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ENTIRE MEAN CURVATURE FLOWS OF GRAPHS

XIAOLI HAN

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We consider the evolution of the graph of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in $\mathbb{R}^n \times \mathbb{R}^n$ by the mean curvature flow. We prove that the flow exists smoothly for all time if the differential of f has a positive lower bound. Moreover, at each time, the flow remains the graph of a map f_t .

1. Introduction

The mean curvature flow deforms the initial surface in the direction of its mean curvature vector. Let $f : \Sigma_1 \rightarrow \Sigma_2$ be a smooth map, and denote the graph of f by Σ which is a submanifold of $\Sigma_1 \times \Sigma_2$ by the embedding $F = \text{id} \times f$. Chen, Li, and Tian [2002], Ecker and Huisken [1989], Li and Li [2003], and Wang [2001; 2002] have studied the deformation of f by the mean curvature flow. The key idea of these papers is to consider the quantity $*\omega$, where ω is the volume form on Σ_1 and $*$ is the Hodge operator with respect to the metric induced on Σ by F . In fact $*\omega$ is the Jacobian of the projection from Σ to Σ_1 , and $*\omega > 0$ if and only if Σ is a graph over Σ_1 , by the implicit function theorem. In [Chen et al. 2002; Li and Li 2003; Wang 2001; 2002], the authors independently obtained the results of long time existence and convergence under the condition $*\omega > 1/\sqrt{2}$. We remark that their results depend on the choice of subspace over which Σ is written as a graph. In this article, we investigate smooth maps of \mathbb{R}^n . The advantage of this kind of map is that we can find a good representation of Σ via Lewy transformation. This technique is used by Yuan [2002] to prove a Bernstein theorem for special Lagrangian graphs. The difficult is that this case is noncompact. We don't know the behavior of solutions Σ_t for large time t . The surfaces Σ_t may diverge to ∞ .

Our result is as follows.

Theorem 1.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth map and Σ be the graph of f . If $Df \geq \varepsilon$ for some $\varepsilon > 0$, then the mean curvature flow of the graph of f remains a graph and exists for all time.*

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2. Evolution equation

First we recall the evolution equation of $*\omega$ along the mean curvature flow. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth map. The graph of f is an embedded submanifold Σ in $\mathbb{R}^n \times \mathbb{R}^n$ by $F = \text{id} \times f$. Let ω be the standard volume form on \mathbb{R}^n . Around any point $p \in \Sigma$, we choose orthonormal frames $\{e_i\}_{i=1,\dots,n}$ for $T\Sigma$ and $\{e_\alpha\}_{\alpha=n+1,\dots,2n}$ for $N\Sigma$. In the sequel, we use Roman letters i, j, k, \dots for tangent indices; we use Greek letters $\alpha, \beta, \gamma, \dots$ for normal indices. We consider the quantity $*\omega$, where $*$ is Hodge operator with respect to the induced metric on Σ . Then $*\omega = \omega(e_1, \dots, e_n) = \omega(\pi_1(e_1), \dots, \pi_1(e_n))$, where $\pi_1 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the projection map from \mathbb{R}^{2n} to the tangent space of the domain \mathbb{R}^n . In fact $*\omega$ is the Jacobian of the projection from $T\Sigma$ to the domain \mathbb{R}^n . By the singular value decomposition (see [Bretschner 1997]), there exist orthonormal bases $\{a_i\}_{i=1,\dots,n}$ for the domain \mathbb{R}^n and $\{a_\alpha\}_{\alpha=n+1,\dots,2n}$ for the target \mathbb{R}^n , such that $\lambda_{i\alpha} = \langle df(a_i), a_\alpha \rangle$ is diagonal. For simplicity, we denote $\lambda_{i(n+i)} = \lambda_i$. Now $\{e_i = (1 + \lambda_{i\alpha}^2)^{-1/2}(a_i + \sum_\alpha \lambda_{i\alpha} a_\alpha)\}$ forms an orthonormal basis for $T\Sigma$ and $\{e_\alpha = (1 + \lambda_{i\alpha}^2)^{-1/2}(a_\alpha - \sum_i \lambda_{i\alpha} a_i)\}$ forms an orthonormal basis for $N\Sigma$. In this setting,

$$*\omega = \frac{1}{\prod_{i=1}^n (1 + \lambda_i^2)}.$$

Define the second fundamental form of Σ as $h_{ij}^\alpha = \langle \nabla_{e_i} e_j, e_\alpha \rangle$. Recall that $*\omega$ satisfies the equation

$$(2-1) \quad \left(\frac{d}{dt} - \Delta\right) *\omega = *\omega \left(\sum_{i,j,\alpha} (h_{ij}^\alpha)^2 - 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{n+i} h_{jk}^{n+j} + 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{n+j} h_{jk}^{n+i} \right);$$

see [Wang 2002; Chen et al. 2002; Li and Li 2003]. This formula plays the important role in these papers.

