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We compute the Galois groups of basic hypergeometric equations.

In this paper q is a complex number such that $0 < |q| < 1$.

1. Basic hypergeometric series and equations

The theory of hypergeometric functions and equations, which dates back at least to Gauss, has long been and still is an integral part of the mathematical literature. In particular, the Galois theory of (generalized) hypergeometric equations has attracted the attention of many authors. See for example [Beukers and Heckman 1989; Beukers et al. 1988; Katz 1990] and references therein. We also single out the papers [Duval and Mitschi 1989; Mitschi 1996], which are devoted to calculating some Galois groups by means of a density theorem (the Ramis theorem).

Here, we focus our attention on the Galois theory of the basic hypergeometric equations, the later being natural q-analogues of the hypergeometric equations.

The *basic hypergeometric series* $\phi(z) = {}_2\phi_1(a, b; c; z)$, with three parameters $(a, b, c) \in (\mathbb{C}^*)^3$, is defined by

$$\begin{aligned}{}_2\phi_1(a, b; c; z) &= \sum_{n=0}^{+\infty} \frac{(a, b; q)_n}{(c, q; q)_n} z^n \\ &= \sum_{n=0}^{+\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-b)(1-bq)\cdots(1-bq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)(1-c)(1-cq)\cdots(1-cq^{n-1})} z^n.\end{aligned}$$

It was first introduced by Heine and was later generalized by Ramanujan. In the subject of functional equations, the basic hypergeometric series provides a solution to the second order q-difference equation, called the *basic hypergeometric equation* with parameters (a, b, c), given by

$$\phi(q^2z) - \frac{(a+b)z-(1+c/q)}{abz-c/q}\phi(qz) + \frac{z-1}{abz-c/q}\phi(z) = 0. \tag{1}$$

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This functional equation is equivalent to a functional system. Indeed, with the notations

$$\lambda(a,b;c;z)=\frac{(a+b)z-(1+c/q)}{abz-c/q} \quad \text{and} \quad \mu(a,b;c;z)=\frac{z-1}{abz-c/q},$$

a function ϕ is solution of (1) if and only if the vector $\Phi(z)=\binom{\phi(z)}{\phi(qz)}$ satisfies the functional system

$$\Phi(qz)=A(a,b;c;z)\Phi(z) \tag{2}$$

with

$$A(a,b;c;z)=\begin{pmatrix} 0 & 1 \\ -\mu(a,b;c;z) & \lambda(a,b;c;z) \end{pmatrix}.$$

This paper focuses on calculating the Galois group of the q-difference Equation (1) or, equivalently, that of the q-difference system (2). A number of authors have developed q-difference Galois theories over the past years; among them are Franke [1963], Etingof [1995], van der Put and Singer [1997], van der Put and Reversat [2005], Chatzidakis and Hrushovski [1999], Sauloy [2003], André [2001]. The exact relations between the existing Galois theories for q-difference equations are partially understood; see [Chatzidakis et al. 2006], and also Remark 1.

We follow the approach of Sauloy (initiated by Etingof in the regular case). Our method for computing the Galois groups of the basic hypergeometric equations is based on a q-analogue of Schlesinger's density theorem stated and established in [Sauloy 2003]. Note that some of these groups were previously computed by Hendriks [1997] using a radically different method (actually, Hendriks dealt with the Galois groups defined by van der Put and Singer, but these do coincide with those defined by Sauloy; see again Remark 1).

In the first part of this paper, we give a brief overview of some results from [Sauloy 2003]. In the second, we compute the Galois groups of the basic hypergeometric equations in all nonresonant (but possibly logarithmic) cases.

2. Galois theory for regular singular q-difference equations

Using analytic tools together with Tannakian duality, Sauloy [2003] developed a Galois theory for regular singular q-difference systems. In this section, we will first recall some notions used there, mainly the Birkhoff matrix and the twisted Birkhoff matrix. Then we will explain briefly how this leads to a Galois theory for regular singular q-difference systems, and we will state a density theorem which will be of main importance in our calculations.

2.1. ***Basic notions.*** Let us consider $A \in \mathrm{Gl}_n(\mathbb{C}(\{z\}))$. Following Sauloy [2003], the q-difference system

$$Y(qz) = A(z)Y(z) \tag{3}$$

is said to be *Fuchsian* at 0 if A is holomorphic at 0 and if $A(0) \in \mathrm{Gl}_n(\mathbb{C})$. Such a system is nonresonant at 0 if in addition $\mathrm{spectrum}(A(0)) \cap q^{\mathbb{Z}^*}\,\mathrm{spectrum}(A(0)) = \varnothing$. Lastly we say that the above q-difference system is *regular singular* at 0 if there exists an $R^{(0)} \in \mathrm{Gl}_n(\mathbb{C}(\{z\}))$ such that the q-difference system defined by $(R^{(0)}(qz))^{-1}A(z)R^{(0)}(z)$ is Fuchsian at 0. Similar notions hold at ∞ using the change of variable $z \leftarrow 1/z$.

In the case of a global system, that is, $A \in \mathrm{Gl}_n(\mathbb{C}(z))$, we will use the following terminology. If $A \in \mathrm{Gl}_n(\mathbb{C}(z))$, then the system (3) is called *Fuchsian* (respectively *Fuchsian and nonresonant*, *regular singular*) if it is Fuchsian (respectively Fuchsian and nonresonant, regular singular) at 0 and at ∞.

For instance, the basic hypergeometric system (2) is Fuchsian.

Local fundamental systems of solutions at 0. Suppose that (3) is Fuchsian and nonresonant at 0, and consider $J^{(0)}$ a Jordan normal form of $A(0)$. According to [Sauloy 2003] there exists an $F^{(0)} \in \mathrm{Gl}_n(\mathbb{C}\{z\})$ such that

$$F^{(0)}(qz)J^{(0)} = A(z)F^{(0)}(z). \tag{4}$$

Therefore, if $e^{(0)}(J^{(0)})$ denotes a fundamental system solving the q-difference system with constant coefficients $X(qz) = J^{(0)}X(z)$, then the matrix-valued function $Y^{(0)} = F^{(0)}e^{(0)}(J^{(0)})$ is a fundamental system of solutions of (3). We are going to describe a possible choice for $e^{(0)}(J^{(0)})$. We denote by θ_q the Jacobi theta function defined by $\theta_q(z) = (q;q)_\infty(z;q)_\infty(q/z;q)_\infty$. This is a meromorphic function over $\mathbb{C}^*$ whose zeros are simple and located on the discrete logarithmic spiral $q^{\mathbb{Z}}$. We also have the functional equation $\theta_q(qz) = -z^{-1}\theta_q(z)$. Now we introduce, for all $\lambda \in \mathbb{C}^*$ such that $|q| \le |\lambda| < 1$, the q-character $e^{(0)}(\lambda) = \theta_q/\theta_{q,\lambda}$ with $\theta_{q,\lambda}(z) = \theta_q(\lambda z)$, and we extend this definition to an arbitrary nonzero complex number $\lambda \in \mathbb{C}^*$ by requiring $e^{(0)}(q\lambda) = ze^{(0)}(\lambda)$. If $D = P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)P^{-1}$ is a semisimple matrix, then we set $e^{(0)}(D) := P\,\mathrm{diag}(e^{(0)}(\lambda_1), \ldots, e^{(0)}(\lambda_n))P^{-1}$. Clearly this does not depend on the chosen diagonalization. Furthermore, consider $\ell_q(z) = -z\theta_q'(z)/\theta_q(z)$ and, if U is a unipotent matrix,

$$e^{(0)}(U) = \sum_{k=0}^{n} \ell_q^{(k)}(U - I_n)^k \quad \text{with} \quad \ell_q^{(k)} = \binom{\ell_q}{k}.$$

If $J^{(0)} = D^{(0)}U^{(0)}$ is the multiplicative Dunford decomposition of $J^{(0)}$, where $D^{(0)}$ is semisimple and $U^{(0)}$ is unipotent, we set $e^{(0)}(J^{(0)}) = e^{(0)}(D^{(0)})e^{(0)}(U^{(0)})$.

