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THE SUPER-VIRASORO ALGEBRAS**

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## LEFT-SYMMETRIC SUPERALGEBRAIC STRUCTURES ON THE SUPER-VIRASORO ALGEBRAS

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**We classify the compatible left-symmetric superalgebraic structures on the super-Virasoro algebras satisfying certain natural conditions.**

### 1. Introduction

Left-symmetric algebras (also called pre-Lie algebras, quasiassociative algebras, or Vinberg algebras) are a class of natural algebraic systems appearing in many fields in mathematics and mathematical physics. They were first mentioned by A. Cayley [1890] as a kind of rooted tree algebra and later arose again from the study of convex homogeneous cones [Vinberg 1963], affine manifolds and affine structures on Lie groups [Koszul 1961], and deformation of associative algebras [Gerstenhaber 1963]. They play an important role in the study of symplectic and complex structures on Lie groups and Lie algebras [Andrada and Salamon 2005; Chu 1974; Dardié and Medina 1996b; 1996a; Lichnerowicz and Medina 1988], phase spaces of Lie algebras [Bai 2006; Kupersmidt 1994; 1999a], certain integrable systems [Bordemann 1990; Svinolupov and Sokolov 1994], classical and quantum Yang–Baxter equations [Diatta and Medina 2004; Etingof and Soloviev 1999; Golubchik and Sokolov 2000; Kupersmidt 1999b], combinatorics [Ebrahimi-Fard 2002], quantum field theory [Connes and Kreimer 1998], vertex algebras [Bakalov and Kac 2003], and operads [Chapoton and Livernet 2001]. [Burde 2006] gives a survey.

The super version of left-symmetric algebras, the left-symmetric superalgebras, also appeared in many other fields; see for example [Chapoton and Livernet 2001; Gerstenhaber 1963; Vasil’eva and Mikhalëv 1996]. To our knowledge, they were first introduced by Gerstenhaber [1963] to study the Hochschild cohomology of associative algebras.

On the other hand, the Virasoro and super-Virasoro algebras are not only a class of important infinite-dimensional Lie algebras and Lie superalgebras but are also of considerable interest in physics. For example, they are the fundamental algebraic

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structures in conformal and superconformal field theory. As Kupershmidt [1999a] pointed out, a compatible left-symmetric algebra structure on the Virasoro algebra can be regarded as the “nature of the Virasoro algebra”; in fact, on the Virasoro algebra  $\mathcal{V}$  given there, such an algebraic structure satisfies

$$(1-1) \quad cc = x_m c = c x_m = 0 \quad \text{and} \quad x_m x_n = f(m, n)x_{m+n} + \omega(m, n)c,$$

where  $f(m, n)$  and  $\omega(m, n)$  are two complex-valued functions, and  $\{x_m, c \mid m \in \mathbb{Z}\}$  is a basis of the Virasoro algebra  $\mathcal{V}$  satisfying

$$(1-2) \quad [c, x_n] = 0 \quad \text{and} \quad [x_m, x_n] = (m - n)x_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

The condition (1-1) is natural since it means that the compatible left-symmetric algebra is still graded and  $c$  is also a central extension given by  $\omega(m, n)$ . Moreover, in [Kong et al. 2007], we proved that any compatible left-symmetric algebraic structure on the Virasoro algebra  $\mathcal{V}$  satisfying (1-1) is isomorphic to one of the examples given in [Kupershmidt 1999a].

In this paper, we study the compatible left-symmetric superalgebraic structures on the super-Virasoro algebras. Motivated by the study in the case of the ordinary Virasoro algebra, we classify such left-symmetric superalgebras satisfying some natural conditions like (1-1). The paper is organized as follows. In Section 2, we give some necessary definitions, notations, and basic results on left-symmetric superalgebras and the super-Virasoro algebras. We also give the classification of compatible left-symmetric algebraic structures on the ordinary Virasoro algebra satisfying (1-1). In Section 3, we study the compatible left-symmetric superalgebraic structures on the centerless super-Virasoro algebras satisfying certain natural conditions. In Section 4, we discuss the nontrivial central extensions of the left-symmetric superalgebras obtained in Section 3 whose supercommutator is a super-Virasoro algebra.

Throughout, all algebras are over the complex field  $\mathbb{C}$  and the indices  $m, n, l \in \mathbb{Z}$  and  $r, s, t \in \mathbb{Z} + \theta$  for  $\theta = 0$  or  $\theta = 1/2$ , unless otherwise stated.

## 2. Preliminaries and fundamental results

Let  $(A, \cdot)$  be an algebra over a field  $\mathbb{F}$ .  $A$  is said to be a superalgebra if the underlying vector space of  $A$  is  $\mathbb{Z}_2$ -graded, that is,  $A = A_{\bar{0}} \oplus A_{\bar{1}}$ , and  $A_{\alpha} \cdot A_{\beta} \subset A_{\alpha+\beta}$ , for  $\alpha, \beta \in \mathbb{Z}_2$ . An element of  $A_{\bar{0}}$  is called even and an element of  $A_{\bar{1}}$  is called odd.

