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An uncertainty principle on Chébli–Trimèche hypergroups is established, as a generalization of Heisenberg inequalities for Jacobi transforms proved in my previous paper. It implies and extends the uncertainty principle for Hankel transforms by M. Rösler and M. Voit. The proof is based on ultracontractive properties of the semigroups generated by a second order differential operator and on the estimate of the heat kernel.

1. Introduction

The classical Heisenberg uncertainty principle states that for $f \in \mathrm{L}^2(\mathbb{R})$,

$$\int_{\mathbb{R}} x^2 |f(x)|^2 dx \cdot \int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 d\xi \geq \tfrac{1}{4} \|f\|^4, \tag{1-1}$$

where

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx.$$

Inequality (1-1) was found in the 1920's and then studied in many situations. Recently, considerable attention has been devoted to discovering new contexts for the uncertainty principle; see the survey [Folland and Sitaram 1997] and the book [Havin and Jöricke 1994] for other forms of the uncertainty principle. In the case of hypergroups, the Heisenberg inequalities were given in [Li and Liu 2005] for some compact Chébli–Trimèche hypergroups, in [Rösler and Voit 1999] for Bessel–Kingman hypergroups, and in [Ma 2007] for Jacobi hypergroups.

This paper generalizes the results of my previous paper [Ma 2007] to noncompact Chébli–Trimèche hypergroups and establishes an inequality analogous to (1-1) and variants of it. The proof is based on ultracontractive properties of the semigroups generated by a second order differential operator and on the estimate of the heat kernel. For further uncertainty principles on Chébli–Trimèche hypergroups, see [Attour and Trimèche 2005; Bouattour and Trimèche 2005; Trimèche 2005].

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The paper is arranged as follows. In Section 2, we recall some basic properties about Chébli–Trimèche hypergroups. In Section 3, we study the Heisenberg uncertainty principle on them.

2. Preliminaries of Chébli–Trimèche hypergroups

We now recall some definitions and properties of Chébli–Trimèche hypergroups; see also [Bloom and Xu 1995; Trimèche 1997].

Suppose that the function A is continuous on $\mathbb{R}_+ = [0, \infty)$, twice continuously differentiable on $\mathbb{R}_+^* = (0, \infty)$, and satisfies the conditions:

(1) $A(0) = 0$ and $A(x) > 0$ for $x > 0$;

(2) A is increasing and unbounded;

(3) $A'(x)/A(x) = (2\alpha + 1)/x + B(x)$ on a neighborhood of 0, where $\alpha > -1/2$ and B is an odd C^∞-function on $\mathbb{R}$;

(4) $A'(x)/A(x)$ is a decreasing C^∞-function on $\mathbb{R}_+^*$, and $\lim_{x\to+\infty} A'(x)/A(x) = 2\rho \geq 0$.

We denote by $(\mathbb{R}_+, *(A))$ the Chébli–Trimèche hypergroup associated with A. It is commutative with neutral element 0 and the identity mapping is the involution. The Haar measure on $(\mathbb{R}_+, *(A))$ is given by $A(x)\,dx$.

Well-known Chébli–Trimèche hypergroups are the Bessel–Kingman hypergroup with $A(x) = x^{2\alpha+1}$ for $\alpha > -1/2$ and the Jacobi hypergroup defined by $A(x) = \sinh^{2\alpha+1} x \, \cosh^{2\beta+1} x$, where $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$.

Let $\mathscr{L} = \mathscr{L}_A$ be the differential operator

$$\mathscr{L} = \frac{d^2}{dx^2} + \Big(\frac{A'(x)}{A(x)}\Big)\frac{d}{dx}.$$

The solutions ϕ_λ for $\lambda \in \mathbb{C}$ of the differential equation $\mathscr{L}v(x) = (\lambda^2 + \rho^2)v(x)$ with boundary conditions $v(0) = 1$ and $v'(0) = 0$ are multiplicative on $(\mathbb{R}_+, *(A))$ in that, for all $x, y \in \mathbb{R}_+^*$,

$$\int_{\mathbb{R}_+} \phi_\lambda(t)(\varepsilon_x * \varepsilon_y)(dt) = \phi_\lambda(x)\phi_\lambda(y),$$

where ε_x denotes the point mass at x and $\varepsilon_x * \varepsilon_y$ is a probability measure that is absolutely continuous with respect to the Haar measure and satisfies

$$\operatorname{supp} \varepsilon_x * \varepsilon_y \subset [|x - y|, x + y].$$

The dual space of $(\mathbb{R}_+, *(A))$ can be identified with the parameter set $\mathbb{R}_+ \cup i[0, \rho]$.

