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ON CHERN–MOSER NORMAL FORMS OF STRONGLY PSEUDOCONVEX HYPERSURFACES WITH HIGH-DIMENSIONAL STABILITY GROUP

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We explicitly describe germs of strongly pseudoconvex nonspherical real-analytic hypersurfaces M at the origin in $\mathbb{C}^{n+1}$ for which the group of local CR-automorphisms preserving the origin has dimension $d_0(M)$ equal to either n^2-2n+1 with $n \geq 2$ or n^2-2n with $n \geq 3$. The description is given in terms of equations defining hypersurfaces near the origin, which are written in the Chern–Moser normal form. These results are motivated by the classification of locally homogeneous Levi nondegenerate hypersurfaces in $\mathbb{C}^3$ with $d_0(M)=1, 2$ due to A. Loboda, and they complement earlier joint work by V. Ezhov and the author for the case $d_0(M) \geq n^2-2n+2$.

1. Introduction

Let M be a strongly pseudoconvex real-analytic hypersurface in $\mathbb{C}^{n+1}$ for $n \geq 1$ passing through the origin and defined near the origin by the equation $r=0$, where r is some real-valued real-analytic function with nowhere vanishing gradient. In some local holomorphic coordinates $z=(z_1, \dots, z_n)$ and $w=u+iv$ in a neighborhood of the origin, M can be given by an equation written in the Chern–Moser [1974] normal form

$$v=|z|^2+\sum_{k,\bar{l}\geq 2} F_{k\bar{l}}(z,\bar{z},u), \tag{1-1}$$

where $|z|$ is the norm of the vector z, and $F_{k\bar{l}}(z,\bar{z},u)$ are polynomials of degree k in z and of degree $\bar{l}$ in $\bar{z}$ whose coefficients are analytic functions of u such that

$$\operatorname{tr} F_{2\bar{2}} \equiv 0, \qquad \operatorname{tr}^2 F_{2\bar{3}} \equiv 0, \qquad \operatorname{tr}^3 F_{3\bar{3}} \equiv 0. \tag{1-2}$$

Here the operator tr is defined as

$$\operatorname{tr} := \sum_{\alpha=1}^{n} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\alpha}.$$

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Everywhere below we assume that the equation of M is given in the normal form.

Let $\mathrm{Aut}_0(M)$ denote the *stability group* of M at the origin, that is, the group of all local CR-automorphisms of M defined near the origin and preserving it. Every element φ of $\mathrm{Aut}_0(M)$ extends to a biholomorphic mapping defined in a neighborhood of the origin in $\mathbb{C}^{n+1}$ and therefore can be written as $z \mapsto f_\varphi(z, w)$ and $w \mapsto g_\varphi(z, w)$, where f_φ and g_φ are holomorphic. We equip $\mathrm{Aut}_0(M)$ with the topology of uniform convergence of the partial derivatives of all orders of the component functions on neighborhoods of the origin in M. The group $\mathrm{Aut}_0(M)$ with this topology is a topological group.

It is shown in [Chern and Moser 1974] that every element $\varphi = (f_\varphi, g_\varphi)$ of $\mathrm{Aut}_0(M)$ is uniquely determined by a set of parameters $(U_\varphi, a_\varphi, \lambda_\varphi, r_\varphi)$, where U_φ lies in the unitary group U_n, $a_\varphi \in \mathbb{C}^n$, $\lambda_\varphi > 0$, and $r_\varphi \in \mathbb{R}$. These parameters are found from the relations

$$\frac{\partial f_\varphi}{\partial z}(0) = \lambda_\varphi U_\varphi, \qquad \frac{\partial f_\varphi}{\partial w}(0) = \lambda_\varphi U_\varphi a_\varphi,$$
$$\frac{\partial g_\varphi}{\partial w}(0) = \lambda_\varphi^2, \qquad \mathrm{Re}\,\frac{\partial^2 g_\varphi}{\partial^2 w}(0) = 2\lambda_\varphi^2 r_\varphi.$$

For results on how local CR-mappings depend on their jets in more general settings, see for example [Baouendi et al. 1998; 1999; Ebenfelt 2001; Zaitsev 2002].

