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We prove that if the outer billiard map around a plane oval is algebraically integrable in a certain nondegenerate sense then the oval is an ellipse.

In this note, an outer billiard table is a compact convex domain in the plane bounded by an oval (closed smooth strictly convex curve) C. Pick a point x outside of C. There are two tangent lines from x to C; choose one of them, say, the right one from the viewpoint of x, and reflect x in the tangency point. One obtains a new point, y, and the transformation $T : x \mapsto y$ is the outer (also called dual) billiard map. We refer to [Tabachnikov and Dogru 2005; Tabachnikov 1995; 2005] for surveys of outer billiards.

If C is an ellipse then the map T possesses a 1-parameter family of invariant curves, the homothetic ellipses; these invariant curves foliate the exterior of C. Conjecturally, if an outer neighborhood of an oval C is foliated by the invariant curves of the outer billiard map, then C is an ellipse — this is an outer version of the famous Birkhoff conjecture concerning the conventional, inner billiards.

In this note we show that ellipses are rigid in a much more restrictive sense of algebraically integrable outer billiards; see [Bolotin 1990] for the case of inner billiards.

We make the following assumptions. Let $f(x, y)$ be a (nonhomogeneous) real polynomial that has zero as a nonsingular value and such that C is a component of its zero level curve. Thus f is the defining polynomial of the curve C, and if a polynomial vanishes on C, then it is a multiple of f (see for example [Clemens 2003; Fischer 2001; Walker 1978]). Assume that a neighborhood of C is foliated by invariant curves of the outer billiard map T and that this foliation is algebraic in that its leaves are components of the level curves of a real polynomial $F(x, y)$. Since C itself is an invariant curve, we assume that $F(x, y) = 0$ on C and that dF is not identically zero on C. Thus $F(x, y) = g(x, y) f(x, y)$ where $g(x, y)$ is a polynomial not identically zero on C. Under these assumptions, our result is as follows.

Theorem 1. *C is an ellipse.*

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Proof. Consider the tangent vector field $v = F_y\ \partial/\partial x - F_x\ \partial/\partial y$ (the symplectic gradient) along C. This vector field is nonzero (except at possibly a finite number of points) and tangent to C. The tangent line to C at point (x, y) is given by $(x+\varepsilon F_y, y-\varepsilon F_x)$, and the condition that F is T-invariant means that the function

$$F(x+\varepsilon F_y, y-\varepsilon F_x) \tag{1}$$

is even in ε for all $(x, y) \in C$. Expand in a series in ε; the first order term in ε vanishes automatically and the first nontrivial condition is cubic in ε:

$$W(F) := F_{xxx}F_y^3 - 3F_{xxy}F_y^2F_x + 3F_{xyy}F_yF_x^2 - F_{yyy}F_x^3 = 0 \tag{2}$$

on C. We claim that this already implies that C is an ellipse. The idea is that otherwise the complex curve $f = 0$ would have an inflection point, in contradiction with identity (2).

Consider the polynomial

$$H(F) = \det\begin{pmatrix} F_y & -F_x \\ F_{yy}F_x - F_{xy}F_y & F_{xx}F_y - F_{xy}F_x \end{pmatrix}.$$

Lemma 2. (i) $v(H(F)) = W(F)$.

(ii) $H(F) = H(gf) = g^3 H(f)$ *on* C.

(iii) *If* C' *is a nonsingular algebraic curve with a defining polynomial* $g(x, y)$, *then* $H(g)(x, y) = 0$ *if and only if* (x, y) *is an inflection point of* C'.

Proof. The first two claims follow from straightforward computations. To prove the third, note that $H(g)$ is the second order term in ε of the Taylor expansion of the function $g(x+\varepsilon g_y, y-\varepsilon g_x)$; see (1). Hence $H(g) = 0$ at the points where the tangent line is second order tangent to the curve, that is, at the inflection points. □

It follows from Lemma 2 and (2) that $H(F) = \text{const}$ on C. Indeed, $v(H(F)) = W(F) = 0$; hence the directional derivative of $H(F)$ along C is zero. Since C is convex, $H(F) \neq 0$. Indeed, if $H(F) = 0$ then, by Lemma 2, $H(f) = 0$ and all points of C are its inflections. Thus we may assume that $H(F) = 1$ on C. It follows that $g^3 H(f) - 1$ vanishes on C and hence

$$g^3 H(f) - 1 = hf, \tag{3}$$

where $h(x, y)$ is some polynomial.

