are defined by

$$\frac{x}{e^{px}-1}\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)e^{jx}=\sum_{k=0}^{\infty}B_{k,p}\frac{x^k}{k!}, \tag{3}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Let k be a positive integer which satisfies

$$k\equiv\frac{p-1}{2}\pmod 2.$$

The generalized Eisenstein series $E_k^0(\tau;\chi_p)$ and $E_k^\infty(\tau;\chi_p)$ are defined by

$$E_k^0(\tau;\chi_p)=\delta_{k,1}-\frac{2k}{B_{k,p}}\sum_{j=1}^{\infty}\frac{j^{k-1}}{1-q^{pj}}\sum_{\ell=1}^{p-1}\left(\frac{\ell}{p}\right)q^{j\ell},\quad\text{and}$$

$$E_k^\infty(\tau;\chi_p)=1-\frac{2k}{B_{k,p}}\sum_{j=1}^{\infty}\left(\frac{j}{p}\right)\frac{j^{k-1}q^j}{1-q^j},$$

where $\delta_{m,n}$ is the Kronecker delta function, defined by

$$\delta_{m,n}=\begin{cases}1 & \text{if } m=n,\\ 0 & \text{if } m\neq n.\end{cases}$$

If p is a prime of the form $p\equiv 3\pmod 4$, let

$$F_1(\tau;p)=E_1^\infty(\tau;\chi_p), \tag{4}$$

and for $k\geq 1$, let

$$F_{2k}(\tau;p)=\frac{E_{2k}(\tau)+(-p)^kE_{2k}(p\tau)}{1+(-p)^k}, \tag{5}$$

$$F_{2k+1}(\tau;p)=E_{2k+1}^\infty(\tau;\chi_p)+(-p)^kE_{2k+1}^0(\tau;\chi_p). \tag{6}$$

For $p=3$, 7, 11 or 23, let

$$z_p=z_p(\tau)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}q^{m^2+mn+(p+1)n^2/4} \tag{7}$$

and

$$\Lambda_p=\Lambda_p(\tau)=\left(\frac{\eta(\tau)\eta(p\tau)}{z_p}\right)^{24/(p+1)}. \tag{8}$$

Furthermore, let

$$z'_{23} = z'_{23}(\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} \tag{9}$$

and

$$\Lambda'_{23} = \Lambda'_{23}(\tau) = \frac{\eta(\tau)\eta(23\tau)}{z'_{23}}. \tag{10}$$

The analogue of the Ramanujan–Mordell Theorem, and the main result of this work, is:

Theorem 1.2. *Suppose $p = 3$, 7, 11 or 23 and let k be a positive integer. Let $F_k(\tau;\, p)$, z_p and Λ_p be defined by (4)–(8). Then*

$$z_p^k = F_k(\tau;\, p) + z_p^k \sum_{1\le j\le (p+1)k/24} c_{p,k,j} \Lambda_p^j,$$

where $c_{p,k,j}$ are numerical constants that depend only on p, k and j.

A similar result holds for z'_{23} and Λ'_{23} defined by (9) and (10), namely

$${z'_{23}}^k = F_k(\tau;\, 23) + {z'_{23}}^k \sum_{1\le j\le k} a_{k,j} {\Lambda'_{23}}^j,$$

where $a_{k,j}$ are numerical constants that depend only on k and j.

A proof of Theorem 1.2 will be given in Section 2. In the remainder of this section we describe some special cases of Theorem 1.2.

Example 1. For $k = 1$ and $p = 3$, 7 or 11, Theorem 1.2 gives

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6\sum_{j=1}^{\infty} \left(\frac{j}{3}\right) \frac{q^j}{1-q^j},$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} = 1 + 2\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{q^j}{1-q^j},$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+3n^2} = 1 + 2\sum_{j=1}^{\infty} \left(\frac{j}{11}\right) \frac{q^j}{1-q^j}.$$

These are equivalent to instances of a general theorem of Dirichlet; see [Landau 1958, Theorem 204]. When $k=1$ and $p=23$, Theorem 1.2 gives