**Definition 2.1.** A Lie superalgebra is a superalgebra  $A = A_{\bar{0}} \oplus A_{\bar{1}}$  with an operation  $[\cdot, \cdot]$  satisfying the conditions

$$(2-1) \quad [a, b] = -(-1)^{\alpha\beta}[b, a],$$

$$(2-2) \quad [a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]$$

where  $a \in A_\alpha$ ,  $b \in A_\beta$ ,  $c \in A$ , and  $\alpha, \beta \in \mathbb{Z}_2$ .

**Definition 2.2.** A superalgebra  $A$  is called a left-symmetric superalgebra if the associator

$$(2-3) \quad (x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$$

of  $A$  satisfies

$$(2-4) \quad (x, y, z) = (-1)^{\alpha\beta}(y, x, z) \quad \text{for all } x \in A_\alpha, y \in A_\beta, z \in A, \alpha, \beta \in \mathbb{Z}_2.$$

Obviously, if  $A = A_{\bar{0}} \oplus A_{\bar{1}}$  is a Lie superalgebra or a left-symmetric superalgebra, then  $A_{\bar{0}}$  is an ordinary Lie algebra or a left-symmetric algebra, respectively. Letting  $A$  be a left-symmetric superalgebra, it is easy to show that the supercommutator

$$(2-5) \quad [x, y] = x \cdot y - (-1)^{\alpha\beta}y \cdot x \quad \text{for all } x \in A_\alpha, y \in A_\beta, \alpha, \beta \in \mathbb{Z}_2$$

defines a Lie superalgebra  $\mathcal{G}(A)$ , which is called the subadjacent Lie superalgebra of  $A$ , and  $A$  is also called the compatible left-symmetric superalgebraic structure on the Lie superalgebra  $\mathcal{G}(A)$ .

On the other hand, we recall the definition of the super-Virasoro algebras. There are two super-Virasoro algebras that correspond to  $N = 1$  [Ramond 1971] and  $N = 2$  [Neveu and Schwarz 1971a; 1971b] superconformal field theory. In fact, let  $\theta = 0$  or  $1/2$ , which corresponds to the Ramond case or the Neveu–Schwarz case, respectively. Let  $\mathcal{SV} = \mathcal{SV}_{\bar{0}} \oplus \mathcal{SV}_{\bar{1}}$  denote a super-Virasoro algebra with a basis  $\{L_m, G_r, c \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \theta\}$ . The superbrackets are defined as

$$(2-6) \quad \begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right)G_{m+r}, \\ [G_r, G_s] &= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s,0}, \\ [\mathcal{SV}_{\bar{0}}, c] &= [\mathcal{SV}_{\bar{1}}, c] = 0, \end{aligned}$$

where the even subspace  $\mathcal{SV}_{\bar{0}}$  is spanned by  $\{L_m, c \mid m \in \mathbb{Z}\}$  and the odd subspace  $\mathcal{SV}_{\bar{1}}$  is spanned by  $\{G_r \mid r \in \mathbb{Z} + \theta\}$ . Obviously,  $\mathcal{SV}_{\bar{0}}$  is nothing but an ordinary Virasoro algebra. A class of compatible left-symmetric algebraic structures on the Virasoro algebra satisfying (1-1) were given in [Kupershmidt 1999a]. Such left-symmetric algebras were classified in [Kong et al. 2007].

**Theorem 2.3** [Kong et al. 2007]. *Any compatible left-symmetric algebraic structure on the Virasoro algebra  $\mathcal{SV}_{\bar{0}}$  satisfying (1-1) is isomorphic to one of the (mutually nonisomorphic) left-symmetric algebras given by the multiplication*

$$(2-7) \quad L_m L_n = \frac{-n(1+\epsilon n)}{1+\epsilon(m+n)} L_{m+n} + \frac{c}{24}(m^3 - m + (\epsilon - \epsilon^{-1})m^2)\delta_{m+n,0}$$

for all  $m, n \in \mathbb{Z}$ , where  $c$  is an annihilator and  $\operatorname{Re} \epsilon > 0$ ,  $\epsilon^{-1} \notin \mathbb{Z}$  or  $\operatorname{Re} \epsilon = 0$ ,  $\operatorname{Im} \epsilon > 0$ .