We assume that M is nonspherical at the origin, that is, M in a neighborhood of the origin is not CR-equivalent to an open subset of the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. In this case for every element $\varphi = (f_\varphi, g_\varphi)$ of $\mathrm{Aut}_0(M)$, we have $\lambda_\varphi = 1$ and the parameters a_φ and r_φ are uniquely determined by the matrix U_φ; moreover, the mapping

$$\Phi : \mathrm{Aut}_0(M) \to GL_n(\mathbb{C}), \quad \varphi \mapsto U_\varphi$$

is a topological group isomorphism between $\mathrm{Aut}_0(M)$ and $G_0(M) := \Phi(\mathrm{Aut}_0(M))$, with $G_0(M)$ being a real algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$. See [Chern and Moser 1974; Belošapka 1979; Loboda 1981; Beloshapka and Vitushkin 1981; Vitushkin and Kruzhilin 1987]. We pull back the Lie group structure from $G_0(M)$ to $\mathrm{Aut}_0(M)$ by means of Φ and let $d_0(M)$ be the dimension of $\mathrm{Aut}_0(M)$. Clearly, $d_0(M) \le n^2$.

We are interested in characterizing hypersurfaces for which $d_0(M)$ is large (certainly positive). We show that in some normal coordinates the equations of such hypersurfaces take a very special form. As will be explained below, results of this kind can potentially be applied to the classification problem for locally CR-homogeneous strongly pseudoconvex hypersurfaces. For $n = 1$, this problem was solved by E. Cartan [1933a; 1933b]. For $n = 2$ with $d_0(M) > 0$, an explicit classification was obtained in [Loboda 2001; 2003]. For $n \ge 3$, there is no such classification even for hypersurfaces with high-dimensional stability group. Note, however, that *globally* homogeneous hypersurfaces have been extensively studied (see for example [Azad et al. 1985] and references therein). We also mention that locally homogeneous hypersurfaces in $\mathbb{C}^3$ with nondegenerate indefinite Levi form and 2-dimensional stability group were classified in [Loboda 2002] and that recently Fels and Kaup [$\geq$ 2008] have determined all locally homogeneous 5-dimensional CR-manifolds with certain degenerate Levi forms.

For a nonspherical hypersurface M, the group $\mathrm{Aut}_0(M)$ is known to be linearizable, that is, in some normal coordinates, every $\varphi \in \mathrm{Aut}_0(M)$ can be written in the form $z \mapsto U_\varphi z$ and $w \mapsto w$; see [Kruzhilin and Loboda 1983]. If all elements of $\mathrm{Aut}_0(M)$ in some coordinates have the above form, we say that $\mathrm{Aut}_0(M)$ is linear in these coordinates. Thus, to describe hypersurfaces M with a particular value of $d_0(M)$, one needs to (a) write M in normal coordinates in which $\mathrm{Aut}_0(M)$ is linear, (b) determine all connected closed subgroups H of U_n of dimension $d_0(M)$, and (c) find all H-invariant real-analytic functions of z, $\bar{z}$ and u that are homogeneous of fixed degrees in each of z and $\bar{z}$. Then every $F_{k\bar{l}}(z, \bar{z}, u)$ in (1-1) is a function of the kind found in (c), and one obtains the general form of M.

