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A FORMULA FOR THE HYPERGEOMETRIC FUNCTION OF TYPE BC_n

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Formulae of Berezin and Karpelevič for the radial parts of invariant differential operators and the spherical function on a complex Grassmann manifold are generalized to the hypergeometric functions associated with root systems of type BC_n under the condition that the multiplicity of the middle roots is zero or one.

Introduction

Berezin and Karpelevič [1958] gave an explicit expression for radial parts of invariant differential operators and spherical functions on $SU(p, q)/S(U(p) \times U(q))$, without proofs; Hoogenboom [1982] gave proofs of these results. Explicit expressions of the Laplace–Beltrami operator and higher order invariant differential operators allows us to construct eigenfunctions by the method of separation of variables. The spherical function can be expressed using the determinant of a matrix whose entries are the Gauss hypergeometric functions.

Heckman and Opdam developed the theory of the hypergeometric function associated with a root system, which is a generalization of the theory of spherical functions on a symmetric space; see [Heckman and Schlichtkrull 1994]. Namely, the radial part of the Laplace–Beltrami operator of a Riemannian symmetric space of the noncompact type consists of data such as the restricted root system, multiplicities of roots. Heckman and Opdam allowed multiplicities of roots to take arbitrary complex numbers (that coincide on every Weyl group orbit), and constructed a commuting family of differential operators and eigenfunctions. For the rank one (single variable) case, their hypergeometric function is the Jacobi function [Koornwinder 1984], which is essentially the same as the Gauss hypergeometric function.

In this paper we prove that the results of Berezin and Karpelevič in [1958] are valid for the hypergeometric function associated with the root system of the type BC_n under the condition that the multiplicity of the middle roots is 1. Though it is an easy generalization of [Berezin and Karpelevič 1958], our results cover integral

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middle multiplicities in conjunction with the hypergeometric shift operator, and include many cases of symmetric spaces.

1. Hypergeometric function associated with a root system

1A. *Notation.* In this section, we review the hypergeometric function associated with a root system. See [Heckman and Schlichtkrull 1994] for details.

Let E be an n-dimensional Euclidean space with inner product $(\cdot\,,\cdot)$. For $\alpha \in E$ with $\alpha \neq 0$, write

$$\alpha^{\vee} = \frac{2\alpha}{(\alpha, \alpha)}.$$

Let $R \subset E$ be a root system of rank n, and W its Weyl group. Let $R_+ \subset R$ be a fixed set of positive roots and $E_+ \subset E$ be the corresponding positive Weyl chamber. Let

$$P = \{\lambda \in E \,:\, (\lambda, \alpha^{\vee}) \in \mathbb{Z}\ \forall\, \alpha \in R\}.$$

Let k_α $(\alpha \in R)$ be complex numbers such that $k_{w\alpha} = k_\alpha$ for all $w \in W$. We call $k = (k_\alpha)_{\alpha \in R}$ a multiplicity function on R. Let K denote the set of multiplicity functions on R. We set

$$\rho(k) = \tfrac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha,$$
$$\delta(k) = \prod_{\alpha \in R_+} \bigl(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\bigr)^{2k_\alpha}.$$

1B. *Commuting family of differential operators.* Let $\xi_1, \dots, \xi_n$ be an orthonormal basis of E and consider the differential operator

$$L(k) = \sum_{j=1}^{n} \partial_{\xi_j}^2 + \sum_{\alpha \in R_+} k_\alpha \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \partial_\alpha \tag{1-1}$$

on E. Here ∂_α denotes the directional derivative along α such that $\partial_\alpha(e^{\lambda}) = (\alpha, \lambda) e^{\lambda}$ for $\alpha,\ \lambda \in E$. We have

$$\delta(k)^{\frac{1}{2}} \circ \bigl(L(k) + (\rho(k), \rho(k))\bigr) \circ \delta(k)^{-\frac{1}{2}} = \sum_{j=1}^{n} \partial_{\xi_j}^2 + \sum_{\alpha \in R_+} \frac{k_\alpha(1 - k_\alpha - 2k_{2\alpha})(\alpha, \alpha)}{\bigl(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\bigr)^2}. \tag{1-2}$$

