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A PRESENTATION FOR THE AUTOMORPHISMS OF THE 3-SPHERE THAT PRESERVE A GENUS TWO HEEGAARD SPLITTING

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Scharlemann constructed a connected simplicial 2-complex Γ with an action by the group $\mathcal{H}_2$ of isotopy classes of orientation-preserving homeomorphisms of S^3 that preserve the isotopy class of an unknotted genus 2 handlebody V. In this paper we prove that the 2-complex Γ is contractible. Therefore we get a finite presentation of $\mathcal{H}_2$.

1. Introduction

Let $\mathcal{H}_g$ be the group of isotopy classes of orientation-preserving homeomorphisms of S^3 that preserve the isotopy class of an unknotted genus g handlebody V. In [1933], Goeritz proved that $\mathcal{H}_2$ is finitely generated. In 1977, Goeritz's theorem was generalized to arbitrary genus $g \geq 2$ by Jerome Powell [1980]. In 2003, Martin Scharlemann noticed that Powell's proof contains a serious gap. Scharlemann [2004] gave a modern proof of Goeritz's theorem by introducing a simplicial 2-complex Γ, with an action by $\mathcal{H}_2$, that deformation retracts onto a graph $\tilde{\Gamma}$. Given any two distinct vertices v, $\tilde{v}$ of Γ, Scharlemann constructed a vertex u in Γ that is adjacent to v and "closer" to $\tilde{v}$ (by "closer" we mean the intersection number of u and $\tilde{v}$; see Definition 1). Hence $\mathcal{H}_2$ acts on the connected graph $\tilde{\Gamma}$ and is generated by the isotopy classes of elements denoted by α, β, γ, and δ (see Section 2 for a complete description). In this paper we study the geometry of Γ by showing that u is essentially unique (for a precise statement see Proposition 2). We derive the following theorem.

Theorem 1. *The graph $\tilde{\Gamma}$ is a tree, and shortest paths can be calculated algorithmically.*

Note that $\tilde{\Gamma}$ is locally infinite. So calculating paths is not trivial. We also get

Theorem 2. (i) *$\mathcal{H}_2$ has generators $[\alpha]$, $[\beta]$, $[\gamma]$, and $[\delta]$ and relations $[\alpha]^2 = [\gamma]^2 = [\delta]^3 = [\alpha\gamma]^2 = [\alpha\delta\alpha\delta^{-1}] = [\alpha\beta\alpha\beta^{-1}] = 1$, $[\gamma\beta\gamma] = [\alpha\beta]$, and $[\delta] = [\gamma\delta^2\gamma]$.*

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(ii) $\mathcal{H}_2 \cong (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \underset{\mathbb{Z}_2 \oplus \mathbb{Z}_2}{*} (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2$

2. Preliminaries

We give a description of the 2-complex Γ introduced in [Scharlemann 2004], to which we refer for details about Γ.

Let V be an unknotted handlebody of genus two in S^3, and let W be the closure of its complement. Let T be the boundary of V. Then T is a genus two Heegaard surface for S^3. Let $\mathcal{H}_2$ denote the group of isotopy classes of orientation-preserving homeomorphisms of S^3 that leave the genus two handlebody V invariant. A sphere P in S^3 is called a *reducing sphere* for T if P intersects T transversely in a simple closed curve which is homotopically nontrivial on T. For any reducing sphere P for T, let c_P denote $P \cap T$, and let v_P denote the isotopy class of c_P on T.

Definition 1. For any two reducing spheres R, Q for T, define the intersection number of v_R and v_Q as

$$v_R \cdot v_Q = \min_{\substack{c_{R'} \in v_R \\ c_{Q'} \in v_Q}} |c_{R'} \cap c_{Q'}|,$$

where $|c_{R'} \cap c_{Q'}|$ is the geometric intersection number of $c_{R'}$ with $c_{Q'}$.

Let Γ be a complex whose vertices are isotopy classes of reducing spheres for T. A collection $P_0, \dots, P_n$ of reducing spheres bounds an n-simplex in Γ if and only if $v_{P_i} \cdot v_{P_j} = 4$ for all $0 \le i \ne j \le n$. In fact $n \le 2$; see [Scharlemann and Thompson 2003, Lemma 2.5]. So Γ is a simplicial 2-complex. See Figure 1 for a local picture of Γ and a picture of three curves forming the vertices of a 2-simplex in Γ. Let $\triangle$ be any 2-simplex of Γ. We denote by $S_\triangle$ the "spine" of $\triangle$, which is the subcomplex of the barycentric subdivision consisting of all closed 1-simplices that contain the barycenter and a vertex of $\triangle$. Clearly $\triangle$ deformation retracts onto $S_\triangle$. Let

$$\tilde{\Gamma} = \bigcup_{\triangle} S_\triangle.$$

So $\tilde{\Gamma}$ is a graph. Since no two 2-simplices of Γ share an edge [Scharlemann and Thompson 2003, Lemma 2.5], the simplicial 2-complex Γ deformation retracts onto the graph $\tilde{\Gamma}$.

A *belt curve* on a genus two surface is a homotopically nontrivial separating simple closed curve. Let P denote a reducing sphere whose intersection with T is a belt curve, which we denote c_P. The reducing sphere P divides S^3 into two 3-balls $B^\pm$ whose intersections with the genus two surface T are two genus one surfaces $T^\pm = T \cap B^\pm$, each having one boundary component. The surface T^- (respectively T^+) contains two simple closed curves B, Z (respectively C, Y)

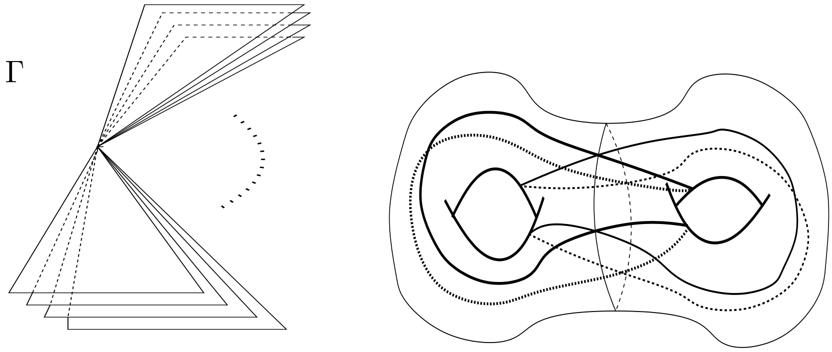


Figure 1. Left: locally Γ. Right: three curves forming the vertices of a 2-simplex in Γ.

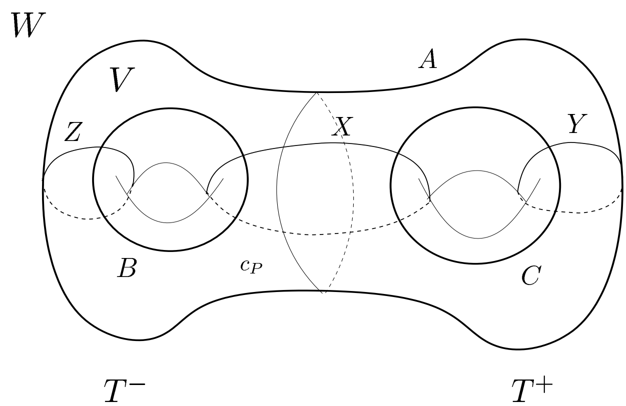


Figure 2. The curves c_P, A, B, C, X, Y, and Z.

meeting at one point. The curve B (respectively C) bounds a nonseparating disc in W which is homotopically nontrivial in V. The curve Z (respectively Y) bounds a nonseparating disc in V which is homotopically nontrivial in W. The genus two surface T contains two disjoint simple closed curves A and X. The curve A is homotopically nontrivial in V, disjoint from B and C, bounds a nonseparating disc in W, and intersects Z and Y at one point. The curve X is homotopically nontrivial in W, disjoint from Z, Y and A, bounds a nonseparating disc in V, and intersects B and C at one point. See Figure 2.

Throughout this paper, unless otherwise stated, whenever we choose a reducing sphere R for T such that $v_R \neq v_P$, we will assume that the curve c_R intersects c_P, B, C, Y, Z transversely and minimally and intersects A transversely.

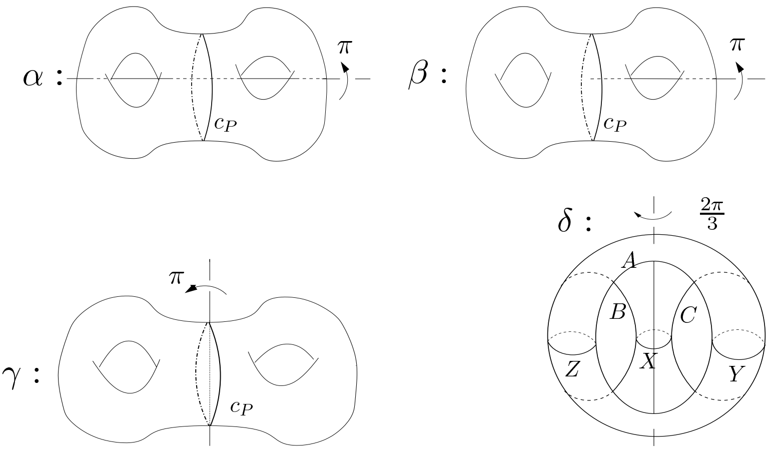


Figure 3. Homeomorphisms α, β, γ and δ.