### 3. Compatible left-symmetric superalgebraic structures on the centerless super-Virasoro algebras

Let  $\widetilde{\mathcal{FV}} = \widetilde{\mathcal{FV}}_{\bar{0}} \oplus \widetilde{\mathcal{FV}}_{\bar{1}}$  be a centerless super-Virasoro algebra with a basis  $\{L_m, G_r \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \theta\}$ , and let the superbrackets be given by (2-6) with  $c = 0$ . Theorem 2.3 motivates us to consider the compatible left-symmetric superalgebraic structures on  $\widetilde{\mathcal{FV}}$  that also satisfy the ‘‘graded’’ condition, that is, the multiplications of the compatible left-symmetric superalgebraic structures on  $\widetilde{\mathcal{FV}}$  that satisfy

$$(3-1) \quad \begin{aligned} L_m \cdot L_n &= f(m, n)L_{m+n}, & L_m \cdot G_r &= g(m, r)G_{m+r}, \\ G_r \cdot L_m &= h(r, m)G_{m+r}, & G_r \cdot G_s &= d(r, s)L_{r+s}, \end{aligned}$$

where  $f, g, h$ , and  $d$  are  $\mathbb{C}$ -valued functions. Then the supercommutators give the super-Virasoro algebra  $\widetilde{\mathcal{FV}}$  if and only if  $f(m, n)$ ,  $g(m, r)$ ,  $h(r, m)$ , and  $d(r, s)$  satisfy

$$(3-2) \quad \begin{aligned} f(m, n) - f(n, m) &= m - n, \\ d(r, s) + d(s, r) &= 2, & g(m, r) - h(r, m) &= m/2 - r. \end{aligned}$$

Furthermore, the functions  $f(m, n)$ ,  $g(m, r)$ ,  $h(r, m)$ , and  $d(r, s)$  define a left-symmetric superalgebra with a basis  $\{L_m, G_r \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \theta\}$  if and only if they satisfy the equations

$$\begin{aligned} (L_m, L_n, L_l) &= (-1)^{0 \cdot 0}(L_n, L_m, L_l), & (L_m, L_n, G_r) &= (-1)^{0 \cdot 0}(L_n, L_m, G_r), \\ (L_m, G_r, L_n) &= (-1)^{0 \cdot 1}(G_r, L_m, L_n), & (L_m, G_r, G_s) &= (-1)^{0 \cdot 1}(G_r, L_m, G_s), \\ (G_r, G_s, L_m) &= (-1)^{1 \cdot 1}(G_s, G_r, L_m), & (G_r, G_s, G_t) &= (-1)^{1 \cdot 1}(G_s, G_r, G_t). \end{aligned}$$

These equations are equivalent to the equations

$$(3-3) \quad \begin{aligned} (m-n)f(m+n, l) &= f(n, l)f(m, n+l) - f(m, l)f(n, m+l), \\ (m-n)g(m+n, r) &= g(n, r)g(m, n+r) - g(m, r)g(n, m+r), \\ (m/2-r)h(m+r, n) &= h(r, n)g(m, n+r) - f(m, n)h(r, m+n), \\ (m/2-r)d(m+r, s) &= d(r, s)f(m, r+s) - g(m, s)d(r, m+s), \\ 2f(r+s, m) &= h(s, m)d(r, m+s) + h(r, m)d(s, m+r), \\ 2g(r+s, t) &= d(s, t)h(r, s+t) + d(r, t)h(s, r+t). \end{aligned}$$

**Proposition 3.1.** *Any compatible left-symmetric superalgebraic structure  $\widetilde{\mathcal{V}}$  on  $\widetilde{\mathcal{FV}}$  satisfies (3-1) if and only if the functions in (3-1) satisfy (3-2) and (3-3).*

By Theorem 2.3, we only need to consider the case that

$$(3-4) \quad f(m, n) = \frac{-n(1+\epsilon n)}{1+\epsilon(m+n)},$$

where  $\operatorname{Re} \epsilon > 0$ ,  $\epsilon^{-1} \notin \mathbb{Z}$  or  $\operatorname{Re} \epsilon = 0$ ,  $\operatorname{Im} \epsilon > 0$ .

**Theorem 3.2.** *For a fixed  $\epsilon$  satisfying  $\operatorname{Re} \epsilon > 0$ ,  $\epsilon^{-1} \notin \mathbb{Z}$  or  $\operatorname{Re} \epsilon = 0$ ,  $\operatorname{Im} \epsilon > 0$  and  $f(m, n)$  satisfying (3-4), there is exactly one solution satisfying (3-2) and (3-3) given by*

$$(3-5) \quad \begin{aligned} g(m, r) &= \frac{-(\frac{m}{2} + r)(1 + 2\epsilon r)}{1 + 2\epsilon(m + r)}, \\ h(r, m) &= \frac{-m(1 + \epsilon m)}{1 + 2\epsilon(m + r)}, \quad d(r, s) = \frac{1 + 2\epsilon s}{1 + \epsilon(r + s)}, \end{aligned}$$

for  $m, n \in \mathbb{Z}$  and  $r, s \in \mathbb{Z} + \theta$ , which define a compatible left-symmetric superalgebra  $\tilde{\mathcal{V}}_\epsilon$  on  $\mathcal{FV}$ .

