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The notion of a Γ-symmetric space is a generalization of the classical notion of a symmetric space, where a general finite abelian group Γ replaces the group $\mathbb{Z}_2$. We approach the classification of Γ-symmetric space in the case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ using recent results on the classification of complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded simple Lie algebras.

The notion of a Γ-symmetric space, introduced by R. Lutz [1981], is a generalization of the classical notion of a symmetric space, where a general finite abelian group Γ replaces the group $\mathbb{Z}_2$. The case $\Gamma = \mathbb{Z}_k$ has also been studied, from the algebraic point of view by V. Kac [1968] and from the point of view of differential geometry by Ledger and Obata [1968], by Kowalski [1980] or by Wolf and Gray [1968] in terms of k-symmetric spaces. In this case, a k-manifold is a homogeneous reductive space and the classification of these varieties is given by the corresponding classification of graded Lie algebras. We approach the classification of Γ-symmetric spaces in the case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ in a similar way, using recent results (see [Bahturin et al. 2005]) on the classification of complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded simple Lie algebras.

1. Introduction

A *symmetric space* is a homogeneous space $M = G/H$, where G is a connected Lie group with an involutive automorphism σ and H is a closed subgroup that lies between the subgroup of all fixed points of σ and its identity component. This automorphism σ induces an involutive *diffeomorphism* σ_0 of M such that $\sigma_0(\pi(x)) = \pi(\sigma(x))$ for every $x \in G$, where $\pi : G \to G/H$ is the canonical projection. It also induces an automorphism γ on the Lie algebra $\mathfrak{g}$ of G. This automorphism satisfies $\gamma^2 = \mathrm{Id}$; hence it is diagonalizable, and the Lie algebra $\mathfrak{g}$ of G admits a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are the eigenspaces of σ corresponding to the eigenvalues 1 and -1. Conversely, every $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ on a Lie algebra $\mathfrak{g}$ makes it into a *symmetric Lie algebra*, that is, a triple $(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g})$, where γ is an involutive automorphism of $\mathfrak{g}$ such that $\gamma(X) = X$ if and only if $X \in \mathfrak{g}_0$ and $\gamma(X) = -X$ for all $X \in \mathfrak{g}_1$. If G is a connected simply

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connected Lie group with Lie algebra $\mathfrak{g}$, then γ induces an automorphism σ of G, and for any subgroup H lying between $G^\sigma = \{x \in G \mid \sigma(x) = x\}$ and its identity component, $(G/H, \sigma)$ is a symmetric space. In the Riemannian case, H is compact and $\mathfrak{g}$ admits an orthogonal symmetric decomposition, that is, the Lie group of linear transformations of $\mathfrak{g}$ generated by $\mathrm{ad}_{\mathfrak{g}} H$ is compact. As a result, the study is reduced to the effective irreducible case, and $\mathfrak{g}$ is semisimple.

In this paper we will look at more general Γ-symmetric homogeneous spaces. They were first introduced by R. Lutz [1981] and by A. Tsagas at a workshop in Bucharest. We propose here to develop the corresponding algebraic structures and to give — using the results on complex simple Lie algebras graded by finite abelian groups [Bahturin et al. 2005; Bahturin and Zaicev 2007; 2006] (see also [Havlíček et al. 1998; 2000]) — the algebraic classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces G/H whose associated Lie algebra $\mathfrak{g}$ is simple of classical type.

2. Group graded Lie algebras

Definition 1. Let P be a group with identity element 1. A Lie algebra $\mathfrak{g}$ over a field F is graded by P if $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p$ $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{pq}$ for all $p, q \in P$.

Definition 2. Given two P-gradings $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p$ and $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}'_p$ of an algebra $\mathfrak{g}$ by a group P, we call them *equivalent* if there exists an automorphism θ of $\mathfrak{g}$ such that $\mathfrak{g}'_p = \theta(\mathfrak{g}_p)$ for all $p \in P$.

An important subset of the grading group is defined as follows.

Definition 3. Given a grading as above, the set $\mathrm{Supp}\, \mathfrak{g} = \{p \in P \mid \mathfrak{g}_p \neq \{0\}\}$ is called the *support* of the grading.

It has been established in [Patera and Zassenhaus 1989] (see also [Bahturin and Zaicev 2006]) that if $\mathfrak{g}$ is complex simple then any two elements in the support of the grading commute. So one can always restrict oneself to the case of abelian groups. In this paper we restrict ourselves to finite abelian grading groups P.

2.1. ***Action of the dual group.*** Let $\Gamma = \widehat{P}$ be the dual group associated to P, that is, the group of characters $\gamma : P \to \mathbb{C}^\star$ of P. If we assume that a Lie algebra $\mathfrak{g}$ is P-graded, then we obtain a natural action of Γ by linear transformations on $\mathfrak{g} \otimes \mathbb{C}$ if, for any homogeneous elements $X \in \mathfrak{g}_p$, we set $\gamma(X) = \gamma(p)X$. Since for $X \in \mathfrak{g}_p$ and $Y \in \mathfrak{g}_q$ we have $[X, Y] \in \mathfrak{g}_{pq}$, it follows that

$$\gamma([X, Y]) = \gamma(pq)[X, Y] = [\gamma(p)X, \gamma(q)Y] = [\gamma(X), \gamma(Y)], \tag{1}$$

that is, Γ acts by Lie automorphisms on $\mathfrak{g}$. In this case, there is a canonical homomorphism

$$\alpha : \Gamma \to \mathrm{Aut}(\mathfrak{g} \otimes \mathbb{C}) \quad \text{given by } \alpha(\gamma)(X) = \gamma(X). \tag{2}$$

If $p^2 = 1$ for any $p \in \operatorname{Supp} \mathfrak{g}$, then the action is defined even on $\mathfrak{g}$ itself, and the above homomorphism maps Γ onto a subgroup of $\operatorname{Aut} \mathfrak{g}$.

Conversely, suppose there is a homomorphism $\alpha : \Gamma \to \operatorname{Aut} \mathfrak{g}$ for a finite abelian group Γ. Then Γ acts on $\mathfrak{g}$, and hence on $\mathfrak{g} \otimes \mathbb{C}$ by automorphisms, if one sets $\gamma(X) = \alpha(\gamma)(X)$. This action extends to $\mathfrak{g} \otimes \mathbb{C}$ and yields a P-grading $\Gamma = \widehat{P}$ of $\mathfrak{g} \otimes \mathbb{C}$ by subspaces $(\mathfrak{g} \otimes \mathbb{C})_p$ for each $p \in P$, defined as

$$(\mathfrak{g} \otimes \mathbb{C})_p = \{X \mid \gamma(X) = \gamma(p)X\}.$$

That $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{pq}$ easily follows from (1). Now the vector space decomposition $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p$ is just a standard weight decomposition under the action of an abelian semisimple group of linear transformations over an algebraically closed field. Again, if $\alpha(\gamma)^2 = 1$ for any $\gamma \in \Gamma$, then we have a P-grading on $\mathfrak{g}$ itself.

Explicitly, let us assume that $\mathfrak{g}$ is a complex Lie algebra and that K is a finite abelian subgroup of $\operatorname{Aut}(\mathfrak{g})$. One can write K as $K = K_1 \times \cdots \times K_p$, where K_i is a cyclic group of order r_i. Let κ_i be a generator of K_i. The automorphisms κ_i satisfy $\kappa_i^{r_i} = \mathrm{Id}$ and $\kappa_i \circ \kappa_j = \kappa_j \circ \kappa_i$ for all $i, j = 1, \dots, p$. These automorphisms are simultaneously diagonalizable. If ξ_i is a primitive root of order r_i of the unity, then the eigenspaces

$$\mathfrak{g}_{s_1,\dots,s_p} = \{X \in \mathfrak{g} \mid \kappa_i(X) = \xi_i^{s_i} X \text{ for } i = 1, \dots, p\}$$

give the following grading of $\mathfrak{g}$ by $P = \mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_p}$:

$$\mathfrak{g} = \bigoplus_{(s_1,\dots,s_p) \in \mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_p}} \mathfrak{g}_{s_1,\dots,s_p}.$$

We can summarize some of what was said above as follows.

Proposition 4. *Let P be a finite abelian group, and let $\Gamma = \widehat{P}$ be the group of complex characters of P.*

(a) *A complex Lie algebra $\mathfrak{g}$ is P-graded if and only if the dual group Γ maps homomorphically onto a finite abelian subgroup of* $\operatorname{Aut}(\mathfrak{g})$ *by the canonical homomorphism α described in* (2).

(b) *A real Lie algebra $\mathfrak{g}$ is P-graded, with $p^2 = 1$ for each $p \in \operatorname{Supp} \mathfrak{g}$, if and only if there is a homomorphism $\alpha : \Gamma \to \operatorname{Aut}(\mathfrak{g})$ such that $\alpha(\gamma)^2 = \mathrm{id}_{\mathfrak{g}}$ for any $\gamma \in \Gamma$.*

(c) *In both cases above,* $\operatorname{Supp} \mathfrak{g}$ *generates P if and only if the canonical mapping α has trivial kernel, that is, if Γ is isomorphic to a* (*finite abelian*) *subgroup of* $\operatorname{Aut}(\mathfrak{g})$.