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+6n^2} = 1+\tfrac{2}{3}\sum_{j=1}^{\infty}\left(\frac{j}{23}\right)\frac{q^j}{1-q^j}+\tfrac{4}{3}q\prod_{j=1}^{\infty}(1-q^j)(1-q^{23j}),$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} = 1+\tfrac{2}{3}\sum_{j=1}^{\infty}\left(\frac{j}{23}\right)\frac{q^j}{1-q^j}-\tfrac{2}{3}q\prod_{j=1}^{\infty}(1-q^j)(1-q^{23j}),$$

and these were proved by F. van der Blij in [1952]. They may be rearranged to give

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+6n^2}+2\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} = 3+2\sum_{j=1}^{\infty}\left(\frac{j}{23}\right)\frac{q^j}{1-q^j},$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+6n^2}-\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} = 2q\prod_{j=1}^{\infty}(1-q^j)(1-q^{23j}).$$

The first of these is equivalent to another instance of Dirichlet's theorem [Landau 1958, Theorem 204], and the second formula was noted by J.-P. Serre in [1977, p. 242].

Example 2. For the case $p=3$, results for $1\le k\le 4$ were given (without proof) by Ramanujan [Andrews and Berndt 2005, pp. 402–403], and results for $3\le k\le 6$ were given by H. Petersson in [1982, p. 90]. For $2\le k\le 6$, these results are:

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^2 = 1+12\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}-36\sum_{j=1}^{\infty}\frac{jq^{3j}}{1-q^{3j}},$$

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^3 = 1-9\sum_{j=1}^{\infty}\left(\frac{j}{3}\right)\frac{j^2q^j}{1-q^j}+27\sum_{j=1}^{\infty}\frac{j^2q^j}{1+q^j+q^{2j}},$$

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^4 = 1+24\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j}+216\sum_{j=1}^{\infty}\frac{j^3q^{3j}}{1-q^{3j}},$$

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^5 = 1+3\sum_{j=1}^{\infty}\left(\frac{j}{3}\right)\frac{j^4q^j}{1-q^j}+27\sum_{j=1}^{\infty}\frac{j^4q^j}{1+q^j+q^{2j}},$$

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^6 = 1+\tfrac{252}{13}\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j}-\tfrac{6804}{13}\sum_{j=1}^{\infty}\frac{j^5q^{3j}}{1-q^{3j}}$$
$$+\tfrac{216}{13}q\prod_{j=1}^{\infty}(1-q^j)^6(1-q^{3j})^6.$$

Results for $p=3$, $1\le k\le 20$, were given by G. Lomadze in [1989a; 1989b]. Lomadze's expansions for $6\le j\le 20$ are different from ours. For example, Lomadze's formula for $k=6$ has

$$\frac{1}{12}\sum_{n=1}^{\infty}\Big(\sum_{x_1^2+x_1y_1+y_1^2+x_2^2+x_2y_2+y_2^2=n} 9x_1^4-9nx_1^2+n^2\Big)q^n$$

in place of

$$q\prod_{j=1}^{\infty}(1-q^j)^6(1-q^{3j})^6,$$

and Lomadze's formulas become more complicated as k increases.

Example 3. For $p=7$, the cases $k=2$ and 3 of Theorem 1.2 give

$$\Big(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}q^{m^2+mn+2n^2}\Big)^2=1+4\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}-28\sum_{j=1}^{\infty}\frac{jq^{7j}}{1-q^{7j}} \tag{11}$$

and

$$\begin{aligned}&\Big(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}q^{m^2+mn+2n^2}\Big)^3 \\ &\quad=1-\tfrac{7}{8}\sum_{j=1}^{\infty}\Big(\frac{j}{7}\Big)\frac{j^2q^j}{1-q^j}+\tfrac{49}{8}\sum_{j=1}^{\infty}\frac{j^2(q^j+q^{2j}-q^{3j}+q^{4j}-q^{5j}-q^{6j})}{1-q^{7j}} \\ &\qquad+\tfrac{3}{4}q\prod_{j=1}^{\infty}(1-q^j)^3(1-q^{7j})^3.\end{aligned} \tag{12}$$

The identity (11) was given by Ramanujan; see [Andrews and Berndt 2005, p. 405, Entry 18.2.15]. See [Chan and Ong 1999; Cooper and Toh 2008; Liu 2003] and [Williams 2006] for other proofs.