For $1 \le p \le \infty$, the Lebesgue space $L^p(\mathbb{R}_+, A(x)\,dx)$ is defined as usual with norm given by $\|\cdot\|_p$. The Fourier transform of f in $L^1(\mathbb{R}_+, A(x)\,dx)$ is given by

$$\widehat{f}(\lambda) := \int_0^\infty f(x)\phi_\lambda(x)A(x)\,dx. \tag{2-1}$$

Theorem 2.1 [Bloom and Heyer 1995]. *There exists a unique nonnegative measure π on $\mathbb{R}_+$ such that the Fourier transform extends to an isometry of $L^2(\mathbb{R}_+, A(x)\,dx)$ onto $L^2(\mathbb{R}_+, \pi)$, and for $f \in L^1 \cap L^2(\mathbb{R}_+, A(x)\,dx)$,*

$$\int_0^\infty |f(x)|^2 A(x)\,dx = \int_0^\infty |\widehat{f}(\lambda)|^2 \pi(d\lambda).$$

As in [Bloom and Xu 1995], here we assume that A satisfies an additional property, which gives a nice behavior for the Plancherel measure π.

Condition 2.1. A function f is said to satisfy this condition if, for some $\mu > 0$ and $x_0 > 0$,

$$\int_{x_0}^\infty x^{\varpi(\mu)}|\zeta(x)|\,dx < \infty, \quad \text{where } \zeta(x) = f(x) - \frac{\mu^2 - 1/4}{x^2},$$

and if $\zeta(x)$ is bounded for $x > x_0$; here $\varpi(\mu) = \mu + 1/2$ if $\mu \ge 1/2$ and $\varpi(\mu) = 1$ otherwise.

Letting

$$G(x) = \frac{1}{4}\Big(\frac{A'(x)}{A(x)}\Big)^2 + \frac{1}{2}\Big(\frac{A'(x)}{A(x)}\Big)' - \rho^2 \quad \text{for } x > 0,$$

we have the following theorem from [Bloom and Xu 1995].

Theorem 2.2. *Suppose that G satisfies Condition 2.1 together with one of the conditions*

(1) $\mu > 1/2$;

(2) $\mu \ne |\alpha|$;

(3) $\mu = \alpha \le \frac{1}{2}$ *and either* $\int_0^\infty x^{1/2-\alpha}\zeta(x)\phi_0(x)A(x)^{1/2}dx \ne -2\alpha\sqrt{M_A}$, *or* $\int_0^\infty x^{1/2+\alpha}\zeta(x)\phi_0(x)A(x)^{1/2}dx = 0$, *where* $M_A = \lim_{x\to 0^+} x^{-2\alpha-1}A(x)$.

Then the Plancherel measure π is absolutely continuous in the Lebesgue measure and has density $|c(\lambda)|^{-2}$, where the function c is such that there exist positive constants C_1, C_2, and K such that, for any $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \le 0$,

$$C_1|\lambda|^{\mu+1/2} \le |c(\lambda)|^{-1} \le C_2|\lambda|^{\mu+1/2} \quad \textit{for } |\lambda| \le K,$$
$$C_1|\lambda|^{\alpha+1/2} \le |c(\lambda)|^{-1} \le C_2|\lambda|^{\alpha+1/2} \quad \textit{for } |\lambda| > K.$$

For $1 \le p \le +\infty$, denote by $L^p(\mathbb{R}_+, d\nu)$ the space of measurable functions f on $[0, \infty)$ such that

$$\|f\|_{p,c} = \left(\int_0^\infty |f(\lambda)|^p |c(\lambda)|^{-2} d\lambda\right)^{1/p} < \infty, \qquad 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{\lambda \ge 0} |f(\lambda)| < \infty, \qquad p = \infty,$$

The generalized translate T_x of a function f is defined by

$$T_x f(y) = \int_0^\infty f(u)(\varepsilon_x * \varepsilon_y)(du).$$

The convolution of two functions f and g is defined by

$$f * g(x) = \int_0^\infty T_x f(y) g(y) A(y)\, dy.$$

The convolution satisfies the following properties (see [Trimèche 1997]).