Local fundamental system of solutions at ∞. Using the variable change $z \leftarrow 1/z$, we have a similar construction at ∞. The corresponding fundamental system of solutions is denoted by $Y^{(\infty)} = F^{(\infty)} e^{(\infty)}(J^{(\infty)})$.

Throughout this section we assume that the system (3) is global and that it is Fuchsian and nonresonant.

Birkhoff matrix. The linear relations between the two fundamental systems of solutions introduced above are given by the Birkhoff matrix (also called the connection matrix) $P = (Y^{(\infty)})^{-1} Y^{(0)}$. Its entries are elliptic functions, that is, meromorphic functions over the elliptic curve $\mathbb{E}_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

Twisted Birkhoff matrix. To describe a Zariki-dense set of generators of the Galois group associated to the system (3), we introduce a "twisted" connection matrix. As in [Sauloy 2003], we choose for all $z \in \mathbb{C}^*$ a group endomorphism g_z of $\mathbb{C}^*$ sending q to z. Before giving an example, we need to introduce more notation. For any fixed $\tau \in \mathbb{C}$ such that $q = e^{-2\pi i \tau}$, write $q^y = e^{-2\pi i \tau y}$ for all $y \in \mathbb{C}$. We also define the (not continuous) function $\log_q$ on the whole punctured complex plane $\mathbb{C}^*$ by $\log_q(q^y) = y$ if $y \in \mathbb{C}^* \setminus \mathbb{R}^+$, and we require that its discontinuity is located just before its branch cut $\mathbb{R}^+$ when turning counterclockwise around 0. An explicit example of the endomorphism g_z is now the function $g_z : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{C}^*$ sending uq^ω to $g_z(uq^\omega) = z^\omega = \exp(-2\pi i \tau \log_q(z)\omega)$ for $(u, \omega) \in \mathbb{U} \times \mathbb{R}$, where $\mathbb{U} \subset \mathbb{C}$ is the unit circle.

Then for all z in $\mathbb{C}^*$, we set $\psi_z^{(0)}(\lambda) = e^{(0)}(\lambda)(z)/g_z(\lambda)$ and define $\psi_z^{(0)}(D^{(0)})$, the *twisted factor* at 0, by $\psi_z^{(0)}(D^{(0)}) = P \operatorname{diag}(\psi_z^{(0)}(\lambda_1), \dots, \psi_z^{(0)}(\lambda_n)) P^{-1}$, where $D^{(0)} = P \operatorname{diag}(\lambda_1, \dots, \lambda_n) P^{-1}$. We have a similar construction at ∞ using the variable change $z \leftarrow 1/z$. We denote the corresponding twisting factor by $\psi_z^{(\infty)}(J^{(\infty)})$.

Finally, the twisted connection matrix $\check{P}(z)$ is

$$\check{P}(z) = \psi_z^{(\infty)}(D^{(\infty)}) P(z) \psi_z^{(0)}(D^{(0)})^{-1}.$$

2.2. ***Definition of the Galois groups.*** The definition of the Galois groups of regular singular q-difference systems given by Sauloy [2003] via a q-analogue of the Riemann-Hilbert correspondence is somewhat technical. Here we describe the underlying idea.

(Global) Galois group. Let us denote by $\mathscr{E}$ the category of regular singular q-difference systems with coefficients in $\mathbb{C}(z)$. This category is naturally equipped with a tensor product $\otimes$ such that $(\mathscr{E}, \otimes)$ satisfies all the axioms of a Tannakian category over $\mathbb{C}$ except for the existence of a *fiber functor*, which is not obvious.

The latter problem can be overcome using an analogue of the Riemann–Hilbert correspondence. For regular singular q-difference systems, this correspondence entails that $\mathscr{E}$ is equivalent to the category $\mathscr{C}$ of connection triples whose objects are triples $(A^{(0)}, P, A^{(\infty)}) \in \mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_n(\mathscr{M}(\mathbb{E}_q)) \times \mathrm{Gl}_n(\mathbb{C})$ (see [Sauloy 2003] for the complete definition of $\mathscr{C}$), and $\mathscr{C}$ can be endowed with a tensor product $\underline{\otimes}$ making the above equivalence of categories compatible with the tensor products. We emphasize that $\underline{\otimes}$ is not the usual tensor product for matrices. Indeed some twisting factors appear because of the bad multiplicative properties of the q-characters $e_{q,c}$, as generally $e_{q,c}e_{q,d} \neq e_{q,cd}$.

The category $\mathscr{C}$ allows us to define a Galois group: $\mathscr{C}$ is a Tannakian category over $\mathbb{C}$. The functor ω_0 from $\mathscr{C}$ to $\mathrm{Vect}_{\mathbb{C}}$ sending an object $(A^{(0)}, P, A^{(\infty)})$ to the underlying vector space $\mathbb{C}^n$ on which $A^{(0)}$ acts is a fiber functor. There is a similar fiber functor ω_∞ at ∞. Following the general formalism of the theory of Tannakian categories (see [Deligne 1990]), the *absolute Galois group* of $\mathscr{C}$ (or, using the above equivalence of categories, of $\mathscr{E}$) is defined as the proalgebraic group $\mathrm{Aut}^{\underline{\otimes}}(\omega_0)$, and the *global Galois group of an object* χ of $\mathscr{C}$ (or, as before, of an object of $\mathscr{E}$) is the complex linear algebraic group $\mathrm{Aut}^{\underline{\otimes}}(\omega_{0|\langle\chi\rangle})$, where $\langle\chi\rangle$ denotes the Tannakian subcategory of $\mathscr{C}$ generated by χ. For simplicity, we will often call $\mathrm{Aut}^{\underline{\otimes}}(\omega_{0|\langle\chi\rangle})$ the *Galois group* of χ (or, as before, of the corresponding object of $\mathscr{E}$).

Local Galois groups. Notions of local Galois groups at 0 and at ∞ are also available. As expected, they are subgroups of the (global) Galois group. Nevertheless, since these groups are of secondary importance in what follows, we omit the details and refer the interested reader to [Sauloy 2003].

Remark 1. Van der Put and Singer [1997] showed that the Galois groups defined using Picard–Vessiot theory can be recovered by means of Tannakian duality: it is the group of tensor automorphisms of some suitable complex-valued fiber functor over $\mathscr{E}$. Since two complex-valued fiber functors on a same Tannakian category are necessarily isomorphic, we conclude that the theories of Sauloy and of Van der Put and Singer coincide.

In the rest of this section we exhibit some natural elements of the Galois group of a given Fuchsian q-difference system and state a density theorem due to Sauloy.