Let $\mathcal{R}$ denote the algebra generated by the functions

$$\frac{1}{1 - e^{-\alpha}} \quad (\alpha \in R_+),$$

which are viewed as a subalgebra of the quotient field of $\mathbb{R}[P]$. Let $S(E)$ denote the symmetric algebra of E. Let $\mathbb{D}_{\mathcal{R}} = \mathcal{R} \otimes S(E)$ denote the algebra of differential operators on E with coefficient in $\mathcal{R}$ and let $\mathbb{D}_{\mathcal{R}}^W$ be the subalgebra of W-invariants in $\mathbb{D}_{\mathcal{R}}$. Let $\gamma(k)$ denote the algebra homomorphism

$$\gamma(k) : \mathbb{D}_{\mathcal{R}} \longrightarrow S(E),$$

defined by

$$\gamma(k)\Big(\frac{1}{1-e^{-\alpha}}\Big) = 1 \quad (\alpha \in R_+).$$

Let

$$\mathbb{D}(k) = \{D \in \mathbb{D}_{\mathcal{R}}^W : [L(k), P] = 0\}$$

denote the commutator of $L(k)$ in $\mathbb{D}_{\mathcal{R}}^W$, and let $S(E)^W$ denote the set of W-invariants in $S(E)$.

Theorem 1.1. *The map*

$$\gamma(k) : \mathbb{D}(k) \longrightarrow S(E)^W$$

is an algebra isomorphism. In particular, $\mathbb{D}(k)$ *is a commutative algebra. Moreover*, *if* $D \in \mathbb{D}_{\mathcal{R}}^W$ *is a differential operator of order* N, *then its principal symbol* $\sigma(D)$ *has constant coefficients and coincides with the homogeneous component of* $\gamma(k)(D)$ *of degree* N.

1C. ***The hypergeometric function.*** Let Q be the root lattice

$$Q = \{\textstyle\sum_{\alpha\in R_+} z_\alpha \alpha : z_\alpha \in \mathbb{Z}_+\}.$$

Set

$$\mathfrak{h} = E_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} E, \quad A = \exp\, E, \quad e = \exp 0, \quad A_+ = \exp E_+.$$

For $\mu \in \mathfrak{h}^*$ and $a \in A$, we write $a^\mu = \exp(\mu(\log a))$.

If $\lambda \in \mathfrak{h}^*$ satisfies the condition

$$-2(\lambda, \mu) + (\mu, \mu) \neq 0 \quad \text{for all } \mu \in Q, \tag{1-3}$$

then the equation

$$L(k)u = \big((\lambda, \lambda) - (\rho(k), \rho(k))\big)u \tag{1-4}$$

has a unique solution on A_+ of the form

$$u(a) = \Phi(\lambda, k; a) = \sum_{\mu\in Q} \Gamma_\mu a^{\lambda-\rho(k)-\mu}, \tag{1-5}$$

with $\Gamma_0 = 1$. The function $\Phi(\lambda, k; a)$ is also a solution of the system of differential equations

$$Du = \gamma(k)(D)(\lambda)u, \quad D \in \mathbb{D}(k). \tag{1-6}$$

If

$$(\lambda, \alpha^\vee) \notin \mathbb{Z} \quad \text{for all } \alpha \in R,$$

then $\Phi(w\lambda, k; a)$ $(w \in W)$ form a basis of the solution space of (1-6).

Define meromorphic functions $\tilde{c}$ and c on $\mathfrak{h} \times K$ by

$$\tilde{c}(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma\big((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha}\big)}{\Gamma\big((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha\big)} \tag{1-7}$$

and

$$c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)}, \tag{1-8}$$

with the convention that $k_{\frac{1}{2}\alpha} \neq 0$ if $\frac{1}{2}\alpha \notin R$. We call the function

$$F(\lambda, k; a) = \sum_{w \in W} c(w\lambda, k)\Phi(w\lambda, k; a)$$

the hypergeometric function associated with R. Let $S \subset K$ denote the set of zeros of $\tilde{c}(\rho(k), k)$.

Theorem 1.2. *Assume that $k \in K \setminus S$. Then the system of differential equations* (1-6) *has a unique solution that is regular at e, W-invariant, and*

$$F(\lambda, k; e) = 1.$$

The function F is holomorphic for $\lambda \in \mathfrak{h}$, $k \in K \setminus S$, and analytic for $a \in A$.

Remark 1.3. Theorems 1.1 and 1.2 were proved by Heckman and Opdam in a series of papers. See [Heckman and Schlichtkrull 1994] and references therein.

Let G/K be a Riemannian symmetric space of the noncompact type, Σ be the restricted root system, and m_α be the root multiplicity (dimension of the root space) of $\alpha \in \Sigma$. Set

$$R = 2\Sigma, \quad k_{2\alpha} = \tfrac{1}{2}m_\alpha.$$

Then (1-1) is the radial part of the Laplace–Beltrami operator on G/K, $\mathbb{D}(k)$ is the algebra of radial parts of invariant differential operators on G/K, and $F(\lambda, k; a)$ is the radial part of the spherical function on G/K. In this case, Theorem 1.1 and Theorem 1.2 were previously proved by Harish-Chandra. See [Helgason 1984] for theory of spherical functions on symmetric spaces.