Now consider the situation in $\mathbb{CP}^2$. We use the notation C' for the complex algebraic curve given by the homogenized polynomial $\bar{f}(x : y : z) = f(x/z, y/z)$. Unless C is a conic, this curve has inflection points (not necessarily real). Let d be the degree of C'.

Lemma 3. *Not all the inflections of* C' *lie on the line at infinity.*

Proof. Consider the Hessian curve given by

$$\det\begin{pmatrix}\bar{f}_{xx} & \bar{f}_{xy} & \bar{f}_{xz}\\ \bar{f}_{yx} & \bar{f}_{yy} & \bar{f}_{yz}\\ \bar{f}_{zx} & \bar{f}_{zy} & \bar{f}_{zz}\end{pmatrix}=0.$$

The intersection points of the curve C' with its Hessian curve are the inflection points of C' (recall that C' is nonsingular). The degree of the Hessian curve is $3(d-2)$ because taking two derivatives lowers the degree by 2 and taking the determinant multiplies terms in threes. By Bézout's theorem, the total number of inflections, counted with multiplicities, is $3d(d-2)$. Also, the order of intersection equals the order of the respective inflection and does not exceed $d-2$, see for example [Walker 1978]. The number of intersection points of C' with a line equals d. Hence the inflection points of C' that lie on a fixed line contribute, at most, $d(d-2)$ to the total of $3d(d-2)$. The remaining inflection points lie off this line. □

To conclude the proof of Theorem 1, consider a finite inflection point of C'. According to Lemma 2, at such a point we have $f = H(f) = 0$, which contradicts (3). This is proves that C is a conic. □

Remarks. First, it would be interesting to remove the nondegeneracy assumptions in Theorem 1.

Second, a more general version of Birkhoff's integrability conjecture is as follows. Let C be a plane oval whose outer neighborhood is foliated by closed curves. For a tangent line ℓ to C, the intersections with the leaves of the foliation define a local involution σ on ℓ. Assume that, for every tangent line, the involution σ is projective. Conjecturally, then C is an ellipse and the foliation consists of ellipses that form a pencil (that is, share four — real or complex — common points). For a pencil of conics, the respective involutions are projective: this is a Desargues theorem; see [Berger 1987]. It would be interesting to establish an algebraic version of this conjecture.

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References

[Berger 1987] M. Berger, *Geometry. I*, Universitext, Springer, Berlin, 1987. Translated from the French by M. Cole and S. Levy. MR 88a:51001a Zbl 0606.51001

[Bolotin 1990] S. V. Bolotin, “Integrable Birkhoff billiards”, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* 2 (1990), 33–36, 105. In Russian; translated in *Mosc. Univ. Mech. Bull.* **45**:2 (1990), 10–13. MR 91e:58143 Zbl 0708.58015

[Clemens 2003] C. H. Clemens, *A scrapbook of complex curve theory*, Second ed., Graduate Studies in Mathematics **55**, American Mathematical Society, Providence, RI, 2003. MR 2003m:14001 Zbl 1030.14010

[Fischer 2001] G. Fischer, *Plane algebraic curves*, Student Mathematical Library **15**, American Mathematical Society, Providence, RI, 2001. MR 2002g:14042 Zbl 0971.14026

[Tabachnikov 1995] S. Tabachnikov, “Billiards”, *Panor. Synth.* 1 (1995), vi+142. MR 96c:58134 Zbl 0833.58001

[Tabachnikov 2005] S. Tabachnikov, *Geometry and billiards*, Student Mathematical Library **30**, American Mathematical Society, Providence, RI, 2005. MR 2006h:51001 Zbl 1119.37001

[Tabachnikov and Dogru 2005] S. Tabachnikov and F. Dogru, “Dual billiards”, *Math. Intelligencer* **27**:4 (2005), 18–25. MR 2006i:37121 Zbl 1088.37014

[Walker 1978] R. J. Walker, *Algebraic curves*, Springer, New York, 1978. MR 80c:14001 Zbl 0399.14016

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