2.3. ***The density theorem.*** Fix a "base point"

$$y_0 \in \Omega = \mathbb{C}^* \setminus \{\text{zeros of } \det(P(z)) \text{ or poles of } P(z)\}.$$

Sauloy [2003] gives the following elements of the (global) Galois group associated to the q-difference system (3):

(Ia) $\gamma_1(D^{(0)})$ and $\gamma_2(D^{(0)})$, where

$$\gamma_1 : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{U} \quad \text{and} \quad \gamma_2 : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{C}^*$$

are respectively the projection onto the first factor, and the map defined by $\gamma_2(uq^\omega) = e^{2\pi i\omega}$;

(Ib) $U^{(0)}$;

(IIa) $\check{P}(y_0)^{-1}\gamma_1(D^{(\infty)})\check{P}(y_0)$ and $\check{P}(y_0)^{-1}\gamma_2(D^{(\infty)})\check{P}(y_0)$;

(IIb) $\check{P}(y_0)^{-1}U^{(\infty)}\check{P}(y_0)$;

(III) $\check{P}(y_0)^{-1}\check{P}(z)$ for $z \in \Omega$.

Theorem 1 [Sauloy 2003]. *The algebraic group generated by the matrices* (Ia) *through* (*III*) *is the* (*global*) *Galois group G of the q-difference system* (3). *The algebraic group generated by the matrices* (Ia) *and* (Ib) *is the local Galois group at* 0 *of the q-difference system* (3). *The algebraic group generated by the matrices* (IIa) *and* (IIb) *is the local Galois group at ∞ of the q-difference system* (3).

The algebraic group generated by the matrices (III) is called the *connection component* of the Galois group G. The following result is easy but very useful. Its proof is left to the reader.

Lemma 1. *The connection component of the Galois group G of a regular singular q-difference system is a subgroup of the identity component G^I of G.*

3. Galois groups of the basic hypergeometric equations: nonresonant and nonlogarithmic cases

We write $a = uq^{\alpha}$, $b = vq^{\beta}$, and $c = wq^{\gamma}$ with $u, v, w \in \mathbb{U}$ and $\alpha, \beta, \gamma \in \mathbb{R}$ (we choose a logarithm of q).

In this section we want to compute the Galois group of the basic hypergeometric system (2) under the assumptions that $a/b \notin q^{\mathbb{Z}}$ and $c \notin q^{\mathbb{Z}}$.

First, we give explicit formulas for the generators of the Galois group of (2) involved in Theorem 1.

Local fundamental system of solutions at 0. We have

$$A(a, b; c; 0) = \begin{pmatrix} 1 & 1 \\ 1 & q/c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q/c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & q/c \end{pmatrix}^{-1}.$$

Hence the system (2) is nonresonant and nonlogarithmic at 0. A fundamental system of solutions at 0 of (2) as described in Section 2.1 is given by $Y^{(0)}(a, b; c; z) = F^{(0)}(a, b; c; z)e^{(0)}(J^{(0)}(c))(z)$ with $J^{(0)}(c) = \mathrm{diag}(1, q/c)$ and

$$F^{(0)}(a, b; c; z) = \begin{pmatrix} {}_2\phi_1(a, b; c; z) & {}_2\phi_1(aq/c, bq/c; q^2/c; z) \\ {}_2\phi_1(a, b; c; qz) & (q/c){}_2\phi_1(aq/c, bq/c; q^2/c; qz) \end{pmatrix}.$$

Generators of the local Galois group at 0. We have two generators

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\gamma} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}.$$

Local fundamental system of solutions at ∞. We have

$$A(a,b;c;\infty)=\begin{pmatrix}1 & 1\\ 1/a & 1/b\end{pmatrix}\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}1 & 1\\ 1/a & 1/b\end{pmatrix}^{-1}.$$

Hence the system (2) is nonresonant and nonlogarithmic at ∞, and a fundamental system of solutions at ∞ of (2) as described in Section 2.1 is given by

$$Y^{(\infty)}(a,b;c;z)=F^{(\infty)}(a,b;c;z)e^{(\infty)}(J^{(\infty)}(a,b))(z)$$

with $J^{(\infty)}(a,b)=\mathrm{diag}(1/a,1/b)$ and

$$F^{(\infty)}(a,b;c;z)=\begin{pmatrix} {}_2\phi_1\big(a,\frac{aq}{c};\frac{aq}{b};\frac{cq}{ab}z^{-1}\big) & {}_2\phi_1\big(b,\frac{bq}{c};\frac{bq}{a};\frac{cq}{ab}z^{-1}\big)\\ \frac{1}{a}\,{}_2\phi_1\big(a,\frac{aq}{c};\frac{aq}{b};\frac{c}{ab}z^{-1}\big) & \frac{1}{b}\,{}_2\phi_1\big(b,\frac{bq}{c};\frac{bq}{a};\frac{c}{ab}z^{-1}\big)\end{pmatrix}.$$

Generators of the local Galois group at ∞. We have two generators,

$$\check{P}(y_0)^{-1}\begin{pmatrix}e^{2\pi i\alpha} & 0\\ 0 & e^{2\pi i\beta}\end{pmatrix}\check{P}(y_0)\quad\text{and}\quad \check{P}(y_0)^{-1}\begin{pmatrix}u & 0\\ 0 & v\end{pmatrix}\check{P}(y_0).$$

Birkhoff matrix. The Barnes–Mellin–Watson formula (see [Gasper and Rahman 2004]) says that $P(z)=(e^{(\infty)}(J^{(\infty)}(a,b))(z))^{-1}M(z)e^{(0)}(J^{(0)}(c))(z)$, where

$$M(z)=\begin{pmatrix}\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)}\\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}\end{pmatrix}. \tag{5}$$

Twisted Birkhoff matrix. We know $\check{P}(z)$ equals

$$\begin{pmatrix}\left(\frac{1}{z}\right)^{-\alpha} & 0\\ 0 & \left(\frac{1}{z}\right)^{-\beta}\end{pmatrix}\begin{pmatrix}\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)}\\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^{1-\gamma}\end{pmatrix}.$$

We now need to consider different cases.

Case 1. $a,b,c,a/b,a/c,b/c\notin q^{\mathbb{Z}}$ and a/b or $c\notin \pm q^{\mathbb{Z}/2}$.

Under this assumption, we have four nonzero numbers

$$\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty},\quad \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty},\quad \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty},\quad \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}.$$

Proposition 1. *Suppose that Case 1 holds. Then the natural action of G^I on $\mathbb{C}^2$ is irreducible.*

Proof. Suppose, at the contrary, that the action of G^I is reducible and let $L \subset \mathbb{C}^2$ be an invariant line. For this and subsequent proofs define a basis in $\mathbb{C}^2$ in the usual way with $\hat{e}_1 = \binom{1}{0}$ and $\hat{e}_2 = \binom{0}{1}$.

Note that L is distinct from $\mathbb{C}\hat{e}_1$ and $\mathbb{C}\hat{e}_2$. Indeed, assume to the contrary that $L = \mathbb{C}\hat{e}_1$ (the case $L = \mathbb{C}\hat{e}_2$ is similar). Because the line $L = \mathbb{C}\hat{e}_1$ is invariant by the connection component, the line generated by $\breve{P}(z)\hat{e}_1$ does not depend on $z \in \Omega$. This yields a contradiction because the ratio of the components of

$$\breve{P}(z)\hat{e}_1 = \begin{pmatrix} \dfrac{(b, c/a; q)_\infty}{(c, b/a; q)_\infty} \dfrac{\theta_q(az)}{\theta_q(z)} z^\alpha \\ \dfrac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty} \dfrac{\theta_q(bz)}{\theta_q(z)} z^\beta \end{pmatrix}$$

depends on z (recall $a/b \notin q^{\mathbb{Z}}$).

On the other hand, since, for all $n \in \mathbb{N}$, the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \gamma n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & w^n \end{pmatrix}$$

belong to G and since G^I is a normal subgroup of G, the lines

$$L_n := \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \gamma n} \end{pmatrix} L \quad \text{and} \quad L'_n := \begin{pmatrix} 1 & 0 \\ 0 & w^n \end{pmatrix} L$$

are also invariant by G^I.