There exist three automorphisms α, β, γ of S^3 with the following properties. The automorphism α is an orientation-preserving homeomorphism of S^3 that preserves V and P and that maps the curves A, B, C to A, B, C respectively by an orientation-reversing map. The homeomorphism α is the hyperelliptic involution that preserves every simple closed curve (up to isotopy). The automorphism β is an orientation-preserving homeomorphism of S^3 that preserves V and P, fixes T^- pointwise, and maps C to C and Y to Y by an orientation-reversing map. Also $|A \cap \beta(X)| = 2$. The automorphism γ preserves V and P and maps the curves c_P to c_P and A to A by an orientation-reversing map. See Figure 3. Scharlemann [2004] showed that $\mathscr{H}_2$ is generated by the isotopy classes $[\alpha]$, $[\beta]$, $[\gamma]$, and $[\delta]$, where δ is any orientation-preserving homeomorphism of S^3 such that $\delta(V) = V$ and $v_P \cdot v_{\delta(P)} = 4$. In this paper we will take δ, as follows. Consider the genus two handlebody V as a regular neighborhood of a sphere, centered at the origin, with three holes. The homeomorphism δ is a $2\pi/3$ rotation of V about the vertical z-axis. See Figure 3.

3. Arc families of reducing spheres on $T^{\pm}$

Definition 2. Denote any oriented curve D on T by $\overrightarrow{D}$ and the curve oriented in the direction opposite to $\overrightarrow{D}$ by $\overleftarrow{D}$.

Orient the curves A, B, C, X, Y, Z in such a way that $\delta^2(\overrightarrow{A}) = \delta(\overrightarrow{B}) = \overrightarrow{C}$ and $\delta^2(\overrightarrow{X}) = \delta(\overrightarrow{Y}) = \overrightarrow{Z}$.

Definition 3. For any oriented properly embedded arc $\nu \subset T^{\pm}$, we may write $[\nu] \in H_1(T^{\pm}, \partial T^{\pm}; \mathbb{Z})$ as $a[\mu] + b[\lambda]$ where $\mu = \overrightarrow{Z}$ and $\lambda = \overrightarrow{B}$ if $\nu \subset T^-$, and $\mu = \overrightarrow{Y}$ and $\lambda = \overrightarrow{C}$ if $\nu \subset T^+$. The slope of ν is defined to be $|a/b| \in \mathbb{Q}^+ \cup \infty$.

Definition 4. For any reducing sphere Q such that $v_Q \neq v_P$, let $N(Q, T^{\pm}, a)$ denote the number of arcs in $Q \cap T^{\pm}$ of slope a.

Definition 5. Up to isotopy, there are natural homeomorphisms $\Omega, \Psi : S^3 \to S^3$, where Ω maps V to W and $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}, \overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{Z}$ to $\overleftarrow{X}, \overleftarrow{Y}, \overleftarrow{Z}, \overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}$, respectively, and Ψ maps W to W and $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}, \overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{Z}$ to $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}, \overleftarrow{X}, \overleftarrow{Y}, \overleftarrow{Z}$, respectively; see Figure 4. Let $\Theta = \Psi\Omega$.

Proposition 1. *Let Q be a reducing sphere for T such that $v_Q \neq v_P$. Then $N(Q, T^-, a) = N(Q, T^+, 1/a)$.*

Proof. Without loss of generality, we may assume that $Q = w(P)$ where w is a word in α, β, γ and δ.

We claim $\Theta(c_Q) = c_Q$. The proof is as follows.

The hyperelliptic involution α preserves the isotopy class of any simple closed curve on T. After an isotopy, we may assume that $\alpha(c_Q) = c_Q$. Let us write w as $a_1 a_2 \cdots a_n$ where $a_i \in \{\alpha, \beta^{\pm 1}, \gamma, \delta^{\pm 1}\}$. The homeomorphism Θ satisfies $\Theta\alpha = \alpha\Theta$, $\Theta\beta = \alpha\beta\Theta$, $\Theta\gamma = \alpha\gamma\Theta$, $\Theta\delta = \delta\Theta$, and $\Theta(c_P) = c_P$. Then $\Theta(c_Q) = \Theta(w(c_P)) = \Theta(a_1 a_2 \cdots a_n(c_P)) = b_1 b_2 \cdots b_n \Theta(c_P)$, where b_i is α if $a_i = \alpha$, $\alpha\beta$ if $a_i = \beta$, $\alpha\gamma$ if $a_i = \gamma$, and δ if $a_i = \delta$. So $b_1 b_2 \cdots b_n \Theta(c_P) = b_1 b_2 \cdots b_n(c_P) = a_1 a_2 \cdots a_n(c_P) = w(c_P) = c_Q$.

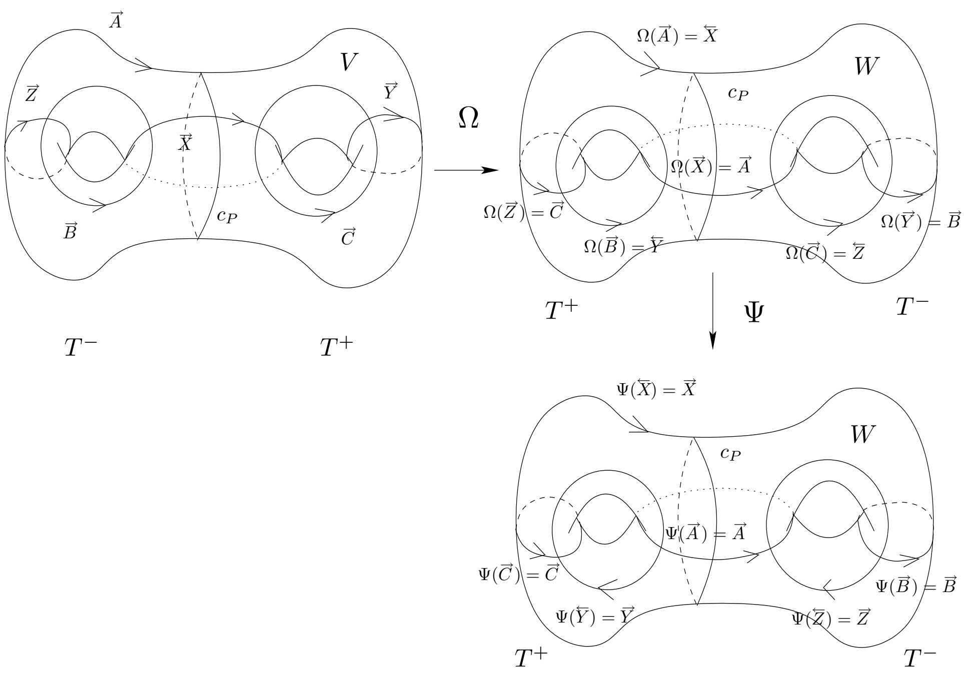


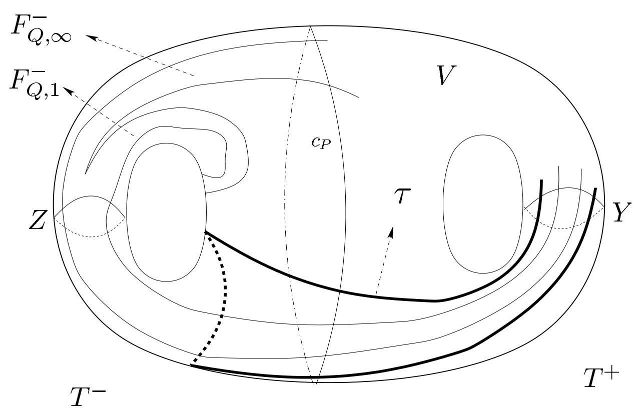
Figure 4. Homeomorphism $\Theta = \Psi\Omega$.

Since Θ maps the curves A, B, C, X, Y, Z to X, Y, Z, A, B, C, respectively, it takes the arcs of c_Q of slope a on T^- to the arcs of c_Q of slope $1/a$ on T^+. □

Definition 6. For any reducing sphere Q for T such that $v_Q \neq v_P$, let $F^{\pm}_{Q,a}$ denote the arc family of c_Q on $T^{\pm}$ of slope a.

Lemma 1. *Suppose Q is any reducing sphere for T such that $v_Q \neq v_P$. Then $N(Q, T^-, 0) \neq N(Q, T^-, \infty)$.*

Proof. Suppose that $N(Q, T^-, 0) = N(Q, T^-, \infty) = m$. The number m cannot be 0 because the curve c_Q must have an arc of slope 0 in either T^- or in T^+ by [Scharlemann and Thompson 2003, Lemma 4]. By Proposition 1, $N(Q, T^+, 0) = N(Q, T^+, \infty) = m$ and $N(Q, T^-, 1) = N(Q, T^+, 1)$. The curve c_Q bounds a disc in V. So c_Q must have a "wave" τ [Volodin et al. 1974] with respect to one of the curves Y or Z. Say it is Y, as illustrated below.



