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Consider the first order linear difference equation

$$\Delta u(k) + p(k)\, u(\tau(k)) = 0, \quad k \in N,$$

where $\Delta u(k) = u(k+1) - u(k)$, $p : N \to \mathbb{R}_+$, $\tau : N \to N$, $\tau(k) \le k-2$ and $\lim_{k\to+\infty} \tau(k) = +\infty$. Optimal conditions for the oscillation of all proper solutions of this equation are established. The results lead to a sharp oscillation condition, when $k - \tau(k) \to +\infty$ as $k \to +\infty$. Examples illustrating the results are given.

1. Introduction

The first systematic study for the oscillation of all solutions to the first order delay differential equation

$$u'(t) + p(t)\, u(\tau(t)) = 0, \tag{1-1}$$

where

$$p \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+),\ \ \tau \in C(\mathbb{R}_+; \mathbb{R}_+),\ \ \tau(t) \le t \text{ for } t \in \mathbb{R}_+ \text{ and } \lim_{t\to+\infty} \tau(t) = +\infty,$$

in the case of constant coefficients and constant delays was made by Myshkis [1972]. For the differential equation (1-1) the problem of oscillation is investigated by many authors. See, for example, [Elbert and Stavroulakis 1995; Koplatadze and Chanturiya 1982; Koplatadze and Kvinikadze 1994; Ladas et al. 1984; Sficas and Stavroulakis 2003] and the references cited therein.

Theorem 1.1 [Koplatadze and Chanturiya 1982]. *Assume that*

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)\, ds > \frac{1}{e}. \tag{1-2}$$

Then all solutions of Equation (1-1) *oscillate.*

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It is to be emphasized that condition (1-2) is optimal in the sense that it cannot be replaced by the condition

$$\liminf_{t\to+\infty}\int_{\tau(t)}^{t} p(s)\,ds \ge \frac{1}{e}. \tag{1-3}$$

For example, if $\tau(t)=t-\delta$ or $\tau(t)=\alpha t$ or $\tau(t)=t^{\alpha}$, where $\delta>0$, $\alpha\in(0,1)$, examples can be given such that condition (1-3) is satisfied, but (1-1) has a nonoscillatory solution.

The discrete analogue of the first order delay differential equation (1-1) is the first order difference equation

$$\Delta u(k)+p(k)\,u(\tau(k))=0, \tag{1-4}$$

where

$$\begin{gathered}\Delta u(k)=u(k+1)-u(k),\qquad p:N\to\mathbb{R}_+,\\ \tau:N\to N,\qquad \tau(k)\le k-1,\qquad \lim_{k\to+\infty}\tau(k)=+\infty.\end{gathered} \tag{1-5}$$

By a *proper solution* of (1-4) we mean a function $u:N_{n_0}\to\mathbb{R}$ with $n_0=\min\{\tau(k):k\in N_n\}$ and $N_n=\{n,n+1,\dots\}$, which satisfies (1-4) on N_n and $\sup\{|u(i)|:i\ge k\}>0$ for $k\in N_{n_0}$.

A proper solution $u:N_{n_0}\to\mathbb{R}$ of (1-4) is said to be *oscillatory* (around zero) if for any positive integer $n\in N_{n_0}$ there exist $n_1,n_2\in N_n$ such that $u(n_1)\,u(n_2)\le 0$. Otherwise, the proper solution is said to be *nonoscillatory*. In other words, a proper solution u is oscillatory if it is neither eventually positive nor eventually negative.

Oscillatory properties of the solutions of (1-4), in the case of a general delay argument $\tau(k)$, have been recently investigated in [Chatzarakis et al. 2008a; 2008b], while the special case when $\tau(k)=k-n$, $n\ge 1$, has been studied rather extensively. See, for example, [Agarwal et al. 2005; Baštinec and Diblik 2005; Chatzarakis and Stavroulakis 2006; Domshlak 1999; Elaydi 1999; Ladas et al. 1989] and the references cited therein. In this particular case, (1-4) becomes

$$\Delta u(k)+p(k)\,u(k-n)=0,\qquad k\in N. \tag{1-6}$$

For this equation Ladas, Philos and Sficas established the following theorem.

Theorem 1.2 [Ladas et al. 1989]. *Assume that*

$$\liminf_{k\to+\infty}\sum_{i=k-n}^{k-1} p(i) > \left(\frac{n}{n+1}\right)^{n+1}. \tag{1-7}$$

Then all proper solutions of (1-6) *oscillate.*

This result is sharp in the sense that the inequality (1-7) cannot be replaced by the nonstrong one for any $n \in N$. Hence, Theorem 1.2 is the discrete analogue of Theorem 1.1 when $\tau(t) = t - \delta$.

An interesting question then arises whether there exists the discrete analogue of Theorem 1.1 for (1-4) in the case of a general delay argument $\tau(k)$ when $\lim_{k\to+\infty}(k - \tau(k)) = +\infty$.

In the present paper optimal conditions for the oscillation of all proper solutions of (1-4) are established and a positive answer to the above question is given.

2. Some auxiliary lemmas

Let $k_0 \in N$. Denote by $\mathbf{U}_{k_0}$ the set of all proper solutions of (1-4) satisfying the condition $u(k) > 0$ for $k \geq k_0$.

Remark 2.1. We will suppose that $\mathbf{U}_{k_0} = \varnothing$, if (1-4) has no solution satisfying the condition $u(k) > 0$ for $k \geq k_0$.