3. Long time existence

Using Equation (2-1), we now begin to prove our theorem.

Theorem 3.1. *If $Df \geq \varepsilon$ for some $\varepsilon > 0$, then the mean curvature flow exists for all time.*

Proof. Step 1: The key idea is to seek a good representation of Σ via Lewy transformation such that $|\lambda_i| \leq 1 - \delta$ on the initial surface for some $\delta > 0$. This is inspired by Yuan’s work [2002]. We rotate the $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ coordinate system to (\bar{x}, \bar{y}) by $\pi/4$, that is, we set $\bar{x} = (x + y)/\sqrt{2}$ and $\bar{y} = (-x + y)/\sqrt{2}$. Then Σ

has a new parametrization

$$\bar{x} = \frac{1}{\sqrt{2}}(x + f(x)) \quad \text{and} \quad \bar{y} = \frac{1}{\sqrt{2}}(-x + f(x)).$$

Since $Df \geq \varepsilon$, we have

$$\begin{aligned} |\bar{x}^2 - \bar{x}^1|^2 &= \frac{1}{2}(|x^2 - x^1|^2 + 2(x^2 - x^1) \cdot (f(x^2) - f(x^1)) + |f(x^2) - f(x^1)|^2) \\ &\geq \frac{1}{2}|x^2 - x^1|^2. \end{aligned}$$

It follows that Σ is still graph over the whole \bar{x} space \mathbb{R}^n . That means Σ has the representation $(\bar{x}, f(\bar{x}))$. Any tangent vector to Σ takes the form

$$\frac{1}{\sqrt{2}}((I + Df(x))e, (-I + Df(x))e),$$

where $e \in \mathbb{R}^n$. It follows that

$$D\bar{f}(\bar{x}) = (I + Df(x))^{-1}(-I + Df(x)).$$

Noting that $Df \geq \varepsilon$, we have

$$-I + \delta \leq (D\bar{f}) \leq I - \delta \quad \text{for some } \delta > 0.$$

For the sake of convenience, we still denote the eigenvalues of $d\bar{f}$ by λ_i . Now we have already shown that $|\lambda_i| \leq 1 - \delta$ for some δ . By the theorem in [Tsui and Wang 2004] we know that this condition can be preserved along the mean curvature flow. After putting this into Equation (2-1), its right side term in parentheses becomes

$$\begin{aligned} &\sum_{i,j,\alpha} (h_{ij}^\alpha)^2 - 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{n+i} h_{jk}^{n+j} + 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{n+j} h_{jk}^{n+i} \\ &= \delta|A|^2 + (1 - \delta) \sum_{i,k} (h_{ik}^{n+i})^2 + (1 - \delta) \sum_{i < j,k} ((h_{jk}^{n+i})^2 + (h_{ik}^{n+j})^2) \\ &\quad - 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{n+i} h_{jk}^{n+j} + 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{n+j} h_{jk}^{n+i} \\ &\geq \delta|A|^2 + (1 - \delta) \sum_{i,k} (h_{ik}^{n+i})^2 + (1 - \delta) \sum_{i < j,k} ((h_{jk}^{n+i})^2 + (h_{ik}^{n+j})^2) \\ &\quad - 2(1 - \delta) \sum_{i < j,k} |h_{ik}^{n+i} h_{jk}^{n+j}| - 2(1 - \delta) \sum_{i < j,k} |h_{ik}^{n+j} h_{jk}^{n+i}| \\ &\geq \delta|A|^2. \end{aligned}$$

Thus we have

$$(3-1) \quad \left(\frac{d}{dt} - \Delta\right) * \omega \geq \delta|A|^2.$$

According to the maximum principle for parabolic equations, $\min_{\Sigma_t} * \omega$ is non-decreasing in time. So $* \omega$ has a positive lower bound, and this implies that Σ_t remains the graph of a map $f_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ whenever the flow smoothly exists.