*Proof.* It is easy to verify that  $f(m, n)$  given in (3-4) and  $g(m, r)$ ,  $h(r, m)$ , and  $d(r, s)$  given in (3-5) satisfy (3-2) and (3-3). On the other hand, set

$$\begin{aligned} G(m, r) &= g(m, r) \frac{1 + 2\epsilon(m + r)}{1 + 2\epsilon r}, \\ H(r, m) &= h(r, m) \frac{1 + 2\epsilon(m + r)}{1 + \epsilon m}, \quad D(r, s) = d(r, s) \frac{1 + \epsilon(r + s)}{1 + 2\epsilon s}. \end{aligned}$$

Then we only need to prove that

$$G(m, r) = -m/2 - r, \quad H(r, m) = -m, \quad D(r, s) = 1.$$

We rewrite equations (3-2) and (3-3) involving  $g(m, r)$ ,  $h(r, m)$ ,  $d(r, s)$  as

$$(3-6) \quad G(m, r)(1 + 2\epsilon r) - H(r, m)(1 + \epsilon m) = (m/2 - r)(1 + 2\epsilon(m + r)),$$

$$(3-7) \quad D(r, s)(1 + 2\epsilon s) + D(s, r)(1 + 2\epsilon r) = 2 + 2\epsilon(r + s),$$

and

$$(3-8) \quad (m - n)G(m + n, r) = G(n, r)G(m, n + r) - G(m, r)G(n, m + r)$$

$$(3-9) \quad (m/2 - r)H(m + r, n) = H(r, n)G(m, n + r) + nH(r, m + n),$$

$$(3-10) \quad (m/2 - r)D(m + r, s) = -(r + s)D(r, s) - G(m, s)D(r, m + s),$$

$$(3-11) \quad -2m = H(s, m)D(r, m + s) + H(r, m)D(s, m + r),$$

$$(3-12) \quad 2G(r + s, t) = D(s, t)H(r, s + t) + D(r, t)H(s, r + t).$$

Letting  $r = s$  in equation (3-7), we have

$$(3-13) \quad D(s, s) = 1 \quad \text{for all } s \in \mathbb{Z} + \theta.$$

In fact,  $D(r, s) \neq 0$  for all  $r, s \in \mathbb{Z} + \theta$ . Otherwise, assume there exist  $r_1$  and  $s_1$  such that  $D(r_1, s_1) = 0$ . Letting  $r = s = r_1$  and  $m = s_1 - r_1$  in (3-11), we have

$$-(s_1 - r_1) = H(r_1, s_1 - r_1)D(r_1, s_1) = 0.$$

Hence  $r_1 = s_1$ . This contradicts (3-13).

Let  $m = 0$  and  $r = s$  in (3-10) and (3-6),  $r = s = -t$  in (3-12),  $m = -2s$ ,  $r = s$  in (3-6) and (3-10), and  $m = -2s \neq 0$ ,  $r = 3s$  in (3-10). We know that

$$(3-14) \quad \begin{aligned} G(0, s) &= -s, & G(2s, -s) &= 0, \\ H(s, 0) &= 0, & H(s, -2s) &= 2s, \\ D(-s, s) &= D(3s, s) = 1 & \text{for all } s \in \mathbb{Z} + \theta. \end{aligned}$$

Letting  $m = -2(n + r)$  in (3-9) and  $m + r + s = 0$  in (3-11), we have

$$(3-15) \quad \begin{aligned} 0 &= (n + 2r)H(-2n - r, n) + nH(r, -n - 2r), \\ -2m &= H(-m - r, m) + H(r, m). \end{aligned}$$

Letting  $r = s$  in (3-11), we have  $-m = H(r, m)D(r, m + r)$ . So

$$(3-16) \quad H(r, m) = -m/D(r, m + r).$$

By (3-13)–(3-16), we have

$$(3-17) \quad \begin{aligned} D(-2n - r, -n - r) &= D(r, -n - r), \\ 1/D(r, m + r) + 1/D(-m - r, -r) &= 2. \end{aligned}$$

Letting  $-n - r = s$  and  $m + r = s$  in (3-17), we have  $D(r, s) = D(2s + r, s)$  and  $D(-s, -r) = D(-s, -2s - r)$  for all  $r, s \in \mathbb{Z} + \theta$ . Thus by induction, we know that

$$(3-18) \quad D(r, s) = D(2ks + r, s) \quad \text{and} \quad D(-s, -r) = D(-s, -2ks - r)$$

for all  $k \in \mathbb{Z}$ . Therefore,  $D(r, r) = D((2k + 1)r, r) = D(r, (2k + 1)r) = 1$  for all  $k \in \mathbb{Z}$ . Let  $r = s$  in (3-12). Then by (3-16) and (3-18), we have

$$G(2s, t) = D(s, t)H(s, s + t) = D(s, t) \frac{-s - t}{D(s, 2s + t)} = -s - t.$$