1D. ***Rank one case.*** For a root system of rank 1, the hypergeometric function is given by the Jacobi function. We will review the Jacobi function; see [Koornwinder 1984] for details.

Assume that $R = \{\pm e_1,\ \pm 2e_1\}$ with $(e_1, e_1) = 1$, and set

$$k_s = k_{e_1}, \quad k_l = k_{2e_1}, \quad \alpha = k_s + k_l - 1/2, \quad \beta = k_l - 1/2. \tag{1-9}$$

We identify $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with $(\lambda, 2e_1) \in \mathbb{C}$ and let $t = e_1(\log a)/2$ be a coordinate on $A \simeq \mathbb{R}$. Then

$$\rho(k) = k_s + 2k_l = \alpha + \beta + 1.$$

The hypergeometric system (1-6) turns out to be the differential equation

$$L(k)F = (\lambda^2 - \rho(k)^2)F, \tag{1-10}$$

where

$$L(k) = \frac{d^2}{dt^2} + 2(k_s \coth t + 2k_l \coth 2t)\frac{d}{dt}, \tag{1-11}$$

and the hypergeometric function $F(\lambda, k; a_t)$ of type BC_1 is given by the Jacobi function

$$F(\lambda, k; a_t) = \varphi^{(\alpha,\beta)}_{\sqrt{-1}\,\lambda}(t) = {}_2F_1\bigl(\tfrac{1}{2}(\rho(k) - \lambda),\ \tfrac{1}{2}(\rho(k) + \lambda);\ \alpha + 1;\ -\sinh^2 t\bigr).$$

Here ${}_2F_1$ is the Gauss hypergeometric function. For $\lambda \neq 1, 2, \dots$, there is another solution (1-5) of (1-10) on $(0, \infty)$ given by

$$\Phi^{(\alpha,\beta)}_{-\sqrt{-1}\,\lambda}(t) = (2\cosh t)^{\lambda-\rho(k)}{}_2F_1\bigl(\tfrac{1}{2}(\rho(k) - \lambda),\ \tfrac{1}{2}(\alpha - \beta + 1 - \lambda);\ 1 - \lambda;\ \cosh^{-2} t\bigr), \tag{1-12}$$

which satisfies

$$\Phi^{(\alpha,\beta)}_{-\sqrt{-1}\,\lambda}(t) = e^{(\lambda-\rho)t}(1 + o(t)) \quad \text{as } t \to \infty. \tag{1-13}$$

For $\lambda \notin \mathbb{Z}$ we have

$$\varphi^{(\alpha,\beta)}_{\sqrt{-1}\,\lambda}(t) = c_{\alpha,\beta}(-\sqrt{-1}\,\lambda)\,\Phi^{(\alpha,\beta)}_{-\sqrt{-1}\,\lambda}(t) + c_{\alpha,\beta}(\sqrt{-1}\lambda)\,\Phi^{(\alpha,\beta)}_{\sqrt{-1}\,\lambda}(t),$$

where

$$c_{\alpha,\beta}(-\sqrt{-1}\,\lambda) = c(\lambda, k) = \frac{2^{\rho(k)-\lambda}\Gamma(\alpha + 1)\Gamma(\lambda)}{\Gamma\bigl(\tfrac{1}{2}(\lambda + \rho(k))\bigr)\Gamma(\tfrac{1}{2}(\lambda + \alpha - \beta + 1))}.$$

2. Hypergeometric function of type BC_n

2A. ***Commuting family of differential operators.*** Let n be a positive integer greater than 1 and R be the root system of type BC_n,

$$R_+ = \{e_p,\ 2e_p,\ e_i \pm e_j \ :\ 1 \le p \le n,\ 1 \le i < j \le n\},$$

where $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $E \simeq \mathbb{R}^n$. We call

$$\pm e_p, \quad \pm(e_i \pm e_j), \quad \pm 2e_p,$$

short, middle, and long roots, respectively. We define

$$k_{e_p} = k_s, \quad k_{e_i+e_j} = k_m, \quad k_{2e_p} = k_l,$$

for the multiplicities of short, middle, and long roots, respectively. Hereafter we assume that $k_m = 0$ or 1. Then the terms corresponding to the roots $e_i \pm e_j$ vanish in (1-2) and we have

$$\delta(k)^{\frac{1}{2}} \circ \big(L(k) + (\rho(k), \rho(k))\big) \circ \delta(k)^{-\frac{1}{2}} = \sum_{j=1}^{n} \Big(\partial_{e_j}^2 + \frac{k_s(1-k_s-2k_l)}{(e^{\frac{1}{2}e_j} - e^{-\frac{1}{2}e_j})^2} + \frac{4k_l(1-k_l)}{(e^{e_j} - e^{-e_j})^2} \Big). \tag{2-1}$$