Then the arc τ of c_Q starts at Y, goes to T^-, and then comes back to Y on the same side without touching Z. So all the arcs of c_Q intersecting Z must intersect the arc on Y that is bounded by ends of τ. Then we get

$$N(Q, T^-, \infty) + N(Q, T^-, 1) + 2 \leq N(Q, T^+, \infty) + N(Q, T^+, 1),$$

a contradiction. □

Notation 1. Let Q be a reducing sphere for T.

- If $N(Q, T^-, 0) = n \neq 0$ then $e_{01}, e_{02}, \ldots, e_{0n}, e_{1n}, e_{1n-1}, \ldots, e_{11}$ will denote consecutive end points on c_P of the arcs in $F^-_{Q,0}$, where e_{0j} and e_{1j} are end points of the same arc; h_{01}, h_{02}, $\ldots$, h_{0n}, h_{1n}, $h_{1n-1}, \ldots, h_{11}$ will denote consecutive end points on c_P of the arcs in $F^+_{Q,\infty}$, where h_{0j} and h_{1j} are end points of the same arc (the existence of h_{ij} is guaranteed by Proposition 1).
- If $N(Q, T^-, \infty) = m \neq 0$ then $g_{01}, g_{02}, \ldots, g_{0m}, g_{1m}, g_{1m-1}, \ldots, g_{11}$ will denote consecutive end points on c_P of the arcs in $F^-_{Q,\infty}$, where g_{0j} are g_{1j}

are end points of the same arc; f_{01}, $f_{02}, \dots, f_{0m}$, f_{1m}, $f_{1m-1}, \dots, f_{11}$ will denote consecutive end points on c_P of the arcs in $F^+_{Q,0}$, where f_{0j} and f_{1j} are end points of the same arc.

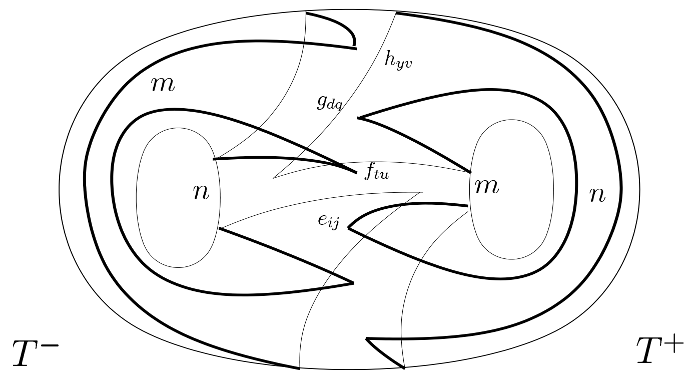
- If $N(Q, T^-, 1) = p \neq 0$ then $k_{01}, k_{02}, \dots, k_{0p}, k_{1p}, k_{1p-1}, \dots, k_{11}$ will denote consecutive end points on c_P of the arcs in $F^-_{Q,1}$ where k_{0j} and k_{1j} are end points of the same arc; l_{01}, $l_{02}, \dots, l_{0p}$, l_{1p}, $l_{1p-1}, \dots, l_{11}$ will denote end points on c_P of the arcs in $F^+_{Q,1}$, where l_{0j} and l_{1j} are end points of the same arc.

Lemma 2. *Let Q be a reducing sphere for T such that*

$$N(Q, T^-, 0) = n > N(Q, T^-, \infty) = m > N(Q, T^-, 1) = 0.$$

Then $\{f_{ij} \mid i = 0, 1$ *and* $j = 1, m\} \subseteq \{e_{ij} \mid i = 0, 1$ *and* $j = 2, \dots, n-1\}$.

Proof. Suppose the contrary, as illustrated below.



Then c_Q does not have a “wave” τ [Volodin et al. 1974] with respect to the curve Y or the curve Z. Therefore c_Q cannot bound a disc in V, a contradiction. □

Proposition 2. *Let v and $\tilde{v}$ be any two distinct vertices of Γ such that $v \cdot \tilde{v} \neq 4$. Then there exists unique vertex u of Γ such that*

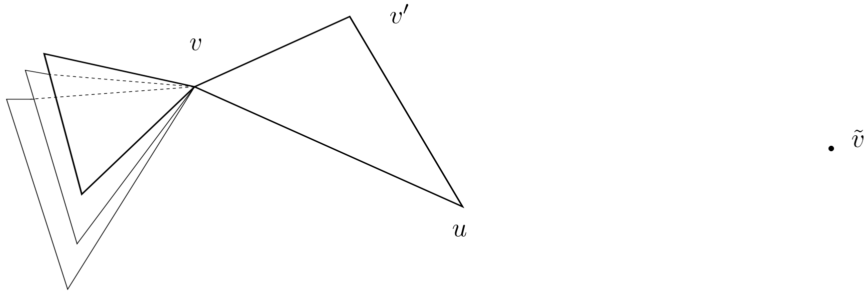
(i) $u \cdot v = 4$,

(ii) $u \cdot \tilde{v} < v \cdot \tilde{v}$, *and*

(iii) $u \cdot \tilde{v} < v' \cdot \tilde{v}$ *for any vertex v' of Γ such that $v' \neq u$ and $v' \cdot v = 4$.*

Moreover, there is at most one vertex v'' of Γ satisfying $v \cdot v'' = 4$ and $u \cdot \tilde{v} < v'' \cdot \tilde{v} \leq v \cdot \tilde{v}$. In this case $v'' \cdot u = 4$.

The proposition is illustrated below.



Proof. Let v and $\tilde{v}$ be any two vertices of Γ such that $v \neq \tilde{v}$ and $v \cdot \tilde{v} \neq 4$. Since the group $\mathcal{H}_2$ is transitive on the vertices of Γ, we may assume that $v = v_P$ and $\tilde{v}$ is a vertex of Γ such that $\tilde{v} \neq v_P$ and $v_P \cdot \tilde{v} \neq 4$. Then some word w in α, γ, β and δ has $w(c_P) \in \tilde{v}$. Let Q denote the reducing sphere $w(P)$. Since Q is not isotopic to P there must be some arcs in $c_Q \cap T^{\pm}$. By [Scharlemann 2004, Lemma 4] there is an arc of c_Q of slope 0 either on T^- or on T^+. Suppose it is on T^-. Let e_{ij}, g_{dq}, k_{rs}, f_{tu}, h_{yv}, l_{wz} denote the end points of the arcs of $c_Q \cap T^{\pm}$ as in Notation 1. Possible cases for the arc families in $c_Q \cap T^{\pm}$ and their configurations, up to the action of a power of β, are the following:

Case I. If

$$N(Q, T^-, 0) = m, \qquad N(Q, T^-, 1/k) = a,$$
$$N(Q, T^-, \infty) = 0, \qquad N(Q, T^-, 1/(k+1)) = b,$$

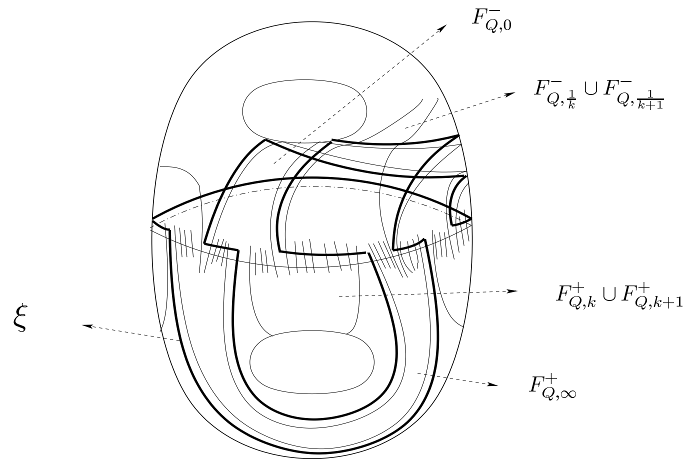
where $k \geq 1$, then

$$N(Q, T^+, \infty) = m, \qquad N(Q, T^+, k) = a,$$
$$N(Q, T^+, 0) = 0, \qquad N(Q, T^+, k+1) = b$$

by Proposition 1. Scharlemann in [2004, Lemma 5] constructs a reducing sphere R satisfying (i) and (ii) (that is, $v_R \cdot v_P = 4$ and $v_R \cdot v_Q < v_P \cdot v_Q$). We will show that up to isotopy the reducing sphere R also satisfies (iii). Scharlemann's reducing sphere will be given explicitly in the various cases of the proof. Let $n = a + b$.

Case I.A: $n \neq 0$. Let us label end points of the arcs in $c_Q \cap T^+$ of slope different from ∞ as $d_1, d_2, \ldots, d_{2n}$. Then it is not hard to show $\{e_{ij}\} \nsubseteq \{d_i\}$ by an argument similar to the proof of Lemma 2.

Case I.A.1: $\{d_i\} \nsubseteq \{e_{ij}\}$. See the figure below. Set $p = |\{e_{ij}\} \cap \{h_{ij}\}|/2$ then $1 \leq p < m$. Consider the curve ξ shown in the figure. It is easy to see that ξ bounds a disc in V and a disc in W. So ξ is the intersection of a reducing sphere S with T. Denote ξ by c_S. The reducing sphere S satisfies $v_S \cdot v_Q \leq |c_S \cap c_Q| = 2(n - m + 2p) < 2(n + m) = v_P \cdot v_Q$ and $v_S \cdot v_P = 4$.



