OPTIMAL OSCILLATION CRITERIA FOR FIRST ORDER DIFFERENCE EQUATIONS WITH DELAY ARGUMENT

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Consider the first order linear difference equation
\[ \Delta u(k) + p(k) u(\tau(k)) = 0, \quad k \in N, \]
where \( \Delta u(k) = u(k + 1) - u(k) \), \( p : N \to \mathbb{R}_+ \), \( \tau : N \to N \), \( \tau(k) \leq k - 2 \) and \( \lim_{k \to +\infty} \tau(k) = +\infty \). Optimal conditions for the oscillation of all proper solutions of this equation are established. The results lead to a sharp oscillation condition, when \( k - \tau(k) \to +\infty \) as \( k \to +\infty \). Examples illustrating the results are given.

1. Introduction

The first systematic study for the oscillation of all solutions to the first order delay differential equation
\[ u'(t) + p(t) u(\tau(t)) = 0, \tag{1-1} \]
where \( p \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+) \), \( \tau \in C(\mathbb{R}_+; \mathbb{R}_+) \), \( \tau(t) \leq t \) for \( t \in \mathbb{R}_+ \) and \( \lim_{t \to +\infty} \tau(t) = +\infty \), in the case of constant coefficients and constant delays was made by Myshkis [1972]. For the differential equation (1-1) the problem of oscillation is investigated by many authors. See, for example, [Elbert and Stavroulakis 1995; Koplatadze and Chanturiya 1982; Koplatadze and Kvinikadze 1994; Ladas et al. 1984; Sficas and Stavroulakis 2003] and the references cited therein.

**Theorem 1.1** [Koplatadze and Chanturiya 1982]. Assume that
\[ \lim_{t \to +\infty} \inf_{t \to +\infty} \int_{\tau(t)}^{t} p(s) \, ds > \frac{1}{e}. \tag{1-2} \]
Then all solutions of Equation (1-1) oscillate.

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It is to be emphasized that condition (1-2) is optimal in the sense that it cannot be replaced by the condition

\begin{equation}
\liminf_{t \to +\infty} \int_{\tau(t)}^{t} p(s) \, ds \geq \frac{1}{e},
\end{equation}

For example, if \( \tau(t) = t - \delta \) or \( \tau(t) = \alpha t \) or \( \tau(t) = t^\sigma \), where \( \delta > 0, \alpha \in (0, 1) \), examples can be given such that condition (1-3) is satisfied, but (1-1) has a nonoscillatory solution.

The discrete analogue of the first order delay differential equation (1-1) is the first order difference equation

\begin{equation}
\Delta u(k) + p(k) u(\tau(k)) = 0,
\end{equation}

where

\begin{equation}
\Delta u(k) = u(k+1) - u(k), \quad p : N \to \mathbb{R}_+,
\end{equation}

\begin{equation}
\tau : N \to N, \quad \tau(k) \leq k - 1, \quad \lim_{k \to +\infty} \tau(k) = +\infty.
\end{equation}

By a **proper solution** of (1-4) we mean a function \( u : N_{n_0} \to \mathbb{R} \) with \( n_0 = \min\{\tau(k) : k \in N_n\} \) and \( N_n = \{n, n+1, \ldots\} \), which satisfies (1-4) on \( N_n \) and \( \sup\{|u(i)| : i \geq k\} > 0 \) for \( k \in N_{n_0} \).

A proper solution \( u : N_{n_0} \to \mathbb{R} \) of (1-4) is said to be **oscillatory** (around zero) if for any positive integer \( n \in N_{n_0} \), there exist \( n_1, n_2 \in N_n \) such that \( u(n_1) u(n_2) \leq 0 \). Otherwise, the proper solution is said to be **nonoscillatory**. In other words, a proper solution \( u \) is oscillatory if it is neither eventually positive nor eventually negative.

Oscillatory properties of the solutions of (1-4), in the case of a general delay argument \( \tau(k) \), have been recently investigated in [Chatzarakis et al. 2008a; 2008b], while the special case when \( \tau(k) = k - n, \ n \geq 1 \), has been studied rather extensively. See, for example, [Agarwal et al. 2005; Baštinec and Diblik 2005; Chatzarakis and Stavroulakis 2006; Domshlak 1999; Elaydi 1999; Ladas et al. 1989] and the references cited therein. In this particular case, (1-4) becomes

\begin{equation}
\Delta u(k) + p(k) u(k - n) = 0, \quad k \in N.
\end{equation}

For this equation Ladas, Philos and Sficas established the following theorem.

**Theorem 1.2** [Ladas et al. 1989]. Assume that

\begin{equation}
\liminf_{k \to +\infty} \sum_{i=k-n}^{k-1} p(i) > \left( \frac{n}{n+1} \right)^{n+1}.
\end{equation}

Then all proper solutions of (1-6) oscillate.
This result is sharp in the sense that the inequality (1-7) cannot be replaced by the nonstrong one for any \( n \in N \). Hence, Theorem 1.2 is the discrete analogue of Theorem 1.1 when \( \tau(t) = t - \delta \).

An interesting question then arises whether there exists the discrete analogue of Theorem 1.1 for (1-4) in the case of a general delay argument \( \tau(k) \) when \( \lim_{k \to +\infty} (k - \tau(k)) = +\infty \).

In the present paper optimal conditions for the oscillation of all proper solutions of (1-4) are established and a positive answer to the above question is given.

2. Some auxiliary lemmas

Let \( k_0 \in N \). Denote by \( U_{k_0} \) the set of all proper solutions of (1-4) satisfying the condition \( u(k) > 0 \) for \( k \geq k_0 \).

**Remark 2.1.** We will suppose that \( U_{k_0} = \emptyset \), if (1-4) has no solution satisfying the condition \( u(k) > 0 \) for \( k \geq k_0 \).