Lemma 2.2. *Assume that* $k_0 \in N$, $\mathbf{U}_{k_0} \neq \varnothing$, $u \in \mathbf{U}_{k_0}$, $\tau(k) \leq k - 1$, τ *is a nondecreasing function and*

$$\liminf_{k\to+\infty} \sum_{i=\tau(k)}^{k-1} p(i) = c > 0. \tag{2-1}$$

Then

$$\limsup_{k\to+\infty} \frac{u(\tau(k))}{u(k+1)} \leq \frac{4}{c^2}. \tag{2-2}$$

Proof. By (2-1), for any $\varepsilon \in (0, c)$, it is clear that

$$\sum_{i=\tau(k)}^{k-1} p(i) \geq c - \varepsilon \quad \text{for} \quad k \in N_{k_0}. \tag{2-3}$$

Since u is a positive proper solution of (1-4), then there exists $k_1 \in N_{k_0}$ such that

$$u(\tau(k)) > 0 \quad \text{for} \quad k \in N_{k_1}.$$

Thus, from (1-4) we have

$$u(k+1) - u(k) = -p(k)u(\tau(k)) \leq 0$$

and so u is an eventually nonincreasing function of positive numbers.

Now from inequality (2-3) it is clear that, there exists $k^* \geq k$ such that

$$\sum_{i=k}^{k^*-1} p(i) < \frac{c-\varepsilon}{2} \quad \text{and} \quad \sum_{i=k}^{k^*} p(i) \geq \frac{c-\varepsilon}{2}. \tag{2-4}$$

This is because in the case where $p(k) < \frac{c-\varepsilon}{2}$, it is clear that there exists $k^* > k$ such that (2-4) is satisfied, while in the case where $p(k) \geq \frac{c-\varepsilon}{2}$, then $k^* = k$, and therefore

$$\sum_{i=k}^{k^*-1} p(i) = \sum_{i=k}^{k-1} p(i) \text{ (by which we mean) } = 0 < \frac{c-\varepsilon}{2}$$

and

$$\sum_{i=k}^{k^*} p(i) = \sum_{i=k}^{k} p(i) = p(k) \geq \frac{c-\varepsilon}{2}.$$

That is, in both cases (2-4) is satisfied.

Now, we will show that $\tau(k^*) \leq k-1$. Indeed, in the case where $p(k) \geq \frac{c-\varepsilon}{2}$, since $k^* = k$, it is obvious that $\tau(k^*) \leq k-1$. In the case where $p(k) < \frac{c-\varepsilon}{2}$, then $k^* > k$. Assume, for the sake of contradiction, that $\tau(k^*) > k-1$. Hence, $k \leq \tau(k^*) \leq k^*-1$ and then

$$\sum_{i=\tau(k^*)}^{k^*-1} p(i) \leq \sum_{i=k}^{k^*-1} p(i) < \frac{c-\varepsilon}{2}.$$

This, in view of (2-3), leads to a contradiction. Thus, in both cases, we have $\tau(k^*) \leq k-1$.

Therefore, it is clear that

$$\sum_{i=\tau(k^*)}^{k-1} p(i) = \sum_{i=\tau(k^*)}^{k^*-1} p(i) - \sum_{i=k}^{k^*-1} p(i) \geq (c-\varepsilon) - \frac{c-\varepsilon}{2} = \frac{c-\varepsilon}{2}. \tag{2-5}$$

Now, summing up (1-4) first from k to k^* and then from $\tau(k^*)$ to $k-1$, and using that the function u is nonincreasing and the function τ is nondecreasing, we have

$$u(k) - u(k^*+1) = \sum_{i=k}^{k^*} p(i)u(\tau(i)) \geq \left(\sum_{i=k}^{k^*} p(i)\right) u(\tau(k^*)) \geq \frac{c-\varepsilon}{2} u(\tau(k^*)),$$

or

$$u(k) \geq \frac{c-\varepsilon}{2} u(\tau(k^*)), \tag{2-6}$$

and then

$$u(\tau(k^*)) - u(k) = \sum_{i=\tau(k^*)}^{k-1} p(i)u(\tau(i)) \geq \left(\sum_{i=\tau(k^*)}^{k-1} p(i)\right) u(\tau(k-1)) \geq \frac{c-\varepsilon}{2} u(\tau(k-1)),$$

or

$$u(\tau(k^*)) \geq \frac{c-\varepsilon}{2} u(\tau(k-1)). \tag{2-7}$$

Combining inequalities (2-6) and (2-7), we obtain

$$\frac{u(\tau(k-1))}{u(k)} \le \frac{4}{(c-\varepsilon)^2}$$

and, for large k, we have

$$\frac{u(\tau(k))}{u(k+1)} \le \frac{4}{(c-\varepsilon)^2}.$$

Hence,

$$\limsup_{k\to+\infty} \frac{u(\tau(k))}{u(k+1)} \le \frac{4}{(c-\varepsilon)^2},$$

which, for arbitrarily small values of ε, implies (2-2). □

Lemma 2.3. *Assume that* $k_0 \in N$, $\mathbf{U}_{k_0} \neq \varnothing$, $u \in \mathbf{U}_{k_0}$, $\tau(k) \le k-1$, τ *is a nondecreasing function and condition* (2-1) *is satisfied. Then*

$$\lim_{k\to+\infty} u(k)\, \exp\left(\lambda \sum_{i=1}^{k-1} p(i)\right) = +\infty \quad \textit{for any} \quad \lambda > \frac{4}{c^2}. \tag{2-8}$$

Proof. Since all the conditions of Lemma 2.2 are satisfied, for any $\gamma > 4/c^2$, there exists $k_1 \in N_{k_0}$ such that

$$\frac{u(\tau(k))}{u(k+1)} \le \gamma \quad \text{for} \quad k \in N_{k_1}. \tag{2-9}$$