In [Ezhov and Isaev 2005], we considered the case $d_0(M) \geq n^2 - 2n + 2$ for $n \geq 2$. It turned out that if $d_0(M) \geq n^2 - 2n + 3$, then $d_0(M) = n^2$, that is, $G_0(M) = \mathrm{U}_n$. Clearly, every U_n-invariant real-analytic function is a function of $|z|^2$ and u, and thus the equation of M in any normal coordinates in which $\mathrm{Aut}_0(M)$ is linear has the form

$$v = |z|^2 + \sum_{p=4}^{\infty} C_p(u)|z|^{2p}, \tag{1-3}$$

where $C_p(u)$ are real-valued analytic functions of u, and for some p we have $C_p(u) \not\equiv 0$. Here the condition $p \geq 4$ comes from identities (1-2).

Further, for $d_0(M) = n^2 - 2n + 2$ we showed that the equation of M in some normal coordinates in which $\mathrm{Aut}_0(M)$ is linear has the form

$$v = |z|^2 + \sum_{p+q \geq 2} C_{pq}(u)|z_1|^{2p}|z|^{2q}, \tag{1-4}$$

where $C_{pq}(u)$ are real-valued analytic functions of u, $C_{pq}(u) \not\equiv 0$ for some p, q with $p > 0$, and C_{pq} for $p + q = 2, 3$ satisfy certain conditions arising from identities (1-2).[1] Equation (1-4) is the most general form of a hypersurface with $d_0(M) = n^2 - 2n + 2$ and cannot be simplified any further without additional assumptions on M. This equation is a consequence of our description of connected closed subgroups of U_n of dimension $n^2 - 2n + 2$ obtained earlier in [Isaev and Krantz 2001].

[1] In [Ezhov and Isaev 2005] we erroneously stated that identities (1-2) imply that $C_{pq} = 0$ for $p + q = 2, 3$. This is in general not the case; see that paper's erratum.

A. Loboda [2001] classified strongly pseudoconvex locally CR-homogeneous hypersurfaces in $\mathbb{C}^3$ with $d_0(M)=2$ (here $n=2$) by means of normal form techniques (see also [Loboda 2002]). Using the homogeneity of M and the condition $d_0(M)=2$, he showed that the equation of M must significantly simplify, which eventually yielded the classification. His arguments avoid using the explicit form of connected closed 2-dimensional subgroups of U_2 (every such subgroup is conjugate to $\mathrm{U}_1\times\mathrm{U}_1$) and, as a result, the special normal form (1-4). It seems that (1-4) can be used to simplify the proof of the main result of [Loboda 2001]. Further, (1-4) may be a useful tool for describing locally CR-homogeneous strongly pseudoconvex hypersurfaces with $d_0(M)=n^2-2n+2$ for arbitrary $n\geq 2$. Overall, the introduction of algebraic arguments into the analysis of normal forms seems to be a fruitful idea.

Observe for comparison that every locally CR-homogeneous strongly pseudoconvex hypersurface with $d_0(M)\geq n^2-2n+3$ and $n\geq 2$ is spherical, since by (1-3) the origin is an umbilic point of M. This is in contrast with hypersurfaces whose Levi form is nondegenerate and indefinite (see [Ezhov and Isaev 2005] for a description of such hypersurfaces with $d_0(M)\geq n^2-2n+3$ and [Loboda 1999] for the homogeneous case with $n=2$).

In this paper we consider the cases $d_0(M)=n^2-2n+1$ with $n\geq 2$, and $d_0(M)=n^2-2n$ with $n\geq 3$. Our result is the following theorem.

Theorem 1.1. *Suppose M is a strongly pseudoconvex real-analytic nonspherical hypersurface in $\mathbb{C}^{n+1}$ passing through the origin.*