Let $t_j = e_j(\log a)/2$ $(j = 1, \dots, n)$ be coordinates of $A \simeq \mathbb{R}^n$, and

$$a_t = \exp(\textstyle\sum_{j=1}^{n} 2t_j e_j).$$

For $\lambda \in \mathfrak{h}^*$ set $\lambda_j = (\lambda, 2e_j)$. Then we have

$$\rho(k)_j = k_s + 2k_l + 2(n-j)k_m.$$

Let Δ_m be the Weyl denominator associated with middle roots:

$$\begin{aligned}\Delta_m(a_t) &= \textstyle\prod_{\alpha \in R_+, \text{middle roots}} \big(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\big) \\ &= 2^{\frac{1}{2}n(n-1)} \prod_{1 \le i < j \le n} (\cosh 2t_i - \cosh 2t_j).\end{aligned}$$

It is easy to see from (2-1) that

$$\Delta_m^{k_m} \circ \big(L(k) + (\rho(k), \rho(k))\big) \circ \Delta_m^{-k_m} = \sum_{j=1}^{n} L_j + n(k_s + 2k_l)^2, \tag{2-2}$$

where

$$L_j = \frac{\partial^2}{\partial t_j^2} + 2(k_s \coth t_j + 2k_l \coth 2t_j) \frac{\partial}{\partial t_j}. \tag{2-3}$$

Theorem 2.1. *If* $k_m = 0$ *or* 1, *then*

$$\mathbb{D}(k) = \{D_p = \Delta_m^{-k_m} \circ p(L_1, \dots, L_n) \circ \Delta_m^{k_m} \;:\; p \in \mathbb{R}[E]^W\}.$$

In particular, $\mathbb{D}(k)$ *is generated by* D_{p_j} $(j = 1, \dots, n)$, *where* p_j *is the* j*-th elementary symmetric function and*

$$D_{p_1} = L(k) + (\rho(k), \rho(k)).$$

Proof. Since $L_1, \dots, L_n$ mutually commute and

$$\gamma(k)\big(\Delta_m^{-k_m} \circ L_j \circ \Delta_m^{k_m}\big) = \partial_{e_j}^2 - (k_s + 2k_l)^2,$$

the theorem follows from Theorem 1.1. □

Remark 2.2. The right hand side of (1-2) has the form of a Schrödinger operator and Theorem 1.1 states that it defines a completely integrable system. Oshima proved, in [1998], the complete integrability of the Schrödinger operator

$$P = -\frac{1}{2}\sum_{j=1}^{n} \partial_{e_j}^2 + \sum_{1 \le i < j \le n} (u(t_i - t_j) + u(t_i + t_j)) + \sum_{1 \le j \le n} v(t_j),$$

with

$$u(x) = C_1\mathcal{P}(x) + C_2$$
$$v(x) = \frac{C_3\mathcal{P}(x)^2 + C_4\mathcal{P}(x)^3 + C_5\mathcal{P}(x)^2 + C_6\mathcal{P}(x) + C_7}{\mathcal{P}'(x)^2}.$$

If $C_1 = 0$, then a result analogous to Theorem 2.1 holds.

Remark 2.3. If R is an arbitrary reduced root system and $k_\alpha = 0$ or 1 for all $\alpha \in R$, then the right hand side of (1-2) is just the Laplacian on the Euclidean space E. In this case, $\mathbb{D}(k)$ (taking the conjugate by $\Delta_m^{k_m}$) consists of constant coefficient differential operators, and the hypergeometric function is expressed by exponential functions. The case in which all multiplicities are equal to 1 is the case of complex semisimple Lie groups in the sense of Remark 1.3. Theorem 2.1 gives another case in which $\mathbb{D}(k)$ has a simple expression.

2B. ***The hypergeometric function.*** If $k_m = 0$ or 1, then the Harish–Chandra series (1-5) is given by a product of the Harish–Chandra series' of the form (1-12) for the root system $R = BC_1$.

