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Here we show that the Carathéodory, Eisenman–Kobayashi, and Kähler–Einstein volume forms are equivalent on Teichmüller space.

Statement of the theorem

Let $\mathcal{T}_g$ denote the Teichmüller space of an orientable compact Riemann surface of genus $g > 1$. Recently Liu, Sun, and Yau [2004; 2005a; 2005b] made some important progress made on some of the geometric aspects of Teichmüller space including a discussion on the equivalence of intrinsic metrics. Here, our goal is to prove the corresponding theorem in the case of intrinsic volume forms:

Main Theorem. *The Carathéodory, Eisenman–Kobayashi, and Kähler–Einstein intrinsic volume forms are equivalent on Teichmüller space.*

This result is also true when the intrinsic measures are defined with respect to the polydisc instead of the unit ball. The arrangement of this paper is as follows. Section 1 describes of the Carathéodory and Eisenman–Kobayashi volume forms. Section 2 gives preliminaries on Teichmüller space. Section 3 proves the main theorem.

1. Definition of the Carathéodory and Eisenman–Kobayashi volume forms

For the basic materials of this subject, refer to [Graham and Wu 1985; Krantz 2001; Pelles 1975; Wong 1977b; 1977a]. Let B_n denote the open unit ball in $\mathbb{C}^n$, and let D be any complex domain in $\mathbb{C}^n$.

Definition. The *Carathéodory volume form* M_D^C is an (n, n)-form

$$\left(\frac{i}{2}\right)^n |M_D^C(z)| \cdot dz_1 \wedge d\bar{z}_1 \cdots dz_n \wedge d\bar{z}_n$$

on D such that $|M_D^C(z)|$ is defined for all $z \in D$ as

$$|M_D^C(z)| = \sup\left\{|\det f'(0)|^2 : f \in \mathrm{Hol}(D, B_n) \text{ and } f(z) = 0\right\},$$

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where the supremum is taken over all f indicated above.

The function $|M_D^C|$ is in general a continuous function. $|M_D^C| > 0$ if D is a bounded domain.

Definition. The *Eisenman–Kobayashi volume form* M_D^E is an (n, n)-form

$$\left(\frac{i}{2}\right)^n |M_D^E(z)| \cdot dz_1 \wedge d\bar{z}_1 \cdots dz_n \wedge d\bar{z}_n$$

on D such that $|M_D^E(z)|$ is defined for all $z \in D$ as

$$|M_D^E(z)| = \inf\left\{\frac{1}{|\det f'(0)|^2} : f \in \mathrm{Hol}(B_n, D) \text{ and } f(0) = z\right\},$$

where the infimum is taken over all f indicated above.

The function $|M_D^E|$ is in general a semicontinuous function. If D is a bounded domain, then $|M_D^E|$ is continuous and $|M_D^E| > 0$ since $|M_D^E| \geq |M_D^C|$ by the generalized Schwarz lemma. Also, both intrinsic measures are volume decreasing under holomorphic mappings and invariant under biholomorphic mappings.

2. Preliminaries on Teichmüller space

Let us begin by discussing some basic properties of Teichmüller space $\mathcal{T}_g$. For the basic definitions and a survey, see [Imayoshi and Taniguchi 1992; Nag 1988]. Let $n = 3g - 3$ unless stated otherwise. Let R be a Riemann surface of genus $g > 1$. The Teichmüller space of R is the collection of Riemann surfaces of genus g with the following equivalence relation. Let S_1 and S_2 be two Riemann surfaces of genus g, and let $g_1 : R \to S_1$ and $g_2 : R \to S_2$ be their respective quasiconformal homeomorphisms. We say that S_1 is equivalent to S_2 if there exists a conformal function $h : S_1 \to S_2$ such that $g_2^{-1} \circ h \circ g_1 : R \to R$ is homotopic to the identity by a homotopy that leaves every point on ∂R fixed if R has boundary. We will denote the Teichmüller space of R by $\mathcal{T}_g$. Let Γ be the discrete group acting on the upper-half plane H such that $R = H/\Gamma$. Then we define $\overline{R}$ to be the Riemann surface conjugate holomorphic to R. The Bers embedding theorem gives rise to a holomorphic embedding $\Phi : \mathcal{T}_g \to \Omega_{\overline{R}}$ where $\Omega_{\overline{R}}$ is the vector space of holomorphic quadratic differentials on the Riemann surface $\overline{R}$ with norm $\|\phi\| = \sup|y^2\phi(z)|$.

We will use the notation $\Phi(\mathcal{T}_g) = T_B(R) \subset \Omega_{\overline{R}}$. $\mathcal{T}_g$ is a bounded domain when identified with its image under Φ and has complex dimension n since $B_n^{1/2} \subset T_B(R) \subset B_n^{3/2}$ [Bers and Ehrenpreis 1964]. Also, $\mathcal{T}_g$ is a domain of holomorphy for $g > 1$ [Bers and Ehrenpreis 1964; Earle 1974], and therefore $\mathcal{T}_g$ has a complete Einstein–Kähler metric of negative Ricci curvature [Cheng and Yau 1980].