Proof. We need only to comment on (c). If $\Lambda \subset \Gamma$, then we denote by $\Lambda^\perp$ the set of all $p \in P$ such that $\lambda(p) = 1$ for all $\lambda \in \Lambda$. Similarly we define $Q^\perp$ for any $Q \subset P$. The sets $\Lambda^\perp$ and $Q^\perp$ are always subgroups in P and Γ, respectively. If Λ and Q are subgroups then $|\Lambda| \cdot |\Lambda^\perp| = |Q| \cdot |Q^\perp| = |\Gamma| = |P|$. We claim that if $\Lambda = \operatorname{Ker} \alpha$, then the subgroup generated by $\operatorname{Supp} \mathfrak{g}$ is $P = \Lambda^\perp$. This follows because for any $p \notin Q$ there is a $\lambda \in \Lambda$ such that $\lambda(p) \neq 1$. If $\mathfrak{g}_p \neq \{0\}$ and $0 \neq X \in \mathfrak{g}_p$, then $\lambda(X) = \lambda(p)X \neq X$ and $\lambda \notin \operatorname{Ker} \alpha$. Conversely, let $\operatorname{Supp} \mathfrak{g} \subset T$, where T is a proper subgroup of $Q = \operatorname{Ker} \alpha^\perp$. Then $T^\perp$ properly contains Λ and for any $\gamma \in T^\perp \setminus \Lambda$ and any $X \in \mathfrak{g}_t$ for $t \in T$, we have $\gamma(X) = \gamma(t)X = X$. Since all such X span $\mathfrak{g}$, we have $\gamma \in \operatorname{Ker} \alpha = \Lambda$, a contradiction. Thus $\operatorname{Supp} \mathfrak{g}$ must generate the whole of Q, proving (c). □

2.2. *Examples.* 1. The gradings of classical simple complex Lie algebras by finite abelian groups have been described in [Bahturin et al. 2001; Bahturin et al. 2005; Bahturin and Zaicev 2006], and [Draper and Viruel 2007] (see also [Havlíček et al. 1998; 2000]). We will use this classification in the case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ to obtain some classification-type results in the theory of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces.

2. In the nonsimple case, the study of gradings is more complicated. Consider, for example, the nilpotent case. In contrast to the simple case, there is no classification of these Lie algebras, except in dimensions up to 7 [Goze and Remm 2007]. Even then, one has to distinguish between two classes of complex nilpotent Lie algebras:

(a) The noncharacteristically nilpotent Lie algebras. These Lie algebras admit a nontrivial abelian subalgebra of the Lie algebra $\operatorname{Der}(\mathfrak{g})$ of derivations whose elements are semisimple. In this case $\mathfrak{g}$ is graded by the roots.

(b) The characteristically nilpotent Lie algebras. Every derivation is nilpotent, and we do not have root decompositions. Nevertheless, these nilpotent Lie algebras can be graded by groups. For example, the following nilpotent Lie algebra, denoted by $(\mathfrak{n}_7^3)$ in [Goze and Remm 2007] and given by

$$\begin{aligned} &[X_1, X_i] = X_{i+1} \quad \text{for } i = 2, \dots, 6,\\ &[X_2, X_3] = X_5 + X_7,\\ &[X_2, X_4] = X_6,\\ &[X_2, X_5] = X_7 \end{aligned}$$

is characteristically nilpotent and admits the grading $\mathfrak{n}_7^3 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is the nilpotent subalgebra generated by $\{X_2, X_4, X_6\}$ and $\mathfrak{g}_1$ is the $\mathfrak{g}_0$-module generated by $\{X_1, X_3, X_5, X_7\}$. On the other hand, the nilpotent Lie algebras $\mathfrak{n}_7^4$ and $\mathfrak{n}_7^5$ do not admit nontrivial group gradings.

3. Γ-symmetric spaces

Definition 5. Let Γ be a finite abelian group. A homogeneous space $M = G/H$ is said to be Γ-*symmetric* if

(1) the Lie group G is connected;

(2) the group G is effective on G/H (that is, the Lie algebra $\mathfrak{h}$ of H does not contain a nonzero proper ideal of the Lie algebra $\mathfrak{g}$ of G);

(3) there is an injective homomorphism $\rho : \Gamma \to \operatorname{Aut} G$ such that if G^Γ is the closed subgroup of all elements of G fixed by $\rho(\Gamma)$ and $(G^\Gamma)_e$ is the identity component of G^Γ, then $(G^\Gamma)_e \subset H \subset G^\Gamma$.

Obviously, in the case of $\Gamma = \mathbb{Z}_2$ we obtain ordinary symmetric spaces.

We denote by ρ_γ the automorphism $\rho(\gamma)$ for any $\gamma \in \Gamma$. If H is connected, we have

$$\begin{aligned} \rho_{\gamma_1} \circ \rho_{\gamma_2} &= \rho_{\gamma_1\gamma_2}, \\ \rho_\varepsilon &= \mathrm{Id}, \\ \rho_\gamma(g) = g \text{ for all } \gamma \in \Gamma &\text{ if and only if } g \in H, \end{aligned}$$

where ε is the identity element of Γ. If $\Gamma = \mathbb{Z}_2$ then a $\mathbb{Z}_2$-symmetric space is a symmetric space; if $\Gamma = \mathbb{Z}_p$ we find again the p-manifolds in the sense of Ledger and Obata [1968].

3.1. $\widehat{\Gamma}$*-grading of the Lie algebra of G.* Let $M = G/H$ be a Γ-symmetric space. Each automorphism ρ_γ of G for $\gamma \in \Gamma$ induces an automorphism τ_γ of $\mathfrak{g}$ given by $\tau_\gamma = (T\rho_\gamma)_e$, where $(Tf)_x$ is the tangent map of f at the point x.

Lemma 6. *The map* $\tau : \Gamma \to \operatorname{Aut}(\mathfrak{g})$ *given by* $\tau(\gamma) = (T\rho_\gamma)_e$ *is an injective homomorphism.*

Proof. Let γ_1 and γ_2 be in Γ. Then $\rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1\gamma_2}$. It follows that $(T\rho_{\gamma_1})_e \circ (T\rho_{\gamma_2})_e = (T\rho_{\gamma_1}\rho_{\gamma_1})_e = (T\rho(\gamma_1\gamma_2))_e$, that is, $\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2)$. Now let us assume that $\tau(\gamma) = \mathrm{Id}_\mathfrak{g}$. Then $(T\rho_\gamma)_e = \mathrm{Id} = (T\rho_\varepsilon)_e$. But ρ_γ is uniquely determined by the corresponding tangent automorphism of $\mathfrak{g}$. Then $\rho_\gamma = \rho_\varepsilon$ and $\gamma = \varepsilon$. □

From this lemma we derive the following.

Proposition 7. *If* $M = G/H$ *is a* Γ*-symmetric space, then the complex Lie algebra* $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$, *where* $\mathfrak{g}$ *is the Lie algebra of* G, *is* $\widehat{\Gamma}$*-graded, and if* $\Gamma = \mathbb{Z}_2^k$, *then the real Lie algebra* $\mathfrak{g}$ *of* G *is* $\widehat{\Gamma}$*-graded. The subgroup of* $\widehat{\Gamma}$ *generated by the support of the grading is* $\widehat{\Gamma}$ *itself.*

Proof. Indeed, by Lemma 6, $\alpha : \Gamma \to \operatorname{Aut} \mathfrak{g}$ is an injective homomorphism, and so all our claims follow by Proposition 4. □

For convenience, recall that if a finite abelian group P satisfies $\widehat{\Gamma} = P$ and if α is the canonical homomorphism introduced in (2), then the components of the grading are given by

$$\mathfrak{g}_p = \{X \in \mathfrak{g} \mid \alpha(\gamma)(X) = \gamma(p)X \text{ for all } \gamma \in \Gamma\}. \tag{3}$$

3.2. Γ*-symmetric spaces and graded Lie algebras.* To study Γ-symmetric spaces, we need to start with the study of P-graded Lie algebras where $\Gamma = \widehat{P}$. But in a general case, if G is a connected Lie group corresponding to $\mathfrak{g}$, the P-grading of $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ does not necessarily give a Γ-symmetric space G/H. Some examples are given in [Berger 1957], even in the symmetric case. Still, if G is simply connected, $\mathrm{Aut}(G)$ is a Lie group isomorphic to $\mathrm{Aut}(\mathfrak{g})$.

Proposition 8. *Let $P = \mathbb{Z}_2^k$ with identity element ε, let $\widehat{\Gamma} = P$, and let $\mathfrak{g}$ be a real P-graded Lie algebra such that the subgroup generated by* $\mathrm{Supp}\,\mathfrak{g}$ *equals P and such that the identity component $\mathfrak{h} = \mathfrak{g}_\epsilon$ of the grading does not contain a nonzero ideal of $\mathfrak{g}$. If G is a connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and if H is a Lie subgroup associated with $\mathfrak{h}$, then the homogeneous space $M = G/H$ is a Γ-symmetric space.*

Proof. By Proposition 4, there is an injective homomorphism $\alpha : \Gamma \to \mathrm{Aut}\,\mathfrak{g}$ defined by this grading. The subgroup $\alpha(\Gamma)$ of $\mathrm{Aut}\,\mathfrak{g}$ is isomorphic to Γ. Choosing for each $\alpha(\gamma)$ a unique automorphism $\rho(\gamma)$ of G such that $(T(\rho(\gamma))_e = \tau(\gamma)$, we obtain an injective homomorphism $\rho : \Gamma \to \mathrm{Aut}\,G$, which makes G/H into a Γ-symmetric space. □

Motivated by Propositions 4, 7, and 8, we introduce the following.