**Lemma 2.2.** Assume that \( k_0 \in N, U_{k_0} \neq \emptyset, u \in U_{k_0}, \tau(k) \leq k - 1, \tau \) is a nondecreasing function and

\[
\liminf_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) = c > 0.
\]

Then

\[
\limsup_{k \to +\infty} \frac{u(\tau(k))}{u(k+1)} \leq \frac{4}{c^2}.
\]

**Proof.** By (2-1), for any \( \varepsilon \in (0, c) \), it is clear that

\[
\sum_{i=\tau(k)}^{k-1} p(i) \geq c - \varepsilon \quad \text{for} \quad k \in N_{k_0}.
\]

Since \( u \) is a positive proper solution of (1-4), then there exists \( k_1 \in N_{k_0} \) such that

\[
u(\tau(k)) > 0 \quad \text{for} \quad k \in N_{k_1}.
\]

Thus, from (1-4) we have

\[
u(k+1) - u(k) = -p(k)u(\tau(k)) \leq 0
\]

and so \( u \) is an eventually nonincreasing function of positive numbers.

Now from inequality (2-3) it is clear that, there exists \( k^* \geq k \) such that

\[
\sum_{i=k}^{k^*-1} p(i) < \frac{c - \varepsilon}{2} \quad \text{and} \quad \sum_{i=k}^{k^*} p(i) \geq \frac{c - \varepsilon}{2}.
\]
This is because in the case where \( p(k) < \frac{c - \varepsilon}{2} \), it is clear that there exists \( k^* > k \) such that (2-4) is satisfied, while in the case where \( p(k) \geq \frac{c - \varepsilon}{2} \), then \( k^* = k \), and therefore

\[
\sum_{i=k}^{k^*-1} p(i) = \sum_{i=k}^{k-1} p(i) \quad \text{(by which we mean)} \quad = 0 < \frac{c - \varepsilon}{2}
\]

and

\[
\sum_{i=k}^{k^*} p(i) = \sum_{i=k}^{k} p(i) = p(k) = \frac{c - \varepsilon}{2}.
\]

That is, in both cases (2-4) is satisfied.

Now, we will show that \( \tau(k^*) \leq k - 1 \). Indeed, in the case where \( p(k) \geq \frac{c - \varepsilon}{2} \), since \( k^* = k \), it is obvious that \( \tau(k^*) \leq k - 1 \). In the case where \( p(k) < \frac{c - \varepsilon}{2} \), then \( k^* > k \). Assume, for the sake of contradiction, that \( \tau(k^*) > k - 1 \). Hence, \( k \leq \tau(k^*) \leq k^* - 1 \) and then

\[
\sum_{i=\tau(k^*)}^{k^*-1} p(i) \leq \sum_{i=k}^{k^*-1} p(i) < \frac{c - \varepsilon}{2}.
\]

This, in view of (2-3), leads to a contradiction. Thus, in both cases, we have \( \tau(k^*) \leq k - 1 \).

Therefore, it is clear that

\[
\sum_{i=\tau(k^*)}^{k^*-1} p(i) = \sum_{i=\tau(k^*)}^{k-1} p(i) - \sum_{i=k}^{k^*-1} p(i) \geq (c - \varepsilon) - \frac{c - \varepsilon}{2} = \frac{c - \varepsilon}{2}.
\]

Now, summing up (1-4) first from \( k \) to \( k^* \) and then from \( \tau(k^*) \) to \( k - 1 \), and using that the function \( u \) is nonincreasing and the function \( \tau \) is nondecreasing, we have

\[
u(k) - u(k^* + 1) = \sum_{i=k}^{k^*} p(i)u(\tau(i)) \geq \left( \sum_{i=k}^{k^*} p(i) \right) u(\tau(k^*)) \geq \frac{c - \varepsilon}{2} u(\tau(k^*)),
\]

or

\[
u(k) \geq \frac{c - \varepsilon}{2} u(\tau(k^*)),
\]

and then

\[
u(\tau(k^*)) - u(k) = \sum_{i=\tau(k^*)}^{k-1} p(i)u(\tau(i)) \geq \left( \sum_{i=\tau(k^*)}^{k-1} p(i) \right) u(\tau(k-1)) \geq \frac{c - \varepsilon}{2} u(\tau(k-1)),
\]

or

\[
u(\tau(k^*)) \geq \frac{c - \varepsilon}{2} u(\tau(k-1)).
\]
Combining inequalities (2-6) and (2-7), we obtain
\[ \frac{u(\tau(k - 1))}{u(k)} \leq \frac{4}{(c - \varepsilon)^2} \]
and, for large \( k \), we have
\[ \frac{u(\tau(k))}{u(k + 1)} \leq \frac{4}{(c - \varepsilon)^2}. \]
Hence,
\[ \limsup_{k \to +\infty} \frac{u(\tau(k))}{u(k + 1)} \leq \frac{4}{(c - \varepsilon)^2}, \]
which, for arbitrarily small values of \( \varepsilon \), implies (2-2). \( \square \)

**Lemma 2.3.** Assume that \( k_0 \in \mathbb{N}, U_{k_0} \neq \emptyset, u \in U_{k_0}, \tau(k) \leq k - 1, \tau \) is a nondecreasing function and condition (2-1) is satisfied. Then

\[ (2-8) \quad \lim_{k \to +\infty} u(k) \exp\left(\lambda \sum_{i=1}^{k-1} p(i)\right) = +\infty \quad \text{for any} \quad \lambda > \frac{4}{c^2}. \]