Letting  $m = 2s$  in (3-6), we have

$$(3-19) \quad H(t, 2s) = -2s \quad \text{for all } s, t \in \mathbb{Z} + \theta.$$

*Case I.*  $\theta = 1/2$ . Then  $D(\theta, \pm\theta) = D(\pm\theta, \theta) = 1$ . Hence  $D(k + \theta, \pm\theta) = D(\pm\theta, k + \theta) = 1$ . That is,  $D(r, \pm 1/2) = D(\pm 1/2, r) = 1$  for all  $r \in \mathbb{Z} + \theta$ . Assume that for any  $|r_1| \leq |s_1|$ , we have  $D(r_1, s_1) = 1$ . Then

$$D(r_1, s_1) = D(2ks_1 + r_1, s_1) = 1 \quad \text{and} \quad D(s_1, r_1) = D(s_1, 2ks_1 + r_1) = 1.$$

For any  $r \in \mathbb{Z} + \theta$ , there exist  $k \in \mathbb{Z}$ ,  $r_1 \in \mathbb{Z} + \theta$ , and  $|r_1| \leq |s_1|$  such that  $r = 2ks_1 + r_1$ . Therefore  $D(r, s_1) = D(s_1, r) = 1$  for all  $r \in \mathbb{Z} + \theta$ . Hence by induction, we know that  $D(r, s) = 1$  for any  $r, s \in \mathbb{Z} + \theta$ . Therefore  $H(r, m) = -m$  and  $G(m, r) = -m/2 - r$  for all  $m \in \mathbb{Z}$  and  $r \in \mathbb{Z} + \theta$ .

*Case II.*  $\theta = 0$ . Let  $m = -2t \neq 0$  and  $s = r = 2t$  in (3-11). Then by (3-19), we have  $2t = H(2t, -2t)D(2t, 0) = 2tD(2t, 0)$ . Therefore

$$(3-20) \quad D(2t, 0) = 1 \quad \text{and} \quad D(0, 2t) = 1 \quad \text{for all } t \in \mathbb{Z}.$$

Letting  $r = 0$  and  $m = s \neq 0$  in (3-10) and (3-11), we have

$$(3-21) \quad \begin{aligned} m/2 &= -mD(0, m) - G(m, m), \\ -2m &= H(m, m) + H(0, m). \end{aligned}$$

So

$$H(m, m) = -2m - H(0, m) = -2m + \frac{m}{D(0, m)} = -2m - \frac{2m^2}{m + 2G(m, m)}.$$

By (3-6), we have  $H(m, m) = -m$  and  $G(m, m) = -3m/2$ , or

$$(3-22) \quad H(m, m) = \frac{-m}{1 + \epsilon m} \quad \text{and} \quad G(m, m) = \frac{-3m - 4\epsilon m^2}{2 + 4\epsilon m}.$$

In fact, the latter case cannot hold for any  $m \neq 0$ . Otherwise, assume that there exists a nonzero integer  $m_1$  satisfying (3-22) with  $m_1$  replacing  $m$ . Then

$$D(0, m_1) = -\frac{1}{2} - \frac{G(m_1, m_1)}{m_1} = \frac{1 + \epsilon m_1}{1 + 2\epsilon m_1}.$$

Letting  $m = -s = m_1$  and  $r = 0$  or  $-m_1$  in (3-10), we have

$$\begin{aligned} \frac{m_1}{2} &= m_1 D(0, -m_1) - G(m_1, -m_1), \\ \frac{3m_1}{2} D(0, -m_1) &= 2m_1 - G(m_1, -m_1)D(-m_1, 0). \end{aligned}$$

Hence  $(3/2)D(0, -m_1) = 2 - D(0, -m_1)D(-m_1, 0) + (1/2)D(-m_1, 0)$ . By (3-7), we have  $D(0, -m_1) = 1$  and  $D(-m_1, 0) = 1$ , or

$$D(0, -m_1) = \frac{3 - \epsilon m_1}{1 - 2\epsilon m_1} \quad \text{and} \quad D(-m_1, 0) = -1 - \epsilon m_1.$$