Note that because Case 1 holds, at least one of the complex numbers w, $e^{2\pi i \gamma}$, u/v, and $e^{2\pi i(\alpha-\beta)}$ is distinct from ± 1.

Now suppose $w \neq \pm 1$. We have seen that $L \neq \mathbb{C}\hat{e}_1, \mathbb{C}\hat{e}_2$; hence L_0, L_1, and L_2 are three distinct lines invariant by the action of G^I. This implies that G^I consists of scalar matrices: this is a contradiction because, for instance, $\mathbb{C}\hat{e}_1$ is not invariant for the action of G^I. Hence for $w \neq \pm 1$, we have proved that G^I acts irreducibly.

The case $e^{2\pi i \gamma} \neq \pm 1$ is similar.

Finally, the proof is analogous in the case $u/v \neq \pm 1$ or $e^{2\pi i(\alpha-\beta)} \neq \pm 1$, as we may then use that, for all $z \in \Omega$, G^I is normalized by $\breve{P}(z)^{-1} \operatorname{diag}(u, v)\breve{P}(z)$ and $\breve{P}(z)^{-1} \operatorname{diag}(e^{2\pi i\alpha}, e^{2\pi i\beta})\breve{P}(z)$ and that there exists a $z \in \Omega$ such that $\breve{P}(z)L$ is distinct from $\mathbb{C}\hat{e}_1$ and $\mathbb{C}\hat{e}_2$. □

Theorem 2. *Suppose that Case 1 holds. Then we have this dichotomy*:

$$\text{If } abq/c \notin q^{\mathbb{Z}} \text{ then } G = \mathrm{Gl}_2(\mathbb{C}); \text{ otherwise } G = \overline{\langle \mathrm{Sl}_2(\mathbb{C}), \sqrt{w} I, e^{\pi i \gamma} I \rangle}.$$

Proof. Since G^I acts irreducibly on $\mathbb{C}^2$, the general theory of algebraic groups entails that G^I is generated by its center $Z(G^I)$ together with its derived subgroup

$G^{I,\mathrm{der}}$ and that $Z(G^I)$ acts as do scalars. Hence $G^{I,\mathrm{der}} \subset \mathrm{Sl}_2(\mathbb{C})$ also acts irreducibly on $\mathbb{C}^2$. Therefore $G^{I,\mathrm{der}} = \mathrm{Sl}_2(\mathbb{C})$. (A connected algebraic group of dimension less than or equal to 2 is solvable; hence $\dim(G^{I,\mathrm{der}}) = 3$ and $G^{I,\mathrm{der}} = \mathrm{Sl}_2(\mathbb{C})$.)

To complete the proof, it is sufficient to determine $\det(G)$. We have

$$\det(\breve{P}(z)) = \frac{(1/z)^{-(\alpha+\beta)} z^{1-\gamma}}{(q^2/c, a/b, c, b/a; q)_\infty} \psi(z),$$

where

$$\psi(z) = \theta_q(b)\theta_q(c/a)\frac{\theta_q(az)}{\theta_q(z)}\frac{\theta_q((bq/c)z)}{\theta_q(z)} - \theta_q(c/b)\theta_q(a)\frac{\theta_q((aq/c)z)}{\theta_q(z)}\frac{\theta_q(bz)}{\theta_q(z)}.$$

A straightforward calculation shows that the function

$$\theta_q(b)\theta_q(c/a)\theta_q(az)\theta_q((bq/c)z) - \theta_q(c/b)\theta_q(a)\theta_q((aq/c)z)\theta_q(bz)$$

vanishes for $z \in q^{\mathbb{Z}}$ and for $z \in (c/abq)q^{\mathbb{Z}}$. On the other hand, ψ is a solution of the first order q-difference equation $y(qz) = (c/abq)y(z)$. Hence, if we suppose that $abq/c \notin q^{\mathbb{Z}}$, we deduce that the ratio $\chi(z) = \psi(z)/(\theta_q((abq/c)z)/\theta_q(z))$ defines a holomorphic elliptic function over $\mathbb{C}^*$. Therefore χ is constant and, evaluating χ at $z = 1/b$, we get $\chi = -b\theta_q(a/b)\theta_q(c)$. Finally, we obtain the identity

$$\det(\breve{P}(z)) = \frac{1-q/c}{1/a - 1/b}(1/z)^{-(\alpha+\beta)} z^{1-\gamma} \frac{\theta_q(\frac{abq}{c}z)}{\theta_q(z)}. \tag{6}$$

By analytic continuation (with respect to the parameters) we see that this formula also holds if $abq/c \in q^{\mathbb{Z}}$.

Consequently, if $abq/c \notin q^{\mathbb{Z}}$, then $\det(\breve{P}(y_0)^{-1}\breve{P}(z))$ for any fixed $y_0 \in \Omega$ is a nonconstant holomorphic function (with respect to z), implying $G = G^I = \mathrm{Gl}_2(\mathbb{C})$. On the other hand, if $abq/c \in q^{\mathbb{Z}}$, then $\det(\breve{P}(y_0)^{-1}\breve{P}(z)) = 1$, so that the connection component of the Galois group is a subgroup of $\mathrm{Sl}_2(\mathbb{C})$ and the Galois group G is the smallest algebraic group that contains $\mathrm{Sl}_2(\mathbb{C})$ and $\{\sqrt{w}I, e^{\pi i\gamma}I\}$. □

We study the case $a, b, c, a/b, a/c, b/c \notin q^{\mathbb{Z}}$ and $a/b, c \in \pm q^{\mathbb{Z}+1/2}$ in two steps.

Case 2. $a, b, c, a/b, a/c, b/c \notin q^{\mathbb{Z}}$ and

$$q^{\mathbb{Z}}a \cup q^{\mathbb{Z}}b \cup q^{\mathbb{Z}}aq/c \cup q^{\mathbb{Z}}bq/c = q^{\mathbb{Z}}a \cup -q^{\mathbb{Z}}a \cup q^{\mathbb{Z}+1/2}a \cup -q^{\mathbb{Z}+1/2}a.$$

We first establish a preliminary result.

Lemma 2. *Suppose that Case 2 holds. Let us consider* $A, B, C, D \in \mathbb{C}$ *and* $n, m, l, k, N, M, L, K \in \mathbb{Z}$ *Then the functional equation* (*in* z)

$$0 = Az^{n/2}\theta_q(q^N az) + Bz^{m/2}\theta_q(-q^M az) + Cz^{l/2}\theta_q(q^L q^{1/2}az) + Dz^{k/2}\theta_q(-q^K q^{1/2}az)$$

holds if and only if $A = B = C = D = 0$.