**Proof.** Since all the conditions of Lemma 2.2 are satisfied, for any \( \gamma > 4/c^2 \), there exists \( k_1 \in N_{k_0} \) such that

\[ (2-9) \quad \frac{u(\tau(k))}{u(k + 1)} \leq \gamma \quad \text{for} \quad k \in N_{k_1}. \]

Also, for any \( n \in N_{k_1} \)
\[ \sum_{k=k_1}^{n} \frac{\Delta u(k)}{u(k + 1)} = \sum_{k=k_1}^{n} \left(1 - \frac{u(k)}{u(k + 1)}\right) = (n - k_1) - \sum_{k=k_1}^{n} \exp\left(\ln \frac{u(k)}{u(k + 1)}\right) \]
\[ \leq (n - k_1) - \sum_{k=k_1}^{n} \left(1 + \ln \frac{u(k)}{u(k + 1)}\right) = -\sum_{k=k_1}^{n} \ln \frac{u(k)}{u(k + 1)} = \ln \frac{u(n + 1)}{u(k_1)}, \]
or
\[ \sum_{k=k_1}^{n} \frac{\Delta u(k)}{u(k + 1)} \leq \ln \frac{u(n + 1)}{u(k_1)}. \]

Moreover, from (1-4), we have
\[ \sum_{k=k_1}^{n} \frac{\Delta u(k)}{u(k + 1)} = -\sum_{k=k_1}^{n} p(k) \frac{u(\tau(k))}{u(k + 1)}. \]

Combining (2-9) with the last two relations, we obtain
\[ u(n + 1) \geq u(k_1) \exp\left(-\gamma \sum_{k=k_1}^{n} p(k)\right). \]
Now, by (2-1), it is obvious that $\sum_{i=0}^{+\infty} p(i) = +\infty$. Therefore, for $\lambda > 4/c^2$, the last inequality yields
\[
\lim_{n \to +\infty} u(n+1) \exp \left( \lambda \sum_{k=k_1}^{n} p(k) \right) = +\infty,
\]
or
\[
\lim_{k \to +\infty} u(k) \exp \left( \lambda \sum_{i=k_1}^{k-1} p(i) \right) = +\infty,
\]
which implies (2-8), since
\[
\sum_{i=1}^{k-1} p(i) \geq \sum_{i=k_1}^{k-1} p(i).
\]
□

Next, consider the difference inequality
\[
(2-10) \quad \Delta u(k) + q(k) u(\sigma(k)) \leq 0,
\]
where
\[
q : N \to \mathbb{R}_+, \quad \sigma : N \to N \quad \text{and} \quad \lim_{k \to +\infty} \sigma(k) = +\infty.
\]

In the sequel the following lemma will be used, which has recently been established in [Chatzarakis et al. 2008a].

**Lemma 2.4.** Assume that (2-1) is satisfied, and for sufficiently large $k$
\[
\sigma(k) \leq \tau(k) \leq k - 1, \quad p(k) \leq q(k)
\]
and $u : N_{k_0} \to (0, +\infty)$ is a positive proper solution of (2-10). Then, there exists $k_1 \in N_{k_0}$ such that $U_{k_1} \neq \emptyset$ and $u_* \in U_{k_1}$ is the solution of (1-4), which satisfies the condition
\[
0 < u_*(k) \leq u(k) \quad \text{for} \quad k \in N_{k_1}.
\]

By virtue of Lemma 2.4, we can formulate Lemma 2.3 in the following more general form, where the function $\tau$ is not required to be nondecreasing.

**Lemma 2.5.** Assume that $k_0 \in N$, $U_{k_0} \neq \emptyset$, $u \in U_{k_0}$, $\tau(k) \leq k - 1$ and condition (2-1) is satisfied. Then, for any $\lambda > 4/c^2$, condition (2-8) holds.

**Proof:** Since $u : N_{k_0} \to (0, +\infty)$ is a solution of (1-4), it is clear that $u$ is a solution of the inequality
\[
\Delta u(k) + p(k) u(\sigma(k)) \leq 0 \quad \text{for} \quad k \in N_{k_1},
\]
where $\sigma(k) = \max\{\tau(i) : 1 \leq s \leq k, s \in N\}$ and $k_1 > k_0$ is a sufficiently large number.
First we will show that

\[ \liminf_{k \to +\infty} \sum_{i=\sigma(k)}^{k-1} p(i) = c. \]  

Assume that (2-11) is not satisfied. Then there exists a sequence \( \{k_i\}_{i=1}^{+\infty} \) of natural numbers such that \( \sigma(k_i) \neq \tau(k_i) \) \( (i = 1, 2, \ldots) \) and

\[ \liminf_{j \to +\infty} \sum_{i=\sigma(k_j)}^{k_j-1} p(i) = c_1 < c. \]  

Also, from the definition of the function \( \sigma \), and in view of \( \sigma(k_i) \neq \tau(k_i) \), for any \( k_i \), there exists \( k'_i < k_i \) such that \( \sigma(k) = \sigma(k_i) \) for \( k'_i \leq k \leq k_i \), \( \lim_{j \to +\infty} k'_i = +\infty \) and \( \sigma(k'_i) = \tau(k'_i) \). Thus

\[ \sum_{j=\tau(k'_i)}^{k'_i-1} p(j) = \sum_{j=\sigma(k'_i)}^{k'_i-1} p(j) = \sum_{j=\sigma(k_i)}^{k_i-1} p(j) \leq \sum_{j=\sigma(k_i)}^{k_i-1} p(j) \quad (i = 1, 2, \ldots), \]

and, by the virtue of (2-12), we have

\[ \liminf_{i \to +\infty} \sum_{j=\tau(k'_i)}^{k'_i-1} p(j) \leq \liminf_{i \to +\infty} \sum_{j=\sigma(k'_i)}^{k'_i-1} p(j) = c_1 < c. \]

In view of (2-1), the last inequality leads to a contradiction. Therefore (2-11) holds.