Since  $\epsilon \neq 0$  and  $\epsilon^{-1} \notin \mathbb{Z}$ , we know that  $1/D(0, m_1) + 1/D(-m_1, 0) \neq 2$ , which contradicts (3-17). Hence  $H(m, m) = -m$  and  $G(m, m) = -3m/2$  for all  $m \in \mathbb{Z}$ . By (3-21) and (3-6), we have  $H(0, m) = -m$  and  $G(m, 0) = -m/2$ . Letting  $r = 0$  and  $m \neq 0$  in (3-8) and (3-9), we have

$$n^2 - m^2 = -nG(m, n) + mG(n, m), \quad (m/2)H(m, n) = -nG(m, n) - n(m + n).$$

So  $H(m, n) + 2G(n, m) = -2(m + n)$ . By (3-6), (3-13), and (3-16), we know that  $H(m, n) = -n$  and  $G(n, m) = -n/2 - m$  and  $D(m, n) = 1$  for all  $m, n \in \mathbb{Z}$ .  $\square$

#### 4. Compatible left-symmetric superalgebraic structures on the super-Virasoro algebras

We now consider the central extensions of the left-symmetric superalgebras obtained in Section 3 whose supercommutator is a super-Virasoro algebra  $\mathcal{SV}$ .

Let  $\tilde{A}$  be a left-symmetric superalgebra, and let  $\omega : \tilde{A} \times \tilde{A} \rightarrow \mathbb{C}$  be a bilinear form. It defines a multiplication on the space  $A = \tilde{A} \oplus \mathbb{C}c$  by the rule

$$(4-1) \quad (x + \lambda c) \cdot (y + \mu c) = x \cdot y + \omega(x, y)c \quad \text{for all } x, y \in A, \lambda, \mu \in \mathbb{C}.$$

Let

$$(4-2) \quad B(x, y, z) := \omega(x \cdot y, z) - \omega(x, y \cdot z).$$

Then it is easy to show that  $A$  is a left-symmetric superalgebra if and only if

$$(4-3) \quad B(x, y, z) = (-1)^{\alpha\beta} B(y, x, z)$$

for all  $x \in \tilde{A}_\alpha$ ,  $y \in \tilde{A}_\beta$ , and  $z \in \tilde{A}$ , where  $\alpha, \beta \in \mathbb{Z}_2$ . The algebra  $A$  is called a central extension of  $\tilde{A}$ . By construction, the bilinear form

$$(4-4) \quad \Omega(x, y) = \omega(x, y) - (-1)^{\alpha\beta} \omega(y, x)$$

for all  $x \in \tilde{A}_\alpha$ ,  $y \in \tilde{A}_\beta$  and  $z \in \tilde{A}$ , where  $\alpha, \beta \in \mathbb{Z}_2$ , defines a central extension of the Lie superalgebra  $\mathcal{G}(A)$ .

Let the left-symmetric superalgebra  $\tilde{V}_\epsilon$  on a centerless Virasoro algebra  $\widehat{\mathcal{SV}}$  be given through Theorem 3.2. Since a super-Virasoro algebra  $\mathcal{SV}$  is a central extension of a centerless super-Virasoro algebra  $\widehat{\mathcal{SV}}$ , it is natural to consider the central extension  $V_\epsilon = \tilde{V}_\epsilon \oplus \mathbb{C}c$  of  $\tilde{V}_\epsilon$  such that  $V_\epsilon$  is a compatible left-symmetric superalgebraic structure on the super-Virasoro algebra  $\mathcal{SV}$  with  $c$  being the annihilator of  $V_\epsilon$ , that is, the products of  $V_\epsilon$  are given by

$$(4-5) \quad \begin{aligned} L_m \cdot L_n &= f(m, n)L_{m+n} + \omega(L_m, L_n)c, \\ L_m \cdot G_r &= g(m, r)G_{m+r} + \omega(L_m, G_r)c, \\ G_r \cdot L_m &= h(r, m)G_{m+r} + \omega(G_r, L_m)c, \\ G_r \cdot G_s &= d(r, s)L_{r+s} + \omega(G_r, G_s)c, \\ c \cdot c &= c \cdot L_m = L_m \cdot c = c \cdot G_r = G_r \cdot c = 0, \end{aligned}$$

where the functions  $f(m, n)$ ,  $g(m, r)$ ,  $h(r, m)$ , and  $d(r, s)$  satisfy (3-4) and (3-5).

For convenience, set

$$(4-6) \quad \begin{aligned} \omega(L_m, L_n) &= \varphi(m, n), & \omega(L_m, G_r) &= \psi(m, r), \\ \omega(G_r, L_m) &= \rho(r, m), & \omega(G_r, G_s) &= \sigma(r, s). \end{aligned}$$

So the supercommutators of  $V_\epsilon$  give a super-Virasoro algebra  $\mathcal{S}^{\mathcal{V}}$  if and only if  $\varphi(m, n)$ ,  $\psi(m, r)$ ,  $\rho(r, m)$ , and  $\sigma(r, s)$  satisfy

$$(4-7) \quad \begin{aligned} \varphi(m, n) - \varphi(n, m) &= \frac{1}{12}(m^3 - m)\delta_{m+n,0}, \\ \sigma(r, s) + \sigma(s, r) &= \frac{1}{12}(4r^2 - 1)\delta_{r+s,0}, \\ \psi(m, r) - \rho(r, m) &= 0. \end{aligned}$$