(A) *If $d_0(M)=n^2-2n+1$ and $n\geq 2$, then in some normal coordinates near the origin in which $\mathrm{Aut}_0(M)$ is linear, the equation of M takes one of the following three forms*:

$$v=|z|^2+\sum_{\substack{p+q\geq 2,\ r+s\geq 2,\\ (p-r)k_1+(q-s)k_2=0}} C_{pqrs}(u)z_1^p z_2^q \bar{z}_1^r \bar{z}_2^s, \tag{1-5}$$

where k_1 and k_2 are nonzero integers with $(k_1,k_2)=1$ and $k_2>0$, $C_{pqrs}(u)$ are real-analytic functions of u, and $C_{pqrs}(u)\not\equiv 0$ for some p,q,r,s with either $p\neq r$ or $q\neq s$ (here $n=2$);

$$v=|z|^2+\sum_{2p+q\geq 2} C_{pq}(u)\left|z_1^2+z_2^2+z_3^2\right|^{2p}|z|^{2q}, \tag{1-6}$$

where $C_{pq}(u)$ are real-valued analytic functions of u and $C_{pq}(u)\not\equiv 0$ for some p,q with $p>0$ (here $n=3$);

$$v=|z|^2+\sum_{p+r,\ q+r\geq 2} C_{pqr}(u)z_1^p\bar{z}_1^q|z|^{2r}, \tag{1-7}$$

where $C_{pqr}(u)$ *are real-analytic functions of* u *and* $C_{pqr}(u) \not\equiv 0$ *for some* p, q, r *with* $p \neq q$.

(B) *If* $d_0(M) = n^2 - 2n$ *and* $n \geq 3$, *then in some normal coordinates near the origin in which* $\mathrm{Aut}_0(M)$ *is linear, the equation of* M *takes one of the following three forms*:

$$v = |z|^2 + \sum_{2p+r\geq 2,\ 2q+r\geq 2} C_{pqr}(u)(z_1^2 + z_2^2 + z_3^2)^p(\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2)^q |z|^{2r}, \tag{1-8}$$

where $C_{pqr}(u)$ *are real-analytic functions of* u *and* $C_{pqr}(u) \not\equiv 0$ *for some* p, q, r *with* $p \neq q$ (*here* $n = 3$);

$$v = |z|^2 + \sum_{p+q+r\geq 2} C_{pqr}(u)|z_1|^{2p}|z_2|^{2q}|z_3|^{2r}, \tag{1-9}$$

where $C_{pqr}(u)$ *are real-valued analytic functions of* u *and* $C_{pqr}(u) \not\equiv 0$ *for some* p, q, r (*here* $n = 3$);

$$v = |z|^2 + \sum_{p+q\geq 2} C_{pq}(u)|z^1|^{2p}|z^2|^{2q}, \tag{1-10}$$

where $z^1 := (z_1, z_2)$ *and* $z^2 := (z_3, z_4)$, $C_{pq}(u)$ *are real-valued analytic functions of* u, *and* $C_{pq}(u) \not\equiv 0$ *for some* p, q (*here* $n = 4$).

Corollary 1.2. *Suppose* M *is a strongly pseudoconvex real-analytic nonspherical hypersurface in* $\mathbb{C}^{n+1}$ *passing through the origin. Assume that* $n \geq 5$ *and that* $d_0(M) \geq n^2 - 2n$. *Then* $d_0(M) \geq n^2 - 2n + 1$. *Also, in some normal coordinates near the origin in which* $\mathrm{Aut}_0(M)$ *is linear, the equation of* M *has the form*

$$v = |z|^2 + \sum_{p+r,\ q+r\geq 2} C_{pqr}(u) z_1^p \bar{z}_1^q |z|^{2r},$$

where $C_{pqr}(u)$ *are real-analytic functions of* u *and* $C_{pqr}(u) \not\equiv 0$ *for some* p, q, r.

Locally CR-homogeneous hypersurfaces in $\mathbb{C}^3$ with $d_0(M) = 1$ (here $n = 2$) were classified in [Loboda 2003]. We believe that Part (A) of Theorem 1.1 can be used to simplify Loboda's arguments.

2. Proof of Theorem 1.1

The main ingredient of the proof is the following proposition.