Now, by Lemma 2.4, we conclude that the equation

\[ \Delta u(k) + p(k) u(\sigma(k)) = 0 \]

has a solution \( u_* \) which satisfies the condition

\[ 0 < u_*(k) \leq u(k) \quad \text{for} \quad k \in N_{k_1}, \]

where \( k_1 > k_0 \) is a sufficiently large number. Hence, taking into account that the function \( \sigma \) is nondecreasing, in view of Lemma 2.3, we have

\[ \lim_{k \to +\infty} u_*(k) \exp\left( \lambda \sum_{i=1}^{k-1} p(i) \right) = +\infty, \]

where \( \lambda > 4/c^2 \). Therefore, by (2-13), we get

\[ \lim_{k \to +\infty} u(k) \exp\left( \lambda \sum_{i=1}^{k-1} p(i) \right) = +\infty \quad \text{for any} \quad \lambda > \frac{4}{c^2}. \]  \( \square \)
Lemma 2.6 (Abel transformation). Let \( \{a_i\}_{i=1}^{+\infty} \) and \( \{b_i\}_{i=1}^{+\infty} \) be sequences of non-negative numbers and

\[
\sum_{i=1}^{+\infty} a_i < +\infty.
\]

Then

\[
\sum_{i=1}^{k} a_i b_i = A_1 b_1 - A_{k+1} b_{k+1} - \sum_{i=1}^{k} A_{i+1} (b_i - b_{i+1}),
\]

where \( A_i = \sum_{j=i}^{+\infty} a_j \).

Proof. Since (2-14) is satisfied, we have

\[
\sum_{i=1}^{k} A_{i+1} (b_i - b_{i+1}) = \sum_{i=1}^{k} A_{i+1} b_i - \sum_{i=2}^{k+1} A_i b_i
\]

\[
= A_2 b_1 - A_{k+1} b_{k+1} + \sum_{i=2}^{k} (A_{i+1} - A_i) b_i
\]

\[
= A_2 b_1 - A_{k+1} b_{k+1} - \sum_{i=2}^{k} a_i b_i
\]

\[
= A_1 b_1 - A_{k+1} b_{k+1} - \sum_{i=1}^{k} a_i b_i,
\]

or

\[
\sum_{i=1}^{k} a_i b_i = A_1 b_1 - A_{k+1} b_{k+1} - \sum_{i=1}^{k} A_{i+1} (b_i - b_{i+1}).
\]

Koplatadze, Kvinikadze and Stavroulakis established the following lemma. For completeness, we present the proof here.

Lemma 2.7 [Koplatadze et al. 2002]. Let \( \varphi, \psi : N \rightarrow (0, +\infty) \), \( \psi \) be nonincreasing and suppose

\[
\lim_{k \to +\infty} \varphi(k) = +\infty,
\]

\[
\liminf_{k \to +\infty} \psi(k) \tilde{\varphi}(k) = 0,
\]

where \( \tilde{\varphi}(k) = \inf\{\varphi(s) : s \geq k, \ s \in N\} \). Then there exists an increasing sequence of natural numbers \( \{k_i\}_{i=1}^{+\infty} \) such that

\[
\lim_{i \to +\infty} k_i = +\infty, \quad \tilde{\varphi}(k_i) = \varphi(k_i), \quad \psi(k) \tilde{\varphi}(k) \geq \psi(k_i) \tilde{\varphi}(k_i)
\]

\[
(k = 1, 2, \ldots, k_i ; \ i = 1, 2, \ldots).
\]
**Proof.** Define the sets $E_1$ and $E_2$ by

\[ k \in E_1 \iff \tilde{\varphi}(k) = \varphi(k), \]
\[ k \in E_2 \iff \tilde{\varphi}(s) \psi(s) \geq \tilde{\varphi}(k) \psi(k) \text{ for } s \in \{1, \ldots, k\}. \]

According to (2-15) and (2-16), it is obvious that

(2-17) \[ \sup E_i = +\infty \quad (i = 1, 2). \]

Show that

(2-18) \[ \sup E_1 \cap E_2 = +\infty. \]

Let $k_0 \in E_2$ be such that $k_0 \notin E_1$. By (2-16) there is $k_1 > k_0$ such that $\tilde{\varphi}(k) = \tilde{\varphi}(k_1)$ for $k = k_0, k_0 + 1, \ldots, k_1$ and $\tilde{\varphi}(k_1) = \varphi(k_1)$. Since $\psi$ is nonincreasing, we have

\[ \tilde{\varphi}(k) \psi(k) \geq \tilde{\varphi}(k_1) \psi(k_1) \text{ for } k = 1, \ldots, k_1. \]

Therefore $k_1 \in E_1 \cap E_2$. The above argument together with (2-17) imply that (2-18) holds. \(\square\)

**Remark 2.8.** The analogue of this lemma for continuous functions $\varphi$ and $\psi$ was proved first in [Koplatadze 1994].

### 3. Necessary conditions of the existence of positive solutions

The results of this section play an important role in establishing sufficient conditions for all proper solutions of (1-4) to be oscillatory.