Also, for any $n \in N_{k_1}$

$$\begin{aligned}\sum_{k=k_1}^{n} \frac{\Delta u(k)}{u(k+1)} &= \sum_{k=k_1}^{n} \left(1 - \frac{u(k)}{u(k+1)}\right) = (n-k_1) - \sum_{k=k_1}^{n} \exp\left(\ln \frac{u(k)}{u(k+1)}\right) \\ &\le (n-k_1) - \sum_{k=k_1}^{n} \left(1 + \ln \frac{u(k)}{u(k+1)}\right) = -\sum_{k=k_1}^{n} \ln \frac{u(k)}{u(k+1)} = \ln \frac{u(n+1)}{u(k_1)},\end{aligned}$$

or

$$\sum_{k=k_1}^{n} \frac{\Delta u(k)}{u(k+1)} \le \ln \frac{u(n+1)}{u(k_1)}.$$

Moreover, from (1-4), we have

$$\sum_{k=k_1}^{n} \frac{\Delta u(k)}{u(k+1)} = -\sum_{k=k_1}^{n} p(k)\, \frac{u(\tau(k))}{u(k+1)}.$$

Combining (2-9) with the last two relations, we obtain

$$u(n+1) \ge u(k_1)\, \exp\left(-\gamma \sum_{k=k_1}^{n} p(k)\right).$$

Now, by (2-1), it is obvious that $\sum^{+\infty} p(i) = +\infty$. Therefore, for $\lambda > 4/c^2$, the last inequality yields

$$\lim_{n\to+\infty} u(n+1)\exp\left(\lambda \sum_{k=k_1}^{n} p(k)\right) = +\infty,$$

or

$$\lim_{k\to+\infty} u(k)\exp\left(\lambda \sum_{i=k_1}^{k-1} p(i)\right) = +\infty,$$

which implies (2-8), since

$$\sum_{i=1}^{k-1} p(i) \geq \sum_{i=k_1}^{k-1} p(i). \qquad \square$$

Next, consider the difference inequality

$$\Delta u(k) + q(k)\, u(\sigma(k)) \leq 0, \tag{2-10}$$

where

$$q : N \to \mathbb{R}_+, \quad \sigma : N \to N \quad \text{and} \quad \lim_{k\to+\infty} \sigma(k) = +\infty.$$

In the sequel the following lemma will be used, which has recently been established in [Chatzarakis et al. 2008a].

Lemma 2.4. *Assume that* (2-1) *is satisfied, and for sufficiently large k*

$$\sigma(k) \leq \tau(k) \leq k-1, \qquad p(k) \leq q(k)$$

and $u : N_{k_0} \to (0, +\infty)$ is a positive proper solution of (2-10)*. Then, there exists $k_1 \in N_{k_0}$ such that $\mathbf{U}_{k_1} \neq \varnothing$ and $u_* \in \mathbf{U}_{k_1}$ is the solution of* (1-4)*, which satisfies the condition*

$$0 < u_*(k) \leq u(k) \quad \textit{for} \quad k \in N_{k_1}.$$

By virtue of Lemma 2.4, we can formulate Lemma 2.3 in the following more general form, where the function τ is not required to be nondecreasing.

Lemma 2.5. *Assume that $k_0 \in N$, $\mathbf{U}_{k_0} \neq \varnothing$, $u \in \mathbf{U}_{k_0}$, $\tau(k) \leq k-1$ and condition* (2-1) *is satisfied. Then, for any $\lambda > 4/c^2$, condition* (2-8) *holds.*

Proof. Since $u : N_{k_0} \to (0, +\infty)$ is a solution of (1-4), it is clear that u is a solution of the inequality

$$\Delta u(k) + p(k)\, u(\sigma(k)) \leq 0 \quad \text{for} \quad k \in N_{k_1},$$

where $\sigma(k) = \max\{\tau(i) : 1 \leq s \leq k,\ s \in N\}$ and $k_1 > k_0$ is a sufficiently large number.

First we will show that

$$\liminf_{k\to+\infty}\sum_{i=\sigma(k)}^{k-1} p(i)=c. \tag{2-11}$$

Assume that (2-11) is not satisfied. Then there exists a sequence $\{k_i\}_{i=1}^{+\infty}$ of natural numbers such that $\sigma(k_i)\neq\tau(k_i)$ $(i=1,2,\dots)$ and

$$\liminf_{j\to+\infty}\sum_{i=\sigma(k_j)}^{k_j-1} p(i)=c_1<c. \tag{2-12}$$

Also, from the definition of the function σ, and in view of $\sigma(k_i)\neq\tau(k_i)$, for any k_i, there exists $k_i'<k_i$ such that $\sigma(k)=\sigma(k_i)$ for $k_i'\le k\le k_i$, $\lim_{i\to+\infty}k_i'=+\infty$ and $\sigma(k_i')=\tau(k_i')$. Thus

$$\sum_{j=\tau(k_i')}^{k_i'-1} p(j)=\sum_{j=\sigma(k_i')}^{k_i'-1} p(j)=\sum_{j=\sigma(k_i)}^{k_i'-1} p(j)\le\sum_{j=\sigma(k_i)}^{k_i-1} p(j)\quad(i=1,2,\dots),$$

and, by the virtue of (2-12), we have

$$\liminf_{i\to+\infty}\sum_{j=\tau(k_i')}^{k_i'-1} p(j)\le\liminf_{i\to+\infty}\sum_{j=\sigma(k_i)}^{k_i-1} p(j)=c_1<c.$$

In view of (2-1), the last inequality leads to a contradiction. Therefore (2-11) holds.