Claim 1. $v_S \cdot v_Q = |c_S \cap c_Q|$.
Claim 2. $v_{\beta^i(S)} \cdot v_Q,\ v_{\beta^i\gamma(S)} \cdot v_Q > 2(n+m)$ for $i \neq 0$.

Proof of Claim 1. It suffices to show that there is no bigon on T formed by the curves c_S and c_Q. We may assume that c_S intersects c_Q in a neighborhood $N \subseteq T$ of c_P where $N \cap (B \cup Z \cup C \cup Y) = \varnothing$. The neighborhood N has two boundary components N^- and N^+. Say $N^\pm \subset T^\pm$. The set $c_S \cap N$ consists of four arcs ν_1, ν_2, ν_3, ν_4. Assume that end points of the arcs ν_1, ν_2, ν_3, and ν_4 on N^- are lined up consecutively as $N^- \cap \nu_1$, $N^- \cap \nu_2$, $N^- \cap \nu_3$, and $N^- \cap \nu_4$. The curve c_S has two arcs a_1 and a_2 on T^- of slope 0 and two arcs b_1 and b_2 on T^+ of slope ∞. Assume that $\nu_i \cap a_1 \neq \varnothing$ for $i = 1, 2$ and $\nu_1 \cap b_1 \neq \varnothing$. See Figure 5. There are eight regions $D_1, \dots, D_8$ on N that can contain a vertex of a bigon. The regions $D_1, \dots, D_8$ are shown in Figure 5. Any bigon should contain two of them. After an isotopy, we may assume that $\alpha(c_Q) = c_Q$ and $\alpha(c_S) = c_S$. Then $\alpha(D_i) = D_{i+2}$ for $i = 1, 2$, and $\Theta(\{D_i \mid i = 1, \dots, 4\}) = \{D_i \mid i = 5, \dots, 8\}$ (see Definition 5 for Θ). So it is enough to check if D_i is a part of a bigon for $i = 1, 2$.

D_1: The region D_1 is part of a region $\widetilde{D}_1$ in T whose four consecutive sides are x, a_1, y, and x', where $y \in F^-_{Q,0}$ and $x, x' \in F^+_{Q,k} \cup F^+_{Q,k+1}$. See Figure 6(a). If $\widetilde{D}_1$ is a bigon then $v_Q \cdot v_P < 2(n+m)$, a contradiction.

D_2:
- If $b = 0$ then $a \neq 0$. Then D_2 is part of a region $\widetilde{D}_2$ whose five sides are x, a_1, y, y', x' where $x, x' \in F^+_{Q,\infty}$ and $y, y' \in F^-_{Q,1/k}$. See Figure 6(b). If $\widetilde{D}_2$ is a bigon then $v_Q \cdot v_P < 2(n+m)$, a contradiction.
- If $a, b \neq 0$ then D_2 is part of a region $\widetilde{D}_2$ whose five sides are x, a_1, y, y', x' where $x, x' \in F^+_{Q,\infty}$, $y \in F^-_{Q,1/(k+1)}$ and $y' \in F^-_{Q,1/k}$. See Figure 6(c). If $\widetilde{D}_2$ is a bigon then $v_Q \cdot v_P < 2(n+m)$, a contradiction.

By the cases above, $v_S \cdot v_Q = |c_S \cap c_Q|$.

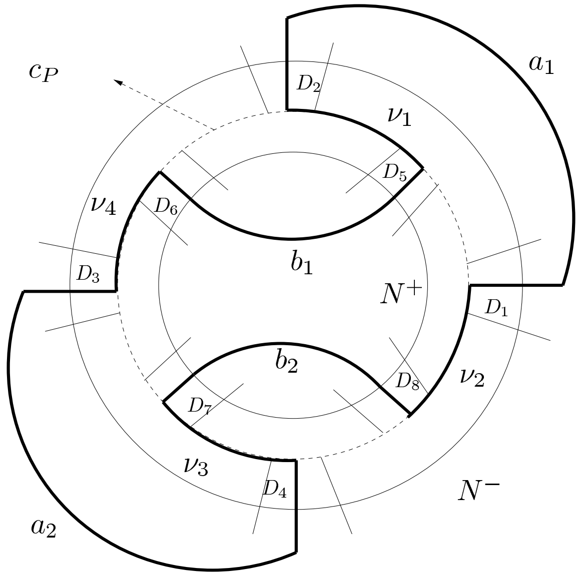


Figure 5

Figure 7 shows the intersection of a reducing sphere R' with the surface T. Notice that $R' \in v_{\gamma S}$ and $v_S \cdot v_{\gamma S} = 4$. By an argument similar to the proof of Claim 1 we can show that $v_{R'} \cdot v_Q = |c_{R'} \cap c_Q| = 4kb + 4(k-1)a + 2m + 2n = v_{\gamma S} \cdot v_Q \geq 2m + 2n$. □

Proof of Claim 2. We will do the calculation for $i = \pm 1$. The general case is similar. We may assume that $\beta^i(c_S)$ and $\beta^i\gamma(c_S)$ intersect c_Q in a neighborhood N described in the proof of Claim 1. By an argument similar to the proof of Claim 1, we get

- $v_{\beta(S)} \cdot v_Q = 4p + 2m + 6n > 2(n+m)$. See Figure 8(a).
- $v_{\beta^{-1}(S)} \cdot v_Q = 6m + 2n - 4p > 2(n+m)$. See Figure 8(b).
- $v_{\beta\gamma(S)} \cdot v_Q = 4kb + 4(k-1)a + 4m + 2n + 2p > 2(n+m)$. See Figure 9(a).
- $v_{\beta^{-1}\gamma(S)} \cdot v_Q = 4kb + 4(k-1)a + 6m + 6n - 4p > 2(n+m)$. See Figure 9(b).

This implies that the vertex $v_R = v_S$ and satisfies the conditions of Proposition 2. □

Case I.A.2: $\{d_i\} \subseteq \{e_{ij}\}$. See Figure 10. Set $p = |\{e_{0j}\} \cap \{h_{0j}\}|$. Then $0 < p \leq m-n$. Either $p < m-n-p$ or $m-n-p < p$. Assume $p < m-n-p$. Consider the curve ξ shown in Figure 10. The curve ξ is an intersection of a reducing sphere S with T. Denote ξ by c_S. Notice that $v_S \cdot v_P = 4$.