By (4-3), we have

$$\begin{aligned} B(L_m, L_n, L_l) &= (-1)^{0\cdot 0} B(L_n, L_m, L_l), \\ B(L_m, L_n, G_r) &= (-1)^{0\cdot 0} B(L_n, L_m, G_r), \\ B(L_m, G_r, L_n) &= (-1)^{0\cdot 1} B(G_r, L_m, L_n), \\ B(L_m, G_r, G_s) &= (-1)^{0\cdot 1} B(G_r, L_m, G_s), \\ B(G_r, G_s, L_m) &= (-1)^{1\cdot 1} B(G_s, G_r, L_m), \\ B(G_r, G_s, G_t) &= (-1)^{1\cdot 1} B(G_s, G_r, G_t). \end{aligned}$$

These are equivalent to the equations

$$(4-8) \quad (m - n)\varphi(m + n, l) = f(n, l)\varphi(m, n + l) - f(m, l)\varphi(n, m + l),$$

$$(4-9) \quad (m - n)\psi(m + n, r) = g(n, r)\psi(m, n + r) - g(m, r)\psi(n, m + r),$$

$$(4-10) \quad \left(\frac{m}{2} - r\right)\rho(m + r, n) = h(r, n)\psi(m, n + r) - f(m, n)\rho(r, m + n),$$

$$(4-11) \quad \left(\frac{m}{2} - r\right)\sigma(m + r, s) = d(r, s)\varphi(m, r + s) - g(m, s)\sigma(r, m + s),$$

$$(4-12) \quad 2\varphi(r + s, m) = h(s, m)\sigma(r, m + s) + h(r, m)\sigma(s, m + r),$$

$$(4-13) \quad 2\psi(r + s, t) = d(s, t)\rho(r, s + t) + d(r, t)\rho(s, r + t).$$

**Proposition 4.1.** *Any compatible left-symmetric superalgebraic structure  $V$  on  $\mathcal{S}^{\mathcal{V}}$  satisfies (4-5) if and only if the functions in (4-5) satisfy (3-4), (3-5) and (4-7)–(4-13).*

If a central extension  $V_\epsilon$  of  $\tilde{V}_\epsilon$  given by  $\omega$  satisfying Equation (4-5) defines a compatible left-symmetric superalgebraic structure on  $\mathcal{S}^{\mathcal{V}}$ , then  $\varphi(m, n)$  defines a central extension of  $\mathcal{S}^{\mathcal{V}}_{\bar{0}}$ . By Theorem 2.3, we know that

$$(4-14) \quad \varphi(m, n) = \frac{1}{24}(m^3 - m + (\epsilon - \epsilon^{-1})m^2)\delta_{m+n,0}.$$

**Theorem 4.2.** *For a fixed  $\epsilon \in \mathbb{C}$  satisfying  $\text{Re } \epsilon > 0$ ,  $\epsilon^{-1} \notin \mathbb{Z}$  or  $\text{Re } \epsilon = 0$ ,  $\text{Im } \epsilon > 0$ , suppose the functions  $f(m, n)$ ,  $g(m, r)$ ,  $h(r, m)$ , and  $d(r, s)$  satisfy (3-4) and (3-5),*

and suppose  $\varphi(m, n)$  satisfies (4-14). Then there is exactly one solution satisfying (4-7)–(4-13). It is given by

$$(4-15) \quad \begin{aligned} \sigma(r, s) &= \frac{1}{24}(4r^2 - 1 + 2(\epsilon - \epsilon^{-1})r)\delta_{r+s,0}, \\ \phi(m, r) &= \rho(r, m) = 0 \end{aligned}$$

for  $m \in \mathbb{Z}$  and  $r, s \in \mathbb{Z} + \theta$ , and it defines a compatible left-symmetric superalgebra  $V_\epsilon$  on  $\mathcal{S}\mathcal{V}$ .

*Proof.* It is easy to verify that the  $\varphi(m, n)$  given in (4-14) and the  $\sigma(r, s)$ ,  $\phi(m, r)$ , and  $\rho(r, m)$  given in (4-15) satisfy (4-7)–(4-13).