Proof. Using the nontrivial monodromy of $z^{1/2}$, we reduce the problem to the case of odd n, m, l, and k. Then, using the functional equation $\theta_q(qz) = -z^{-1}\theta_q(z)$, we can assume without loss of generality that $n = l = m = k = 0$. The expansion of θ_q as an infinite Laurent series $\theta_q(z) = \sum_{j\in\mathbb{Z}} q^{j(j-1)/2}(-z)^j$ ensures that

$$A(q^N)^j + B(-q^M)^j + C(q^{L+1/2})^j + D(-q^{K+1/2})^j = 0$$

for all $j \in \mathbb{Z}$. Considering the associated generating series, this implies

$$\frac{A}{1-q^N z} + \frac{B}{1+q^M z} + \frac{C}{1-q^{L+1/2} z} + \frac{D}{1+q^{K+1/2} z} = 0.$$

Considering the poles of this rational fraction, we obtain $A = B = C = D = 0$. □

Proposition 2. *Suppose Case 2 holds. Then the natural action of* G^I *on* $\mathbb{C}^2$ *is irreducible.*

Proof. Suppose to the contrary that the action of G^I is reducible and consider an invariant line $L \subset \mathbb{C}^2$. In particular, L is invariant under the action of the connection component. Consequently, the line $\breve{P}(z)L$ does not depend on $z \in \Omega$. However, this is impossible by Lemma 2. (The cases $L = \mathbb{C}\hat{e}_1$ or $\mathbb{C}\hat{e}_2$ are excluded by direct calculation; for the remaining cases consider the ratio of the coordinates of a generator of L and apply Lemma 2.) □

Theorem 3. *If Case 2 holds then we have this dichotomy*:

$$\text{If } abq/c \notin q^{\mathbb{Z}} \text{ then } G = \mathrm{Gl}_2(\mathbb{C})\text{; otherwise } G = \overline{\langle \mathrm{Sl}_2(\mathbb{C}), \sqrt{w}I, e^{\pi i\gamma}I\rangle}.$$

Proof. The proof follows the same lines as that of Theorem 2. □

The remaining subcases are $b \in -aq^{\mathbb{Z}}$ and $c \in -q^{\mathbb{Z}}$; $b \in -aq^{\mathbb{Z}+1/2}$ and $c \in -q^{\mathbb{Z}+1/2}$; and $b \in aq^{\mathbb{Z}+1/2}$ and $c \in q^{\mathbb{Z}+1/2}$.

Case 3. $a, b, c, a/b, a/c, b/c \notin q^{\mathbb{Z}}$, $b \in -aq^{\mathbb{Z}}$ and $c \in -q^{\mathbb{Z}}$.

We use the notations $b = -aq^{\delta}$ and $c = -q^{\gamma}$ with $\delta = \beta - \alpha$, $\gamma \in \mathbb{Z}$.

The twisted connection matrix takes the form

$$\breve{P}(z) = (1/z)^{-\alpha} \begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)} z^{1-\gamma} \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)} z^{\delta} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)} z^{1+\delta-\gamma} \end{pmatrix} = \left(\frac{1}{z}\right)^{-\alpha}$$

$$\times \begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty} q^{\frac{\gamma(1-\gamma)}{2}} a^{\gamma-1}\frac{\theta_q(-az)}{\theta_q(z)} \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty} q^{\frac{-\delta(\delta-1)}{2}} a^{-\delta}\frac{\theta_q(-az)}{\theta_q(z)} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty} q^{-\frac{(\delta-\gamma+1)(\delta-\gamma)}{2}}(-a)^{\gamma-\delta-1}\frac{\theta_q(az)}{\theta_q(z)} \end{pmatrix}$$

Theorem 4. *Suppose that Case 3 holds. For some*

$$R = \begin{pmatrix} 1 & 1 \\ C & -C \end{pmatrix}, \quad \textit{where } C \in \mathbb{C}^*,$$

we have $G = R\,\mathrm{diag}(\mathbb{C}^*, \mathbb{C}^*)R^{-1} \cup \mathrm{diag}(1, -1)R\,\mathrm{diag}(\mathbb{C}^*, \mathbb{C}^*)R^{-1}$,

Proof. Note there exist two nonzero constants A and B such that, for all $z \in \Omega$,

$$\check{P}(-1/a)^{-1}\check{P}(z) = (-a)^{\alpha}\,\frac{\theta_q(-1/a)}{\theta_q(-1)}\,(1/z)^{-\alpha}\begin{pmatrix} \frac{\theta_q(az)}{\theta_q(z)} & A\frac{\theta_q(-az)}{\theta_q(z)} \\ B\frac{\theta_q(-az)}{\theta_q(z)} & \frac{\theta_q(az)}{\theta_q(z)} \end{pmatrix}$$

$$= (-a)^{\alpha}\,\frac{\theta_q(-1/a)}{\theta_q(-1)}\,(1/z)^{-\alpha}R\begin{pmatrix} \frac{\theta_q(az)}{\theta_q(z)} + \sqrt{BA}\,\frac{\theta_q(-az)}{\theta_q(z)} & 0 \\ 0 & \frac{\theta_q(az)}{\theta_q(z)} - \sqrt{BA}\,\frac{\theta_q(-az)}{\theta_q(z)} \end{pmatrix}R^{-1},$$

where $R = \begin{pmatrix} 1 & 1 \\ \sqrt{B/A} & -\sqrt{B/A} \end{pmatrix}$.

We claim that the functions

$$X(z) := (1/z)^{-\alpha}\Big(\frac{\theta_q(az)}{\theta_q(z)} + \sqrt{BA}\,\frac{\theta_q(-az)}{\theta_q(z)}\Big),$$
$$Y(z) := (1/z)^{-\alpha}\Big(\frac{\theta_q(az)}{\theta_q(z)} - \sqrt{BA}\,\frac{\theta_q(-az)}{\theta_q(z)}\Big)$$

do not satisfy any nontrivial relation of the form $X^rY^s = 1$ with $(r, s) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Indeed, suppose to the contrary that such a relation holds. Then

$$\frac{((1/z)^{-\alpha}(\theta_q(az) + \sqrt{BA}\theta_q(-az)))^r}{((1/z)^{-\alpha}(\theta_q(az) - \sqrt{BA}\theta_q(-az)))^s} = \theta_q(z)^{s-r}.$$

Let us first exclude the case $r \neq s$. If $s > r$, then $\theta_q(az) + \sqrt{BA}\theta_q(-az)$ must vanish on $q^{\mathbb{Z}}$. In particular, $\theta_q(a)+\sqrt{BA}\theta_q(-a)=0$ and $\theta_q(aq)+\sqrt{BA}\theta_q(-aq) = -(az)^{-1}(\theta_q(az) - \sqrt{BA}\theta_q(-az)) = 0$, and so $\theta_q(a) = 0$, that is, $a \in q^{\mathbb{Z}}$. This is a contradiction. The case $r > s$ is similar by symmetry. Hence we have $r = s$, and so

$$\Big(\frac{\theta_q(az) + \sqrt{BA}\theta_q(-az)}{\theta_q(az) - \sqrt{BA}\theta_q(-az)}\Big)^r = 1.$$

Since $r \neq 0$, the function in parentheses is constant. This is clearly impossible and our claim is proved.

This ensures that the connection component of G^I, generated by the matrices $\check{P}(-1/a)^{-1}\check{P}(z)$ for $z \in \Omega$, is equal to $R\,\mathrm{diag}(\mathbb{C}^*, \mathbb{C}^*)R^{-1}$. Thus G is generated as an algebraic group by $R\,\mathrm{diag}(\mathbb{C}^*, \mathbb{C}^*)R^{-1}$, $\mathrm{diag}(1, -1)$,

$$\check{P}(-1/a)^{-1}\,\mathrm{diag}(u, -u)\check{P}(-1/a) = \mathrm{diag}(u, -u),$$
$$\check{P}(-1/a)^{-1}\,\mathrm{diag}(e^{2\pi i\alpha}, e^{2\pi i\alpha})\check{P}(-1/a) = \mathrm{diag}(e^{2\pi i\alpha}, e^{2\pi i\alpha}). \qquad \square$$

The case $b \in -aq^{\mathbb{Z}+1/2}$ and $c \in -q^{\mathbb{Z}+1/2}$ and the case $b \in aq^{\mathbb{Z}+1/2}$ and $c \in q^{\mathbb{Z}+1/2}$ are similar.