Proposition 2.1. *Let* H *be a connected closed subgroup of* U_n *with* $n \geq 2$. *If* $\dim H = n^2 - 2n + 1$, *then* H *is conjugate in* U_n *to one of these three subgroups*:

(i) $e^{i\mathbb{R}}\mathrm{SO}_3(\mathbb{R})$ (*here* $n = 3$);

(ii) $\mathrm{U}_1 \times \mathrm{SU}_{n-1}$ *realized as the subgroup of all matrices* $\operatorname{diag}(e^{i\theta}, A)$, *where* $\theta \in \mathbb{R}$ *and* $A \in \mathrm{SU}_{n-1}$, *for* $n \geq 3$;

(iii) *the subgroup* $H^n_{k_1,k_2}$ *of all matrices* $\operatorname{diag}(a, A)$, *where* k_1 *and* k_2 *are fixed integers with* $(k_1, k_2) = 1$ *and* $k_2 > 0$, $a \in (\det A)^{k_1/k_2} := \exp(k_1/k_2 \log(\det A))$, *and* $A \in \mathrm{U}_{n-1}$.[2]

If $\dim H = n^2 - 2n$, *then* H *is conjugate in* U_n *to one of these four subgroups*:

(iv) $\mathrm{SO}_3(\mathbb{R})$ *(here* $n = 3$*);*

(v) $\mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{U}_1$ *realized as diagonal matrices in* U_3 *(here* $n = 3$*);*

(vi) $\mathrm{U}_2 \times \mathrm{U}_2$ *realized as block-diagonal matrices in* U_4 *(here* $n = 4$*);*

(vii) SU_{n-1} *realized as the subgroup of all matrices* $\operatorname{diag}(1, A)$ *for* $A \in \mathrm{SU}_{n-1}$.

Proof. The proof consists of two parts corresponding to the two different group dimensions.

Part I. Suppose first that $\dim H = n^2 - 2n + 1$. Since H is compact, it is completely reducible, that is, $\mathbb{C}^n$ splits into the sum $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m$ of H-invariant pairwise orthogonal complex subspaces such that the restriction H_j of H to each V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence $n_1 + \cdots + n_m = n$), and let U_{n_j} be the group of unitary transformations of V_j. Clearly $H_j \subset \mathrm{U}_{n_j}$, and therefore $\dim H \leq n_1^2 + \cdots + n_m^2$. On the other hand, $\dim H = n^2 - 2n + 1$, which shows that $m \leq 2$. We will now consider two cases.

Case 1. Let H be reducible, that is, $m = 2$. Then there exists a unitary change of coordinates in $\mathbb{C}^n$ such all elements of H take the form $\operatorname{diag}(a, A)$, where $a \in \mathrm{U}_1$ and $A \in \mathrm{U}_{n-1}$. If $\dim H_1 = 0$, then $H_1 = \{1\}$, and therefore $H_2 = \mathrm{U}_{n-1}$. In this case we obtain the group $H^n_{0,1}$. Suppose now $\dim H_1 = 1$, that is, $H_1 = \mathrm{U}_1$. Then $n^2 - 2n \leq \dim H_2 \leq n^2 - 2n + 1$. If $\dim H_2 = n^2 - 2n$, then $H_2 = \mathrm{SU}_{n-1}$, and hence H is conjugate to $\mathrm{U}_1 \times \mathrm{SU}_{n-1}$ for $n \geq 3$ and to $H^2_{0,1}$ for $n = 2$. Now let $\dim H_2 = n^2 - 2n + 1$, that is, $H_2 = \mathrm{U}_{n-1}$. Consider the Lie algebra $\mathfrak{h}$ of H. Up to conjugation, it consists of matrices of the form $\operatorname{diag}(l(\mathfrak{A}), \mathfrak{A})$, where $\mathfrak{A} \in \mathfrak{u}_{n-1}$ and $l(\mathfrak{A}) \not\equiv 0$ is a linear function of the matrix elements of $\mathfrak{A}$ ranging in $i\mathbb{R}$. Clearly, $l(\mathfrak{A})$ must vanish on the derived algebra of $\mathfrak{u}_{n-1}$, which is $\mathfrak{su}_{n-1}$. Hence matrices $\operatorname{diag}(l(\mathfrak{A}), \mathfrak{A})$ form a Lie algebra if and only if $l(\mathfrak{A}) = c \cdot \operatorname{tr} \mathfrak{A}$, where $c \in \mathbb{R} \setminus \{0\}$. Such an algebra can be the Lie algebra of a closed subgroup of $\mathrm{U}_1 \times \mathrm{U}_{n-1}$ only if $c \in \mathbb{Q} \setminus \{0\}$. Hence H is conjugate to $H^n_{k_1,k_2}$ for some $k_1, k_2 \in \mathbb{Z}$, where one can always assume that $k_2 > 0$ and $(k_1, k_2) = 1$.