Proposition 2.4. *Assume that* $k_m = 0$ *or* 1 *and let* $\alpha = k_s + k_l - 1/2,\ \beta = k_l - 1/2$. *If* λ *satisfies condition* (1-3), *then*

$$\Phi(\lambda, k; a) = \Delta_m(a_t)^{-k_m} \prod_{j=1}^{n} \Phi^{(\alpha,\beta)}_{-\sqrt{-1}\,\lambda_j}(t_j). \tag{2-4}$$

Proof. In view of (1-11), (1-12), (2-2), and (2-3), the right hand side of (2-4) is a solution of (1-4), where α and β are given by (1-9). We can see by elementary computations of power series that the right hand side of (2-4) has a series expansion of the form (1-5), analogous to the form used in the proof of [Hoogenboom 1982, Theorem 1]. By the uniqueness of the Harish–Chandra series, (2-4) follows. □

By virtue of Proposition 2.4, the hypergeometric function has a simple expression.

Theorem 2.5. *Let*

$$\alpha = k_s + k_l - \tfrac{1}{2}, \quad \beta = k_l - \tfrac{1}{2},$$

and assume that $\alpha \neq 0, -1, -2, \ldots$.

If $k_m = 1$, *then*

$$F(\lambda, k; a_t) = \frac{B}{\prod_{1 \le i < j \le n}(\lambda_i^2 - \lambda_j^2)} \cdot \frac{\det\bigl(\varphi^{(\alpha,\beta)}_{\sqrt{-1}\lambda_i}(t_j)\bigr)_{1 \le i,\, j \le n}}{\Delta_m(a_t)}, \tag{2-5}$$

where B *is given by*

$$B = (-1)^{\frac{1}{2}n(n-1)} 2^{2n(n-1)} \prod_{i=1}^{n-1}\bigl((\alpha+i)^{n-i} i!\bigr). \tag{2-6}$$

If $k_m = 0$, *then*

$$F(\lambda, k; a_t) = \frac{1}{n!}\operatorname{perm}\bigl(\varphi^{(\alpha,\beta)}_{\sqrt{-1}\lambda_i}(t_j)\bigr)_{1 \le i,\, j \le n}, \tag{2-7}$$

where $\operatorname{perm}(M)$ *denotes the permanent*

$$\sum_{\sigma \in S_n} m_{1\sigma(1)} \cdots m_{n\sigma(n)}$$

of matrix $M = (m_{ij})_{1 \le i, j \le n}$.

Proof. First notice that the Weyl group of type BC_n is given by

$$W = \{w = (\varepsilon, \sigma) \in \{-1\}^n \times S_n : w(t_1, \ldots, t_n) = (\varepsilon_1 t_{\sigma(1)}, \ldots, \varepsilon_n t_{\sigma(n)})\}.$$

Assume that $k_m = 1$. The c-function for the middle roots (the product being taken over the middle roots in (1-7)) is given by

$$\begin{aligned}\tilde{c}_m(\lambda, k) &= \prod_{1 \le i < j \le n} \frac{\Gamma\bigl(\frac{1}{2}(\lambda_i + \lambda_j)\bigr)\Gamma\bigl(\frac{1}{2}(\lambda_i - \lambda_j)\bigr)}{\Gamma\bigl(\frac{1}{2}(\lambda_i + \lambda_j) + 1\bigr)\Gamma\bigl(\frac{1}{2}(\lambda_i - \lambda_j) + 1\bigr)} \\ &= \frac{2^{n(n-1)}}{\prod_{1 \le i < j \le n}(\lambda_i^2 - \lambda_j^2)}.\end{aligned}$$

The c-function for e_j and $2e_j$ is given by

$$\begin{aligned}\tilde{c}_{e_j}(\lambda,k)\,\tilde{c}_{2e_j}(\lambda,k) &= \frac{2^{-\lambda_j-k_s+1}\Gamma(\lambda_j)}{\Gamma\big(\frac{1}{2}(\lambda_j+k_s+1)\big)\Gamma\big(\frac{1}{2}(\lambda_j+k_s+2k_l)\big)}\\ &= 2^{-2k_s-2k_l+1}\Gamma\big(k_s+k_l+\tfrac{1}{2}\big)^{-1}c_{\alpha,\beta}(-\sqrt{-1}\,\lambda_j).\end{aligned}$$

We now have

$$\begin{aligned}\tilde{c}(\lambda,k) &= \tilde{c}_m(\lambda,k)\prod_{j=1}^{n}\tilde{c}_{e_j}(\lambda,k)\,\tilde{c}_{2e_j}(\lambda,k)\\ &= \frac{2^{n(n-2k_s-2k_l)}}{\Gamma\big(k_s+k_l+\frac{1}{2}\big)^n\prod_{1\le i<j\le n}(\lambda_i^2-\lambda_j^2)}\prod_{j=1}^{n}c_{\alpha,\beta}(\lambda_j).\end{aligned}$$