By an argument similar to the proof of Case I.A.1, we get

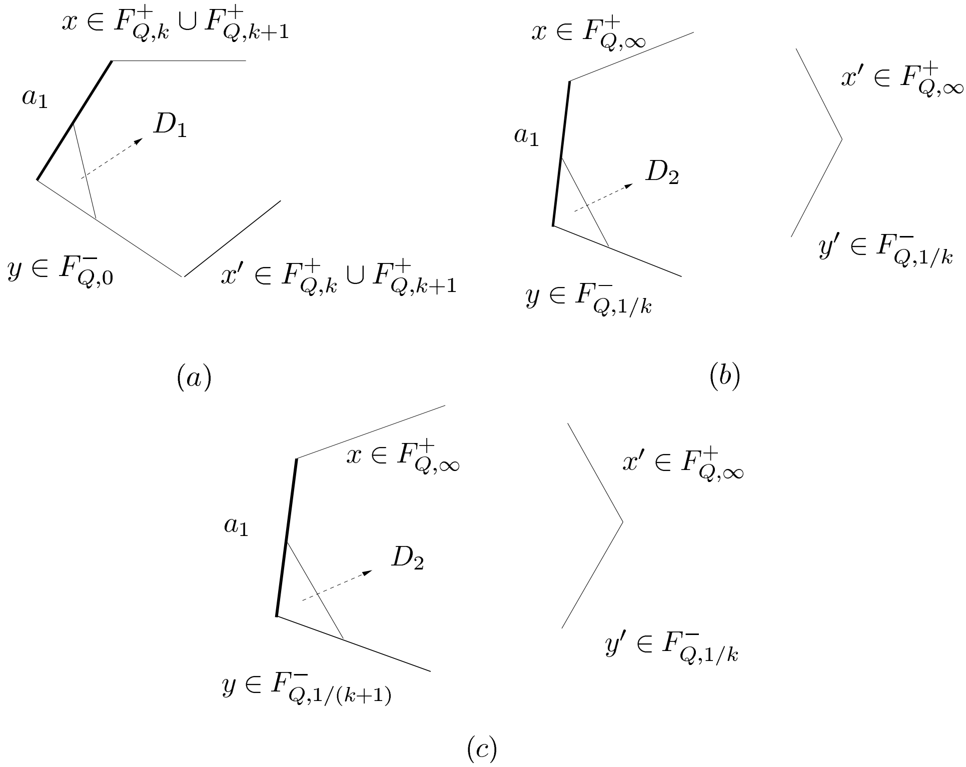


Figure 6

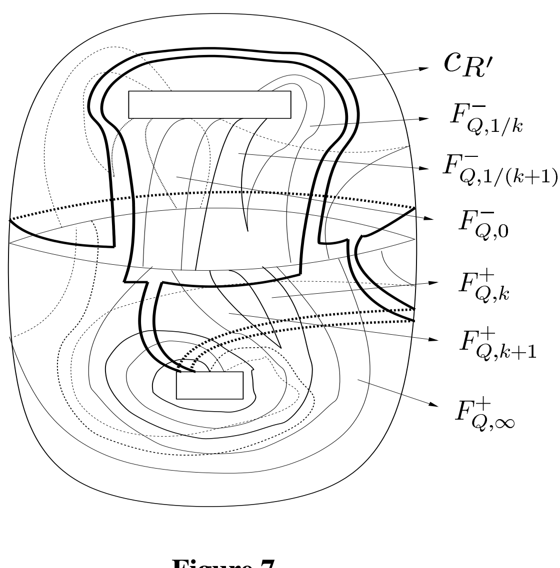


Figure 7

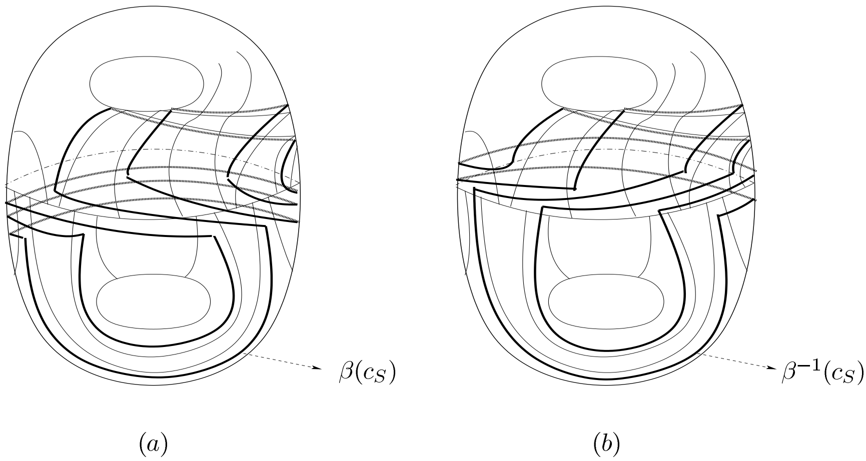


Figure 8

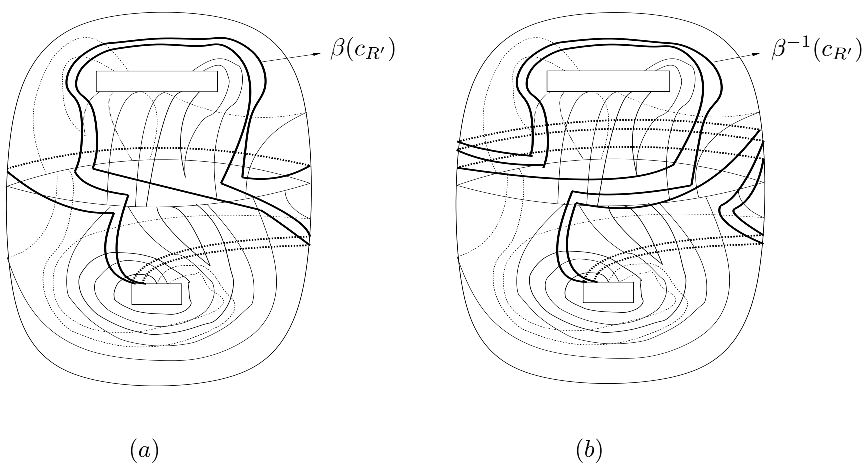


Figure 9

- $v_S \cdot v_Q = |c_S \cap c_Q| = 2(m-n-2p) < v_P \cdot v_Q = 2(n+m)$;
- $v_S \cdot v_{\gamma(S)} = 4$;
- $v_{\gamma(S)} \cdot v_Q = 4kb + 4(k-1)a + 2(m+n) \geq 2(m+n)$ (see Figure 11);
- $v_{\beta^i(S)} \cdot v_Q$, $v_{\beta^i \gamma(S)} \cdot v_Q > 2(n+m)$ for $i \neq 0$.

This implies that the vertex $v_R = v_S$ and satisfies the conditions of Proposition 2.

Case I.B: $n = 0$. This is a special case of Case I.A.2.

Case II: $N(Q, T^-, 0) = m$ and $N(Q, T^-, \infty) = n \neq 0 = N(Q, T^-, 1)$. In this case, $N(Q, T^+, 0) = n$ and $N(Q, T^+, \infty) = m \neq 0 = N(Q, T^+, 1)$ by Proposition 1.

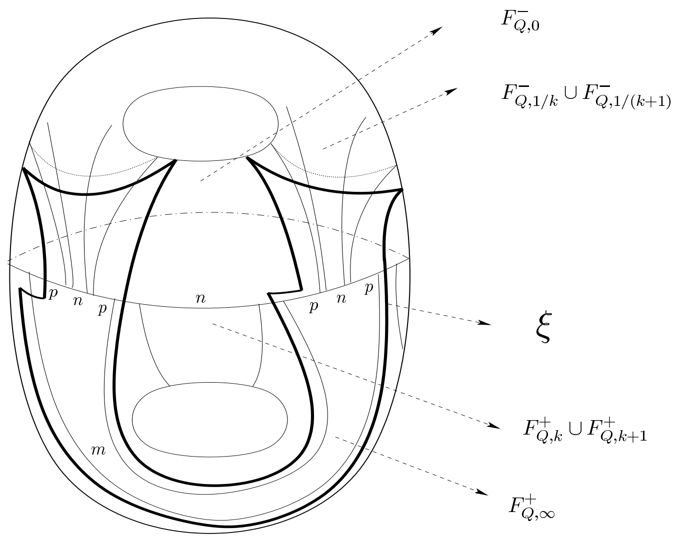


Figure 10

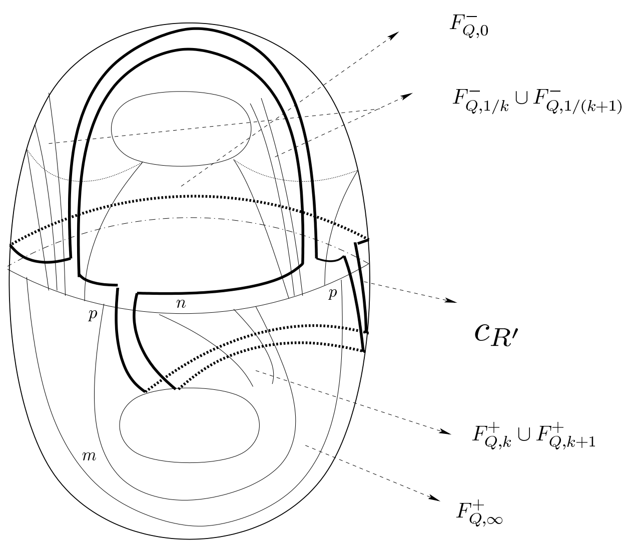


Figure 11. The curve $c_{R'}$ in the figure is $R' \cap T$ for some reducing sphere R' for T satisfying $R' \in v_{\gamma S}$.

By Lemma 1, $m \neq n$. Suppose $m < n$. By Lemma 2,

$$\{e_{ij} \mid i = 0, 1 \text{ and } j = 1, \ldots, m\} \subseteq \{f_{ij} \mid i = 0, 1 \text{ and } j = 2, \ldots, n-1\}.$$

By the argument in [Scharlemann 2004, Lemma 5], we get two nonisotopic reducing spheres for T that satisfy (i) and (ii). Let us call S the one having an arc on

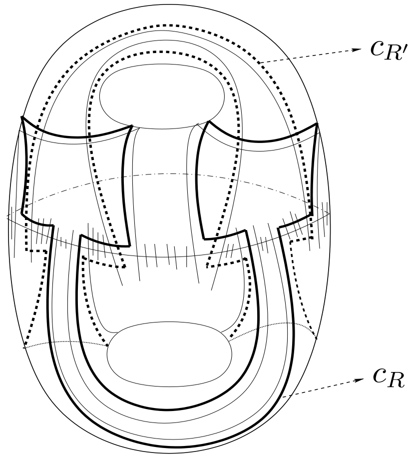


Figure 12

T^- of slope 0 and S' the one having an arc on T^+ of slope 0. Figure 12 shows the intersections of two reducing spheres R and R' with T. It is easy to see that $R \in v_S$ and $R' \in v_{S'}$.

Let $p = |\{g_{0j}\} \cap \{f_{0j}\}|$. Then $0 < p \leq m-n$. Either $p < m-n-p$ or $m-n-p < p$. Assume $p < m-n-p$. Then by an argument similar to the proof of Case I.A.1, we can show that $2n+2m = v_P \cdot v_Q > v_R \cdot v_Q = 2n-2m > v_{R'} \cdot v_Q = 2(n-m-2p)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > 2n+2m$ for $i \neq 0$.

Case III: $N(Q, T^-, 0) = m$, $N(Q, T^-, \infty) = n$, and $N(Q, T^-, 1) = p$ where $m, n, p \neq 0$. In this case, $N(Q, T^+, 0) = n$, $N(Q, T^+, \infty) = m$, $N(Q, T^+, 1) = p$ by Proposition 1. By Lemma 1, $m \neq n$. Say $m > n$.