On the other hand, let  $m = 0$  and  $r + s \neq 0$  in (4-11). Then we have

$$-r\sigma(r, s) = d(r, s)\varphi(0, r+s) - g(0, s)\sigma(r, s) = s\sigma(r, s).$$

Hence  $\sigma(r, s) = 0$  for all  $r + s \neq 0$ . Letting  $r = s$  and  $m = -2s$  in Equation (4-12), we have  $2\varphi(2s, -2s) = h(s, -2s)\sigma(s, -s) + h(s, -2s)\sigma(s, -s) = 4s\sigma(s, -s)$ . So  $\sigma(s, -s) = (1/24)(4s^2 - 1 + 2(\epsilon - \epsilon^{-1})s)$ . Thus

$$\sigma(r, s) = \frac{1}{24}(4r^2 - 1 + 2(\epsilon - \epsilon^{-1})r)\delta_{r+s,0} \quad \text{for all } r, s \in \mathbb{Z} + \theta.$$

Next, we prove that  $\psi(m, r) = \rho(r, m) = 0$  for all  $m \in \mathbb{Z}$  and  $r \in \mathbb{Z} + \theta$ . There are two cases.

*Case I.*  $\theta = 1/2$ . Letting  $m = n = 0$  in (4-10), we have

$$-r\rho(r, 0) = h(r, 0)\psi(0, r) - f(0, 0)\rho(r, 0) = 0.$$

So  $\rho(r, 0) = 0$ . By (4-7), we know that  $\psi(0, r) = 0$ . Letting  $n = 0$  in (4-9), we have  $m\psi(m, r) = \psi(m, r)g(0, r) - \psi(0, m+r)g(m, r) = -r\psi(m, r)$ . Hence  $(m+r)\psi(m, r) = 0$ . Therefore, we have

$$\psi(m, r) = \rho(r, m) = 0 \quad \text{for all } m \in \mathbb{Z}, r \in \mathbb{Z} + 1/2.$$

*Case II.*  $\theta = 0$ . Letting  $n = 0$  and  $m = -r \neq 0$  in (4-10), we have  $\psi(0, 0) = \rho(0, 0) = 0$ . Letting  $m = n = 0$  and  $r \neq 0$  in (4-10) and letting  $m = r = 0$  and  $n \neq 0$  in (4-9), we have

$$\rho(r, 0) = \psi(0, r) = 0 \quad \text{and} \quad \psi(n, 0) = \rho(0, n) = 0 \quad \text{for all } r, n \in \mathbb{Z}, r, n \neq 0.$$

Let  $r = 0, m, n \neq 0$  in (4-9) and (4-10), we have

$$\begin{aligned} \psi(m, n)\frac{n}{1+2\epsilon n} - \psi(n, m)\frac{m}{1+2\epsilon m} &= 0, \\ \frac{m}{2}\psi(n, m) + \psi(m, n)\frac{n(1+\epsilon n)}{1+2\epsilon n} &= 0. \end{aligned}$$

Since  $\epsilon \neq 0$  and  $\epsilon^{-1} \notin \mathbb{Z}$ , we have  $\psi(n, m) = \rho(n, m) = 0$  for all  $m, n \in \mathbb{Z}$ .  $\square$

By Theorem 2.3, it is easy to show that the  $V_\epsilon$  are mutually nonisomorphic for all  $\epsilon \in \mathbb{C}$  satisfying  $\operatorname{Re} \epsilon > 0$ ,  $\epsilon^{-1} \notin \mathbb{Z}$  or  $\operatorname{Re} \epsilon = 0$ ,  $\operatorname{Im} \epsilon > 0$ . Altogether, from Theorem 2.3, Proposition 3.1, Theorem 3.2, Proposition 4.1, and Theorem 4.2, we have the following conclusion.

**Theorem 4.3.** *Any compatible left-symmetric superalgebra on a super-Virasoro algebra satisfying (4-5) is isomorphic to one of the following (mutually nonisomorphic) left-symmetric superalgebras given by the multiplications*

$$\begin{aligned} L_m \cdot L_n &= -\frac{n(1+\epsilon n)}{1+\epsilon(m+n)}L_{m+n} + \frac{c}{24}(m^3 - m + (\epsilon - \epsilon^{-1})m^2)\delta_{m+n,0}, \\ L_m \cdot G_r &= -\frac{(m/2+r)(1+2\epsilon r)}{1+2\epsilon(m+r)}G_{m+r}, \\ G_r \cdot L_m &= -\frac{m(1+\epsilon m)}{1+2\epsilon(m+r)}G_{m+r}, \\ G_r \cdot G_s &= \frac{1+2\epsilon s}{1+\epsilon(r+s)}L_{r+s} + \frac{c}{24}(4r^2 - 1 + 2(\epsilon - \epsilon^{-1})r)\delta_{r+s,0}, \end{aligned}$$

where  $m, n \in \mathbb{Z}$ ,  $r, s \in \mathbb{Z} + \theta$ ,  $c$  is an annihilator, and  $\operatorname{Re} \epsilon > 0$ ,  $\epsilon^{-1} \notin \mathbb{Z}$  or  $\operatorname{Re} \epsilon = 0$ ,  $\operatorname{Im} \epsilon > 0$ .

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