Now, by Lemma 2.4, we conclude that the equation

$$\Delta u(k)+p(k)\,u(\sigma(k))=0$$

has a solution u_* which satisfies the condition

$$0<u_*(k)\le u(k)\quad\text{for}\quad k\in N_{k_1}, \tag{2-13}$$

where $k_1>k_0$ is a sufficiently large number. Hence, taking into account that the function σ is nondecreasing, in view of Lemma 2.3, we have

$$\lim_{k\to+\infty}u_*(k)\,\exp\biggl(\lambda\sum_{i=1}^{k-1}p(i)\biggr)=+\infty,$$

where $\lambda>4/c^2$. Therefore, by (2-13), we get

$$\lim_{k\to+\infty}u(k)\,\exp\biggl(\lambda\sum_{i=1}^{k-1}p(i)\biggr)=+\infty\quad\text{for any}\quad\lambda>\frac{4}{c^2}.\qquad\square$$

Lemma 2.6 (Abel transformation). *Let* $\{a_i\}_{i=1}^{+\infty}$ *and* $\{b_i\}_{i=1}^{+\infty}$ *be sequences of non-negative numbers and*

$$\sum_{i=1}^{+\infty} a_i < +\infty. \tag{2-14}$$

Then

$$\sum_{i=1}^{k} a_i\, b_i = A_1\, b_1 - A_{k+1}\, b_{k+1} - \sum_{i=1}^{k} A_{i+1}(b_i - b_{i+1}),$$

where $A_i = \sum_{j=i}^{+\infty} a_j$.

Proof. Since (2-14) is satisfied, we have

$$\begin{aligned}\sum_{i=1}^{k} A_{i+1}(b_i - b_{i+1}) &= \sum_{i=1}^{k} A_{i+1} b_i - \sum_{i=2}^{k+1} A_i b_i \\ &= A_2 b_1 - A_{k+1} b_{k+1} + \sum_{i=2}^{k} (A_{i+1} - A_i) b_i \\ &= A_2 b_1 - A_{k+1} b_{k+1} - \sum_{i=2}^{k} a_i b_i \\ &= A_1 b_1 - A_{k+1} b_{k+1} - \sum_{i=1}^{k} a_i b_i,\end{aligned}$$

or

$$\sum_{i=1}^{k} a_i b_i = A_1 b_1 - A_{k+1} b_{k+1} - \sum_{i=1}^{k} A_{i+1}(b_i - b_{i+1}). \qquad \square$$

Koplatadze, Kvinikadze and Stavroulakis established the following lemma. For completeness, we present the proof here.

Lemma 2.7 [Koplatadze et al. 2002]. *Let* $\varphi, \psi : N \to (0, +\infty)$, ψ *be nonincreasing and suppose*

$$\lim_{k \to +\infty} \varphi(k) = +\infty, \tag{2-15}$$

$$\liminf_{k \to +\infty} \psi(k)\, \widetilde{\varphi}(k) = 0, \tag{2-16}$$

where $\widetilde{\varphi}(k) = \inf\{\varphi(s) : s \geq k,\ s \in N\}$. *Then there exists an increasing sequence of natural numbers* $\{k_i\}_{i=1}^{+\infty}$ *such that*

$$\lim_{i \to +\infty} k_i = +\infty, \quad \widetilde{\varphi}(k_i) = \varphi(k_i), \quad \psi(k)\, \widetilde{\varphi}(k) \geq \psi(k_i)\, \widetilde{\varphi}(k_i)$$
$$(k = 1, 2, \ldots, k_i\ ;\ i = 1, 2, \ldots).$$

Proof. Define the sets E_1 and E_2 by

$$k \in E_1 \iff \widetilde{\varphi}(k) = \varphi(k),$$
$$k \in E_2 \iff \widetilde{\varphi}(s)\,\psi(s) \geq \widetilde{\varphi}(k)\,\psi(k) \text{ for } s \in \{1, \dots, k\}.$$

According to (2-15) and (2-16), it is obvious that

$$\sup E_i = +\infty \qquad (i = 1, 2). \tag{2-17}$$

Show that

$$\sup E_1 \cap E_2 = +\infty. \tag{2-18}$$

Let $k_0 \in E_2$ be such that $k_0 \notin E_1$. By (2-16) there is $k_1 > k_0$ such that $\widetilde{\varphi}(k) = \widetilde{\varphi}(k_1)$ for $k = k_0, k_0 + 1, \dots, k_1$ and $\widetilde{\varphi}(k_1) = \varphi(k_1)$. Since ψ is nonincreasing, we have

$$\widetilde{\varphi}(k)\,\psi(k) \geq \widetilde{\varphi}(k_1)\,\psi(k_1) \quad \text{for} \quad k = 1, \dots, k_1.$$

Therefore $k_1 \in E_1 \cap E_2$. The above argument together with (2-17) imply that (2-18) holds. □

Remark 2.8. The analogue of this lemma for continuous functions φ and ψ was proved first in [Koplatadze 1994].

3. Necessary conditions of the existence of positive solutions

The results of this section play an important role in establishing sufficient conditions for all proper solutions of (1-4) to be oscillatory.

Theorem 3.1. *Assume that* $k_0 \in N$, $\mathbf{U}_{k_0} \neq \varnothing$, (1-5) *is satisfied*,

$$\liminf_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) = c > 0, \tag{3-1}$$

and

$$\limsup_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) < +\infty. \tag{3-2}$$

Then there exists $\lambda \in [1, 4/c^2]$ *such that*

$$\limsup_{\varepsilon \to 0+} \left(\liminf_{k \to +\infty} \exp\left((\lambda + \varepsilon) \sum_{i=1}^{k-1} p(i) \right) \sum_{i=k}^{+\infty} p(i) \exp\left(-(\lambda + \varepsilon) \sum_{l=1}^{\tau(i)-1} p(l) \right) \right) \leq 1. \tag{3-3}$$

Proof. Since $U_{k_0} \neq \varnothing$, Equation (1-4) has a positive solution $u : N_{k_0} \to (0, +\infty)$. First we show that

$$\limsup_{k\to+\infty} u(k) \exp\left(\sum_{i=1}^{k-1} p(i)\right) < +\infty. \tag{3-4}$$