The curves A, B, C, and c_P divide T into four punctured discs T_f^-, T_b^-, T_f^+, and T_b^+, where $T_f^- \cup T_b^- = T^-$ and $T_f^+ \cup T_b^+ = T^+$. This division also gives two pairs of pants $T_f^- \cup T_f^+ = P_f$ and $T_b^- \cup T_b^+ = P_b$. Let $c_f = P_f \cap c_P$ and $c_b = P_b \cap c_P$.

Let K be a reducing sphere intersecting the interior of T^- in a simple closed curve parallel to c_P. The reducing sphere K divides T into two parts. Denote the one containing the curve B by t^- and the one containing the curve C by t^+. Let $c_K^f = T_f^- \cap K$ and $c_K^b = T_b^- \cap K$.

Suppose that

$$F_{Q,0}^- \cap t^- \cap A = F_{Q,1}^- \cap t^- \cap A = \varnothing,$$
$$|F_{Q,\infty}^- \cap (c_K^f \setminus A)| = |F_{Q,\infty}^- \cap (c_K^b \setminus A)| = |F_{Q,\infty}^- \cap t^- \cap A| = n$$

and that $k'_{01}, k'_{02}, \ldots, k'_{0p}$, $e'_{01}, e'_{02}, \ldots, e'_{0m}$, and $g'_{01}, g'_{02}, \ldots, g'_{0n}$ are consecutive intersection points of the arcs in $F_{Q,1}^-$, $F_{Q,0}^-$ and $F_{Q,\infty}^-$ with c_K^f, respectively. Locate

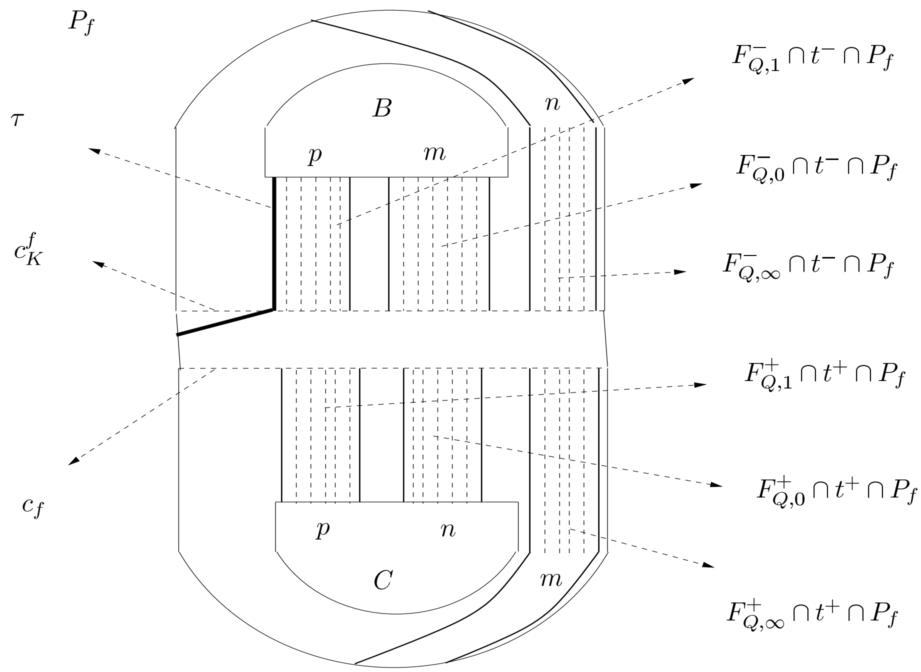


Figure 13

arcs of c_Q on T^+ so that

$$|F_{Q,\infty}^+ \cap (c_P^f \setminus A)| = |F_{Q,\infty}^+ \cap (c_P^b \setminus A)| = |F_{Q,\infty}^+ \cap A| = m,$$
$$|F_{Q,0}^+ \cap A| = |F_{Q,1}^+ \cap A| = 0.$$

Suppose that $l_{01}, \ldots, l_{0p}$, $f_{01}, \ldots, f_{0n}$, and $h_{01}, \ldots, h_{0m}$ are consecutive intersection points of the arcs in $F_{Q,1}^+$, $F_{Q,0}^+$, and $F_{Q,\infty}^+$ with c_f, respectively. Suppose τ is an arc in $F_{Q,1}^-$ whose intersection with c_K^f is k'_{01}. Suppose $\tau \cap (t^+ \setminus T^+) \cap A \neq \varnothing$. See Figure 13. By applying a power of β, we can assume $2 \leq |c_Q \cap A \cap (t^+ \setminus T^+)| < 2(p+n+m)$. By the argument in [Scharlemann 2004, Lemma 5], we get two nonisotopic reducing spheres for T that satisfy (i) and (ii). Let us call S the one having an arc on T^- of slope 0 and S' the one having an arc on T^+ of slope 0.

The figures below show intersections of two reducing spheres R, R' with T. It is easy to see that $R \in v_S$ and $R' \in v_{S'}$.

Case III.A: $\{g_{ij}\} \subseteq \{h_{ij}\}$. See Figure 14. Let $x = |\{h_{ij}\} \cap \{k_{ij}\}|/2$. Arguing as in Case I.A.1, we get $2(n+m+p) = v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(m+p-n) > v_R \cdot v_Q = 2(m+p-n-2x)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

Case III.B: $\{g_{ij}\} \cap \{h_{ij}\} \neq \varnothing$ and $\{g_{ij}\} \cap \{f_{ij}\} \neq \varnothing$, $\{e_{ij}\} \cap \{h_{ij}\} = \varnothing$. See Figure 15. Let $x = |\{k_{ij}\} \cap \{h_{ij}\}|/2$. Arguing as in Case I.A.1, we get $2(n+m+p) =$

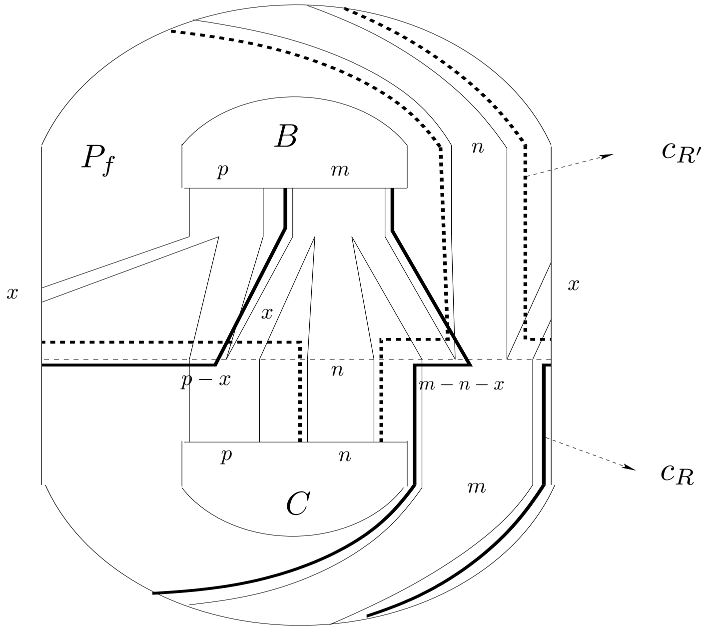


Figure 14

$v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(p+n-m+2x) > v_R \cdot v_Q = 2(p+n-m)$, $v_R \cdot v_{R\prime} = 4$, and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

Case III.C: $\{g_{ij}\} \cap \{h_{ij}\} \neq \varnothing$, $\{g_{ij}\} \cap \{f_{ij}\} \neq \varnothing$, and $\{e_{ij}\} \cap \{h_{ij}\} \neq \varnothing$. See Figure 16. Let $x = |\{f_{ij}\} \cap \{g_{ij}\}|/2$. Arguing as in Case I.A.1, we get $2(n+m+p) = v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(m-n+2x+p) > v_R \cdot v_Q = 2(m-n-p+2x)$, $v_R \cdot v_{R\prime} = 4$, and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

Case III.D: $\{g_{ij}\} \cap \{f_{ij}\} \neq \varnothing$, $\{g_{ij}\} \cap \{l_{ij}\} \neq \varnothing$, $\{e_{ij}\} \cap \{l_{ij}\} \neq \varnothing$, and $\{e_{ij}\} \cap \{h_{ij}\} = \varnothing$. See Figure 17. Let $x = |\{g_{ij}\} \cap \{l_{ij}\}|/2$. Arguing as in Case I.A.1, we get $2(n+m+p) = v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(p+n+m-2x) > v_R \cdot v_Q = 2(p+n-m)$, $v_R \cdot v_{R\prime} = 4$, and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

Case III.E: $\{g_{ij}\} \cap \{f_{ij}\} \neq \varnothing$, $\{g_{ij}\} \cap \{l_{ij}\} \neq \varnothing$, $\{e_{ij}\} \cap \{l_{ij}\} \neq \varnothing$, and $\{e_{ij}\} \cap \{h_{ij}\} \neq \varnothing$. See Figure 18. Let $x = |\{g_{ij}\} \cap \{l_{ij}\}|/2$. Arguing as in Case I.A.1, we get $v_{R'} \cdot v_Q = 2(m+n+p-2x)$, $v_R \cdot v_Q = 2(m+n-p+2x)$ and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > 2(n+m+p)$ for $i \neq 0$. So $v_{R'} \cdot v_Q = v_R \cdot v_Q$ if and only if $p = 2x$. If p is equal to $2x$, then by an argument given in the proof of Lemma 2, we can show that c_Q does not bound a disc in V. Therefore either $v_{R'} \cdot v_Q > v_R \cdot v_Q$ or $v_{R'} \cdot v_Q < v_R \cdot v_Q$. Notice that $v_R \cdot v_{R\prime} = 4$.