Case 4. $a \in q^{\mathbb{N}^*}$.

In this case, the twisted connection matrix is lower triangular:

$$\check{P}(z) = \begin{pmatrix} \dfrac{(b, c/a; q)_\infty}{(c, b/a; q)_\infty}(-1)^\alpha q^{-\alpha(\alpha-1)/2} & 0 \\ \dfrac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty}\dfrac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta} & \dfrac{(aq/c, q/b; q)_\infty}{(q^2/c, a/b; q)_\infty}\dfrac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma} \end{pmatrix}$$

Let

$$G_{\mathrm{I}} = \begin{pmatrix} 1 & 0 \\ \mathbb{C} & \mathbb{C}^* \end{pmatrix}, \quad G_{\mathrm{II}} = \begin{pmatrix} 1 & 0 \\ \mathbb{C} & \langle w, e^{2\pi i\gamma}\rangle \end{pmatrix}, \quad G_{\mathrm{III}} = \begin{pmatrix} 1 & 0 \\ 0 & \langle w, e^{2\pi i\gamma}\rangle \end{pmatrix}.$$

Theorem 5. *Suppose Case 4 holds. We have the following trichotomy* :

(I) *if* $b/c \notin q^{\mathbb{Z}}$ *then* $G = G_{\mathrm{I}}$;

(II) *if* $c/b \in q^{\mathbb{N}^*}$ *then* $G = G_{\mathrm{II}}$;

(III) *if* $bq/c \in q^{\mathbb{N}^*}$ *then* $G = G_{\mathrm{III}}$.

Proof. In each case for all $z \in \Omega$, we have

$$\check{P}(1/b)^{-1}\check{P}(z) = \begin{pmatrix} 1 & 0 \\ X_1 & Y_1 \end{pmatrix},$$

where

$$X_1 = A\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta} \quad \text{and} \quad Y_1 = B\frac{\theta_q((bq/c)z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma},$$

for some constants A, B with $B \neq 0$. Hence the connection component is a subgroup of G_{I}.

Now assume case (I), that is, $b/c \notin q^{\mathbb{Z}}$. Then $A \neq 0$ and we claim that the connection component is equal to G_1. Indeed, for all $n \in \mathbb{Z}$, the matrix

$$(\check{P}(1/b)^{-1}\check{P}(z))^n = \begin{pmatrix} 1 & 0 \\ X_1\dfrac{1-Y_1^n}{1-Y_1} & Y_1^n \end{pmatrix}$$

belongs to the connection component. Consider a polynomial in two variables $K(X, Y) \in \mathbb{C}[X, Y]$ such that $K(X_1(1-Y_1^n)/(1-Y_1), Y_1^n) = 0$. If K were nonzero then we could assume $K(X,0)\neq 0$. But, for all $z\in\Omega$ in a neighborhood of $c/(bq)$, we have $|Y_1|<1$; hence letting n tend to $+\infty$, we would get $K(X_1/(1-Y_1),0)=0$, which would imply $K(X,0)=0$. This proves that $K=0$. In other words the only algebraic subvariety of $\mathbb{C}\times\mathbb{C}^*$ containing $(X_1(1-Y_1^n)/(1-Y_1),\,Y_1^n)$ for all $n\in\mathbb{Z}$ is $\mathbb{C}\times\mathbb{C}^*$ itself. In particular, the algebraic group generated by the matrix $(\breve{P}(1/b)^{-1}\breve{P}(z))^n$ for all $n\in\mathbb{Z}$ is G_{I}, hence the connection component is equal to G_{I}. It is now straightforward that $G=G_{\mathrm{I}}$.

Now assume for case (II) that $c/b\in q^{\mathbb{N}^*}$. Then Y_1 is constant in z. Hence

$$\breve{P}(1/b)^{-1}\breve{P}(z)=\begin{pmatrix}1&0\\X_1&1\end{pmatrix}$$

with $A\neq 0$ (in X_1). The connection component is equal to $\left(\begin{smallmatrix}1&0\\\mathbb{C}&1\end{smallmatrix}\right)$ and the whole Galois group G is equal to G_{II}.

Finally for case (II), suppose $bq/c\in q^{\mathbb{N}^*}$. Then $A=0$ and the Y_1 is constant, hence $G=G_{\mathrm{III}}$. □

Case 5. $a\in q^{-\mathbb{N}}$.

In this case, the twisted connection matrix is upper triangular:

$$\breve{P}(z)=\begin{pmatrix}\dfrac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}(-1)^\alpha q^{-\frac{\alpha(\alpha-1)}{2}} & \dfrac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\dfrac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)}(1/z)^{-\alpha}z^{1-\gamma}\\ 0 & \dfrac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\dfrac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}\end{pmatrix}.$$

For any set of matrices G, denote by ${}^tG$ the set of transposed elements of G.

Theorem 6. *Suppose that Case 5 holds. We have the following trichotomy*:

(I) *if* $b/c\notin q^{\mathbb{Z}}$ *then* $G={}^tG_{\mathrm{I}}$;

(II) *if* $bq/c\in q^{\mathbb{N}^*}$ *then* $G={}^tG_{\mathrm{II}}$;

(III) *if* $c/b\in q^{\mathbb{N}^*}$ *then* $G={}^tG_{\mathrm{III}}$.

Proof. We argue as in Theorem 5. □

The case $b\in q^{\mathbb{Z}}$ or $a/c\in q^{\mathbb{Z}}$ or $b/c\in q^{\mathbb{Z}}$ is similar to the case $a\in q^{\mathbb{Z}}$. We omit the details.

4. Galois groups of the basic hypergeometric equations: logarithmic cases

We write $a=uq^\alpha,\ b=vq^\beta$, and $c=wq^\gamma$ with $u,v,w\in\mathbb{U}$ and $\alpha,\beta,\gamma\in\mathbb{R}$ (we choose a logarithm of q).

4.1. ***The first logarithmic case.*** Here we compute the Galois group of the basic hypergeometric system (2) under the assumption that $c=q$ and $a/b\notin q^{\mathbb{Z}}$.

Local fundamental system of solutions at 0. We have

$$A(a,b;q;0)=\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}^{-1}.$$

Consequently, we are in the nonresonant logarithmic case at 0. We consider this situation as a degenerate case as c tends to q, but $c\neq q$.

More precisely, we consider the limit as c tends to q, with $c\neq q$, of the matrix-valued function

$$F^{(0)}(a,b;c;z)\begin{pmatrix}1&1\\1&q/c\end{pmatrix}^{-1}\begin{pmatrix}1&0\\1&1\end{pmatrix}$$
$$=\frac{-c}{c-q}\begin{pmatrix}\left(\frac{q}{c}-1\right){}_2\phi_1(a,b;c;z) & {}_2\phi_1\left(\frac{aq}{c},\frac{bq}{c};\frac{q^2}{c};z\right)-{}_2\phi_1(a,b;c;z)\\ \left(\frac{q}{c}-1\right){}_2\phi_1(a,b;c;qz) & \frac{q}{c}\,{}_2\phi_1\left(\frac{aq}{c},\frac{bq}{c};\frac{q^2}{c};qz\right)-{}_2\phi_1(a,b;c;qz)\end{pmatrix}$$