The hypergeometric function is given by

$$\begin{aligned}&\Delta_m(a_t)F(\lambda,k;a_t)\\ &\quad= \tilde{c}(\rho(k),k)^{-1}\sum_{w\in W}\tilde{c}(w\lambda,k)\Delta_m(a_t)\Phi(w\lambda,k,a_t)\\ &\quad= B\sum_{\sigma\in S_n,\,\varepsilon\in\{-1\}^n}\frac{1}{\prod_{i<j}(\lambda_{\sigma(i)}^2-\lambda_{\sigma(j)}^2)}\prod_{l=1}^{n}c_{\alpha,\beta}(-\sqrt{-1}\varepsilon_l\lambda_{\sigma(l)})\Phi^{(\alpha,\beta)}_{-\sqrt{-1}\varepsilon_l\lambda_{\sigma(l)}}(t_l)\\ &\quad= B\frac{1}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)}\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\prod_{l=1}^{n}\varphi^{(\alpha,\beta)}_{\sqrt{-1}\,\lambda_{\sigma(l)}}(t_l) = B\frac{\det\big(\varphi^{(\alpha,\beta)}_{\sqrt{-1}\,\lambda_i}(t_j)\big)_{i,j}}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)},\end{aligned}$$

where

$$B = \frac{2^{n(n-1)}}{\tilde{c}(\rho(k),k)\big(2^{2k_s+2k_l-1}\Gamma\big(k_s+k_l+\frac{1}{2}\big)\big)^n}.$$

The formula for B can be obtained by explicit computations.

Next suppose $k_m=0$. Then

$$c_m(\lambda,k) = \lim_{k_m\to 0}\frac{\tilde{c}(\lambda,k)}{\tilde{c}(\rho(k),k)} = \frac{1}{n!}.$$

Here $c_m(\lambda,k)$ is the c-function for the middle roots (the product is taken over the middle roots in (1-7)). (2-7) follows by direct computation, similar to the method used in deriving (2-5). □

Remark 2.6. Let p and q $(p\le q)$ be positive integers and set $k_s=q-p$, $k_m=1$, and $k_l=1/2$. Then the hypergeometric function $F(\lambda,k;a_t)$ is the radial part of the spherical function on $SU(p,q)/S(U(p)\times U(q))$. In this case Theorem 2.1, Proposition 2.4, and Theorem 2.5 were given in [Berezin and Karpelevič 1958] without proof and a complete proof appeared in [Hoogenboom 1982].

We give two corollaries of our results.

First we give a limiting case of the hypergeometric function. We replace (t, λ) by $(\epsilon t, \epsilon^{-1}\lambda)$ and let $\epsilon \searrow 0$. Then the hypergeometric equation (1-10) of type BC_1 becomes

$$\frac{d^2u}{dt^2} + \frac{2\alpha+1}{t}\frac{d^2u}{dt^2} = \lambda^2 u. \tag{2-8}$$

Here we set $\alpha = k_s + k_l + 1/2$. There exists a unique even solution of (2-8) that is regular at 0 and $u(0) = 1$, which is given by

$$\mathcal{J}_\alpha(\sqrt{-1}\lambda t) = 2^\alpha \Gamma(\alpha+1)(\sqrt{-1}\lambda t)^{-\alpha} J_\alpha(\sqrt{-1}\lambda t),$$

where J_α denotes the usual Bessel function. Then it is known [Koornwinder 1984, § 2.3] that

$$\lim_{\epsilon\searrow 0} \varphi^{(\alpha,\beta)}_{\sqrt{-1}\epsilon^{-1}\lambda}(\epsilon t) = \mathcal{J}_\alpha(\sqrt{-1}\lambda t). \tag{2-9}$$

The limit of operator (1-1) becomes

$$L(k)^{\mathrm{rat}} = \sum_{j=1}^{n} \partial_{\xi_j}^2 + \sum_{\alpha\in R_+} \frac{2k_\alpha}{\alpha}\partial_\alpha, \tag{2-10}$$

and we have

$$\lim_{\epsilon\searrow 0} \epsilon^{-n(n-1)} \Delta_m(a_{\epsilon t}) = \prod\nolimits_{\alpha\in R_+,\text{middle roots}} \alpha(\log a_t).$$

We denote the right hand side of the above equation by $\Delta_{m,\mathrm{rat}}(a_t)$. Set

$$L_j^{\mathrm{rat}} = \frac{\partial^2}{\partial t_j^2} + \frac{2k_s+2k_l+2}{t_j}\frac{\partial}{\partial t_j}.$$

Then we have the following explicit expression of a commuting family of differential operators including $L(k)^{\mathrm{rat}}$.