Case III.F: $\{g_{ij}\} \cap \{f_{ij}\} \neq \varnothing$, $\{g_{ij}\} \cap \{l_{ij}\} \neq \varnothing$, and $\{g_{ij}\} \cap \{h_{ij}\} \neq \varnothing$. See Figure 19. Let $x = |\{g_{ij}\} \cap \{f_{ij}\}|/2$. Arguing as in Case I.A.1, we get $2(n+m+p) =$

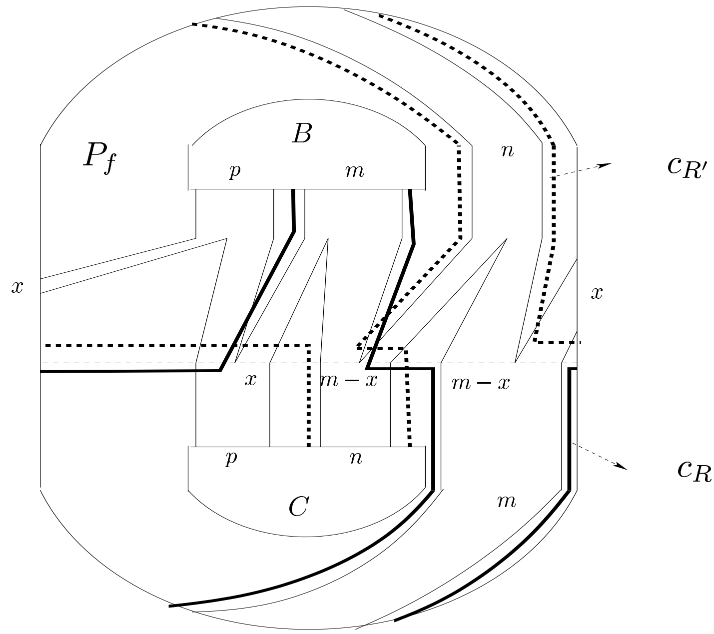


Figure 15

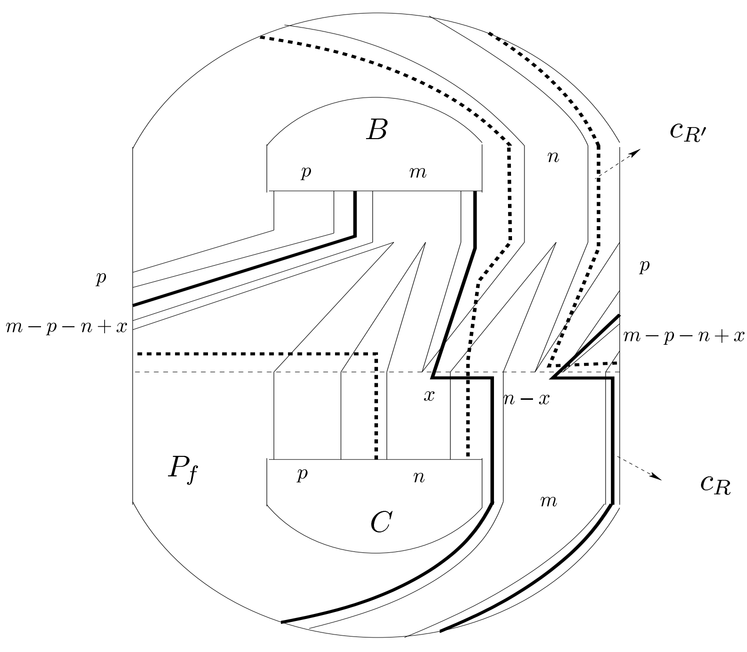


Figure 16

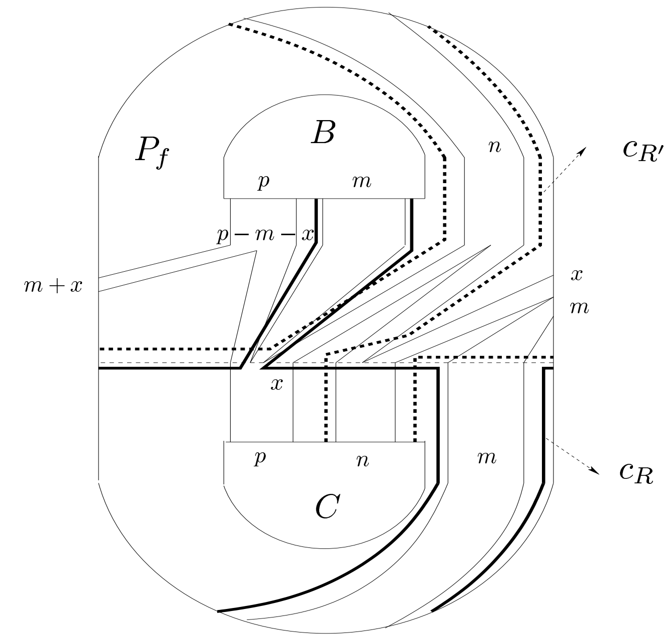


Figure 17

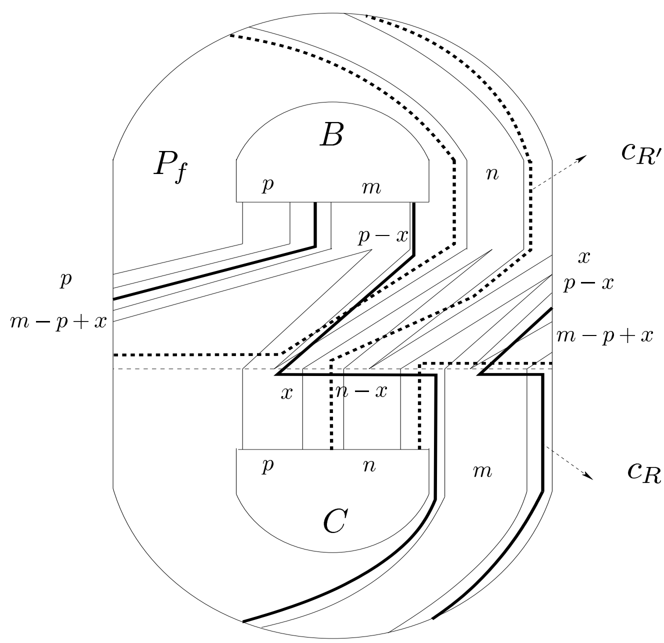


Figure 18

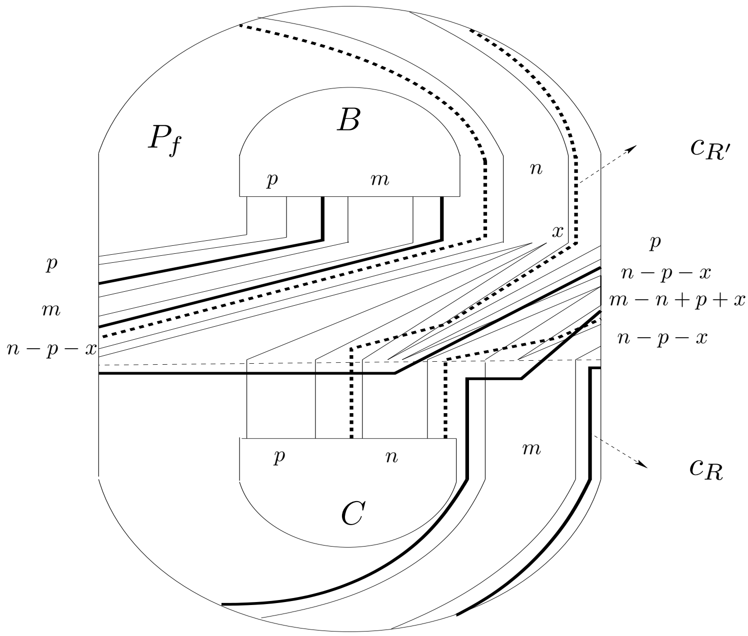


Figure 19

$v_P \cdot v_Q > v_R \cdot v_Q = 2(m+x+3p-n+x) > v_{R'} \cdot v_Q = 2(m+x+p-n+x)$, $v_R \cdot v_{R'} = 4$, and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

Case III.G: $\{e_{ij}, g_{ij}\} \subseteq \{l_{ij}\}$. See Figure 20. Let $x = |\{k_{ij}\} \cap \{l_{ij}\}|/2$. Arguing as in Case I.A.1, we get $v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(p+m-n) > v_R \cdot v_Q = 2(p-m+n)$, $v_R \cdot v_{R'} = 4$, and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

Case III.H: $\{g_{ij}\} \subseteq \{l_{ij}\}$ and $\{e_{ij}\} \cap \{h_{ij}\} \neq \varnothing$. This case is eliminated by an argument given in proof of Lemma 2 (the curve c_Q does not bound a disc in V).