**Theorem 3.1.** Assume that $k_0 \in \mathbb{N}$, $U_{k_0} \neq \emptyset$, (1-5) is satisfied,

(3-1) \[ \liminf_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) = c > 0, \]

and

(3-2) \[ \limsup_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) < +\infty. \]

Then there exists $\lambda \in [1, 4/c^2]$ such that

(3-3) \[ \limsup_{\varepsilon \to 0+} \left( \liminf_{k \to +\infty} \left( \lambda + \varepsilon \sum_{i=1}^{k-1} p(i) \right)^{+\infty} \sum_{i=k}^{\tau(i)-1} p(i) \exp \left( -(\lambda + \varepsilon) \sum_{l=1}^{\tau(i)-1} p(l) \right) \right) \leq 1. \]
Proof. Since $U_{k_0} \neq \emptyset$, Equation (1-4) has a positive solution $u : N_{k_0} \to (0, +\infty)$. First we show that

$$\limsup_{k \to +\infty} u(k) \exp\left( \sum_{i=1}^{k-1} p(i) \right) < +\infty. \tag{3-4}$$

Indeed, if $k_1 \in N_{k_0}$, we have

$$\sum_{i=k_1}^{k} \frac{\Delta u(i)}{u(i)} = \sum_{i=k_1}^{k} \frac{u(i+1)}{u(i)} - (k - k_1) = \sum_{i=k_1}^{k} \exp\left( \ln \frac{u(i+1)}{u(i)} \right) - (k - k_1)$$

$$\geq \sum_{i=k_1}^{k} \left( 1 + \frac{u(i+1)}{u(i)} \right) - (k - k_1) = \ln \frac{u(k+1)}{u(k_1)},$$

or

$$\sum_{i=k_1}^{k} \frac{\Delta u(i)}{u(i)} \geq \ln \frac{u(k+1)}{u(k_1)}.$$

By (1-4), and taking into account that the function $u$ is nonincreasing, we have

$$\sum_{i=k_1}^{k} \frac{\Delta u(i)}{u(i)} = - \sum_{i=k_1}^{k} p(i) \frac{u(\tau(i))}{u(i)} \leq - \sum_{i=k_1}^{k} p(i).$$

Combining the last two inequalities, we obtain

$$u(k+1) \exp\left( \sum_{i=k_1}^{k} p(i) \right) \leq u(k_1),$$

that is, (3-4) is fulfilled. On the other hand, since all the conditions of Lemma 2.5 are satisfied, we conclude that condition (2-8) holds for any $\lambda > 4/c^2$. Denote by $\Lambda$ the set of all $\lambda$ for which

$$\lim_{k \to +\infty} u(\tau(k)) \exp\left( \lambda \sum_{i=1}^{\tau(k)-1} p(i) \right) = +\infty \tag{3-5}$$

and $\lambda_0 = \inf \Lambda$. In view of (1-5), (2-8) and (3-4), it is obvious that $\lambda_0 \in [1, 4/c^2]$. Thus, it suffices to show that, for $\lambda = \lambda_0$ the inequality (3-3) holds. First, we will show that for any $\varepsilon > 0$

$$\lim_{k \to +\infty} u(\tau(k)) \exp\left( (\lambda_0 + \varepsilon) \sum_{i=1}^{\tau(k)-1} p(i) \right) = +\infty. \tag{3-6}$$
Indeed, if \( \lambda_0 \in \Lambda \), it is obvious from (3-5) that condition (3-6) is fulfilled. If \( \lambda_0 \not\in \Lambda \), according to the definition of \( \lambda_0 \), there exists \( \lambda_k > \lambda_0 \) such that \( \lambda_k \to \lambda_0 \) when \( k \to +\infty \) and \( \lambda_k \in \Lambda \), \( k = 1, 2, \ldots \). Thus, condition (3-5) holds for any \( \lambda = \lambda_k \). However, for any \( \varepsilon > 0 \), there exists \( \lambda_k = \lambda_k(\varepsilon) \) such that \( \lambda_0 < \lambda_k \leq \lambda_0 + \varepsilon \).

This insures the validity of (3-5) and (3-6) for any \( \varepsilon > 0 \).

Similarly, we show that for any \( \varepsilon > 0 \),

\[
\liminf_{k \to +\infty} u(\tau(k)) \exp\left( (\lambda_0 - \varepsilon) \sum_{i=1}^{\tau(k)-1} p(i) \right) = 0.
\]

Hence, by virtue of (1-5), (3-6) and (3-7), it is clear that for any \( \varepsilon > 0 \), the functions

\[
\varphi(k) = u(\tau(k)) \exp\left( (\lambda_0 + \varepsilon) \sum_{i=1}^{\tau(k)-1} p(i) \right)
\]

and

\[
\psi(k) = \exp\left( -2\varepsilon \sum_{i=1}^{k-1} p(i) \right)
\]

satisfy the conditions of Lemma 2.7 for sufficiently large \( k \). Hence, there exists an increasing sequence \( \{k_i\}_{i=1}^{+\infty} \) of natural numbers satisfying \( \lim_{i \to +\infty} k_i = +\infty \),

\[
\psi(k_i) \varphi(k_i) \leq \psi(k) \varphi(k) \quad \text{for} \quad k^* \leq k \leq k_i,
\]

where \( k^* \) is a sufficiently large number, and

\[
\varphi(k_i) = \varphi(k_i) \quad (i = 1, 2, \ldots),
\]

Now, given that

\[
u(\tau(i)) \exp\left( (\lambda_0 + \varepsilon) \sum_{l=1}^{\tau(i)-1} p(l) \right) \geq \inf\left\{ u(\tau(s)) \exp(\lambda_0 + \varepsilon) \sum_{l=1}^{\tau(s)-1} p(l) : s \geq i, \ s \in N \right\}
\]

\[
= \varphi(i),
\]