Definition 9. Given a real or complex Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a finite abelian subgroup $\Gamma \subset \mathrm{Aut}\,\mathfrak{g}$, we say that $(\mathfrak{g}/\mathfrak{h}, \Gamma)$ is a *local* Γ*-symmetric space* if $\mathfrak{h} = \mathfrak{g}^\Gamma$, the set of all fixed points of $\mathfrak{g}$ under the action of Γ. We call $\mathfrak{h}$ the *isotropy subalgebra* of $(\mathfrak{g}/\mathfrak{h}, \Gamma)$.

Any Γ-symmetric space gives rise to a local Γ-symmetric space, and, in the case of connected simply connected groups, the converse is also true. If $\Gamma = \mathbb{Z}_2^k$, then $(\mathfrak{g}/\mathfrak{h}, \Gamma)$ is a local Γ-symmetric space if and only if $\mathfrak{g}$ is P-graded, where $P = \widehat{\Gamma}$ and the isotropy subalgebra $\mathfrak{h}$ is the identity component of the grading. If Γ is a more general finite abelian group, then the grading by $P = \widehat{\Gamma}$ arises only on the complexification $\mathfrak{g} \otimes \mathbb{C}$, and still $\mathfrak{h} \otimes \mathbb{C}$ is both the set of fixed points of Γ and the identity component of the grading. Again, the study of local complex Γ-symmetric spaces amounts to the study of P-graded Lie algebras, where $P = \widehat{\Gamma}$.

3.3. Γ*-symmetries on the homogeneous space* $M = G/H$. Given a Γ-symmetric space $(G/H, \Gamma)$ it is easy to construct, for each point x of the homogeneous space $M = G/H$, a subgroup of the group $\mathrm{Diff}(M)$ of diffeomorphisms of M that is isomorphic to Γ and has x as an isolated fixed point. We denote by $\bar{g}$ the class of $g \in G$ in M. If e is the identity of G and $\gamma \in \Gamma$, we set $s_{(\gamma,\bar{e})}(\bar{g}) = \overline{\rho_\gamma(g)}$. If $\bar{g}$ satisfies $s_{(\gamma,\bar{e})}(\bar{g}) = \bar{g}$, then $\overline{\rho_\gamma(g)} = \bar{g}$, that is, $\rho_\gamma(g) = gh_\gamma$ for $h_\gamma \in H$. Thus $h_\gamma = g^{-1}\rho_\gamma(g)$. But $\Gamma \cong \widehat{\Gamma}$ is a finite abelian group. If p_γ is the order of γ, then $\rho_{\gamma^{p_\gamma}} = \mathrm{Id}$. Then

$$h_\gamma^2 = g^{-1}\rho_\gamma(g)\rho_\gamma(g^{-1})\rho_{\gamma^2}(g) = g^{-1}\rho_{\gamma^2}(g).$$

Proceeding inductively, and considering $h^m \in H$ for any m, we have $(h_\gamma)^m = g^{-1}\rho_{\gamma^m}(g)$. For $m = p_\gamma$ we obtain $(h_\gamma)^{p_\gamma} = e$. If g is near the identity element of G, then h_γ is also close to the identity, and $h_\gamma^{p_\gamma} = e$ implies $h_\gamma = e$. Then $\rho_\gamma(g) = g$. This is true for all $\gamma \in \Gamma$, and thus $g \in H$. It follows that $\bar{g} = \bar{e}$ and that the only fixed point of $s_{(\gamma,\bar{e})}$ is $\bar{e}$. In conclusion, the family $\{s_{(\gamma,\bar{e})}\}_{\gamma\in\Gamma}$ of diffeomorphisms of M satisfy

$$s_{(\gamma_1,\bar{e})} \circ s_{(\gamma_2,\bar{e})} = s_{(\gamma_1\gamma_2,\bar{e})},$$
$$s_{(\gamma,\bar{e})}(\bar{g}) = \bar{g} \text{ for all } \gamma \in \Gamma \text{ implies } \bar{g} = \bar{e}.$$

Thus, $\Gamma_{\bar{e}} = \{s_{(\gamma,\bar{e})} \mid \gamma \in \Gamma\}$ is a finite abelian subgroup of $\mathrm{Diff}(M)$ isomorphic to Γ, for which $\bar{e}$ is an isolated fixed point.

In another point $\bar{g}_0$ of M we put $s_{(\gamma,\bar{g}_0)}(\bar{g}) = g_0(s_{(\gamma,\bar{e})})(g_0^{-1}\bar{g})$. As above, we can see that

$$s_{(\gamma_1,\bar{g}_0)} \circ s_{(\gamma_2,\bar{g}_0)} = s_{(\gamma_1\gamma_2,\bar{g}_0)},$$
$$s_{(\gamma,\bar{g}_0)}(\bar{g}) = \bar{g} \text{ for all } \gamma \in \Gamma \text{ implies } \bar{g} = \bar{g}_0.$$

and $\Gamma_{\bar{g}_0} = \{s_{(\gamma,\bar{g}_0)} \mid \gamma \in \Gamma\}$ is a finite abelian subgroup of $\mathrm{Diff}(M)$ isomorphic to Γ, for which $\bar{g}_0$ is an isolated fixed point.

Thus for each $\bar{g} \in M$ we have a finite abelian subgroup $\Gamma_{\bar{g}}$ of $\mathrm{Diff}(M)$ isomorphic to Γ, for which $\bar{g}$ is an isolated fixed point.

Definition 10. Let $(G/H, \Gamma)$ be a Γ-symmetric space. For any point $x \in M = G/H$ the subgroup $\Gamma_x \subset \mathrm{Diff}(M)$ is called *the group of symmetries* of M at x.

Since for every $x \in M$ and $\gamma \in \Gamma$, the map $s_{(\gamma,x)}$ is a diffeomorphism of M such that $s_{(\gamma,x)}(x) = x$, the tangent linear map $(Ts_{(\gamma,x)})_x$ is in $\mathrm{GL}(T_xM)$. For every $x \in M$, we obtain a linear representation $S_x : \Gamma \to \mathrm{GL}(T_xM)$ defined by $S_x(\gamma) = (Ts_{(\gamma,x)})_x$. Thus for every $\gamma \in \Gamma$ the map $S(\gamma) : M \to T(M)$ defined by $S(\gamma)(x) = S_x(\gamma)$ is a $(1, 1)$-tensor on M satisfying these properties:

(1) the map $S(\gamma)$ is of class $\mathscr{C}^\infty$;

(2) for every $x \in M$, $\{X_x \in T_x(M) \mid S_x(\gamma)(X_x) = X_x \text{ for all } \gamma \in \Gamma\} = \{0\}$. In fact, this last remark is a consequence of the property that $s_{(\gamma,x)}(y) = y$ for every γ implies $y = x$.

Definition 11. Let M be a real differential manifold, and let Γ be a finite abelian group. We denote by $T_x M$ the tangent space to M at the point x.

A Γ-symmetric structure on M is given for all $x \in M$ by a linear representation ρ_x of Γ on the vector space $T_x M$ such that

(1) for every $\gamma \in \Gamma$, the map $x \in M \to \rho_x(\gamma)$ is of class $\mathscr{C}^\infty$;

(2) for every $x \in M$, $\{X_x \in T_x(M) \mid \rho_x(\gamma)(X_x) = X_x \text{ for all } \gamma\} = \{0\}$.

Proposition 12. *If* $(G/H, \Gamma)$ *is a* Γ*-symmetric space, the family* $\{S_x\}_{x\in M}$ *is a* Γ*-symmetric structure on the homogeneous space* $M = G/H$.

3.4. *Canonical connections on the homogeneous space G/H.* Let $(G/H, \Gamma)$ be a Γ-symmetric space. As we learned, the compexified Lie algebra $\mathfrak{g}\otimes\mathbb{C}$ of G is then P-graded, $P = \widehat{\Gamma}$, and $\mathfrak{g}\otimes\mathbb{C} = \oplus_{p\in P}\ (\mathfrak{g}\otimes\mathbb{C})_p$. If ε is the identity element of P, then the component $\mathfrak{h} = (\mathfrak{g}\otimes\mathbb{C})_1$ is a Lie subalgebra of $\mathfrak{g}\otimes\mathbb{C}$ of the form $\mathfrak{h}\otimes\mathbb{C}$ where $\mathfrak{h} = \mathfrak{g}^\Gamma$, which is the set of fixed points of the action of Γ on $\mathfrak{g}$ and is also the Lie algebra of the subgroup H. Let us consider the subspace $\mathfrak{g}'$ of $\mathfrak{g}$ given by $\mathfrak{g}' = \bigoplus_{p\neq 1}\mathfrak{g}_p$. For every $1 \neq p \in P$, if $p^2 = 1$ then $(\mathfrak{g}\otimes\mathbb{C})_p = \mathfrak{g}_p\otimes\mathbb{C}$, where $\mathfrak{g}_p$ is given by (3); if $p^2 \neq 1$, then $(\mathfrak{g}\otimes\mathbb{C})_p \oplus (\mathfrak{g}\otimes\mathbb{C})_{p^{-1}} = \widetilde{\mathfrak{g}}_p\otimes\mathbb{C}$, where $\widetilde{\mathfrak{g}}_p$ is the subspace of $\mathfrak{g}$ spanned by the real and imaginary parts of the vectors in $(\mathfrak{g}\otimes\mathbb{C})_p$.