(1) For all $f, g \in L^1(\mathbb{R}_+, A(x)\,dx)$, the function $f * g$ is defined almost everywhere on $\mathbb{R}_+$, belongs to $L^1(\mathbb{R}_+, A(x)\,dx)$, and satisfies

$$\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda).$$

(2) Let f and g be in $L^2(\mathbb{R}_+, A(x)\,dx)$. Then the function $f * g$ belongs to $L^2(\mathbb{R}_+, A(x)\,dx)$ if and only if the function $\hat{f} \cdot \hat{g}$ belongs to $L^2(\mathbb{R}_+, d\nu)$ and

$$\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda).$$

(3) Suppose $f \in L^p(\mathbb{R}_+, A(x)\,dx)$ and $g \in L^q(\mathbb{R}_+, A(x)\,dx)$ with $1 \le p, q, r \le \infty$ and $1/p + 1/q - 1 = 1/r$. Then $f * g \in L^r(\mathbb{R}_+, A(x)\,dx)$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

3. Uncertainty principle on Chébli–Trimèche hypergroups

In the sequel, we assume that A satisfies the Condition 2.1 in Theorem 2.2. In addition, we also suppose that in the case $\rho = 0$, there exists a $\beta \in (-1/2, \mu]$ such that $A(x) = O(x^{2\beta+1})$ as $x \to \infty$.

Let $D_\rho := [(\beta + 1)/(\mu + 1), 1]$ if $\rho = 0$. If $0 < \rho < 1$, let $D_\rho := (1/2, 1]$; otherwise let $D_\rho := [1/2, 1]$.

Theorem 3.1. *Assume $a, b > 0$ and $\gamma \in D_\rho$. Then there exists a constant $C > 0$ such that*

$$\|x^{\gamma a} f\|_2^{b/(a+b)} \cdot \|(\lambda^2 + \rho^2)^{b/2} \widehat{f}\|_{2,c}^{a/(a+b)} \ge C \|f\|_2, \tag{3-1}$$

for all $f \in L^2(\mathbb{R}_+, A(x)\,dx)$.

Remarks. (1) For Bessel–Kingman hypergroups, $\rho = 0$ and $\beta = \mu = \alpha$, So (3-1) becomes

$$\|x^a f\|_2^{b/(a+b)} \cdot \|\lambda^b \widehat{f}\|_{2,c}^{a/(a+b)} \geq C\|f\|_2,$$

which extends the result in [Rösler and Voit 1999].

(2) In the case of Jacobi hypergroups, $\rho > 0$ and $\mu = 1/2$. It's just the uncertainty principle for Jacobi transform, which improves the original theorem in [Ma 2007].

(3) If there exists a $\delta > 0$ such that for all $x \in [x_0, \infty)$ (for some $x_0 > 0$),

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\delta x} D(x), & \text{if } \rho > 0, \\ (2\alpha+1)/x + e^{-\delta x} D(x) & \text{if } \rho = 0, \end{cases}$$

where D and its derivatives are bounded C^∞ functions, we have $\mu = 1/2$ if $\rho > 0$, and $\beta = \mu = \alpha$ if $\rho = 0$ and $\alpha > 0$; see [Trimèche 1997].

Lemma 3.1 [Bloom and Xu 1995]. *$A(x) \sim x^{2\alpha+1}$ as $x \to 0^+$, and if $\rho > 0$ then $A(x) \sim e^{2\rho x}$ as $x \to +\infty$. Here $f \sim g$ means that there exist positive constants C_1 and C_2 such that $C_1 g \leq f \leq C_2 g$.*

In our proof, the heat kernel $h_t(x)$, given by

$$h_t(x) = \int_0^\infty e^{-t(\lambda^2+\rho^2)} \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda \quad \text{for } t > 0$$

plays an important role. For all $t > 0$ and $f \in L^2(\mathbb{R}_+, A(x)\,dx)$, we have

$$\widehat{h_t * f}(\lambda) = e^{-t(\lambda^2+\rho^2)} \hat{f}(\lambda).$$

We can now prove nice estimates for $\|h_t\|_2$.