Case 2. Let H be irreducible, that is, $m = 1$. We shall proceed as in the proof of [Isaev 2007, Lemma 1.4]. Let $\mathfrak{h}^{\mathbb{C}} := \mathfrak{h} + i\mathfrak{h} \subset \mathfrak{gl}_n$ be the complexification of $\mathfrak{h}$, where $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C})$. The algebra $\mathfrak{h}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and by a theorem of

[2]The group $H^n_{k_1,k_2}$ is a k_2-sheeted cover of U_{n-1} (note that $k_2 = 1$ if $k_1 = 0$).

E. Cartan, $\mathfrak{h}^{\mathbb{C}}$ is either semisimple or the direct sum of the center $\mathfrak{c}$ of $\mathfrak{gl}_n$ and a semisimple ideal $\mathfrak{t}$. Clearly, the action of the ideal $\mathfrak{t}$ on $\mathbb{C}^n$ is irreducible. We will now separately consider each of these situations.

Case 2.1. Assume first that $\mathfrak{h}^{\mathbb{C}}$ is semisimple, and let $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$ be its decomposition into the direct sum of simple ideals. Then the natural irreducible n-dimensional representation of $\mathfrak{h}^{\mathbb{C}}$ (given by the embedding of $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{gl}_n$) is the tensor product of some irreducible faithful representations of the $\mathfrak{h}_j$. Let n_j be the dimension of the corresponding representation of $\mathfrak{h}_j$ for $j = 1, \ldots, k$. Then $n_j \geq 2$, $\dim_{\mathbb{C}} \mathfrak{h}_j \leq n_j^2 - 1$, and $n = n_1 \times \cdots \times n_k$.

It is straightforward to show that if $n = n_1 \times \cdots \times n_k$ with $k \geq 2$ and $n_j \geq 2$ for $j = 1, \ldots, k$, then $\sum_{j=1}^k n_j^2 \leq n^2 - 2n$. Since $\dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}} = n^2 - 2n + 1$, it then follows that $k = 1$, that is, $\mathfrak{h}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations V of complex simple Lie algebras $\mathfrak{s}$ are well known and shown in the following table (see for example [Onishchik and Vinberg 1990]).

$\mathfrak{s}$	$\dim V$	$\dim \mathfrak{s}$
$\mathfrak{sl}_k,\ k \geq 2$	k	$k^2 - 1$
$\mathfrak{o}_k,\ k \geq 7$	k	$k(k-1)/2$
$\mathfrak{sp}_{2k},\ k \geq 2$	$2k$	$2k^2 + k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

It is straightforward to see that none of these dimensions is compatible with the condition $\dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}} = n^2 - 2n + 1$.