Corollary 2.7. *If $k_m = 0$ or 1, then*

$$\{D_p^{\mathrm{rat}} = \Delta_{m,\mathrm{rat}}^{-k_m} \circ p(L_1, \ldots, L_n) \circ \Delta_{m,\mathrm{rat}}^{k_m} \,:\, p \in \mathbb{R}[E]^W\}$$

forms a commutative algebra of differential operators, which is generated by

$$\Delta_{m,\mathrm{rat}}^{-k_m} \circ p_j(L_1, \ldots, L_n) \circ \Delta_{m,\mathrm{rat}}^{k_m}, \quad (j = 1, \ldots, n),$$

where p_j is the j-th elementary symmetric function. $D_{p_1}^{\mathrm{rat}} = L(k)^{\mathrm{rat}}$ and the principal symbol of $D_{p_j}^{\mathrm{rat}}$ is p_j for $j = 1, \ldots, n$.

By Theorem 2.5 and (2-9), we have the following limit behavior.

Corollary 2.8. *Let $\alpha = k_s + k_l - 1/2$, and assume that $\alpha \neq 0, -1, -2, \ldots$ and $\lambda_j \neq 0$, $t_j \neq 0$, for $j = 1, \ldots, n$.*

If $k_m = 1$, then

$$\lim_{\epsilon \searrow 0} F(\epsilon^{-1}\lambda, k; a_{\epsilon t}) = \frac{B}{\prod_{1 \le i < j \le n}(\lambda_i^2 - \lambda_j^2)} \cdot \frac{\det\bigl(\mathcal{J}_\alpha(\sqrt{-1}\,\lambda_i t_j)\bigr)_{1 \le i,\, j \le n}}{\Delta_{m,\mathrm{rat}}(a_t)}, \tag{2-11}$$

where B is given by (2-6). *If $k_m = 0$, then*

$$\lim_{\epsilon \searrow 0} F(\epsilon^{-1}\lambda, k; a_{\epsilon t}) = \frac{1}{n!} \operatorname{perm}\bigl(\mathcal{J}_\alpha(\sqrt{-1}\,\lambda_i t_j)\bigr)_{1 \le i,\, j \le n}. \tag{2-12}$$

Remark 2.9. In the group case that we mentioned in Remark 2.6, (2-11) was proved by Meaney in [1986]. It gives contraction of spherical functions between symmetric spaces of the noncompact type and the Euclidean type.

The right-hand sides of (2-11) and (2-12) give explicit expressions for the Bessel function of type BC_n from [Opdam 1993, Definition 6.9]. The Bessel function of type BC_n for $k_m = 0$ or 1 is a W-invariant C^∞ joint-eigenfunction of the commuting family of differential operators, given in Corollary 2.7, being equal to 1 at the origin.

The type of limit transition in Corollary 2.8 was given also by Ben Saïd and Ørsted in [2005a; 2005b], and by de Jeu in [2006].

Finally, we give a formula for a Θ-spherical function. Let Ψ denote the set of simple roots in R_+,

$$\Psi = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}.$$

For a subset $\Theta \subset \Psi$, let

$$\langle \Theta \rangle = R \cap \sum_{\alpha \in \Theta} \mathbb{Z}\alpha,$$

and define $\tilde{c}_\Theta(\lambda, k)$ by the product of the form (1-7), where the product is taken over $R_+ \cap \langle \Theta \rangle$, and let

$$c_\Theta(\lambda, k) = \frac{\tilde{c}_\Theta(\lambda, k)}{\tilde{c}_\Theta(\rho(k), k)}.$$

We make a sum

$$F_\Theta(\lambda, k; a) = \sum_{w \in W_\Theta} c_\Theta(w\lambda, k)\, \Phi(w\lambda, k; a). \tag{2-13}$$

The sum of the form (2-13) is important in harmonic analysis of the spherical function on symmetric spaces; see [Ólafsson and Pasquale 2002; Schlichtkrull 1984, Chapter 6; Shimeno 1994].

By Proposition 2.4, we can derive formulae for $F_\Theta(\lambda, k; a)$. For

$$\Theta = \Psi \setminus \{e_1 - e_2, \ldots, e_{j-1} - e_j\} \quad (2 \le j \le n),$$

we have a formula for $F_\Theta(\lambda, k; a_t)$ that is similar to the formula for $F(\lambda, k; a_t)$ in Theorem 2.5.

If $\Theta = \{e_1 - e_2, \dots, e_{n-1} - e_n\}$, then $\langle\Theta\rangle$ is a root system of type A_{n-1}, and we have the following result.