This simple claim follows because $\overline{\gamma(u+vi)} = \gamma(\overline{u+vi})$, where the action of Γ on $\mathfrak{g}\otimes\mathbb{C}$ is given by $\gamma(u+vi) = \gamma(u)+\gamma(v)i$. Thus if $p^2 = 1$ and $u+vi \in (\mathfrak{g}\otimes\mathbb{C})_p$, then $\overline{\gamma(u+vi)} = \gamma(p)(u-vi)$ and $\gamma(\overline{u+vi}) = \gamma(u) - \gamma(v)i$. Since $\gamma(p)$ is real, $\gamma(u) = \gamma(p)(u)$ and $\gamma(v) = \gamma(p)(v)$, proving that $u, v \in \mathfrak{g}_p$ and $(\mathfrak{g}\otimes\mathbb{C})_p = \mathfrak{g}_p\otimes\mathbb{C}$. But if $p^2 \neq 1$ and again $u+vi \in (\mathfrak{g}\otimes\mathbb{C})_p$, then

$$\overline{\gamma(u+vi)} = \overline{\gamma(p)}(u-vi) = \gamma(p^{-1})\overline{u+vi} = \gamma(u-vi),$$

showing that complex conjugation leaves invariant $(\mathfrak{g}\otimes\mathbb{C})_p \oplus (\mathfrak{g}\otimes\mathbb{C})_{p^{-1}}$. Then, of course, our claim follows.

If we set $\mathfrak{m}$ to the sum of all $\mathfrak{g}_p$ and $\widetilde{\mathfrak{g}}_p$ for $p \neq 1$, then $\mathfrak{g}\otimes\mathbb{C} = \mathfrak{g}_1\otimes\mathbb{C}\oplus\mathfrak{m}\otimes\mathbb{C}$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{m}$. We also have $[\mathfrak{g}_1, \mathfrak{m}] \subset \mathfrak{m}$, and so $\mathfrak{m}$ is an $\operatorname{ad}\mathfrak{g}_1$-invariant subspace. If H is a connected Lie group, then $[\mathfrak{g}_1, \mathfrak{m}] \subset \mathfrak{m}$ is equivalent to $(\operatorname{ad} H)(\mathfrak{m}) \subset \mathfrak{m}$, that is, $\mathfrak{m}$ is an $\operatorname{ad} H$-invariant subspace. This property is true without any conditions on H.

Lemma 13. *Any* Γ*-symmetric space* $(G/H, \Gamma)$ *is reductive.*

Proof. Let us consider the associated local Γ-symmetric space $(\mathfrak{g}/\mathfrak{h}, \Gamma)$. We need to find a decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ such that $\mathfrak{m}$ is invariant under the adjoint action of the isotropy subgroup H or, which is the same, under the action of the isotropy subalgebra $\mathfrak{h}$. Now since $\mathfrak{h}\otimes\mathbb{C} = (\mathfrak{g}\otimes\mathbb{C})_1$ and $\mathfrak{m}\otimes\mathbb{C} = \oplus_{p\neq 1}(\mathfrak{g}\otimes\mathbb{C})_p$, we have $[\mathfrak{h}\otimes\mathbb{C}, \mathfrak{m}\otimes\mathbb{C}] \subset \mathfrak{m}\otimes\mathbb{C}]$. Then, of course, also $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. ☐

We now deduce from [Kobayashi and Nomizu 1996, Chapter X] that $M = G/H$ admits two G-invariant canonical connections denoted by ∇ and $\overline{\nabla}$. The *first canonical connection*, ∇, satisfies, for all $X, Y \in \mathfrak{m}$,

$$R(X, Y) = -\operatorname{ad}([X, Y]_{\mathfrak{h}}), \qquad T(X, Y)_{\bar{e}} = -[X, Y]_{\mathfrak{m}},$$
$$\nabla R = 0, \qquad \nabla T = 0,$$

where T and R are the torsion and the curvature tensors of ∇. The tensor T is trivial if and only if $[X, Y]_{\mathfrak{m}} = 0$ for all $X, Y \in \mathfrak{m}$. This means that $[X, Y] \in \mathfrak{h}$, that is, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Then the Lie algebra $\mathfrak{g}$ is $\mathbb{Z}_2$-graded, and the homogeneous space G/H is symmetric. If the grading of $\mathfrak{g}$ is given by Γ, where Γ is not isomorphic to $\mathbb{Z}_2$, then $[\mathfrak{m}, \mathfrak{m}]$ need not be a subset of in $\mathfrak{h}$, and then the torsion T need not vanish. In this case another connection, $\overline{\nabla}$, will be defined if one sets $\overline{\nabla}_X Y = \nabla_X Y - T(X, Y)$. This is an affine invariant torsion-free connection on G/H which has the same geodesics as ∇. This connection is called the *second canonical connection* or the *torsion-free canonical connection*. For example, if $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ then these connections can be distinct, as one can see from several examples in Section 4.

Remark. Actually, there is another way of writing the canonical affine connection of a Γ-symmetric space without any reference to Lie algebras. This is done by an intrinsic construction of Γ-symmetric spaces proposed by Lutz [1981].

4. Classification of local $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric complex spaces

We have seen that the classification of Γ-symmetric spaces $(G/H, \Gamma)$, when G is connected and simply connected, corresponds to the classification of Lie algebras graded by $P = \widehat{\Gamma}$. Below we establish the classification of local $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces $(\mathfrak{g}, \Gamma)$ in the case where the corresponding Lie algebra $\mathfrak{g}$ is simple complex and classical.

We recall Definition 2 that, given two P-gradings $\mathfrak{g} = \oplus_{p \in P} \mathfrak{g}_p$ and $\mathfrak{g} = \oplus_{p \in P} \mathfrak{g}'_p$ of an algebra $\mathfrak{g}$ by a group P, we call them *equivalent* if there exists an automorphism α of $\mathfrak{g}$ such that $\mathfrak{g}'_p = \alpha(\mathfrak{g}_p)$. To make the classification even more compact we will use another equivalence relation on the gradings. We will call two P-gradings $\mathfrak{g} = \oplus_{p \in P} \mathfrak{g}_p$ and $\mathfrak{g} = \oplus_{p \in P} \mathfrak{g}'_a$ of an algebra $\mathfrak{g}$ by a group P *weakly equivalent* if there exists an automorphism π of $\mathfrak{g}$ and an automorphism ω of P such that $\mathfrak{g}'_p = \pi(\mathfrak{g}_{\omega(p)})$. So the classification we are about to produce will be *up to the weak equivalence*.

4.1. ***Introductory remarks about the $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings.*** In this section we let $P = \{e, a, b, c\}$ be the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ with identity e and the relations $a^2 = b^2 = c^2 = e$ and $ab = c$. We will consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on a complex simple Lie algebra $\mathfrak{g}$ to be one of the types A_l for $l \geq 1$, B_l for $l \geq 2$, C_l for $l \geq 3$, and D_l for $l \geq 4$. We will use some results from [Bahturin et al. 2005; Bahturin and Zaicev 2007]. That is why in the rest of the paper we denote by e the identity of the grading group P.

Note that in those papers we did not consider the case of so(8). All fine group gradings on this algebra have been described in [Draper and Viruel 2007]. We thank the referee for pointing out that one can use this classification to construct all $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings on D_4 by [Draper and Martín 2006, Proposition 2]. At the same time, we note that if one considers the P-gradings of $\mathfrak{g} = \mathrm{so}(8)$, where P is an elementary abelian 2-group, then each grading is equivalent to a grading induced from a grading of the matrix algebra M_8. This quickly follows from the description of the outer automorphisms of order 2. If we fix a canonical realization of D_4 in M_8 and a basis of the root system for D_4, then one of the three diagram automorphisms of order 2 can be given as the conjugation by an appropriate nonsingular matrix in M_8, while the others are the conjugates of this fixed one by the diagram automorphisms of order 3; see for example [Jacobson 1962, Chapter III].

4.1.1. According to [Bahturin et al. 2005], any P-grading of a simple Lie algebra $\mathfrak{g} = \mathrm{so}(2l+1)$ for $l \geq 2$, $\mathfrak{g} = \mathrm{so}(2l)$ for $l > 4$ and $\mathrm{sp}(2l)$ for $l \geq 3$ is induced from a P-grading of the respective associative matrix algebra $R = M_{2l+1}$ in the first case, or M_{2l} in the second and the third case. As we just explained, this is also true for so(8) provided that $P = \mathbb{Z}_2 \times \mathbb{Z}_2$. Two kinds of P-grading on the associative matrix algebra $M_n = R = \oplus_{p \in P} R_a$ are of special importance:

Elementary gradings. Each such grading is defined by an n-tuple $(p_1, \dots, p_n)$ of elements of P in such a way that all matrix units E_{ij} are homogeneous with $E_{ij} \in R_p$ if and only if $p = p_i^{-1} p_j$.