3. Proof of the main theorem

We say that two volume forms M^1 and M^2 on $D \subset \mathbb{C}^n$ are equivalent if there exists a constant $C > 0$ such that

$$\frac{1}{C}|M_D^2| \le |M_D^1| \le C|M_D^2|,$$

where $M_D^j = (i/2)^n |M_D^j| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$. Let us denote by M_D^{KE} the volume form determined by the Kähler–Einstein metric on a bounded domain of holomorphy D. The following theorems characterize the distance-decreasing property with respect to volume forms.

Theorem 3.1 [Chern 1968]. *Let $f : B_n \to N$ be a holomorphic mapping where B_n is the unit n-ball with the standard Kähler metric and where N is an n-dimensional Hermitian Einsteinian manifold with scalar curvature $\le -n(n+1)$. Then f is volume-decreasing.*

Theorem 3.2 [Mok and Yau 1983; Yau 1978]. *Let M be a complete Hermitian manifold with scalar curvature bounded from below by $-K_1$, and let N be a complex manifold of the same dimension with a volume form V_N such that the Ricci form is negative definite and $V_N = (\frac{i}{2})^n |V_N| dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n$ is almost Einstein with respect to volume*; *that is*,

$$\left(\frac{i}{2}\,\partial\bar{\partial}\ln|V_N|\right)^n \ge K_2 V_N.$$

Suppose $f : M \to N$ is a holomorphic map and the Jacobian is nonvanishing at one point. Then $K_1 > 0$ and

$$\sup \frac{f^* V_N}{V_M} \le \frac{K_1^n}{n^n K_2},$$

where V_M is the volume form associated to the given Kähler metric on M.

The proof of the main theorem will depend on the next two lemmas.

Lemma 3.1. *Let D be a bounded domain of holomorphy. Then*

$$|M_D^C| \le |M_D^{KE}| \le |M_D^E|.$$

Proof. Let's choose the scalar curvature of the Kähler–Einstein metric on both B_n and D to be $-n(n+1)$. We are going to apply Theorems 3.1 and 3.2 to these two metrics.

(i) Suppose $f : B_n \to D$ is any holomorphic mapping with $f(0) = z$. By Theorem 3.1, $M = B_n$, $N = D$, and $f^* M_D^{KE} \le M_{B_n}$. Also

$$|\det f'(0)|^2 |M_D^{KE}(z)| \le |M_{B_n}(0)| \qquad \text{and} \qquad |M_D^{KE}(z)| \le 1/|\det f'(0)|^2.$$

Taking the infimum over all such holomorphic mappings f, we find $|M_D^{KE}| \le |M_D^E|$.
(ii) Suppose $f : D \to B_n$ is any holomorphic mapping with $f(z) = 0$. By Theorem 3.2, in this case $M = D$, $N = B_n$, $K_1 = n(n+1)$, and $K_2 = n+1$, and we obtain an inequality $f^* M_{B_n} \le M_D^{KE}$, so that $|\det f'(z)|^2 \le |M_D^{KE}(z)|$. Taking the supremum over all such holomorphic mappings f, we have $|M_D^C| \le |M_D^{KE}|$. □

Lemma 3.2. $|M^E_{\mathcal{T}_g}(X)| \le k |M^C_{\mathcal{T}_g}(X)|$, *where k is a positive constant and X is any point in $\mathcal{T}_g$.*

Proof. Let the Bers embedding be denoted by $\mathcal{B} : \mathcal{T}_g \to \Omega_{\overline{R}}$, where $\mathcal{T}_g$ is identified with $T_B(R)$ for some Riemann surface R and $\overline{R}$ is the Riemann surface conjugate holomorphic to R. There exists a biholomorphism $F : T_B(R) \to T_B(X) \subset \Omega_{\overline{X}}$ such that $F(\mathcal{B}(X)) = 0$ [Earle 1974; Imayoshi and Taniguchi 1992] (comparing with earlier notation, we have $\mathcal{B}(\mathcal{T}_g) = T_B(R) \subset \Omega_{\overline{R}}$). The image of this map has the property that $B_n^{1/2} \subset T_B(X) \subset B_n^{3/2}$. Now,

$$\begin{aligned} |M^C_{T_B(X)}(0)| &\ge |M^C_{B_n^{3/2}}(\pi(0))| = \frac{1}{9^n} |M^C_{B_n^{1/2}}(0)| \\ &= \frac{1}{9^n} |M^E_{B_n^{1/2}}(0)| \ge \frac{1}{9^n} |M^E_{T_B(X)}(\iota(0))| = \frac{1}{9^n} |M^E_{T_B(X)}(0)|, \end{aligned}$$

since the intrinsic measures are volume decreasing under the holomorphic inclusion mappings $\iota : B_n^{1/2} \to T_B(X)$ and $\pi : T_B(X) \to B_n^{3/2}$, and they are equal on any Euclidean ball in $\mathbb{C}^n$.

The composition $F \circ \mathcal{B}$ is a biholomorphic mapping from $\mathcal{T}_g$ to $T_B(X) \subset \Omega_{\overline{X}}$. Hence $|M^E_{\mathcal{T}_g}(X)| \le 9^n |M^C_{\mathcal{T}_g}(X)|$, since our volume measures are invariant under biholomorphic mappings. □

Proof of Main Theorem. The proof now follows from Lemmas 3.1 and 3.2. □

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