Equation (1-4) implies

\[
u(\tau(k_j)) \geq \sum_{i=\tau(k_j)}^{+\infty} p(i) u(\tau(i)) \geq \sum_{i=\tau(k)}^{+\infty} p(i) \varphi(i) \exp\left( - (\lambda_0 + \varepsilon) \sum_{l=1}^{\tau(i)-1} p(l) \right)
\]
that is,

\[
u(\tau(k_j)) \geq \sum_{i=\tau(k_j)}^{k_j-1} p(i) \tilde{\varphi}(i) \exp \left( -2\varepsilon \sum_{l=1}^{i-1} p(l) \right) \exp \left( 2\varepsilon \sum_{l=1}^{i-1} p(l) \right) \times \exp \left( -\lambda_0 - \varepsilon \sum_{l=1}^{i-1} p(l) \right) + \sum_{\tau(i)=1}^{\tau(k_j)-1} \tilde{\varphi}(i) \exp \left( -\lambda_0 - \varepsilon \sum_{l=1}^{\tau(i)-1} p(l) \right),
\]

for \( j = 1, 2, \ldots \). Thus, by (3-9), and using the fact that the function \( \tilde{\varphi} \) is non-decreasing, the last inequality yields

\[
(3-11) \quad u(\tau(k_j)) \geq \tilde{\varphi}(k_j) \exp \left( -2\varepsilon \sum_{l=1}^{k_j-1} p(l) \right) \times \sum_{i=\tau(k_j)}^{k_j-1} p(i) \exp \left( 2\varepsilon \sum_{l=1}^{i-1} p(l) \right) \exp \left( -\lambda_0 - \varepsilon \sum_{l=1}^{i-1} p(l) \right) + \tilde{\varphi}(k_j) \sum_{i=\tau(k_j)}^{k_j-1} p(i) \exp \left( -\lambda_0 - \varepsilon \sum_{l=1}^{i-1} p(l) \right),
\]

Also, in view of Lemma 2.6, we have

\[
(3-12) \quad I(k_j, \varepsilon) = \sum_{i=\tau(k_j)}^{k_j-1} p(i) \exp \left( 2\varepsilon \sum_{l=1}^{i-1} p(l) \right) \exp \left( -\lambda_0 - \varepsilon \sum_{l=1}^{i-1} p(l) \right) \times \sum_{i=\tau(k_j)}^{k_j-1} p(i) \exp \left( -\lambda_0 - \varepsilon \sum_{l=1}^{i-1} p(l) \right) - \sum_{i=\tau(k_j)}^{k_j-1} p(i) \exp \left( -\lambda_0 - \varepsilon \sum_{l=1}^{i-1} p(l) \right) \times \sum_{i=1}^{+\infty} p(i) \exp \left( -\lambda_0 - \varepsilon \sum_{l=1}^{i-1} p(l) \right),
\]

for \( j = 1, 2, \ldots \).
Given that
\[
\exp\left(2\varepsilon \sum_{l=1}^{i} p(l)\right) - \exp\left(2\varepsilon \sum_{l=1}^{i-1} p(l)\right) \geq 0,
\]
inequality (3-12) becomes
\[
I(k, \varepsilon) \geq \exp\left(2\varepsilon \sum_{i=\tau(k)}^{\tau(k_j)-1} p(i) \exp\left(-\left(\lambda_0 + \varepsilon\right) \sum_{l=1}^{\tau(i)-1} p(l)\right)\right) - \exp\left(2\varepsilon \sum_{i=1}^{k} p(i) \exp\left(-\left(\lambda_0 + \varepsilon\right) \sum_{l=1}^{\tau(i)-1} p(l)\right)\right) \times +\infty \sum_{i=\tau(k_j)}^{\tau(k_j)-1} p(i) \exp\left(-\left(\lambda_0 + \varepsilon\right) \sum_{l=1}^{\tau(i)-1} p(l)\right) \times +\infty \sum_{i=\tau(k_j)}^{\tau(k_j)-1} p(i) \exp\left(-\left(\lambda_0 + \varepsilon\right) \sum_{l=1}^{\tau(i)-1} p(l)\right) \cdot
\]
Therefore, by (3-11), we take
\[
u(\tau(k_j)) \geq \tilde{\varphi}(k_j) \exp\left(-2\varepsilon \sum_{i=\tau(k_j)}^{k_j-1} p(l)\right) \exp\left(2\varepsilon \sum_{l=1}^{1} p(l)\right) \times +\infty \sum_{i=\tau(k_j)}^{\tau(k_j)-1} p(i) \exp\left(-\left(\lambda_0 + \varepsilon\right) \sum_{l=1}^{\tau(i)-1} p(l)\right) \cdot
\]
Thus, (3-8) and (3-10) imply
\[
\exp\left(\lambda_0 + \varepsilon\right) \sum_{i=1}^{\tau(k_j)-1} p(i) \exp\left(-\left(\lambda_0 + \varepsilon\right) \sum_{l=1}^{\tau(i)-1} p(l)\right) \leq \exp\left(2\varepsilon \sum_{i=1}^{\tau(k_j)} p(i)\right) \cdot
\]
From the last inequality, and taking into account that (3-2) is satisfied, we have
\[
\lim sup_{j \to +\infty} \exp\left(\lambda_0 + \varepsilon\right) \sum_{i=1}^{\tau(k_j)-1} p(i) \exp\left(-\left(\lambda_0 + \varepsilon\right) \sum_{l=1}^{\tau(i)-1} p(l)\right) \leq \exp(2\varepsilon M),
\]
where
\[
M = \lim sup_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i).
\]
Hence, for any \(\varepsilon > 0\), (3-13) gives
\[
\lim inf_{k \to +\infty} \exp\left(\lambda_0 + \varepsilon\right) \sum_{i=1}^{k-1} p(i) \exp\left(-\left(\lambda_0 + \varepsilon\right) \sum_{l=1}^{\tau(i)-1} p(l)\right) \leq \exp(2\varepsilon M),
\]
which implies
\[
\limsup_{\epsilon \to 0^+} \left( \liminf_{k \to +\infty} \exp \left( (\lambda_0 + \epsilon) \sum_{i=1}^{k-1} p(i) \right) \sum_{i=k}^{\infty} p(i) \exp \left( -(\lambda_0 + \epsilon) \sum_{l=1}^{\tau(i)-1} p(l) \right) \right) \leq 1. \]
\[
\square
\]

**Remark 3.2.** Condition (3-2) is not a limitation since, as proved in [Chatzarakis et al. 2008a], if \( \tau \) is a nondecreasing function and
\[
\limsup_{k \to +\infty} k \sum_{i=\tau(k)}^{k-1} p(i) > 1,
\]
then \( U_{k_0} = \emptyset \), for any \( k_0 \in \mathbb{N} \).