Corollary 2.10. *Assume that $k_m = 0$ or 1 and let $\Theta = \{e_1 - e_2, \dots, e_{n-1} - e_n\}$ and $\alpha = k_s + k_l - 1/2$, $\beta = k_l - 1/2$. Then $F_\Theta(\lambda, k; a_t)$ is holomorphic in λ in the region $\operatorname{Re}\lambda_i > 0$, where $i = 1, \dots, n$. Moreover, we have the following results.*

(i) *Suppose $k_m = 1$ and set $\pi(x_1, \dots, x_n) = \prod_{1\le i<j\le n}(x_i - x_j)$. Then we have*

$$F_\Theta(\lambda, k; a_t) = \frac{\pi(\rho(k))}{\pi(\lambda)} \cdot \frac{\det\big(\Phi^{(\alpha,\beta)}_{\sqrt{-1}\,\lambda_i}(t_j)\big)_{1\le i,\, j\le n}}{\Delta_m(a_t)}. \tag{2-14}$$

Moreover, if $\operatorname{Re}\lambda_i > 0$ $(i = 1, \dots, n)$, then

$$\lim_{u\to\infty} e^{(\rho(k)-\lambda)(\log a_{(u,\dots,u)})} F_\Theta(\lambda, k; a_{(t_1+u,\dots,t_n+u)}) = \frac{\pi(\rho(k))}{\pi(\lambda)} \cdot \frac{\det(e^{\lambda_i t_j})_{1\le i,\, j\le n}}{\pi(e^{2t_1}, \dots, e^{2t_n})}. \tag{2-15}$$

(ii) *If $k_m = 0$, then*

$$F_\Theta(\lambda, k; a_t) = \frac{1}{n!} \operatorname{perm}\big(\Phi^{(\alpha,\beta)}_{\sqrt{-1}\,\lambda_i}(t_j)\big)_{1\le i,\, j\le n}. \tag{2-16}$$

Moreover, if $\operatorname{Re}\lambda_i > 0$ $(i = 1, \dots, n)$, then

$$\lim_{u\to\infty} e^{(\rho(k)-\lambda)(\log a_{(u,\dots,u)})} F_\Theta(\lambda, k; a_{(t_1+u,\dots,t_n+u)}) = \frac{1}{n!} \operatorname{perm}(e^{\lambda_i t_j})_{1\le i,\, j\le n}. \tag{2-17}$$

Proof. $F_\Theta(\lambda, k; a_t)$ is holomorphic in the region $\operatorname{Re}\lambda_i > 0$ $(i = 1, \dots, n)$ by [Ólafsson and Pasquale 2002, Theorem 8]. (2-14) and (2-16) follow by simple computations. (2-15) and (2-17) follow from (1-13). □

Remark 2.11. (i) The right hand sides of (2-15) and (2-17) are hypergeometric functions of type A_{n-1} with the multiplicity 1 and 0 respectively. Namely, the right hand side of (2-15) is the spherical function on $SL(n, \mathbb{C})/SU(n)$ (see [Helgason 1984, Chapter IV Theorem 5.7]) and (2-17) is the normalized average of the exponential function $e^{(\lambda,t)}$ under the action of the symmetric group.

(ii) By [Shimeno 1994, Proposition 2.6, Remark 6.13], the spherical function for a one-dimensional K-type $(\tau_{-\ell_1}, \tau_{-\ell_2})$ on $SU(p, q)$ can be written as the hypergeometric function $F(\lambda, k; a_t)$ with $k_s = m/2 - \ell_2$, $k_m = 1$, $k_l = 1/2 - \ell_1 - \ell_2$. Here, $m = 1$ and $\ell_1 = \ell_2$ if $p \ne q$, and $m = 0$ if $p = q$. Thus, spherical functions for one-dimensional K-types on $SU(p, q)$ are given by Theorem 2.1. Conversely, by considering the universal covering group of $SU(p, q)$, we can take ℓ_1, ℓ_2 arbitrary complex numbers; hence the hypergeometric function (2-5) for any k_s and k_l corresponds to a spherical function on $\widetilde{SU(p, p)}$.

By the preceding observation, the Plancherel formula for the integral transform with the kernel $F(\lambda, k; a)$ with $k_m = 1$ is a special case of [Shimeno 1994, Theorem 6.11]. Notice that low dimensional spectra including discrete spectra appear in general. It seems to be possible to give an alternative proof of the Plancherel formula by rank one reduction as in [Meaney 1986, Theorem 22].

(iii) In Theorem 2.1 we give an explicit formula for the hypergeometric function of type BC_n with $k_m = 0, 1$ and k_s, k_l arbitrary. We obtain a formula of the hypergeometric function for $k_m \in \mathbb{Z}$ by applying Opdam's hypergeometric shift operator corresponding to the middle roots, which is a differential operator of order $n(n-1)/2$; see [Heckman and Schlichtkrull 1994, Definition 3.2.1].

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