Using the notations

$$\zeta(a,b;z)=\left.\frac{d}{dc}\right|_{c=q}{}_2\phi_1(a,b;c;z)\text{ and }\xi(a,b;z)=\left.\frac{d}{dc}\right|_{c=q}{}_2\phi_1(aq/c,bq/c;q^2/c;z),$$

the above limit is equal to

$$\begin{pmatrix}{}_2\phi_1(a,b;q;z) & -q(\xi(a,b;z)-\zeta(a,b;z))\\ {}_2\phi_1(a,b;q;qz) & {}_2\phi_1(a,b;q;qz)-q(\xi(a,b;qz)-\zeta(a,b;qz))\end{pmatrix},$$

a matrix we denote by $F^{(0)}(a,b;q;z)$. From (4) we deduce that this satisfies $F^{(0)}(a,b;q;qz)J^{(0)}(q)=A(a,b;c;z)F^{(0)}(a,b;q;z)$, where $J^{(0)}(q)=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. As this matrix is invertible as a matrix in the field of meromorphic functions, the matrix-valued function $Y^{(0)}(a,b;q;z)=F^{(0)}(a,b;q;z)e^{(0)}(J^{(0)}(q))(z)$ is a fundamental system of solutions of the basic hypergeometric equation with $c=q$. Recall that

$$e^{(0)}(J^{(0)}(q))(z)=\begin{pmatrix}1&\ell_q(z)\\0&1\end{pmatrix}.$$

Generators of the local Galois group at 0. There is one generator, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

Local fundamental system of solutions at ∞. The situation is as in Section 3. Hence we are in the nonresonant and nonlogarithmic case at ∞, and a fundamental system of solutions at ∞ of (2) (see Section 2.1) is given by $Y^{(\infty)}(a,b;q;z)=F^{(\infty)}(a,b;q;z)e^{(\infty)}(J^{(\infty)}(a,b))(z)$ with $J^{(\infty)}(a,b)=\mathrm{diag}(1/a,1/b)$ and

$$F^{(\infty)}(a,b;q;z)=\begin{pmatrix}{}_2\phi_1(a,a;aq/b;\frac{q^2}{ab}z^{-1}) & {}_2\phi_1(b,b;bq/a;\frac{q^2}{ab}z^{-1})\\ \frac{1}{a}\,{}_2\phi_1(a,a;aq/b;\frac{q}{ab}z^{-1}) & \frac{1}{b}\,{}_2\phi_1(b,b;bq/a;\frac{q}{ab}z^{-1})\end{pmatrix}.$$

Generators of the local Galois group at ∞. We have two generators

$$\breve{P}(y_0)^{-1}\begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\beta}\end{pmatrix}\breve{P}(y_0) \quad \text{and} \quad \breve{P}(y_0)^{-1}\begin{pmatrix} u & 0 \\ 0 & v\end{pmatrix}\breve{P}(y_0).$$

Connection matrix. The connection matrix is the limit as c tends to q of

$$(e^{(\infty)}(J^{(\infty)}(a,b))(z))^{-1}M(z)\begin{pmatrix} 1 & 1 \\ 1 & q/c\end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}e^{(0)}(J^{(0)}(q))(z),$$

where $M(z)$ is as in (5). This is equal to

$$P(z) := (e^{(\infty)}(J^{(\infty)}(a,b))(z))^{-1}M_2(z)e^{(0)}(J^{(0)}(q))(z),$$

where $M_2(z)$ equals

$$\begin{pmatrix} u(a,b;q)\dfrac{\theta_q(az)}{\theta_q(z)} & q(u_c(a,b;q)-v_c(a,b;q))\dfrac{\theta_q(az)}{\theta_q(z)}+azv(a,b;q)\dfrac{\theta_q'(az)}{\theta_q(z)} \\ w(a,b;q)\dfrac{\theta_q(bz)}{\theta_q(z)} & q(w_c(a,b;q)-y_c(a,b;q))\dfrac{\theta_q(bz)}{\theta_q(z)}+bzy(a,b;q)\dfrac{\theta_q'(bz)}{\theta_q(z)}\end{pmatrix}$$

and

$$u(a,b;c)=\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}; \qquad v(a,b;c)=\frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty};$$
$$w(a,b;c)=\frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}; \qquad y(a,b;c)=\frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty},$$

and where the subscript c means that we take the derivative with respect to the third variable.

Twisted connection matrix. For this we have

$$\breve{P}(z)=\begin{pmatrix} (1/z)^{-\alpha} & 0 \\ 0 & (1/z)^{-\beta}\end{pmatrix}M_2(z)\begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1\end{pmatrix}.$$

We need to consider different cases.

Case 6. $a \notin q^{\mathbb{Z}}$ and $b \notin q^{\mathbb{Z}}$.

Subject to this condition, the complex numbers $u(a,b;q)$, $v(a,b;q)$, $w(a,b;q)$, and $y(a,b;q)$ are nonzero.

Proposition 3. *If Case 6 holds then the natural action of G^I on $\mathbb{C}^2$ is irreducible.*

Proof. Assume to the contrary that the action of G^I is reducible and let L be an invariant line.

First note $L \neq \mathbb{C}\hat{e}_1$ (in particular, G^I does not consist of scalar matrices). Indeed, if not, $\mathbb{C}\hat{e}_1$ would be stabilized by the connection component, and the line spanned by $\breve{P}(z)\hat{e}_1$ would be independent of $z \in \Omega$, but this is clearly false.

Because the group G^I is normalized by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ (since G^I is a normal subgroup of G), the lines $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^n L$ are also invariant by the action of G^I. These lines being distinct (since $L \neq \mathbb{C}\hat{e}_1$), we conclude that G^I consists of scalar matrices and we get a contradiction. $\square$

As a consequence we have the following theorem.

Theorem 7. *Suppose Case 6 holds. Then we have the following dichotomy*:

$$\text{if } ab \notin q^{\mathbb{Z}} \text{ then } G = \mathrm{Gl}_2(\mathbb{C}); \quad \text{otherwise } G = \mathrm{Sl}_2(\mathbb{C}).$$

Proof. Using the irreducibility of the natural action of G^I and arguing as in the proof of Theorem 2, we obtain the equality $G^{I,\mathrm{der}} = \mathrm{Sl}_2(\mathbb{C})$. From (6) we deduce that the determinant of the twisted connection matrices when $c = q$ is equal to the limit as c tends to q of

$$\frac{-1}{1/a - 1/b}(1/z)^{-(\alpha+\beta)} z^{1-\gamma} \frac{\theta_q(\frac{abq}{c} z)}{\theta_q(z)}.$$

If $ab \notin q^{\mathbb{Z}}$, then this determinant is a nonconstant holomorphic function and consequently $G = \mathrm{Gl}_2(\mathbb{C})$.

If $ab \in q^{\mathbb{Z}}$, then this determinant does not depend on z. This implies that the connection component of the Galois group is a subgroup of $\mathrm{Sl}_2(\mathbb{C})$. Furthermore, $ab \in q^{\mathbb{Z}}$ entails that $uv = 1$ and $\alpha + \beta \in \mathbb{Z}$, that is, $e^{2\pi i(\alpha+\beta)} = 1$. Consequently, the local Galois groups are subgroups of $\mathrm{Sl}_2(\mathbb{C})$ and the global Galois group G is therefore a subgroup of $\mathrm{Sl}_2(\mathbb{C})$. $\square$

Case 7. $b \in q^{\mathbb{N}^*}$.