Fine gradings. These gradings are characterized by the property that $\dim R_p = 1$ for every $p \in \mathrm{Supp}(R)$, where $\mathrm{Supp}(R) = \{p \in P \mid \dim R_p \neq 0\}$. In the case $P = \mathbb{Z}_2 \times \mathbb{Z}_2$, each fine grading is either trivial or weakly equivalent to the grading on $R \cong M_2$ given by the Pauli matrices

$$X_e = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in such a way that the graded component of degree p is spanned by X_p for $p = e, a, b, c$. Notice [Bahturin et al. 2001] that the support of a fine grading of a simple associative algebra is always a subgroup of P.

According to [Bahturin et al. 2001; Bahturin and Zaicev 2003] any P-grading of $R = M_n$ can be written as the tensor product of two graded matrix subalgebras $R = A \otimes B$, where $A \cong M_k$ and its grading is (equivalent to) an elementary grading, and the grading of $B \cong M_l$ is fine with $\mathrm{Supp}\, A \cap \mathrm{Supp}\, B = \{e\}$ for $kl = n$. Thus the only cases possible, when $P = \mathbb{Z}_2 \times \mathbb{Z}_2$, are

(1) $B = \mathbb{C}$ and $R = A \otimes \mathbb{C} = A$;

(2) $B = M_2$ and the grading on A is trivial.

If R is graded by P as above, then an involution $*$ of $R = M_n$ is said to be *graded* if $(R_p)^* = R_p$ for any $p \in P$. In the case of such involution, the spaces $K(R, *) = \{X \in R \mid X^* = -X\}$ of elements skew-symmetric under $*$ and $H(R, *) = \{X \in R \mid X^* = X\}$ of elements symmetric under $*$ are graded, and the first is a simple Lie algebra of one of the types B, C, D under the bracket $[X, Y] = XY - YX$.

It is proved in [Bahturin et al. 2005] that $\mathfrak{g}$ as a P-graded algebra is isomorphic to $K(R, *) = \{X \in R \mid X^* = -X\}$ for an appropriate graded involution $*$.

In general the involution does not need to respect A and B. But this is definitely the case when $P = \mathbb{Z}_2 \times \mathbb{Z}_2$. In fact, since the grading of B is fine, using the support conditions, we see that either $B = \mathbb{C}$ and $R = A \otimes \mathbb{C} = A$ is respected by $*$ or $B = M_2$ and the grading of A is trivial. Since B is the centralizer of A in R, it follows that B is also invariant under the involution. For the details of the above claims see [Bahturin and Zaicev 2007].

Now any involution has the form $* : X \to X^* = \Phi^{-1} X^t \Phi$ for a nonsingular matrix Φ which is either symmetric in the orthogonal case and skew-symmetric in the symplectic case. Since the elementary and fine components are invariant under the involution, $\Phi = \Phi_1 \otimes \Phi_2$, where Φ_1 defines a graded involution on A and Φ_2 on B.

First we recall the description of the graded involutions for the elementary gradings. Given an element p of a group P and a natural number n we denote by $p^{(n)}$ the n-tuple $p, \ldots, p$. Using the argument of [Bahturin and Zaicev 2007], one may assume that this grading is given by an n-tuple

$$\nu = (p_1^{(k_1)}, p_2^{(k_2)}, \ldots, p_{s+2t}^{(k_{s+2t})}) \quad \text{where } n = k_1 + \cdots + k_{s+2t},$$

$p_i \in \Gamma$ are pairwise different, $k_{s+2i-1} = k_{s+2i}$ for $i = 1, \ldots, t$, and there is p_0 such that $p_0 = p_1^2 = \cdots = p_s^2 = p_{s+1} p_{s+2} = \cdots = p_{s+2t-1} p_{s+2t}$. However, since $p^2 = e$ for the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, we have $p_0 = e$. If $st \neq 0$, then $p_{s+1} p_{s+2} = e$; this implies $p_{s+1} = p_{s+2}$, a contradiction. Then in this case either $t = 0$ and the $\mathbb{Z}_2 \times \mathbb{Z}_2$-elementary grading corresponds to $e = a^2 = b^2 = c^2$ and is given by the p-tuple $(e^{(k_1)}, a^{(k_2)}, b^{(k_3)}, c^{(k_4)})$ or $s = 0$, and the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading corresponds to $a = ae = bc$ with $k_1 = k_2$ and $k_3 = k_4$.

In the general case, let I_k be the identity matrix of order k, and let

$$S_{2l} = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

Then the matrix Φ defining the involution has the form

$$\Phi = \operatorname{diag}\left(I_{k_1}, \dots, I_{k_s}, \begin{pmatrix} 0 & I_{k_{s+1}} \\ I_{k_{s+1}} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & I_{k_{s+2t-1}} \\ I_{k_{s+2t-1}} & 0 \end{pmatrix}\right)$$

if $*$ is a transpose involution, that is, Φ is symmetric. In the case of a skew-symmetric Φ,

$$\Phi = \operatorname{diag}\left(S_{k_1}, \dots, S_{k_p}, \begin{pmatrix} 0 & I_{k_{p+1}} \\ -I_{k_{p+1}} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & I_{k_{p+2s-1}} \\ -I_{k_{p+2s-1}} & 0 \end{pmatrix}\right).$$

If $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, when we consider the case $t = 0$, the matrices of Φ are the identity in the symmetric case and $\Phi = \operatorname{diag}(S_{k_1}, \dots, S_{k_4})$ in the skew-symmetric case, and if $s = 0$, then

$$\Phi = \left(\begin{pmatrix} 0 & I_{k_1} \\ I_{k_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{k_3} \\ I_{k_3} & 0 \end{pmatrix}\right)$$

if Φ is symmetric and

$$\Phi = \operatorname{diag}\left(\begin{pmatrix} 0 & I_{k_1} \\ -I_{k_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{k_3} \\ -I_{k_3} & 0 \end{pmatrix}, \right)$$

if Φ is skew-symmetric.

If $R = A \otimes B$ and $B \neq \mathbb{C}$, then $\operatorname{Supp}(A) = \{e\}$. We have $\Phi = \Phi_1 \otimes \Phi_2$, and the involution on R defines involutions on A and B. It follows that Φ is symmetric if and only if either Φ_1 and Φ_2 are both symmetric or they are both skew-symmetric. Similarly, Φ is skew-symmetric if one of Φ_1 and Φ_2 is symmetric, and the other is skew-symmetric. It was proved in [Bahturin et al. 2005] that M_2 with graded involution is isomorphic to M_2 with Γ-graded basis $\{X_e, X_a, X_b, X_c\}$ and the graded involution is given by one of $\Phi = X_e, X_a, X_b, X_c$.

4.1.2. Now let $\mathfrak{g}$ be a simple Lie algebra of type A_l. We view $\mathfrak{g}$ as the set $\mathfrak{g} = \mathrm{sl}(n)$ of all matrices of trace zero in the matrix algebra $R = M_n(\mathbb{C})$, where $n = l + 1$. In this case any P-grading of $\mathfrak{g}$ belongs to one of the following two classes; see [Bahturin and Zaicev 2007].

- For Class I gradings, any grading of $\mathfrak{g}$ is induced from a P-grading of $R = \bigoplus_{p\in P} R_p$, and one simply has to set $\mathfrak{g}_p = R_p$ for $p \neq e$ and $\mathfrak{g}_e = R_e \cap \mathfrak{g}$ otherwise. For $P = \mathbb{Z}_2 \times \mathbb{Z}_2$ we still have to distinguish between the cases $R = A \otimes \mathbb{C}$ with an elementary grading on $A = M_n$ or $R = A \otimes B$ with trivial grading on A and fine $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on $B = M_2$.

- For Class II gradings, we have to fix an element q of order 2 in P and an involution P-grading $R = \bigoplus_{p\in P} R_p$. Then for any $p \in P$ one has

$$\mathfrak{g}_p = K(R_p, *) \oplus H(R_{pq}, *) \cap \mathfrak{g}.$$

The involution grading on M_n have been discussed just before in Section 4.1.1. It should be noted that in the case where $B \neq \mathbb{C}$ in $R = A \otimes B$ we have

$$K(R_p, *) = K(A, *) \otimes H(B_p, *) \oplus H(A, *) \otimes K(B_p, *),$$
$$H(R_p, *) = H(A, *) \otimes H(B_p, *) \oplus K(A, *) \otimes K(B_p, *).$$

As noted above, $\Phi = \Phi_1 \otimes \Phi_2$ with $\Phi_2 = X_p$ for $p = e, a, b, c$. If X_p is symmetric with respect to Φ_2, then the last two equations become

$$K(R_p, *) = K(A, *) \otimes X_p \qquad \text{and} \qquad H(R_p, *) = H(A, *) \otimes X_p.$$

On the other hand, if X_p is skew-symmetric with respect to Φ_2, then they become

$$K(R_p, *) = H(A, *) \otimes X_p \qquad \text{and} \qquad H(R_p, *) = K(A, *) \otimes X_p.$$

All these remarks allow us to determine up to the weak equivalence the pairs $(\mathfrak{g}, \mathfrak{g}_e)$ inside the respective matrix algebra M_n. This gives a local classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric homogeneous spaces G/H, where G is simple classical connected Lie group.