Lemma 3.2.

$$\|h_t\|_2 \sim \begin{cases} t^{-(\alpha+1)/2} & \text{if } t \leq 1 \\ e^{-t\rho^2} t^{-(\mu+1)/2} & \text{if } t > 1. \end{cases}$$

Proof.

$$\|h_t\|_2 = \Big(\int_0^\infty e^{-2t(\lambda^2+\rho^2)} |c(\lambda)|^{-2} d\lambda\Big)^{1/2} = e^{-t\rho^2} \Big(\int_0^\infty e^{-2t\lambda^2} |c(\lambda)|^{-2} d\lambda\Big)^{1/2}.$$

Using the estimates of the c-function,

$$|c(\lambda)|^{-2} \sim \begin{cases} \lambda^{2\mu+1} & \text{if } t \leq K, \\ \lambda^{2\alpha+1} & \text{if } t > K. \end{cases}$$

Then

$$\|h_t\|_2 \sim e^{-t\rho^2} \Big(\int_0^K e^{-2t\lambda^2} \lambda^{2\mu+1} d\lambda + \int_K^\infty e^{-2t\lambda^2} \lambda^{2\alpha+1} d\lambda\Big)^{1/2}$$

$$\sim e^{-t\rho^2} \bigg(t^{-(\mu+1)} \int_0^{2tK^2} e^{-s} s^\mu\, ds + t^{-(\alpha+1)} \int_{2tK^2}^\infty e^{-s} s^\alpha\, ds\bigg)^{1/2},$$

where the second equivalence is obtained by a change of variables. Because $t \sim 0$, the first term in curly braces is bounded, and the second is equivalent to $t^{-(\alpha+1)}$. When t tends to infinity, the first term is equivalent to $t^{-(\mu+1)}$. By the relation between the incomplete gamma function $\Gamma(\,\cdot\,,\cdot\,)$ and the confluent hypergeometric function $G(\,\cdot\,,\cdot\,,\cdot\,)$ (see [Nikiforov and Uvarov 1988, pages 401–413]), we know that the second term equals

$$t^{-(\alpha+1)}\Gamma(\alpha+1, 2tK^2) = t^{-(\alpha+1)}e^{-2tK^2}(2tK^2)^{\alpha+1}G(1, 2+\alpha, 2tK^2) \sim e^{-2tK^2}G(1, 2+\alpha, 2tK^2) \sim e^{-2tK^2}t^{-1} \quad \text{as } t \to \infty$$

The lemma is proved by the above equivalences. □

The constant C below is not fixed and might change appropriately in different equalities or inequalities.

Lemma 3.3. *Assume $\gamma \in D_\rho$. Then for all $f \in L^2(\mathbb{R}_+, A(x)\,dx)$ we have*

$$\|h_t * f\|_2 \le Ct^{-a/2}\|x^{\gamma a} f\|_2, \tag{3-2}$$

where $a > 0$ and satisfies $\gamma a < \alpha+1$ if $\rho > 0$ and $\gamma a < \min\{\alpha+1, \beta+1\}$ otherwise.

Proof. We only give the proof for $\rho > 0$; the case $\rho = 0$ can be got similarly. For $r > 0$, let $f_r = f\chi_{[0,r]}$ and $f^r = f - f_r$. Then, since $|f^r(x)| \le r^{-\gamma a}|x^{\gamma a} f(x)|$,

$$\|h_t * f^r\|_2 = \|e^{-t(\lambda^2+\rho^2)}\widehat{f^r}(\lambda)\|_{2,\,c} \le \|\widehat{f^r}\|_{2,\,c} = \|f^r\|_2 \le r^{-\gamma a}\|x^{\gamma a} f\|_2.$$

On the other hand, we have

$$\|f_r * h_t\|_2 \le \|f_r\|_1\|h_t\|_2 \le \|h_t\|_2\Big(\int_0^r x^{-2\gamma a}A(x)\,dx\Big)^{1/2}\|x^{\gamma a} f\|_2.$$