Indeed, if $k_1 \in N_{k_0}$, we have

$$\begin{aligned}\sum_{i=k_1}^{k} \frac{\Delta u(i)}{u(i)} &= \sum_{i=k_1}^{k} \frac{u(i+1)}{u(i)} - (k-k_1) = \sum_{i=k_1}^{k} \exp\Big(\ln \frac{u(i+1)}{u(i)}\Big) - (k-k_1)\\ &\geq \sum_{i=k_1}^{k} \Big(1 + \ln \frac{u(i+1)}{u(i)}\Big) - (k-k_1) = \ln \frac{u(k+1)}{u(k_1)},\end{aligned}$$

or

$$\sum_{i=k_1}^{k} \frac{\Delta u(i)}{u(i)} \geq \ln \frac{u(k+1)}{u(k_1)}.$$

By (1-4), and taking into account that the function u is nonincreasing, we have

$$\sum_{i=k_1}^{k} \frac{\Delta u(i)}{u(i)} = -\sum_{i=k_1}^{k} p(i) \frac{u(\tau(i))}{u(i)} \leq -\sum_{i=k_1}^{k} p(i).$$

Combining the last two inequalities, we obtain

$$u(k+1) \exp\left(\sum_{i=k_1}^{k} p(i)\right) \leq u(k_1),$$

that is, (3-4) is fulfilled. On the other hand, since all the conditions of Lemma 2.5 are satisfied, we conclude that condition (2-8) holds for any $\lambda > 4/c^2$. Denote by Λ the set of all λ for which

$$\lim_{k\to+\infty} u(\tau(k)) \exp\Big(\lambda \sum_{i=1}^{\tau(k)-1} p(i)\Big) = +\infty \tag{3-5}$$

and $\lambda_0 = \inf \Lambda$. In view of (1-5), (2-8) and (3-4), it is obvious that $\lambda_0 \in [1, 4/c^2]$. Thus, it suffices to show, that for $\lambda = \lambda_0$ the inequality (3-3) holds. First, we will show that for any $\varepsilon > 0$

$$\lim_{k\to+\infty} u(\tau(k)) \exp\Big((\lambda_0 + \varepsilon) \sum_{i=1}^{\tau(k)-1} p(i)\Big) = +\infty. \tag{3-6}$$

Indeed, if $\lambda_0 \in \Lambda$, it is obvious from (3-5) that condition (3-6) is fulfilled. If $\lambda_0 \notin \Lambda$, according to the definition of λ_0, there exists $\lambda_k > \lambda_0$ such that $\lambda_k \to \lambda_0$ when $k \to +\infty$ and $\lambda_k \in \Lambda$, $k = 1, 2, \ldots$. Thus, condition (3-5) holds for any $\lambda = \lambda_k$. However, for any $\varepsilon > 0$, there exists $\lambda_k = \lambda_k(\varepsilon)$ such that $\lambda_0 < \lambda_k \le \lambda_0 + \varepsilon$. This insures the validity of (3-5) and (3-6) for any $\varepsilon > 0$.

Similarly, we show that for any $\varepsilon > 0$,

$$\liminf_{k\to+\infty} u(\tau(k)) \exp\left((\lambda_0 - \varepsilon) \sum_{i=1}^{\tau(k)-1} p(i) \right) = 0. \tag{3-7}$$

Hence, by virtue of (1-5), (3-6) and (3-7), it is clear that for any $\varepsilon > 0$, the functions

$$\varphi(k) = u(\tau(k)) \exp\left((\lambda_0 + \varepsilon) \sum_{i=1}^{\tau(k)-1} p(i) \right) \tag{3-8}$$

and

$$\psi(k) = \exp\left(-2\varepsilon \sum_{i=1}^{k-1} p(i) \right)$$

satisfy the conditions of Lemma 2.7 for sufficiently large k. Hence, there exists an increasing sequence $\{k_i\}_{i=1}^{+\infty}$ of natural numbers satisfying $\lim_{i\to+\infty} k_i = +\infty$,

$$\psi(k_i)\, \widetilde{\varphi}(k_i) \le \psi(k)\, \widetilde{\varphi}(k) \quad \text{for} \quad k^* \le k \le k_i, \tag{3-9}$$

where k^* is a sufficiently large number, and

$$\widetilde{\varphi}(k_i) = \varphi(k_i) \quad (i = 1, 2, \ldots), \tag{3-10}$$

Now, given that

$$u(\tau(i)) \exp\left((\lambda_0 + \varepsilon) \sum_{l=1}^{\tau(i)-1} p(l) \right) \ge \inf\left\{ u(\tau(s)) \exp(\lambda_0 + \varepsilon) \sum_{l=1}^{\tau(s)-1} p(l) : s \ge i,\ s \in N \right\}$$
$$= \widetilde{\varphi}(i),$$

Equation (1-4) implies

$$u(\tau(k_j)) \ge \sum_{i=\tau(k_j)}^{+\infty} p(i)\, u(\tau(i)) \ge \sum_{i=\tau(k)}^{+\infty} p(i)\, \widetilde{\varphi}(i) \exp\left(-(\lambda_0 + \varepsilon) \sum_{l=1}^{\tau(i)-1} p(l) \right)$$

that is,

$$u(\tau(k_j)) \geq \sum_{i=\tau(k_j)}^{k_j-1} p(i)\,\widetilde{\varphi}(i) \exp\biggl(-2\varepsilon \sum_{l=1}^{i-1} p(l)\biggr) \exp\biggl(2\varepsilon \sum_{l=1}^{i-1} p(l)\biggr) \times \exp\biggl(-(\lambda_0+\varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)\biggr) + \sum_{i=k_j}^{+\infty} p(i)\,\widetilde{\varphi}(i) \exp\biggl(-(\lambda_0+\varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)\biggr),$$

for $j = 1, 2, \dots$. Thus, by (3-9), and using the fact that the function $\widetilde{\varphi}$ is nondecreasing, the last inequality yields

$$u(\tau(k_j)) \geq \widetilde{\varphi}(k_j)\, \exp\biggl(-2\varepsilon \sum_{l=1}^{k_j-1} p(l)\biggr) \times \sum_{i=\tau(k_j)}^{k_j-1} p(i) \exp\biggl(2\varepsilon \sum_{l=1}^{i-1} p(l)\biggr) \exp\biggl(-(\lambda_0+\varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)\biggr) + \widetilde{\varphi}(k_j) \sum_{i=k_j}^{+\infty} p(i) \exp\biggl(-(\lambda_0+\varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)\biggr) \quad (j = 1, 2, \dots). \tag{3-11}$$