The identity (12) is a consequence of the formulas for $E_3^{\infty}(q;\chi_7)$ and $E_3^{0}(q;\chi_7)$ in [Chan and Cooper 2008]. In [Chan et al. 2008], it was shown that

$$q\prod_{j=1}^{\infty}(1-q^j)^3(1-q^{7j})^3=\tfrac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\Big(m+n\Big(\frac{1+i\sqrt{7}}{2}\Big)\Big)^2q^{m^2+mn+2n^2}.$$

Another result for z_7^3 can be obtained by combining two of Ramanujan's results, [Andrews and Berndt 2005, p. 404, Entry 18.2.14] and [Berndt 1991, p. 467, Entry 5 (i)]:

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}\right)^3 = \prod_{j=1}^{\infty}\frac{(1-q^j)^7}{(1-q^{7j})} + 13q\prod_{j=1}^{\infty}(1-q^j)^3(1-q^{7j})^3 + 49q^2\prod_{j=1}^{\infty}\frac{(1-q^{7j})^7}{(1-q^j)}. \tag{13}$$

Other proofs of (13) have been given by H. H. Chan and Y. L. Ong in [1999, Lemma 2.2] and Z.-G. Liu in [2003].

The remainder of this paper is organized as follows. We shall give a proof of Theorem 1.2 in Section 2. The proof depends on three transformation formulas (Lemmas 2.1–2.3) for $\Gamma_0(p)$, as well as a result that says certain bounded functions must be constant (Lemma 2.4). A proof of the identity (13) using the same technique is also given. Some applications to convolution sums are given in Section 3.

2. Proofs

Let

$$\Gamma = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad-bc=1\right\},$$

$$\Gamma_0(p) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad-bc=1,\ c\equiv 0 \pmod{p}\right\}.$$

For $p = 3$, 7, 11 or 23, define

$$\eta_p(\tau) = (\eta(\tau)\eta(p\tau))^{24/(p+1)}. \tag{14}$$

The proof of Theorem 1.2 hinges on the following four lemmas.

Lemma 2.1. *Let $p = 3$, 7, 11 or 23 and let $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(p)$. Then, for $\eta_p(\tau)$ defined by (14), we have*

$$\eta_p\left(\frac{a\tau+b}{c\tau+d}\right) = \left(\frac{d}{p}\right)^{24/(p+1)} (c\tau+d)^{24/(p+1)}\eta_p(\tau)$$

and

$$\eta_p\left(\frac{-1}{\tau\sqrt{p}}\right) = (-i\tau)^{24/(p+1)}\eta_p\left(\frac{\tau}{\sqrt{p}}\right).$$

Proof. These follow from the transformation formula for the Dedekind eta-function [Apostol 1990, p. 52, Theorem 3.4]. □

Lemma 2.2. *Let $p = 3$, 7, 11 or 23 and let $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(p)$. Then, for $z_p(\tau)$ defined by (7), we have*

$$z_p\left(\frac{a\tau+b}{c\tau+d}\right) = \left(\frac{d}{p}\right)(c\tau+d)z_p(\tau)$$

and

$$z_p\Big(\frac{-1}{\tau\sqrt{p}}\Big) = -i\tau z_p\Big(\frac{\tau}{\sqrt{p}}\Big).$$

The same transformation formulas hold when z_{23} is replaced with z'_{23}.

Proof. The first result follows from [Schoeneberg 1974, p. 217, Theorem 4] by taking $r = 1$, $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & (p+1)/2 \end{smallmatrix}\right)$, $h = (0, 0)$, $k = 0$ and $P_k = 1$. The corresponding result for z'_{23} follows by taking $A = \left(\begin{smallmatrix} 4 & 1 \\ 1 & 6 \end{smallmatrix}\right)$, with the other parameters being the same as for the case $p = 23$.