By Lemma 3.1, $A(x) \sim e^{2\rho x}$ when x is big, and $A(x) \sim x^{2\alpha+1}$ when $x \sim 0$. Under the assumption on γ, we have that there exists some positive constant d such that

$$\Big(\int_0^r x^{-2\gamma a}A(x)\,dx\Big)^{1/2} \le dr^{-\gamma a}V(r), \quad \text{where} \quad V(r) = \begin{cases} r^{\alpha+1} & \text{if } r \le 1, \\ r^{1/2}e^{\rho r} & \text{if } r > 1. \end{cases}$$

So

$$\begin{aligned}\|h_t * f\|_2 &\le \|h_t * f_r\|_2 + \|h_t * f^r\|_2 \\ &\le Cr^{-\gamma a}(1 + \|h_t\|_2 V(r))\,\|x^{\gamma a} f\|_2.\end{aligned}$$

Choosing $r = t^{1/(2\gamma)}$ for $\gamma \in D_\rho$, we obtain (3-2) by Lemma 3.2. □

Proof of Theorem 3.1. We only prove the case $\rho > 0$, as it can be proved similarly for $\rho = 0$. Assume that $\gamma \in D_\rho$ for $\gamma a < \alpha + 1$. If $b \le 2$, it suffices to prove that

$$\|x^{\gamma a} f\|_2^{b/(a+b)} \cdot \|(\lambda^2+\rho^2)^{b/2}\widehat{f}(\lambda)\|_{2,\,c}^{a/(a+b)} \ge C\|f\|_2.$$

for f and $x^{\gamma a} f$ in $L^2(R_+, A(x)\,dx)$ and $(\lambda^2+\rho^2)^{b/2}\widehat{f}(\lambda)$ in $L^2(R_+, d\nu)$.

By (3-2),

$$\begin{aligned}\|f\|_2 &\le \|h_t * f\|_2 + \|f - h_t * f\|_2 \\ &\le Ct^{-a/2}\|x^{\gamma a} f\|_2 + \|(1-e^{-t(\lambda^2+\rho^2)})\hat{f}(\lambda)\|_{2,c} \\ &= Ct^{-a/2}\|x^{\gamma a} f\|_2 + \|(1-e^{-t(\lambda^2+\rho^2)})(t(\lambda^2+\rho^2))^{-b/2}(t(\lambda^2+\rho^2))^{b/2}\hat{f}(\lambda)\|_{2,c}.\end{aligned}$$

The size of last term is controlled by $t^{b/2}\|(\lambda^2+\rho^2)^{b/2}\hat{f}(\lambda)\|_{2,c}$, since the function $(1-e^{-s})s^{-b/2}$ is bounded for $s \ge 0$ if $b \le 2$. Therefore

$$\|f\|_2 \le C(t^{-a/2}\|x^{\gamma a} f\|_2 + t^{b/2}\|(\lambda^2+\rho^2)^{b/2}\hat{f}(\lambda)\|_{2,c}),$$

from which, optimizing in t, we obtain (3-1) for $\gamma a < \alpha+1$ and $b \le 2$.

If $b > 2$, let $b' \le 2$. For $u \ge 0$ and $b' < b$, we have $u^{b'} \le 1 + u^b$ which, for $u = ((\lambda^2+\rho^2)/\epsilon)^{1/2}$ gives the inequality $((\lambda^2+\rho^2)/\epsilon)^{b'/2} \le 1 + ((\lambda^2+\rho^2)/\epsilon)^{b/2}$ for all $\epsilon > 0$.

It follows that

$$\|(\lambda^2+\rho^2)^{b'/2}\widehat{f}\|_{2,c} \le \epsilon^{b'/2}\|f\|_2 + \epsilon^{(b'-b)/2}\|(\lambda^2+\rho^2)^{b/2}\widehat{f}\|_{2,c}.$$

Optimizing in ϵ, we get

$$\|(\lambda^2+\rho^2)^{b'/2}\widehat{f}\|_{2,c} \le \|f\|_2^{1-b'/b}\|(\lambda^2+\rho^2)^{b/2}\widehat{f}\|_{2,c}^{b'/b}.$$

Together with (3-1) for b', we get the result for $b > 2$.

If $\gamma a \ge \alpha+1$, let $\gamma a' < \alpha+1$. Then using

$$\frac{x^{\gamma a'}}{\epsilon^{\gamma a'}} \le 1 + \frac{x^{\gamma a}}{\epsilon^{\gamma a}} \quad \text{for } \epsilon > 0,$$

we get the result similarly. □

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