Also, in view of Lemma 2.6, we have

$$\begin{aligned} I(k_j,\varepsilon) &= \sum_{i=\tau(k_j)}^{k_j-1} p(i) \exp\biggl(2\varepsilon \sum_{l=1}^{i-1} p(l)\biggr) \exp\biggl(-(\lambda_0+\varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)\biggr) \\ &= \exp\biggl(2\varepsilon \sum_{i=1}^{\tau(k_j)-1} p(i)\biggr) \sum_{i=\tau(k_j)}^{+\infty} p(i) \exp\biggl(-(\lambda_0+\varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)\biggr) \\ &\quad - \exp\biggl(2\varepsilon \sum_{i=1}^{k_j-1} p(i)\biggr) \sum_{i=k_j}^{+\infty} p(i) \exp\biggl(-(\lambda_0+\varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)\biggr) \\ &\quad + \sum_{i=k_j}^{k_j-1} \biggl(\exp\biggl(2\varepsilon \sum_{l=1}^{i} p(l)\biggr) - \exp\biggl(2\varepsilon \sum_{l=1}^{i-1} p(l)\biggr)\biggr) \\ &\quad \times \sum_{i=1}^{+\infty} p(i) \exp\biggl(-(\lambda_0+\varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)\biggr) \qquad (j = 1, 2, \dots). \end{aligned} \tag{3-12}$$

Given that

$$\exp\Bigl(2\varepsilon\sum_{l=1}^{i}p(l)\Bigr)-\exp\Bigl(2\varepsilon\sum_{l=1}^{i-1}p(l)\Bigr)\geq 0,$$

inequality (3-12) becomes

$$\begin{aligned}I(k_j,\varepsilon)\ \geq\ \exp\Bigl(2\varepsilon\sum_{i=1}^{\tau(k_j)-1}p(i)\Bigr)\sum_{i=\tau(k_j)}^{+\infty}p(i)\exp\Bigl(-(\lambda_0+\varepsilon)\sum_{l=1}^{\tau(i)-1}p(l)\Bigr)\\ -\exp\Bigl(2\varepsilon\sum_{i=1}^{k_j-1}p(i)\Bigr)\sum_{i=k_j}^{+\infty}p(i)\exp\Bigl(-(\lambda_0+\varepsilon)\sum_{l=1}^{\tau(i)-1}p(l)\Bigr).\end{aligned}$$

Therefore, by (3-11), we take

$$\begin{aligned}u(\tau(k_j))\geq\widetilde{\varphi}(k_j)\exp\Bigl(-2\varepsilon\sum_{l=1}^{k_j-1}p(l)\Bigr)\exp\Bigl(2\varepsilon\sum_{l=1}^{\tau(k_j)-1}p(l)\Bigr)\\ \times\sum_{i=\tau(k_j)}^{+\infty}p(i)\exp\Bigl(-(\lambda_0+\varepsilon)\sum_{l=1}^{\tau(i)-1}p(l)\Bigr).\end{aligned}$$

Thus, (3-8) and (3-10) imply

$$\begin{aligned}\exp\Bigl((\lambda_0+\varepsilon)\sum_{i=1}^{\tau(k_j)-1}p(i)\Bigr)\sum_{i=\tau(k_j)}^{+\infty}p(i)\exp\Bigl(-(\lambda_0+\varepsilon)\sum_{l=1}^{\tau(i)-1}p(l)\Bigr)\\ \leq\exp\Bigl(2\varepsilon\sum_{i=\tau(k_j)}^{k_j-1}p(i)\Bigr).\end{aligned}$$

From the last inequality, and taking into account that (3-2) is satisfied, we have

$$\begin{aligned}\limsup_{j\to+\infty}\exp\Bigl((\lambda_0+\varepsilon)\sum_{i=1}^{\tau(k_j)-1}p(i)\Bigr)\sum_{i=\tau(k_j)}^{+\infty}p(i)\exp\Bigl(-(\lambda_0+\varepsilon)\sum_{l=1}^{\tau(i)-1}p(l)\Bigr)\\ \leq\exp(2\varepsilon M),\end{aligned}\tag{3-13}$$

where

$$M=\limsup_{k\to+\infty}\sum_{i=\tau(k)}^{k-1}p(i).$$

Hence, for any $\varepsilon>0$, (3-13) gives

$$\liminf_{k\to+\infty}\exp\Bigl((\lambda_0+\varepsilon)\sum_{i=1}^{k-1}p(i)\Bigr)\sum_{i=k}^{+\infty}p(i)\exp\Bigl(-(\lambda_0+\varepsilon)\sum_{l=1}^{\tau(i)-1}p(l)\Bigr)\leq\exp(2\varepsilon M),$$

which implies

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{k\to+\infty}\exp\left((\lambda_0+\varepsilon)\sum_{i=1}^{k-1}p(i)\right)\sum_{i=k}^{+\infty}p(i)\exp\left(-(\lambda_0+\varepsilon)\sum_{l=1}^{\tau(i)-1}p(l)\right)\right)\le 1.$$

□

Remark 3.2. Condition (3-2) is not a limitation since, as proved in [Chatzarakis et al. 2008a], if τ is a nondecreasing function and

$$\limsup_{k\to+\infty}\sum_{i=\tau(k)}^{k}p(i)>1,$$

then $\mathbf{U}_{k_0}=\varnothing$, for any $k_0\in N$.