The second result is a direct consequence of [Schoeneberg 1974, p. 205, (5)]. □

Lemma 2.3. *Let $p \equiv 3 \pmod 4$ be prime and let n be a positive integer. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$. Then for $F_k(\tau;\, p)$ defined by (4)–(6), we have*

$$F_k\Big(\frac{a\tau+b}{c\tau+d};\, p\Big) = \Big(\frac{d}{p}\Big)^k (c\tau+d)^k F_k(\tau;\, p)$$

and

$$F_k\Big(\frac{-1}{\tau\sqrt{p}};\, p\Big) = (-i\tau)^k F_k\Big(\frac{\tau}{\sqrt{p}};\, p\Big).$$

Proof. For odd values of k, these follow from [Cooper 2008, Theorem 6.1] or [Kolberg 1968, (1.8)–(1.12)]. For even values of k with $k \geq 4$, these follow from the well-known transformation formulas for $E_{2k}(\tau)$, for example, see [Serre 1973, pp. 83, 92, 95–96]. For $k = 2$, the results are most easily proved by appealing to the transformation formulas for the function $\big(\frac{\eta(p\tau)}{\eta(\tau)}\big)^{24}$ in [Apostol 1990, pp. 84–85, Theorems 4.7 and 4.8], and then applying logarithmic differentiation. □

Lemma 2.4. *Let $f(\tau)$ be analytic and bounded in the upper half plane* $\mathrm{Im}(\tau) > 0$, *and suppose it satisfies the transformation property*

$$f\Big(\frac{a\tau+b}{c\tau+d}\Big) = f(\tau) \quad \textit{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p). \tag{15}$$

Then f is constant.

Proof. This is Theorem 4.4 in [Apostol 1990, p. 79]. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $p = 3$, 7, 11 or 23, and let k be a positive integer. Let ℓ be the smallest integer that satisfies $\frac{24\ell}{p+1} \geq k$. Consider the functions

$$\varphi(\tau) \;=\; \varphi_{p,k}(\tau) = \frac{F_k(\tau;\, p)}{(z_p(\tau))^k}\Big(\frac{z_p(\tau)}{\eta(\tau)\eta(p\tau)}\Big)^{24\ell/(p+1)} \quad \text{and}$$
$$\psi(\tau) \;=\; \psi_p(\tau) = \Big(\frac{z_p(\tau)}{\eta(\tau)\eta(p\tau)}\Big)^{24/(p+1)}.$$

By Lemmas 2.1–2.3, $\varphi(\tau)$ and $\psi(\tau)$ satisfy the transformation property (15). Furthermore, φ and ψ are both analytic in the upper half plane $0 < \mathrm{Im}(\tau) < \infty$, as $\eta(\tau)$ does not vanish in this region. Let us analyze the behavior at $\tau = i\infty$. From the q-expansions, we find that

$$\varphi(\tau) = \frac{(1+O(q))}{(1+O(q))^k}\left(\frac{1+O(q)}{q+O(q^2)}\right)^{\ell} = q^{-\ell} + O(q^{-\ell+1}) \quad \text{as } \tau \to i\infty.$$

Therefore $\varphi(\tau)$ has a pole of order ℓ at $i\infty$. Similarly, we find that $\psi(\tau)$ has a pole of order 1 at $i\infty$. It follows that there exist constants $b_1, \ldots, b_\ell$, such that the function

$$\lambda(\tau) := \varphi(\tau) - \sum_{j=1}^{\ell} b_j\, (\psi(\tau))^j \tag{16}$$

has no pole at $i\infty$. That is to say,

$$\lambda(\tau) = b_0 + O(q) \quad \text{as } \tau \to i\infty$$

for some constant b_0. Let us consider the behavior of $\lambda(\tau)$ at $\tau = 0$. By the second result in each of Lemmas 2.1–2.3, we find that

$$\varphi\Big(\frac{-1}{\tau\sqrt{p}}\Big) = \varphi\Big(\frac{\tau}{\sqrt{p}}\Big) \quad \text{and} \quad \psi\Big(\frac{-1}{\tau\sqrt{p}}\Big) = \psi\Big(\frac{\tau}{\sqrt{p}}\Big).$$

Therefore

$$\lambda(\tau) = \lambda\Big(\frac{-1}{p\tau}\Big) \longrightarrow b_0 \quad \text{as} \quad \tau \to 0.$$

It follows from the description of the fundamental region for $\Gamma_0(p)$ given in [Apostol 1990, p. 76, Theorem 4.2] that $\lambda(\tau)$ is bounded in the upper half plane. Hence by Lemma 2.4, $\lambda(\tau)$ is constant, that is, $\lambda(\tau) \equiv b_0$. Therefore, from (16) we have

$$\varphi(\tau) = \sum_{j=0}^{\ell} b_j\, (\psi(\tau))^j\,.$$

Using the fact that $\psi(\tau) = 1/\Lambda_p(\tau)$, this is equivalent to

$$F_k(\tau;\, p) = z_p^k \sum_{j=0}^{\ell} b_j \Lambda_p^{\ell-j} = z_p^k \sum\nolimits_{0\le j\le (p+1)k/24} c_j \Lambda_p^j,$$

where $c_j = b_{\ell-j}$. Letting $q = 0$ on both sides we deduce that $c_0 = 1$.