Case 2.2. Suppose now that $\mathfrak{h}^{\mathbb{C}} = \mathfrak{c} \oplus \mathfrak{t}$, where $\dim \mathfrak{t} = n^2 - 2n$. Then, if $n = 2$, we obtain that H coincides with the center of U_2, which is impossible since its action on $\mathbb{C}^2$ is then not irreducible. Assuming that $n \geq 3$ and repeating the above argument for $\mathfrak{t}$ in place of $\mathfrak{h}^{\mathbb{C}}$, we see that $\mathfrak{t}$ can only be isomorphic to $\mathfrak{sl}_{n-1}$. But $\mathfrak{sl}_{n-1}$ does not have an irreducible n-dimensional representation unless $n = 3$.

Thus, $n = 3$ and $\mathfrak{h}^{\mathbb{C}} \simeq \mathbb{C} \oplus \mathfrak{sl}_2 \simeq \mathbb{C} \oplus \mathfrak{so}_3$. Further, we observe that every irreducible 3-dimensional representation of $\mathfrak{so}_3$ is equivalent to its defining representation. This implies that H is conjugate in $\mathrm{GL}_3(\mathbb{C})$ to $e^{i\mathbb{R}}\mathrm{SO}_3(\mathbb{R})$. Since $H \subset \mathrm{U}_3$, it is straightforward to show that the conjugating element can be chosen to belong to U_3. This completes the proof of the proposition in the case $\dim H = n^2 - 2n + 1$.

Part II. Let $\dim H = n^2 - 2n$. Arguing as in Part I, we see that either $m \le 2$ or, for $n = 3$, we have $m = 3$. As before, we consider two cases.

Case 1. Let H be reducible. If $n = 3$ and $m = 3$, then H is conjugate in U_3 to $\mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{U}_1$. Suppose that $m = 2$. Then either $n = 4$ and H is conjugate in U_4 to $\mathrm{U}_2 \times \mathrm{U}_2$, or there exists a unitary change of coordinates in $\mathbb{C}^n$ such all elements of H take the form $\mathrm{diag}(a, A)$, where $a \in \mathrm{U}_1$ and $A \in \mathrm{U}_{n-1}$. If $\dim H_1 = 0$, then $H_1 = \{1\}$, and therefore $H_2 = \mathrm{SU}_{n-1}$. Assume now that $\dim H_1 = 1$, that is, $H_1 = \mathrm{U}_1$. Then $n \ge 3$ and $n^2 - 2n - 1 \le \dim H_2 \le n^2 - 2n$. [Isaev 2007, Lemma 1.4] shows that the possibility $\dim H_2 = n^2 - 2n - 1$ cannot in fact occur, and thus we have $\dim H_2 = n^2 - 2n$. Then $H_2 = \mathrm{SU}_{n-1}$, and hence H is conjugate to a codimension 1 subgroup of the group of all matrices of the form $\mathrm{diag}(a, A)$ with $A \in \mathrm{SU}_{n-1}$. Consider the Lie algebra $\mathfrak{h}$ of H. Up to conjugation, it consists of matrices of the form $\mathrm{diag}(l(\mathfrak{A}), \mathfrak{A})$, where $\mathfrak{A} \in \mathfrak{su}_{n-1}$ and $l(\mathfrak{A}) \not\equiv 0$ is a linear function of the matrix elements of $\mathfrak{A}$ ranging in $i\mathbb{R}$. Clearly, $l(\mathfrak{A})$ must vanish on the derived algebra of $\mathfrak{su}_{n-1}$, which is $\mathfrak{su}_{n-1}$ itself. This contradiction shows that the possibility $\dim H_1 = 1$ does not in fact realize.

Case 2. Let H be irreducible. Then $n \ge 3$, and we argue as in Part I. If $\mathfrak{h}^{\mathbb{C}}$ is semisimple, it follows as before that $\mathfrak{h}^{\mathbb{C}}$ is in fact simple. A glance at the table of minimal dimensions of irreducible faithful representations of complex simple Lie algebras now yields that $n = 3$ and $\mathfrak{h}^{\mathbb{C}} \simeq \mathfrak{sl}_2 \simeq \mathfrak{so}_3$, and hence H is conjugate in U_3 to $\mathrm{SO}_3(\mathbb{R})$. Finally, if $\mathfrak{h}^{\mathbb{C}} = \mathfrak{c} \oplus \mathfrak{t}$, where $\dim \mathfrak{t} = n^2 - 2n - 1$, we see that $\mathfrak{t}$ must be simple and obtain a contradiction with the above table.