4.2. *Classification: B_l, C_l, D_l cases.* We have that $\mathfrak{g} = K(M_n, \Phi)$, where Φ is symmetric for the cases B_l and D_l and Φ is skew-symmetric for C_l. In the case B_l, we have $n = 2l + 1$. In the case of C_l and D_l we have $n = 2l$.

4.2.1. *Lie gradings corresponding to the elementary grading of M_n.* Since we are interested in the gradings only up to the weak equivalence, it is sufficient to consider the following tuples defining the elementary gradings:

$$\nu_1 = (e^{(k_1)}, a^{(k_2)}), \quad \nu_2 = (e^{(k_1)}, a^{(k_2)}, b^{(k_3)}), \quad \nu_3 = (e^{(k_1)}, a^{(k_2)}, b^{(k_3)}, c^{(k_4)}).$$

Notice that the $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings corresponding to ν_1 coincide with $\mathbb{Z}_2$-gradings, and thus the corresponding homogeneous spaces are symmetric in the classical sense. The matrices defining the graded transpose involution in the case of ν_1 are $\Phi_1 = \operatorname{diag}(I_{k_1}, I_{k_2})$ and

$$\Phi_1' = \begin{pmatrix} 0 & I_{k_1} \\ I_{k_1} & 0 \end{pmatrix}.$$

If the case is ν_2, then $\Phi_2 = \operatorname{diag}(I_{k_1}, I_{k_2}, I_{k_3})$. Finally, in the case of ν_3 we have

$$\Phi_3 = \operatorname{diag}(I_{k_1}, I_{k_2}, I_{k_3}, I_{k_4}) \quad \text{or} \quad \Phi_3' = \operatorname{diag}\left(\begin{pmatrix} 0 & I_{k_1} \\ I_{k_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{k_3} \\ I_{k_3} & 0 \end{pmatrix}\right).$$

If the involution is symplectic, then the respective matrices, in the case of ν_1, are

$$\overline{\Phi}_1 = \operatorname{diag}(S_{k_1}, S_{k_2}) \quad \text{and} \quad \overline{\Phi}_1' = \begin{pmatrix} 0 & I_{k_1} \\ -I_{k_1} & 0 \end{pmatrix}.$$

$\mathfrak{g}$	$\mathfrak{g}_e$
$\mathrm{so}(k_1+k_2)$	$\mathrm{so}(k_1)\oplus\mathrm{so}(k_2)$
$\mathrm{so}(k_1+k_2+k_3)$	$\mathrm{so}(k_1)\oplus\mathrm{so}(k_2)\oplus\mathrm{so}(k_3)$
$\mathrm{so}(k_1+k_2+k_3+k_4)$	$\mathrm{so}(k_1)\oplus\mathrm{so}(k_2)\oplus\mathrm{so}(k_3)\oplus\mathrm{so}(k_4)$
$\mathrm{sp}(k_1+k_2)$	$\mathrm{sp}(k_1)\oplus\mathrm{sp}(k_2)$
$\mathrm{sp}(k_1+k_2+k_3)$	$\mathrm{sp}(k_1)\oplus\mathrm{sp}(k_2)\oplus\mathrm{sp}(k_3)$
$\mathrm{sp}(k_1+k_2+k_3+k_4)$	$\mathrm{sp}(k_1)\oplus\mathrm{sp}(k_2)\oplus\mathrm{sp}(k_3)\oplus\mathrm{sp}(k_4)$
$\mathrm{so}(2m)$	$\mathrm{gl}(m)$
$\mathrm{so}(2(k_1+k_2))$	$\mathrm{gl}(k_1)\oplus\mathrm{gl}(k_2)$
$\mathrm{sp}(2m)$	$\mathrm{gl}(m)$
$\mathrm{sp}(2(k_1+k_2))$	$\mathrm{gl}(k_1)\oplus\mathrm{gl}(k_2)$

Table 1. Pairs $(\mathfrak{g}, \mathfrak{g}_e)$ giving $\mathbb{Z}_2\times\mathbb{Z}_2$-symmetric spaces for the elementary gradings of M_n.

In the case of ν_2, $\overline{\Phi}_2=\mathrm{diag}(S_{k_1}, S_{k_2}, S_{k_3})$, and in the case of ν_3,

$$\overline{\Phi}_3=\mathrm{diag}(S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4}) \quad \text{or} \quad \overline{\Phi}'_3=\mathrm{diag}\left(\begin{pmatrix}0 & I_{k_1}\\ -I_{k_1} & 0\end{pmatrix}, \begin{pmatrix}0 & I_{k_3}\\ -I_{k_3} & 0\end{pmatrix}\right).$$

In the cases (ν_1, Φ_1) and $(\nu_2, \overline{\Phi}_1)$ we have $\mathfrak{g}_e=\mathrm{so}(k_1)\oplus\mathrm{so}(k_2)$ and $\mathfrak{g}_e=\mathrm{sp}(k_1)\oplus\mathrm{sp}(k_2)$, respectively.

In the cases (ν_1, Φ'_1) and $(\nu_3, \overline{\Phi}'_1)$, we have $\mathfrak{g}_e=\{\mathrm{diag}(U_1, -U_1^t)\mid U_1\in M_{k_1}\}$. So for both D_{k_1} and C_{k_1} cases, we will have $\mathfrak{g}_e=\mathrm{gl}(k_1)$.

In the cases (ν_2, Φ_2) and $(\nu_2, \overline{\Phi}_2)$, we have $\mathfrak{g}_e=\mathrm{so}(k_1)\oplus\mathrm{so}(k_2)\oplus\mathrm{so}(k_3)$ and $\mathfrak{g}_e=\mathrm{sp}(k_1)\oplus\mathrm{sp}(k_2)\oplus\mathrm{sp}(k_3)$, respectively.

In the cases (ν_3, Φ_3) and $(\nu_3, \overline{\Phi}_3)$ we have $\mathfrak{g}_e=\mathrm{so}(k_1)\oplus\mathrm{so}(k_2)\oplus\mathrm{so}(k_3)\oplus\mathrm{so}(k_4)$ and $\mathfrak{g}_e=\mathrm{sp}(k_1)\oplus\mathrm{sp}(k_2)\oplus\mathrm{sp}(k_3)\oplus\mathrm{sp}(k_4)$, respectively.

In the cases (ν_3, Φ'_3) and $(\nu_3, \overline{\Phi}'_3)$ we have $\mathfrak{g}_e=\mathrm{diag}(U_1, -U_1^t, U_2, -U_2^t)$ for $U_1\in M_{k_1}$ and $U_2\in M_{k_2}$. So for both the case $D_{k_1+k_3}$ and $C_{k_1+k_3}$, we have $\mathfrak{g}_e=\mathrm{gl}(k_1)\oplus\mathrm{gl}(k_3)$.

Table 1 lists those pairs $(\mathfrak{g}, \mathfrak{g}_e)$ considered so far that have corresponding $\mathbb{Z}_2\times\mathbb{Z}_2$-symmetric spaces.

In all the above cases the $\mathbb{Z}_2\times\mathbb{Z}_2$-gradings of $\mathfrak{g}$ are very easy to compute using explicit matrices of the involutions. For example, the gradings of $\mathrm{so}(n)$ are given in [Bouyakoub et al. 2006].

4.2.2. *Gradings corresponding to $R=A\otimes B$, with nontrivial B.* The algebra B is endowed with a fine grading given by the Pauli matrices X_e, X_a, X_b, X_c. Also

$\mathfrak{g}$	$\mathfrak{g}_e$
so$(2m)$	so(m)
so$(4m)$	sp$(2m)$
sp$(4m)$	sp$(2m)$
sp$(2m)$	so(m)

Table 2. Pairs $(\mathfrak{g}, \mathfrak{g}_e)$ giving $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces for gradings with $R = A \otimes B$, with nontrivial B.

$A = M_m$ and $\Phi = \Phi_1 \otimes \Phi_2$. We have

$$\mathfrak{g} = K(R, \Phi) = K(A, \Phi_1) \otimes H(B, \Phi_2) \oplus H(A, \Phi_1) \otimes K(B, \Phi_2).$$

In particular, $\mathfrak{g}_e = K(R_e, \Phi_1) \otimes I_2$. If Φ_1 is symmetric, then $\mathfrak{g}_e = \mathrm{so}(m)$, and if Φ_1 is skew symmetric, $g_e = \mathrm{sp}(m)$.

We give the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces corresponding to the pairs $(\mathfrak{g}, \mathfrak{g}_e)$ given in Table 2.