Case III.I: $\{g_{ij}\} \cap \{l_{ij}\} \neq \varnothing$ and $\{g_{ij}\} \cap \{h_{ij}\} \neq \varnothing$. After applying β^{-1} to c_Q we can assume that c_Q is as in Figure 21. Let $x = |\{k_{ij}\} \cap \{l_{ij}\}|/2$ then by arguing as in Case I.A.1, we have $2(n+m+p) = v_P \cdot v_Q > v_R \cdot v_Q = 2(m-n+3p-2x) > v_{R'} \cdot v_Q = 2(m-n+p)$, $v_R \cdot v_{R'} = 4$, and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$. □

4. A presentation for $\mathcal{H}_2$

We will first prove Theorem 1. Then by using Bass–Serre theory we will prove Theorem 2.

Proof of Theorem 1. Suppose that $\tilde{\Gamma}$ is not a tree. Then there is a nontrivial loop in $\tilde{\Gamma}$. For any loop ξ in $\tilde{\Gamma}$, let $NV(\xi)$ denote the number of vertices of ξ. Then

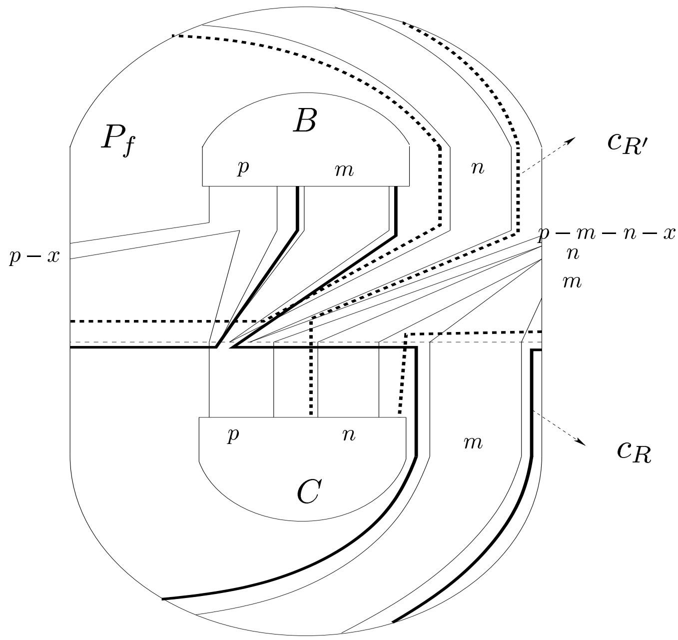


Figure 20

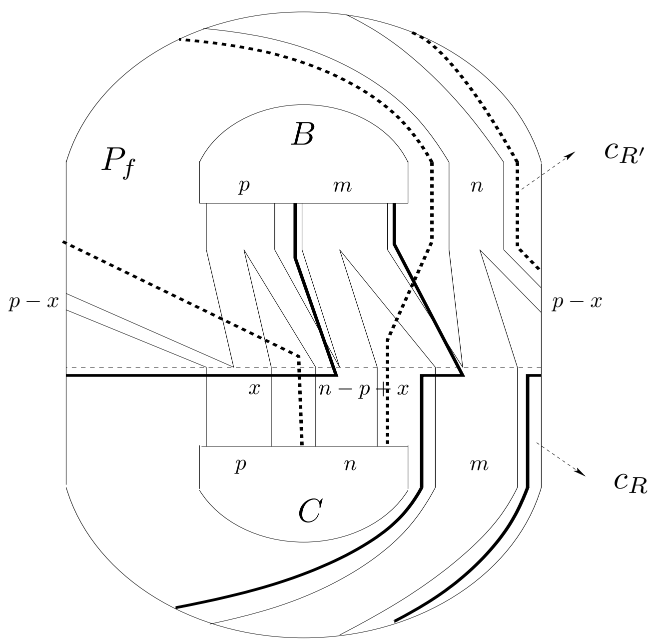


Figure 21

$\alpha_0 = \min\{NV(\xi) \mid \xi$ is a nontrivial loop in $\tilde{\Gamma}\} > 0$. Since each edge of Γ lies on a single 2-simplex, $\alpha_0 \geq 8$. Let ξ_0 be a nontrivial loop in $\tilde{\Gamma}$ such that $NV(\xi_0) = \alpha_0$. Since ξ_0 is of minimal length all its vertices are distinct. Let v_0 be any vertex of ξ_0, and let $v_0, v_1, v_2, v_3, \ldots, v_{\alpha_0-1}$ be the consecutive vertices of ξ_0. We may suppose that $v_0 \in \Gamma$. Then $v_0, v_2, v_4, \ldots, v_{\alpha_0-2}$ are vertices of Γ, and $v_k \cdot v_{k+2} = v_{\alpha_0-2} \cdot v_0 = 4$ for $k \in \{0, 2, 4, \ldots, \alpha_0 - 4\}$.

We claim $v_k \cdot v_0 < v_{k+2} \cdot v_0$ for $k \in \{0, 2, 4, \ldots, \alpha_0 - 4\}$. The proof will be by induction on the index k, as follows.

If $k = 0$, then $v_0 \cdot v_0 = 0 < v_2 \cdot v_0 = 4$. Assume $v_k \cdot v_0 < v_{k+2} \cdot v_0$ for $k \in \{0, 2, \ldots, \alpha_0 - 6\}$. If $v_{k+4} \cdot v_0 \leq v_{k+2} \cdot v_0$, then $v_k \cdot v_{k+4} = 4$ by Proposition 2. Since $v_k \cdot v_{k+2} = v_{k+2} \cdot v_{k+4} = 4$, the vertices v_k, v_{k+2}, v_{k+4} form a 2-simplex $\triangle$ in Γ. Then we get a loop ξ in $\tilde{\Gamma}$ with vertices $v_0, v_1, \ldots, v_k, u, v_{k+4}, v_{k+5}, \ldots, v_{\alpha_0-2}, v_{\alpha_0-1}$, where u is the barycenter of $\triangle$. This contradicts the minimality of α_0.

By the claim above, we get $v_0 \cdot v_{\alpha_0-4} < v_0 \cdot v_{\alpha_0-2}$. But $4 < v_0 \cdot v_{\alpha_0-4}$ and $v_0 \cdot v_{\alpha_0-2} = 4$, a contradiction. □

Proof of Theorem 2. Let v_M be a vertex of $\tilde{\Gamma}$ corresponding to the barycenter of the 2-simplex whose vertices are v_P, $v_{\delta(P)}$ and $v_{\delta^2(P)}$. Let E be the edge of $\tilde{\Gamma}$ whose vertices are v_P and v_M. Let H_P be the subgroup of $\mathcal{H}_2$ generated by the elements that stabilize v_P. Let H_M be the subgroup of $\mathcal{H}_2$ generated by the elements that preserve v_M. Let H_E be the group of elements of $\mathcal{H}_2$ that stabilize the edge E.

- Scharlemann in [2004, Lemma 2] presents H_P as

$$\begin{aligned} H_P &= \langle [\alpha], [\beta], [\gamma] \mid [\alpha]^2 = [\gamma]^2 = [\alpha\gamma]^2 = [\alpha\beta\alpha\beta^{-1}] = 1,\ [\gamma\beta\gamma] = [\alpha\beta] \rangle \\ &\cong (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2. \end{aligned}$$

- The subgroup H_M fixes the set $\{v_P, v_{\delta(P)}, v_{\delta^2(P)}\}$. Therefore

$$\begin{aligned} H_M = \big\langle [\delta], [\alpha], [\gamma] \mid\ & [\delta]^3 = [\alpha]^2 = [\gamma]^2 = [\alpha\delta\alpha^{-1}\delta^{-1}] = [\alpha\gamma]^2 = 1, \\ & [\delta] = [\gamma\delta^2\gamma] \big\rangle \\ \cong\ & (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2. \end{aligned}$$

- An element h of $\mathcal{H}_2$ fixes the sets $\{v_P\}$ and $\{v_{\delta(P)}, v_{\delta^2(P)}\}$ if and only if $h \in H_E$. Hence

$$\begin{aligned} H_E &= \langle [\alpha], [\gamma] \mid [\alpha]^2 = [\gamma]^2 = [\alpha\gamma]^2 = 1 \rangle \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

The action of $\mathcal{H}_2$ on the 2-complex Γ induces an action of $\mathcal{H}_2$ on the tree $\tilde{\Gamma}$. The subgroups H_P and H_M are the isotropy subgroups of $\mathcal{H}_2$ fixing the vertices v_P and v_M, respectively. By the standard Bass–Serre theory [Serre 2003], the group $\mathcal{H}_2$ is thus a free product of the subgroups H_P and H_M amalgamated over the subgroup H_E:

$$\mathcal{H}_2 \cong H_P \underset{H_E}{*} H_M \cong (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \underset{\mathbb{Z}_2 \oplus \mathbb{Z}_2}{*} (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2. \qquad \square$$

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