Remark 3.3. In (3-1), without loss of generality, we may assume that $c\le 1$. Otherwise, for any $k_0\in N$, we have $\mathbf{U}_{k_0}=\varnothing$ [Chatzarakis et al. 2008a].

Theorem 3.4. *Assume that all the conditions of Theorem 3.1 are satisfied. Then*

$$\liminf_{k\to+\infty}\sum_{i=\tau(k)}^{k-1}p(i)\le\frac{1}{e}. \tag{3-14}$$

Proof. Since all the conditions of Theorem 3.1 are satisfied, there exists $\lambda=\lambda_0\in[1,\,4/c^2]$ such that the inequality (3-3) holds.

Assume that the condition (3-14) does not hold. Then, there exists $k_1\in N$ and $\varepsilon_0>0$ such that

$$\sum_{i=\tau(k)}^{k-1}p(i)\ge\frac{1+\varepsilon_0}{e}\quad\text{for}\quad k\in N_{k_1}.$$

Therefore, for any $\varepsilon>0$,

$$\begin{aligned}I(k,\varepsilon)&=\exp\left((\lambda_0+\varepsilon)\sum_{i=1}^{k-1}p(i)\right)\sum_{i=k}^{+\infty}p(i)\exp\left(-(\lambda_0+\varepsilon)\sum_{l=1}^{\tau(i)-1}p(l)\right)\\&\ge\exp\left(\frac{(\lambda_0+\varepsilon)(1+\varepsilon_0)}{e}\right)\exp\left((\lambda_0+\varepsilon)\sum_{i=1}^{k-1}p(i)\right)\\&\qquad\times\sum_{i=k}^{+\infty}p(i)\exp\left(-(\lambda_0+\varepsilon)\sum_{l=1}^{i-1}p(l)\right)\qquad\text{for } k\in N_{k_1}.\end{aligned}\tag{3-15}$$

Defining $\sum_{l=1}^{i-1}p(l)=a_{i-1}$, we will show that

$$\liminf_{k\to+\infty}\exp((\lambda_0+\varepsilon)a_{k-1})\sum_{i=k}^{+\infty}p(i)\exp(-(\lambda_0+\varepsilon)a_{i-1})\ge\frac{1}{\lambda_0+\varepsilon}.$$

Indeed, since

$$\liminf_{k\to+\infty} \sum_{i=\tau(k)}^{k-1} p(i) = c > 0,$$

it is obvious that $\sum_{i=1}^{+\infty} p(i) = +\infty$, that is, $\lim_{i\to+\infty} a_i = +\infty$. Therefore

$$\begin{aligned}
\exp((\lambda_0+\varepsilon)a_{k-1}) &\sum_{i=k}^{+\infty} p(i)\exp(-(\lambda_0+\varepsilon)a_{i-1}) \\
&= \exp((\lambda_0+\varepsilon)a_{k-1}) \sum_{i=k}^{+\infty} (a_i - a_{i-1})\exp(-(\lambda_0+\varepsilon)a_{i-1}) \\
&= \exp((\lambda_0+\varepsilon)a_{k-1}) \sum_{i=k}^{+\infty} \exp(-(\lambda_0+\varepsilon)a_{i-1}) \int_{a_{i-1}}^{a_i} ds \\
&\geq \exp((\lambda_0+\varepsilon)a_{i-1}) \sum_{i=k}^{+\infty} \int_{a_{i-1}}^{a_i} \exp(-(\lambda_0+\varepsilon)s)ds \\
&= \exp((\lambda_0+\varepsilon)a_{i-1}) \int_{a_{i-1}}^{+\infty} \exp(-(\lambda_0+\varepsilon)s)ds = \frac{1}{\lambda_0+\varepsilon}.
\end{aligned}$$

Hence, by (3-15), we obtain

$$\limsup_{\varepsilon\to0+}\Big(\liminf_{k\to+\infty} I(k,\varepsilon)\Big) \geq \frac{1}{\lambda_0}\cdot\exp\Big(\frac{\lambda_0(1+\varepsilon_0)}{e}\Big) \geq 1+\varepsilon_0.$$

This contradicts (3-3) for $\lambda = \lambda_0$. □

4. Sufficient conditions of the proper solutions to be oscillatory

Theorem 4.1. *Assume that conditions* (1-5), (3-1), (3-2) *are satisfied and that, for any* $\lambda \in [1, 4/c^2]$,

(4-1)

$$\limsup_{\varepsilon\to0+}\left(\liminf_{k\to+\infty}\left(\exp\left((\lambda+\varepsilon)\sum_{i=1}^{k-1} p(i)\right)\sum_{i=k}^{+\infty} p(i)\exp\left(-(\lambda+\varepsilon)\sum_{l=1}^{\tau(i)-1} p(l)i\right)\right)\right) > 1.$$

Then all proper solutions of Equation (1-4) *oscillate.*

Proof. Assume that $u : N_{k_0} \to (0, +\infty)$ is a positive proper solution of (1-4). Then $\mathbf{U}_{k_0} \neq \varnothing$. Thus, in view of Theorem 3.1, there exists $\lambda_0 \in [1, 4/c^2]$ such that the condition (3-3) is satisfied for $\lambda = \lambda_0$. But this contradicts (4-1). □

Using Theorem 3.4, we can similarly prove:

Theorem 4.2. *Assume that conditions* (1-5) *and* (3-2) *are satisfied and*

$$\liminf_{k\to+\infty}\sum_{i=\tau(k)}^{k-1} p(i) > \frac{1}{e}. \tag{4-2}$$

Then all proper solutions of Equation (1-4) *oscillate.*

Remark 4.3. It is to be pointed out that Theorem 4.2 is the discrete analogue of Theorem 1.1 for the first order difference equation (1-4) in the case of a general delay argument $\tau(k)$.