The proof of the proposition is complete. □

To finish the proof of Theorem 1.1 we now need to determine polynomials in z and $\bar{z}$ with coefficients depending on u that are invariant under each of the groups listed in (i)–(vii) of Proposition 2.1. Note that by the assumption of Theorem 1.1, we have $n \ge 2$ for case (iii) and $n \ge 3$ for case (vii).

(i) and (iv). It is known from the classical invariant theory (see [Weyl 1997, Theorem 2.9.A]) that every $\mathrm{SO}_3(\mathbb{R})$-invariant polynomial is a polynomial in $z_1^2 + z_2^2 + z_3^2$, $\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2$ and $|z|^2$; see also [Huebschmann 1996, Section 5]. If in addition such a polynomial is $e^{i\mathbb{R}}$-invariant, it depends only on $|z_1^2 + z_2^2 + z_3^2|^2$ and $|z|^2$. These observations lead to forms (1-6) and (1-8).

(ii). Every $\mathrm{U}_1 \times \mathrm{SU}_{n-1}$-invariant polynomial for $n \ge 3$ is a polynomial in $|z_1|^2$ and $|z'|^2$, where $z' := (z_2, \cdots, z_n)$. Therefore, such polynomials are in fact $\mathrm{U}_1 \times \mathrm{U}_{n-1}$-invariant and hence lead to hypersurfaces with $d_0(M) \ge n^2 - 2n + 2$; this rules out case (ii).

(iii). Every $H^n_{0,1}$-invariant polynomial is a polynomial in z_1, $\bar{z}_1$ and $|z'|^2$, which leads to form (1-7). Let $k_1 \ne 0$ and $n \ge 3$. Taking $A \in \mathrm{SU}_{n-1}$ and $a = 1$ in $\mathrm{diag}(a, A)$, we see that every $H^n_{k_1,k_2}$-invariant polynomial is a polynomial in z_1, $\bar{z}_1$ and $|z'|^2$. Further, setting A to be a scalar matrix yields that every $H^n_{k_1,k_2}$-invariant polynomial is in fact a polynomial in $|z_1|^2$ and $|z'|^2$ and hence is $\mathrm{U}_1 \times \mathrm{U}_{n-1}$-invariant. As in case (ii) above, such polynomials lead to hypersurfaces with $d_0(M) \geq n^2 - 2n + 2$, thus ruling out case (iii) for $k_1 \neq 0$, $n \geq 3$. Finally, invariance under the group $H^2_{k_1,k_2}$ with $k_1 \neq 0$ (here $n = 2$) leads to form (1-5).

(v). Every $\mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{U}_1$-invariant polynomial is a polynomial in $|z_1|^2$, $|z_2|^2$, $|z_3|^3$, which leads to form (1-9).

(vi). Every $\mathrm{U}_2 \times \mathrm{U}_2$-invariant polynomial is a polynomial in $|z^1|^2$, $|z^2|^2$, where $z^1 := (z_1, z_2)$ and $z^2 := (z_3, z_4)$; this leads to form (1-10).

(vii). Every SU_{n-1}-invariant polynomial for $n \geq 3$ is a polynomial in z_1, $\bar{z}_1$ and $|z'|^2$, and hence is in fact U_{n-1}-invariant. Therefore, such polynomials lead to hypersurfaces with $d_0(M) \geq n^2 - 2n + 1$, thus ruling out case (vii).

The proof of Theorem 1.1 is complete. $\square$

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