In this case, the twisted connection matrix simplifies to

$$\breve{P}(z) = \begin{pmatrix} u(a, b; q)\dfrac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & p_{12} \\ 0 & p_{22} \end{pmatrix} \begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1 \end{pmatrix},$$

where

$$p_{12} = q(u_c(a, b; q) - v_c(a, b; q))\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} + azv(a, b; q)\frac{\theta_q'(az)}{\theta_q(z)}(1/z)^{-\alpha},$$
$$p_{22} = q(w_c(a, b; q) - y_c(a, b; q))(-1)^{\beta} q^{-\beta(\beta-1)/2}.$$

Theorem 8. *Suppose Case 7 holds. Then*

$$G = \begin{pmatrix} \mathbb{C}^* & \mathbb{C} \\ 0 & 1 \end{pmatrix}.$$

Proof. Fix a point $y_0 \in \Omega$ such that $\breve{P}(y_0)$ is of the form $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ with $A, C \neq 0$. There exists a constant $D \in \mathbb{C}^*$ such that

$$\breve{P}(y_0)^{-1}\breve{P}(z) = \begin{pmatrix} D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & * \\ 0 & 1 \end{pmatrix}.$$

Since G^I is normalized by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (remember that G^I is a normal subgroup of G), it contains, for all $n \in \mathbb{Z}$, the matrix

$$\begin{pmatrix} D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & * + n\big(D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} - 1\big) \\ 0 & 1 \end{pmatrix}.$$

Because $a \notin q^{\mathbb{Z}}$, the function $D(\theta_q(az)/\theta_q(z))(1/z)^{-\alpha} - 1$ is not identically equal to zero over $\mathbb{C}^*$ and therefore G^I contains, for all $z \in \Omega$,

$$\begin{pmatrix} D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & \mathbb{C} \\ 0 & 1 \end{pmatrix}.$$

The element whose upper right entry is zero belongs to G^I, so that diag($\mathbb{C}^*$, 1) is a subgroup of G^I and $\left(\begin{smallmatrix} \mathbb{C}^* & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right) \subset G$. The converse inclusion is clear. □

Case 8. $b \in q^{-\mathbb{N}}$.

Using the identity $bz\theta_q'(bz)/\theta_q(z) = (-\beta - \ell_q(z))(-1)^\beta q^{-\beta(\beta-1)/2}$, we see that in this case the twisted connection matrix $\breve{P}(z)$ equals

$$\begin{pmatrix} 0 & q(u_c(a,b;q) - v_c(a,b;q))\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} \\ w(a,b,q)(-1)^\beta q^{-\frac{\beta(\beta-1)}{2}} & q(w_c(a,b;q) - y_c(a,b;q) - \beta/q)(-1)^\beta q^{-\frac{\beta(\beta-1)}{2}} \end{pmatrix}$$

Theorem 9. *Suppose Case 8 holds. Then*

$$G = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & \mathbb{C}^* \end{pmatrix}.$$

Proof. Fix a base point $y_0 \in \Omega$. There exist three constants $C, C', C'' \in \mathbb{C}$ with $C \neq 0$, such that

$$\breve{P}(y_0)^{-1}\breve{P}(z) = \begin{pmatrix} 1 & C'\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} + C'' \\ 0 & C\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} \end{pmatrix}$$

for all $z \in \Omega$. The rest is similar to the proof of Theorem 8. □

The remaining case $a \in q^{\mathbb{Z}}$ is similar to Case 8.

The case $a = b$ and $c \notin q^{\mathbb{Z}}$ is similar to the case treated in this section.

4.2. ***The second logarithmic case.*** Here we compute the Galois group of the basic hypergeometric system (2) under the assumption that $a = b$ and $c = q$.

Local fundamental system of solutions at 0. The situation is the same as in the case $c = q$ and $a/b \notin q^{\mathbb{Z}}$.

Generator of the local Galois group at 0. The generator is $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$.

Local fundamental system of solutions at ∞. We have

$$A(a, a; q; z) = \begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix} \begin{pmatrix} 1/a & 1 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix}^{-1}.$$

Thus, we are in the nonresonant logarithmic case at ∞. We consider the case $a = b$ and $c = q$ as a degenerate case of the situation $c = q$ as a tends to b for $a/b \neq 1$.

We consider the matrix-valued function

$$F^{(\infty)}(a, b; q; z) \begin{pmatrix} 1 & 1 \\ 1/a & 1/b \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix}.$$

A straightforward but tedious calculation, which we omit, shows that this matrix-valued function does admit a limit $F^{(\infty)}(a, a; q; z)$ as a tends to b. A fundamental system of solutions at ∞ of (2) (see Section 2.1) is given by $Y^{(\infty)}(a, a; q; z) = F^{(\infty)}(a, a; q; z)e^{(\infty)}(J^{(\infty)}(a, a))(z)$ with $J^{(\infty)}(a, a) = \operatorname{diag}(1/a, 1/a)$.

Generator of the local Galois group at ∞. We have the generators

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \qquad \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}, \qquad \check{P}(y_0)^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \check{P}(y_0).$$

Birkhoff matrix. This matrix is $(e^{(\infty)}(J^{(\infty)}(a, a))(z))^{-1} Q e^{(0)}(J^{(0)}(q))(z)$, where Q is the limit as a tends to b of

$$\begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1/a & 1/b \end{pmatrix} M_2(z).$$

It has the form

$$Q := \begin{pmatrix} C\frac{\theta_q(az)}{\theta_q(z)} + az\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q'(az)}{\theta_q(z)} & * \\ -(1/a)\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q(az)}{\theta_q(z)} & C'\frac{\theta_q(az)}{\theta_q(z)} - z\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q'(az)}{\theta_q(z)} \end{pmatrix},$$

where $*$ denotes some meromorphic function.

The twisted Birkhoff matrix. This matrix is of the form

$$(1/z)^{-\alpha} \begin{pmatrix} 1 & -a\ell_q(z) \\ 0 & 1 \end{pmatrix} Q \begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1 \end{pmatrix}.$$

We need to consider different cases.

Case 9. $a \notin q^{\mathbb{Z}}$.

Proposition 4. *Suppose Case 9 holds. Then the natural action of G^I on $\mathbb{C}^2$ is irreducible.*

Proof. Note $\mathbb{C}\hat{e}_1$ is not an invariant line. Indeed, if not, it would be invariant by the action of the connection component, and hence the line spanned by $\breve{P}(z)\hat{e}_1$ would be independent of $z \in \Omega$. Considering the ratio of the coordinates of this line, this would imply the existence of some constant $A \in \mathbb{C}$ making the functional equation

$$C\frac{\theta_q(az)}{\theta_q(z)} + az\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q'(az)}{\theta_q(z)} + \frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q(az)}{\theta_q(z)}\ell_q(z) = A\frac{\theta_q(az)}{\theta_q(z)}.$$

true on $\mathbb{C}^*$. The fact that $\theta_q(az)$ vanishes identically to first order at $z = 1/a$, yields a contradiction.

The rest is similar to the proof of Proposition 3. □

Theorem 10. *If Case 9 holds then we have the dichotomy*

$$\textit{if } a^2 \notin q^{\mathbb{Z}} \textit{ then } G = \mathrm{Gl}_2(\mathbb{C}); \quad \textit{otherwise } G = \mathrm{Sl}_2(\mathbb{C}).$$

Proof. The proof follows the same reasoning as that of Theorem 2. □

Case 10. $a \in q^{\mathbb{Z}}$.

In this case, the connection matrix simplifies, for some constants $C, C' \in \mathbb{C}$, to

$$\begin{pmatrix} C & * \\ 0 & C' \end{pmatrix}.$$

Theorem 11. *Suppose Case 10 holds. Then*

$$G = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}.$$

Proof. The local Galois group at 0 is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, hence G contains $\left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$.

Since the twisted connection matrix is upper triangular with constant diagonal entries, the connection component is a subgroup of $\left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$. The generators of the local Galois group at 0 and at ∞ also lie in $\left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$. Therefore G is a subgroup of $\left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$. □

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