Remark 4.4. The condition (4-2) is optimal for (1-4) under the assumption that

$$\lim_{k\to+\infty}(k-\tau(k)) = +\infty,$$

since in this case the set of natural numbers increases infinitely in the interval $[\tau(k), k-1]$ for $k\to+\infty$.

Now, we are going to present two examples to show that the condition (4-2) is optimal, in the sense that it cannot be replaced by the nonstrong inequality.

Example 4.5. Consider (1-4), where

$$\begin{aligned} &\tau(k) = [\alpha k], \quad p(k) = (k^{-\lambda} - (k+1)^{-\lambda})[\alpha k]^{\lambda}, \\ &\alpha\in(0,1), \quad \lambda = -\ln^{-1}\alpha, \end{aligned} \tag{4-3}$$

with $[\alpha k]$ the integer part of αk.

It is obvious that

$$k^{1+\lambda}(k^{-\lambda} - (k+1)^{-\lambda}) \to \lambda \quad \text{for} \quad k\to+\infty.$$

Therefore

$$k(k^{-\lambda} - (k+1)^{-\lambda})[\alpha k]^{\lambda} \to \frac{\lambda}{e} \quad \text{for} \quad k\to+\infty. \tag{4-4}$$

Hence, in view of (4-3) and (4-4), we have

$$\begin{aligned} \liminf_{k\to+\infty}\sum_{i=\tau(k)}^{k-1} p(i) &= \frac{\lambda}{e}\liminf_{k\to+\infty}\sum_{i=[\alpha k]}^{k-1}\frac{e}{\lambda}\, i(i^{-\lambda} - (i+1)^{-\lambda})[\alpha i]^{\lambda}\frac{1}{i} \\ &= \frac{\lambda}{e}\liminf_{k\to+\infty}\sum_{i=[\alpha k]}^{k-1}\frac{1}{i} = \frac{\lambda}{e}\ln\frac{1}{\alpha} = \frac{1}{e} \end{aligned}$$

or

$$\liminf_{k\to+\infty}\sum_{i=\tau(k)}^{k-1} p(i) = \frac{1}{e}.$$

Observe that all the conditions of Theorem 4.2 are satisfied except the condition (4-2). In this case, it is not guaranteed that all solutions of (1-4) oscillate. Indeed, it is easy to see that the function $u = k^{-\lambda}$ is a positive solution of (1-4).

Example 4.6. Consider (1-4), where

$$\begin{aligned} &\tau(k) = [k^\alpha], \quad p(k) = (\ln^{-\lambda} k - \ln^{-\lambda}(k+1))\ \ln^\lambda [k^\alpha], \\ &\alpha \in (0, 1), \quad \lambda = -\ln^{-1}\alpha, \end{aligned} \tag{4-5}$$

with $[k^\alpha]$ the integer part of k^α.

It is obvious that

$$k \ln^{1+\lambda} k (\ln^{-\lambda} k - \ln^{-\lambda}(k+1)) \to \lambda \quad \text{for } k \to +\infty.$$

Therefore

$$k\ \ln k\ \ln^\lambda[k^\alpha](\ln^{-\lambda} k - \ln^{-\lambda}(k+1)) \to \frac{\lambda}{e} \quad \text{for } k \to +\infty. \tag{4-6}$$

On the other hand,

$$\sum_{i=[k^\alpha]}^{k-1} \frac{1}{i\ \ln i} \geq \sum_{i=[k^\alpha]}^{k-1} \int_i^{i+1} \frac{ds}{s\ \ln s} = \int_{[k^\alpha]}^k \frac{ds}{s\ \ln s} = \ln \frac{\ln k}{\ln [k^\alpha]},$$

which tends to $\ln(1/\alpha)$ as $k \to +\infty$, and

$$\sum_{i=[k^\alpha]}^{k-1} \frac{1}{i\ \ln i} \leq \sum_{i=[k^\alpha]}^{k-1} \int_{i-1}^{i} \frac{ds}{s\ \ln s} = \int_{[k^\alpha]-1}^{k-1} \frac{ds}{s\ \ln s} = \ln \frac{\ln(k-1)}{\ln [k^\alpha]-1},$$

which also tends to $\ln(1/\alpha)$ as $k \to +\infty$. Together these two bounds imply

$$\lim_{k\to+\infty} \sum_{i=[k^\alpha]}^{k-1} \frac{1}{i\ \ln i} = \ln\frac{1}{\alpha}.$$

Hence, in view of (4-5) and (4-6), we obtain

$$\begin{aligned} \liminf_{k\to+\infty} \sum_{i=[k^\alpha]}^{k-1} p(i) &= \liminf_{k\to+\infty} \sum_{i=[k^\alpha]}^{k-1} \ln^\lambda[i^\alpha](\ln^{-\lambda} i - \ln^{-\lambda}(i+1)) \\ &= \frac{\lambda}{e} \liminf_{k\to+\infty} \sum_{i=[k^\alpha]}^{k-1} \frac{e}{\lambda}\, i\ \ln i\ \ln^\lambda[i^\alpha](\ln^{-\lambda} i - \ln^{-\lambda}(i+1)) \frac{1}{i\ \ln i} \\ &= \frac{\lambda}{e} \liminf_{k\to+\infty} \sum_{i=[k^\alpha]}^{k-1} \frac{1}{i\ \ln i} = \frac{\lambda}{e}\ \ln\frac{1}{\alpha} = \frac{1}{e}. \end{aligned}$$

We again observe that all the conditions of Theorem 4.2 are satisfied except (4-2). In this case, it is not guaranteed that all solutions of (1-4) oscillate. Indeed, it is easy to see that the function $u = \ln^{-\lambda} k$ is a positive solution of (1-4).

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