In these cases we will describe the components of the gradings explicitly. If $\mathfrak{g} = \mathrm{so}(2m)$, then Φ is symmetric and is of one of the form

$$\Psi_1 = I_m \otimes I_2, \quad \Psi_2 = I_m \otimes X_a, \quad \Psi_3 = I_m \otimes X_b, \quad \Psi_4 = S_m \otimes X_c.$$

Then for Ψ_1, Ψ_2, Ψ_3, and Ψ_4, we respectively have

$$\mathfrak{g} = \mathfrak{g}(\Psi_1) = K(A, I_m) \otimes I \oplus K(A, I_m) \otimes X_a \oplus K(A, I_m) \otimes X_b \oplus H(A, I_m) \otimes X_c,$$
$$\mathfrak{g} = \mathfrak{g}(\Psi_2) = K(A, I_m) \otimes I \oplus K(A, I_m) \otimes X_a \oplus H(A, I_m) \otimes X_b \oplus K(A, I_m) \otimes X_c,$$
$$\mathfrak{g} = \mathfrak{g}(\Psi_3) = K(A, I_m) \otimes I \oplus H(A, I_m) \otimes X_a \oplus K(A, I_m) \otimes X_b \oplus K(A, I_m) \otimes X_c,$$
$$\mathfrak{g} = \mathfrak{g}(\Psi_4) = K(A, S_m) \otimes I \oplus H(A, S_m) \otimes X_a \oplus H(A, S_m) \otimes X_b \oplus H(A, S_m) \otimes X_c.$$

Conjugation by $I_m \otimes \left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)$ maps $\mathfrak{g}(\Psi_1)$ to $\mathfrak{g}(\Psi_2)$, while mapping $K(A, I_m) \otimes I$ and $K(A, I_m) \otimes X_a$ into themselves, $K(A, I_m) \otimes X_b$ into $K(A, I_m) \otimes X_c$ and $H(A, I_m) \otimes X_c$ into $H(A, I_m) \otimes X_b$. If we apply an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ changing places of b and c, we will see that the first and the second gradings are weakly equivalent. Quite similarly, the conjugation by $I_m \otimes 1/\sqrt{2}\left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ maps $\mathfrak{g}(\Psi_1)$ to $\mathfrak{g}(\Psi_3)$ while mapping $K(A, I_m) \otimes I$ into itself, $K(A, I_m) \otimes X_a$ into $K(A, I_m) \otimes X_b$, $K(A, I_m) \otimes X_b$ into $K(A, I_m) \otimes X_c$, and $H(A, I_m) \otimes X_c$ into $H(A, I_m) \otimes X_a$. It remains to apply an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ mapping a to c, c to b, and b to a to make sure that the first and the third gradings are weakly equivalent. Thus, the first three gradings are all weakly equivalent. None of them is weakly equivalent to the fourth one because in these cases $\mathfrak{g}_e \cong \mathrm{so}(m)$, while in the fourth case we have $\mathfrak{g}_e \cong \mathrm{sp}(m)$.

The matrix form of the first and the fourth gradings are given below.

For Ψ_1 we have

$$\mathfrak{g}=\left\{\begin{pmatrix}U_1-U_2 & U_3-V\\ U_3+V & U_1+U_2\end{pmatrix}\,\middle|\, U_1,U_2,U_3\in \mathrm{so}(m),\ V^t=V\right\}.$$

The components $\mathfrak{g}_e\oplus\mathfrak{g}_a\oplus\mathfrak{g}_b\oplus g_c$ are

$$\left\{\begin{pmatrix}U_1 & 0\\ 0 & U_1\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}-U_2 & 0\\ 0 & U_2\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0 & U_3\\ U_3 & 0\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0 & -V\\ V & 0\end{pmatrix}\right\}.$$

For Ψ_4 we have

$$\mathfrak{g}=\left\{\begin{pmatrix}P-Q_1 & Q_2-Q_3\\ Q_2-Q_3 & P+Q_1\end{pmatrix}\,\middle|\, P\in \mathrm{sp}(m),\ Q_1,Q_2,Q_3\in H(M_m,S_m)\right\}.$$

The components $\mathfrak{g}_e\oplus\mathfrak{g}_a\oplus\mathfrak{g}_b\oplus g_c$ are

$$\left\{\begin{pmatrix}P & 0\\ 0 & P\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}-Q_1 & 0\\ 0 & Q_1\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0 & Q_2\\ Q_2 & 0\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0 & -Q_3\\ -Q_3 & 0\end{pmatrix}\right\}.$$

If $\mathfrak{g}=\mathrm{sp}(2m)$, then Φ is skew-symmetric and is of one of the form

$$\overline{\Psi}_1=S_m\otimes I_2,\quad \overline{\Psi}_2=S_m\otimes X_a,\quad \overline{\Psi}_3=S_m\otimes X_b,\quad \overline{\Psi}_4=I_m\otimes X_c.$$

For $\overline{\Psi}_1$, $\overline{\Psi}_2$, $\overline{\Psi}_3$, $\overline{\Psi}_4$, we respectively have

$$\mathfrak{g}=\mathfrak{g}(\overline{\Psi}_1)=K(A,S_m)\otimes I\oplus K(A,S_m)\otimes X_a\oplus K(A,S_m)\otimes X_b\oplus H(A,S_m)\otimes X_c,$$
$$\mathfrak{g}=\mathfrak{g}(\overline{\Psi}_2)=K(A,S_m)\otimes I\oplus K(A,S_m)\otimes X_a\oplus H(A,S_m)\otimes X_b\oplus K(A,S_m)\otimes X_c,$$
$$\mathfrak{g}=\mathfrak{g}(\overline{\Psi}_3)=K(A,S_m)\otimes I\oplus H(A,I_m)\otimes X_a\oplus K(A,I_m)\otimes X_b\oplus K(A,I_m)\otimes X_c,$$
$$\mathfrak{g}=\mathfrak{g}(\overline{\Psi}_4)=K(A,S_m)\otimes I\oplus H(A,S_m)\otimes X_a\oplus H(A,S_m)\otimes X_b\oplus H(A,S_m)\otimes X_c.$$

The same argument as before shows that the first three gradings are weakly equivalent and that none of them is weakly equivalent to the fourth one. In the first three cases, we have $\mathfrak{g}_e\cong\mathrm{sp}(m)$, while in the fourth case we have $\mathfrak{g}_e\cong\mathrm{so}(m)$.

Again we give the matrix form of the first and the fourth gradings. If Φ is skew symmetric, then $\mathfrak{g}=\mathrm{sp}(2m)$ and Φ is of one of the form

$$\overline{\Psi}_1=S_m\otimes I_2,\quad \overline{\Psi}_2=S_m\otimes X_a,\quad \overline{\Psi}_3=S_m\otimes X_b,\quad \overline{\Psi}_4=I_m\otimes X_c.$$

For $\overline{\Psi}_1$ we have

$$\mathfrak{g}=\left\{\begin{pmatrix}U_1-U_2 & U_3-V\\ U_3+V & U_1+U_2\end{pmatrix}\,\middle|\, U_1,U_2,U_3\in \mathrm{sp}(m),\ V\in H(A,S_m)\right\}.$$

The components $\mathfrak{g}_e\oplus\mathfrak{g}_a\oplus\mathfrak{g}_b\oplus g_c$ are

$$\left\{\begin{pmatrix}U_1 & 0\\ 0 & U_1\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}-U_2 & 0\\ 0 & U_2\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0 & U_3\\ U_3 & 0\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0 & -V\\ V & 0\end{pmatrix}\right\}.$$

For $\overline{\Psi}_4$ we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} P - Q_1 & Q_2 - Q_3 \\ Q_2 - Q_3 & P + Q_1 \end{pmatrix} \,\middle|\, P \in \mathrm{so}(m),\ Q_1, Q_2, Q_3 \in H(M_m, I_m) \right\}.$$

The components $\mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus g_c$ are

$$\left\{ \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} -Q_1 & 0 \\ 0 & Q_1 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & Q_2 \\ Q_2 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & -Q_3 \\ Q_3 & 0 \end{pmatrix} \right\}.$$

4.3. ***Classification of Class I gradings on A_l-type Lie algebras.*** If no fine component is present in $R = M_n \supset \mathfrak{g} = \mathrm{sl}(n)$, where $n = l + 1$, then all is defined by the n-tuples

$$\nu_1 = (e^{(k_1)}, a^{(k_2)}), \quad \nu_2 = (e^{(k_1)}, a^{(k_2)}, b^{(k_3)}), \quad \nu_3 = (e^{(k_1)}, a^{(k_2)}, b^{(k_3)}, c^{(k_4)}).$$

In the case of ν_1, the gradings correspond to a symmetric decomposition; in fact we obtain the symmetric pair $(\mathrm{sl}(n), \mathrm{sl}(k_1) \oplus \mathrm{sl}(k_2) \oplus \mathbb{C})$ (or $\mathbb{R}$ if we are in the real case).

In the case of ν_2,

$$\mathfrak{g}_e = \mathrm{diag}\{X, Y, Z \mid X \in M_{k_1},\ Y \in M_{k_2},\ Z \in M_{k_3},\ \mathrm{tr}(X + Y + Z) = 0\}$$

and $\mathfrak{g}_e = \mathrm{sl}(k_1) \oplus \mathrm{sl}(k_2) \oplus \mathrm{sl}(k_3) \oplus \mathbb{C}^2$.

In the case of ν_3,

$$\mathfrak{g}_e = \mathrm{diag}\{X, Y, Z, T \mid X \in M_{k_1},\ Y \in M_{k_2},\ Z \in M_{k_3},\ T \in M_{k_4},\ \mathrm{tr}(X+Y+Z+T) = 0\}$$

and $\mathfrak{g}_e = \mathrm{sl}(k_1) \oplus \mathrm{sl}(k_2) \oplus \mathrm{sl}(k_3) \oplus \mathrm{sl}(k_4) \oplus \mathbb{C}^3$.