If we replace z_{23} and Λ_{23} by z'_{23} and Λ'_{23}, respectively, at every step in the proof, we establish the result for z'_{23} and Λ'_{23}.

This completes the proof of Theorem 1.2. □

Remarks. For $p = 3, 7, 11$ or 23, the genus of the normalizer of $\Gamma_0(p)$ in $SL_2(\mathbb{R})$ (denoted by $\Gamma_0(p)+$) is 0. It turns out that for each p, the field of functions invariant under $\Gamma_0(p)+$ is generated by $\psi_p(\tau)$, which has a simple pole at $\tau = i\infty$. Since $\varphi_{p,k}(\tau)$ has a pole of order ℓ at $\tau = i\infty$ and $\varphi_{p,k}(\tau)$ is a function on $\Gamma_0(p)+$, it follows that $\varphi_{p,k}(\tau)$ is a polynomial in $\psi_p(\tau)$ with degree exactly ℓ. This explains the existence of relation (16).

The identity (13) may be proved similarly.

Proof of (13). Let

$$F(\tau) = \frac{z_7^3}{\eta^3(\tau)\eta^3(7\tau)} \quad \text{and} \quad G(\tau) = \frac{\eta^4(\tau)}{\eta^4(7\tau)}.$$

Lemmas 2.1 and 2.2 imply $F(\tau)$ satisfies the transformation formula (15). Furthermore, [Apostol 1990, p. 87, Theorem 4.9] implies that $G(\tau)$ also satisfies the transformation formula (15). The q-expansions are

$$F(\tau) = \frac{1}{q} + O(1) \quad \text{and} \quad G(\tau) = \frac{1}{q} + O(1) \quad \text{as} \quad \tau \to i\infty. \tag{17}$$

Hence $F(\tau)$ and $G(\tau)$ both have a pole of order 1 at $\tau = i\infty$.

By the second parts of Lemmas 2.1 and 2.2, and by the transformation formula for the Dedekind eta-function [Apostol 1990, p. 52, Theorem 3.4], we have

$$F\left(\frac{-1}{\tau}\right) = F(\tau) \quad \text{and} \quad G\left(\frac{-1}{\tau}\right) = \frac{49}{G(\tau)}. \tag{18}$$

Therefore at the point $\tau = 0$, $F(\tau)$ has a pole of order 1 and $G(\tau)$ has a zero of order 1.

Let

$$H(\tau) := F(\tau) - aG(\tau) - \frac{b}{G(\tau)},$$

where a and b are constants that will be chosen so that $H(\tau)$ has no pole at 0 or $i\infty$. In order for there to be no pole at $\tau = i\infty$, (17) implies $a = 1$. In order for there to be no pole at $\tau = 0$, (17) and (18) imply $b = 49$. It follows that the function $H(\tau)$ with these values of a and b is bounded in the upper half plane, and Lemma 2.4 implies that it is constant. That is,

$$\frac{z_7^3}{\eta^3(\tau)\eta^3(7\tau)} = c + \frac{\eta^4(\tau)}{\eta^4(7\tau)} + 49\frac{\eta^4(7\tau)}{\eta^4(\tau)},$$

for some constant c. If we multiply by $\eta^3(\tau)\eta^3(7\tau)$ and compare coefficients of q on both sides, we find that $c = 13$. This completes the proof of (13). $\square$

3. Application to convolution sums

Let $\sigma_j(n)$ denote the sum of the j-th powers of the divisors of n, and let $\sigma(n) = \sigma_1(n)$. The convolution sum

$$W_k(n) = \sum_{1 \le m < n/k} \sigma(m)\sigma(n - km)$$

has been evaluated for $1 \le k \le 14$ and $k = 16$, 18 and 24. See [Alaca et al. 2007] and [Royer 2007] for references. In this section, we show how Theorem 1.2 leads to an evaluation of $W_k(n)$ for the cases $k = 3$, 7, 11 and 23. The case $k = 23$ is new. Let

$$\begin{aligned} P(q) = E_2(\tau) &= 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \\ Q(q) = E_4(\tau) &= 1 + 240\sum_{j=1}^{\infty} \frac{j^3q^j}{1-q^j}, \\ S(q) = -\frac{q}{24}\frac{d}{dq}P(q) &= \sum_{j=1}^{\infty} \frac{j^2q^j}{(1-q^j)^2}. \end{aligned}$$