**Remark 3.3.** In (3-1), without loss of generality, we may assume that \( c \leq 1 \). Otherwise, for any \( k_0 \in \mathbb{N} \), we have \( U_{k_0} = \emptyset \) [Chatzarakis et al. 2008a].

**Theorem 3.4.** Assume that all the conditions of Theorem 3.1 are satisfied. Then
\[
\liminf_{k \to +\infty} \sum_{i=1}^{k-1} p(i) \leq \frac{1}{e}.
\]

**Proof.** Since all the conditions of Theorem 3.1 are satisfied, there exists \( \lambda = \lambda_0 \in [1, 4/c^2] \) such that the inequality (3-3) holds.

Assume that the condition (3-14) does not hold. Then, there exists \( k_1 \in \mathbb{N} \) and \( \epsilon_0 > 0 \) such that
\[
\sum_{i=\tau(k)}^{k-1} p(i) \geq \frac{1 + \epsilon_0}{e} \quad \text{for} \quad k \in N_{k_1}.
\]
Therefore, for any \( \epsilon > 0 \),
\[
I(k, \epsilon) = \exp \left( (\lambda_0 + \epsilon) \sum_{i=1}^{k-1} p(i) \right) \sum_{i=k}^{\infty} p(i) \exp \left( -(\lambda_0 + \epsilon) \sum_{l=1}^{\tau(i)-1} p(l) \right)
\geq \exp \left( \frac{(\lambda_0 + \epsilon)(1 + \epsilon_0)}{e} \right) \exp \left( (\lambda_0 + \epsilon) \sum_{i=1}^{k-1} p(i) \right)
\times \sum_{i=k}^{\infty} p(i) \exp \left( -(\lambda_0 + \epsilon) \sum_{l=1}^{i-1} p(l) \right) \quad \text{for} \quad k \in N_{k_1}.
\]

Defining \( \sum_{l=1}^{i-1} p(l) = a_{i-1} \), we will show that
\[
\liminf_{k \to +\infty} \exp((\lambda_0 + \epsilon)a_{k-1}) \sum_{i=k}^{\infty} p(i) \exp(-(\lambda_0 + \epsilon)a_{i-1}) \geq \frac{1}{\lambda_0 + \epsilon}.
\]
Indeed, since
\[
\liminf_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) = c > 0,
\]
it is obvious that \(\sum_{i=1}^{+\infty} p(i) = +\infty\), that is, \(\lim_{i \to +\infty} a_i = +\infty\). Therefore
\[
\exp((\lambda_0 + \varepsilon)a_{k-1}) \sum_{i=k}^{+\infty} p(i) \exp(-(\lambda_0 + \varepsilon)a_{i-1})
\]
\[
= \exp((\lambda_0 + \varepsilon)a_{k-1}) \sum_{i=k}^{+\infty} (a_i - a_{i-1}) \exp(-(\lambda_0 + \varepsilon)a_{i-1})
\]
\[
= \exp((\lambda_0 + \varepsilon)a_{k-1}) \sum_{i=k}^{+\infty} \exp(-(\lambda_0 + \varepsilon)a_{i-1}) \int_{a_{i-1}}^{a_i} ds
\]
\[
\geq \exp((\lambda_0 + \varepsilon)a_{i-1}) \sum_{i=k}^{+\infty} \int_{a_{i-1}}^{a_i} \exp(-(\lambda_0 + \varepsilon)s) ds
\]
\[
= \exp((\lambda_0 + \varepsilon)a_{i-1}) \int_{a_{i-1}}^{+\infty} \exp(-(\lambda_0 + \varepsilon)s) ds = \frac{1}{\lambda_0 + \varepsilon}.
\]
Hence, by (3-15), we obtain
\[
\limsup_{\varepsilon \to 0+} \left( \liminf_{k \to +\infty} I(k, \varepsilon) \right) \geq \frac{1}{\lambda_0} \cdot \exp\left(\frac{\lambda_0(1 + \varepsilon_0)}{e}\right) \geq 1 + \varepsilon_0.
\]
This contradicts (3-3) for \(\lambda = \lambda_0\). \(\square\)

4. Sufficient conditions of the proper solutions to be oscillatory

**Theorem 4.1.** Assume that conditions (1-5), (3-1), (3-2) are satisfied and that, for any \(\lambda \in [1, 4/c^2]\),
\[\tag{4-1}\]
\[
\limsup_{\varepsilon \to 0+} \left( \liminf_{k \to +\infty} \left( \exp\left( (\lambda + \varepsilon) \sum_{i=1}^{k-1} p(i) \right) \sum_{i=k}^{+\infty} p(i) \exp\left( - (\lambda + \varepsilon) \sum_{l=1}^{\tau(i)-1} p(l)i \right) \right) \right) > 1.
\]
Then all proper solutions of Equation (1-4) oscillate.

**Proof.** Assume that \(u : N_{k_0} \to (0, +\infty)\) is a positive proper solution of (1-4). Then \(U_{k_0} \neq \emptyset\). Thus, in view of **Theorem 3.1**, there exists \(\lambda_0 \in [1, 4/c^2]\) such that the condition (3-3) is satisfied for \(\lambda = \lambda_0\). But this contradicts (4-1). \(\square\)

Using **Theorem 3.4**, we can similarly prove:
Theorem 4.2. Assume that conditions (1-5) and (3-2) are satisfied and

(4-2) \[ \liminf_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) > \frac{1}{e}. \]

Then all proper solutions of Equation (1-4) oscillate.