In all these case the grading is obvious. If $R = A \otimes B = M_n,\ A = M_m,\ n = 2m$, with a trivial grading on A, then $\mathfrak{g}_e = \{\mathrm{diag}(X, X) \mid X \in \mathrm{sl}(m)\}$ and the grading is given by

$$\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus g_c$$
$$= \mathfrak{g}_e \oplus \left\{ \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \,\middle|\, X \in M_m \right\} \oplus \left\{ \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \,\middle|\, X \in M_m \right\} \oplus \left\{ \begin{pmatrix} 0 & -X \\ X & 0 \end{pmatrix} \,\middle|\, X \in M_m) \right\}.$$

We obtain $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces for the pairs $(\mathfrak{g}, \mathfrak{g}_e)$ given in Table 3.

4.4. ***Classification of Class II gradings on A_l-type Lie algebras.*** The general approach described in Section 4.1.2 enables one to classify the Class II gradings on $\mathfrak{g} = \mathrm{sl}(n)$ for any $n \geq 2$ and any grading group P. However, in the case $P = \mathbb{Z}_2 \times \mathbb{Z}_2$, the amount of work can be significantly reduced if one uses the results of [Bahturin and Zaicev 2006; 2007]. In the former paper, in the case of outer gradings of $\mathrm{sl}(n)$, the authors showed that the dual Γ of the grading group P decomposes as the direct product $\langle \varphi \rangle \times \Lambda$, where φ is an antiautomorphism of order 2^m and Λ acts by inner

$\mathfrak{g}$	$\mathfrak{g}_e$
$\mathrm{sl}(2n)$	$\mathrm{sl}(n)$
$\mathrm{sl}(k_1+k_2)$	$\mathrm{sl}(k_1)\oplus\mathrm{sl}(k_2)\oplus\mathbb{C}$
$\mathrm{sl}(k_1+k_2+k_3)$	$\mathrm{sl}(k_1)\oplus\mathrm{sl}(k_2)\oplus\mathrm{sl}(k_3)\oplus\mathbb{C}^2$
$\mathrm{sl}(k_1+k_2+k_3+k_4)$	$\mathrm{sl}(k_1)\oplus\mathrm{sl}(k_2)\oplus\mathrm{sl}(k_3)\oplus\mathrm{sl}(k_4)\oplus\mathbb{C}^3$

Table 3. Pairs $(\mathfrak{g}, \mathfrak{g}_e)$ giving $\mathbb{Z}_2\times\mathbb{Z}_2$-symmetric spaces for Class I gradings on A_l-type Lie algebras.

matrix automorphisms. If $H=\Lambda^\perp$, then H is a subgroup of order 2 and the induced G/H-grading of $\mathrm{sl}(n)$ is induced from M_n. To obtain the G-grading of $\mathfrak{g}$, one has to refine the G/H-grading by intersecting them with the eigenspaces of φ. In the case $P=\mathbb{Z}_2\times\mathbb{Z}_2$, φ is (the negative of) a graded involution of M_n, Λ is generated by an automorphism λ of order 2, and the generators of P are a and b such that $\lambda(a)=-1$, $\lambda(b)=1$, $\varphi(a)=1$, and $\varphi(b)=-1$. In the latter paper the authors described all graded involutions on graded matrix algebras. In our particular case the gradings by $\mathbb{Z}_2\times\mathbb{Z}_2$ on $\mathrm{sl}(n)$ correspond to $\mathbb{Z}_2$-graded eigenspaces of (the negative of) a graded involution φ on a $\mathbb{Z}_2$-graded associative algebra $R=M_n$. Any $\mathbb{Z}_2$-grading on R is elementary and given by a tuple $\nu_1=(e^{(k_1)}, a^{(k_2)})$, where a is the generator of $\mathbb{Z}_2$ and $k_1+k_2=n$. Now, as described in [Bahturin and Zaicev 2007, Theorem 3], any graded involution is graded equivalent to $\omega(X)=\Phi^{-1}X^t\Phi$ where Φ is one of the types

$$\Phi_1=\mathrm{diag}(I_{k_1}, I_{k_2}),\qquad \overline{\Phi}_1=\mathrm{diag}(S_{k_1}, S_{k_2}),$$
$$\Phi_1'=\begin{pmatrix}0 & I_{k_1}\\ I_{k_1} & 0\end{pmatrix}\qquad \overline{\Phi}_1'=\begin{pmatrix}0 & I_{k_1}\\ -I_{k_1} & 0\end{pmatrix}.$$

Now it remains to apply [Bahturin and Zaicev 2006, Corollary 5.6], where $K=\langle a\rangle$, to obtain that all Class II gradings of $\mathfrak{g}$ have are drawn from the forms

$$\mathfrak{g}_e=K(R_e,\Phi),\quad \mathfrak{g}_a=K(R_a,\Phi),\quad \mathfrak{g}_b=H(R_e,\Phi),\quad \mathfrak{g}_c=H(R_a,\Phi).$$

In all four cases, depending on the choice of Φ, we have

$$R=\left\{\begin{pmatrix}U & V\\ W & T\end{pmatrix}\,\middle|\, U\in M_{k_1},\ T\in M_{k_2}\right\}\quad\text{and}\quad \mathfrak{g}=\left\{\begin{pmatrix}U & V\\ W & T\end{pmatrix}\,\middle|\,\mathrm{tr}\,U+\mathrm{tr}\,T=0\right\},$$

and in the last two cases, we additionally have that $k_1=k_2$. It easily follows that for the $\mathbb{Z}$-grading we have

$$R_e=\left\{\begin{pmatrix}U & 0\\ 0 & T\end{pmatrix}\right\}\subset R\quad\text{and}\quad R_a=\left\{\begin{pmatrix}0 & V\\ W & 0\end{pmatrix}\right\}\subset R.$$

Now we can explicitly write all the gradings in this case. In what follows we keep the following notation. For any X of size $q \times p$, we denote by X^* the ordinary transpose of X, except in the case of Φ_1' where it means $S_p^{-1} X^t S_q$. We use the symbol "U" and its adornments to denote matrices with the property that $X^* = -X$. Likewise V, V_1, and so on denote matrices with $X^* = X$. Finally, matrices with "W" are any matrices of appropriate size. All matrices must be in $\mathfrak{g}$.

In both cases $\Phi = \Phi_1$ and $\overline{\Phi}_1$, we then have

$$\mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus g_c = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & W \\ -W^* & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \right\}.$$

However, because we have transpose involution in the $\Phi = \Phi_1$ case, and symplectic matrices in the other case, we obtain two inequivalent local symmetric spaces

$$\mathrm{sl}(k_1 + k_2)/(\mathrm{so}(k_1) \oplus \mathrm{so}(k_2)) \quad \text{and} \quad \mathrm{sl}(k_1 + k_2)/(\mathrm{sp}(k_1) \oplus \mathrm{sp}(k_2)).$$

It should be noted that k_1 is always nonzero, while we could have $k_2 = 0$. In this case we actually have a $\mathbb{Z}_2$-grading rather than a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading. It is well known that the respective local symmetric spaces are

$$\mathrm{sl}(n)/\,\mathrm{so}(n) \quad \text{and} \quad \mathrm{sl}(2m)/\,\mathrm{sp}(2m).$$

In the case $\Phi = \Phi_1'$, we have

$$\mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus g_c = \left\{ \begin{pmatrix} W & 0 \\ 0 & -W^t \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & U_1 \\ U_2 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} W & 0 \\ 0 & W^t \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix} \right\}.$$

Finally, in the case $\Phi = \overline{\Phi}_1'$ we have

$$\mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus g_c = \left\{ \begin{pmatrix} W & 0 \\ 0 & -W^t \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} W & 0 \\ 0 & W^t \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & U_1 \\ U_2 & 0 \end{pmatrix} \right\}.$$

Obviously, the latter two gradings are weakly equivalent, and the weak equivalence is achieved by an automorphism of P that interchanges a and c. So, we obtain the third local symmetric space $\mathrm{sl}(2k)/\mathrm{gl}(k)$.

We obtain the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces corresponding to the pairs $(\mathfrak{g}, \mathfrak{g}_e)$ given in Table 4.

We summarize the results obtained in Section 4 as follows.

Theorem 14. *All local complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces in cases A_l for $l \geq 1$, B_l for $l \geq 2$, C_l for $l \geq 3$ or D_l for $l \geq 4$ are given in Tables 1, 2, 3, 4. Each space $\mathfrak{g}/\mathfrak{g}_e$ is uniquely, defined by $\mathfrak{g}$ and $\mathfrak{g}_e$ up to a weak equivalence of the respective $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.*

$\mathfrak{g}$	$\mathfrak{g}_e$
$\mathrm{sl}(2k)$	$\mathrm{gl}(k)$
$\mathrm{sl}(n)$	$\mathrm{so}(n)$
$\mathrm{sl}(2m)$	$\mathrm{sp}(2m)$
$\mathrm{sl}(k_1+k_2)$	$\mathrm{so}(k_1)\oplus\mathrm{so}(k_2)$
$\mathrm{sl}(2(k_1+k_2))$	$\mathrm{sp}(2k_1)\oplus\mathrm{sp}(2k_2)$

Table 4. Pairs $(\mathfrak{g}, \mathfrak{g}_e)$ giving $\mathbb{Z}_2\times\mathbb{Z}_2$-symmetric spaces for Class II gradings on A_l-type Lie algebras.

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