Theorem 3.1. *For $p = 3$, 7, 11 and 23 we have*

$$P(q)P(q^p) = \frac{1}{p^2+1}(Q(q)+p^2Q(q^p)) - \frac{144}{p}(S(q)+p^2S(q^p)) - 576\, z_p^4 u_p(\Lambda_p),$$

where

$$\begin{aligned} u_3(\Lambda_3) &= 0, \\ u_7(\Lambda_7) &= \tfrac{1}{70}\Lambda_7, \\ u_{11}(\Lambda_{11}) &= \tfrac{1}{671}(15\Lambda_{11} - 17\Lambda_{11}^2), \\ u_{23}(\Lambda_{23}) &= \tfrac{1}{2438}(77\Lambda_{23} - 222\Lambda_{23}^2 + 201\Lambda_{23}^3 - 30\Lambda_{23}^4). \end{aligned}$$

Proof. By Theorem 1.2 with $k = 2$ and 4, we have

$$\frac{pP(q^p) - P(q)}{p-1} = z_p^2\Bigl(1 - \sum_{1 \le j \le (p+1)/12} c_{p,j}\Lambda_p^j\Bigr), \tag{19}$$

$$\frac{p^2Q(q^p) + Q(q)}{p^2+1} = z_p^4\Bigl(1 - \sum_{1 \le j \le (p+1)/6} d_{p,j}\Lambda_p^j\Bigr), \tag{20}$$

for some constants $c_{p,j}$ and $d_{p,j}$. If we square (19) and subtract the result from (20), we obtain

$$\frac{p^2Q(q^p) + Q(q)}{p^2+1} - \frac{(pP(q^p) - P(q))^2}{(p-1)^2} = z_p^4 \sum_{1 \le j \le (p+1)/6} d'_{p,j}\Lambda_p^j,$$

for some constants $d'_{p,j}$. This may be rewritten as

$$P(q)P(q^p) = \frac{1}{2p}(p^2P^2(q^p) + P^2(q)) - \frac{(p-1)^2}{2p(p^2+1)}(p^2Q(q^p) + Q(q)) \\ + z_p^4 \sum_{1\le j\le (p+1)/6} d''_{p,j}\Lambda_p^j,$$

for some constants $d''_{p,j}$. Now use the result (see [Chan 2007; Glaisher 1885] or [Ramanujan 2000, p. 142, Eq. (30)])

$$P^2(q) = Q(q) - 288S(q)$$

to get

$$P(q)P(q^p) = \frac{1}{p^2+1}(Q(q) + p^2Q(q^p)) - \frac{144}{p}(S(q) + p^2S(q^p)) \\ + z_p^4 \sum_{1\le j\le (p+1)/6} d''_{p,j}\Lambda_p^j.$$

The values of the coefficients $d''_{p,j}$ may be determined by expanding in powers of q and equating coefficients of q^j for $1 \le j \le (p+1)/6$. In this way we obtain the polynomials $u_p(\Lambda_p)$ given in the statement of the theorem. This completes the proof. □

Theorem 3.2. *For $p = 3, 7, 11$ and 23 we have*

$$W_p(n) = \frac{5}{12(p^2+1)}\Big(\sigma_3(n) + p^2\sigma_3\Big(\frac{n}{p}\Big)\Big) \\ + \Big(\frac{1}{24} - \frac{n}{4p}\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{n}{4}\Big)\sigma\Big(\frac{n}{p}\Big) - c_p(n).$$

Here $c_p(n)$ is defined by

$$\sum_{n=1}^{\infty} c_p(n)q^n = z_p^4 u_p(\Lambda_p),$$

and $u_p(\Lambda_p)$ is as in Theorem 3.1.

Proof. Equate coefficients of q^n on both sides of the identity in Theorem 3.1. □

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