Remark 4.3. It is to be pointed out that Theorem 4.2 is the discrete analogue of Theorem 1.1 for the first order difference equation (1-4) in the case of a general delay argument \( \tau(k) \).

Remark 4.4. The condition (4-2) is optimal for (1-4) under the assumption that \( \lim_{k \to +\infty} (k - \tau(k)) = +\infty \). since in this case the set of natural numbers increases infinitely in the interval \([\tau(k), k-1]\) for \( k \to +\infty \).

Now, we are going to present two examples to show that the condition (4-2) is optimal, in the sense that it cannot be replaced by the nonstrong inequality.

Example 4.5. Consider (1-4), where

(4-3) \[ \tau(k) = [\alpha k], \quad p(k) = (k^{-\lambda} - (k + 1)^{-\lambda})[\alpha k], \]

\[ \alpha \in (0, 1), \quad \lambda = -\ln^{-1} \alpha, \]

with \( [\alpha k] \) the integer part of \( \alpha k \).

It is obvious that

\[ k^{1+\lambda}(k^{-\lambda} - (k + 1)^{-\lambda}) \to \lambda \] \[ \text{for} \quad k \to +\infty. \]

Therefore

(4-4) \[ k(k^{-\lambda} - (k + 1)^{-\lambda})[\alpha k]^\lambda \to \frac{\lambda}{e} \] \[ \text{for} \quad k \to +\infty. \]

Hence, in view of (4-3) and (4-4), we have

\[ \liminf_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) = \frac{\lambda}{e} \liminf_{k \to +\infty} \sum_{i=[\alpha k]}^{k-1} \frac{1}{i} \left( i^{-\lambda} - (i + 1)^{-\lambda} \right) [\alpha i]^\lambda \]

\[ = \frac{\lambda}{e} \liminf_{k \to +\infty} \sum_{i=[\alpha k]}^{k-1} \frac{1}{i} = \frac{\lambda}{e} \ln \frac{1}{\alpha} = \frac{1}{e} \]

or

\[ \liminf_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) = \frac{1}{e}. \]
Observe that all the conditions of Theorem 4.2 are satisfied except the condition (4-2). In this case, it is not guaranteed that all solutions of (1-4) oscillate. Indeed, it is easy to see that the function \( u = k^{-\lambda} \) is a positive solution of (1-4).

**Example 4.6.** Consider (1-4), where

\[
\tau(k) = \lfloor k^\alpha \rfloor, \quad p(k) = (\ln^{-\lambda} k - \ln^{-\lambda} (k + 1)) \ln^\lambda [k^\alpha],
\]

\( \alpha \in (0, 1), \quad \lambda = -\ln^{-1} \alpha, \)

with \( [k^\alpha] \) the integer part of \( k^\alpha \).

It is obvious that

\[
k \ln^{1+\lambda} k (\ln^{-\lambda} k - \ln^{-\lambda} (k + 1)) \to \lambda \quad \text{for} \quad k \to +\infty.
\]

Therefore

\[
k \ln k \ln^\lambda [k^\alpha] (\ln^{-\lambda} k - \ln^{-\lambda} (k + 1)) \to \frac{\lambda}{e} \quad \text{for} \quad k \to +\infty.
\]

On the other hand,

\[
\sum_{i=[k^\alpha]}^{k-1} \frac{1}{i \ln i} \geq \sum_{i=[k^\alpha]}^{k-1} \int_i^{i+1} \frac{ds}{s \ln s} = \int_{[k^\alpha]}^{k} \frac{ds}{s \ln s} = \ln \frac{k}{\ln [k^\alpha]},
\]

which tends to \( \ln(1/\alpha) \) as \( k \to +\infty \), and

\[
\sum_{i=[k^\alpha]}^{k-1} \frac{1}{i \ln i} \leq \sum_{i=[k^\alpha]}^{k-1} \int_{i-1}^{i} \frac{ds}{s \ln s} = \int_{[k^\alpha]-1}^{k-1} \frac{ds}{s \ln s} = \ln \frac{\ln(k-1)}{\ln [k^\alpha]-1},
\]

which also tends to \( \ln(1/\alpha) \) as \( k \to +\infty \). Together these two bounds imply

\[
\lim_{k \to +\infty} \sum_{i=[k^\alpha]}^{k-1} \frac{1}{i \ln i} = \ln \frac{1}{\alpha}.
\]

Hence, in view of (4-5) and (4-6), we obtain

\[
\liminf_{k \to +\infty} \sum_{i=[k^\alpha]}^{k-1} p(i) = \frac{\lambda \ln^\frac{\ln^\lambda [i^\alpha] (\ln^{-\lambda} i - \ln^{-\lambda} (i + 1))}{\lambda}}{e^{k \ln i}} = \frac{\lambda}{e^{k \ln i}} = \frac{\lambda}{e^{\ln \frac{1}{\alpha}}} = \frac{\lambda}{e^{\frac{1}{\alpha}}}
\]
We again observe that all the conditions of Theorem 4.2 are satisfied except (4-2). In this case, it is not guaranteed that all solutions of (1-4) oscillate. Indeed, it is easy to see that the function $u = \ln^{-i} k$ is a positive solution of (1-4).

References


[Baštinec and Diblik 2005] J. Baštinec and J. Diblik, “Remark on positive solutions of discrete equation $\Delta u(k + n) = -p(k) u(k)$”, *Nonlinear Anal.* 63 (2005), e2145–e2151.


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