Step 2: The remaining proof is routine. Fix any point (X_0, t_0) . Then the backward heat kernel $\rho(F, X_0, t, t_0)$ at (X_0, t_0) is given by

$$\rho(F, X_0, t, t_0) = \frac{1}{(4\pi(t_0-t))^{n/2}} \exp\left(\frac{-|F(x, t) - X_0|^2}{4(t_0-t)}\right).$$

First we find a weighted monotonicity formula for $\int_{\Sigma_t} (1/* \omega)\rho(F, X_0, t, t_0)d\mu_t$ (see [Chen and Li 2004]). Recalling that

$$\begin{aligned} \frac{d}{dt}d\mu_t &= -|H|^2d\mu_t, \\ \left(\frac{\partial}{\partial t} + \Delta\right)\rho &= -\left(|H + \frac{(F - X_0)^\perp}{2(t_0-t)}| - |H|^2\right), \end{aligned}$$

and combining these equations and Equation (3-1), we get

$$\begin{aligned} &\frac{d}{dt} \int_{\Sigma_t} \frac{1}{* \omega} \rho(F, X_0, t, t_0) \\ &\leq \int_{\Sigma_t} \phi \rho \Delta \frac{1}{* \omega} - \delta \int_{\Sigma_t} \frac{|A|^2}{* \omega^2} \rho - 2 \int_{\Sigma_t} \frac{|\nabla * \omega|^2}{* \omega^3} \rho \\ &\quad - \int_{\Sigma_t} \frac{1}{* \omega} \left(\Delta \rho + \left(\left| H + \frac{(F - X_0)^\perp}{2(t_0-t)} \right|^2 - |H|^2 \right) \rho \right) \\ &\quad - \int_{\Sigma_t} \frac{1}{* \omega} \rho |H|^2 \\ &\leq - \int_{\Sigma_t} \rho \left(\frac{2}{* \omega^3} |\nabla * \omega|^2 + \frac{1}{* \omega} \left| H + \frac{(F - X_0)^\perp}{2(t_0-t)} \right|^2 + \delta \frac{|A|^2}{* \omega^2} \right) \\ &\quad + \int_{\Sigma_t} \left(\rho \Delta \frac{1}{* \omega} - \frac{1}{* \omega} \Delta \rho \right) \\ (3-2) \quad &\leq - \left(\int_{\Sigma_t} \frac{1}{* \omega} \rho(F, X_0, t, t_0) \left| H + \frac{(F - X_0)^\perp}{2(t_0-t)} \right|^2 d\mu_t \right. \\ &\quad \left. + \delta \int_{\Sigma_t} \frac{|A|^2}{* \omega^2} \rho(F, X_0, t, t_0) d\mu_t \right. \\ &\quad \left. + 2 \int_{\Sigma_t} \frac{|\nabla * \omega|^2}{* \omega^3} \rho(F, X_0, t, t_0) d\mu_t \right). \end{aligned}$$

From this we see that $\lim_{t \rightarrow t_0} \int_{\Sigma_t} (1/* \omega)\rho$ exists.

Let λ_i be positive numbers tending to ∞ as $i \rightarrow \infty$ and let F_i be the diverging sequence obtained by translating F by X_0 and then dilating parabolically by λ_i , that is, by taking $(F, t) \rightarrow (\lambda_i(F - X_0), \lambda_i^2(t - t_0))$. Denote the new time parameter

by s . Then $t = t_0 + s/\lambda_i^2$. Thus,

$$F_i(x, s) = \lambda_i(F(x, t_0 + \lambda_i^{-2}s) - X_0).$$

Let $d\mu_s^i$ denote the induced volume form on Σ_s^i by F_i . Notice that $*\omega$ is invariant under the parabolic dilation. It is clear that

$$\int_{\Sigma_t} \frac{1}{*\omega} \rho(F, X_0, t, t_0) d\mu_t = \frac{1}{*\omega} \int_{\Sigma_s^i} \rho(F_i, 0, s, 0) d\mu_s^i.$$

Therefore we get

$$\begin{aligned} \frac{d}{ds} \int_{\Sigma_s^i} \frac{1}{*\omega} \rho(F_i, 0, s, 0) d\mu_s^i &\leq - \left(\int_{\Sigma_s^i} \frac{1}{*\omega} \rho(F_i, 0, s, 0) \left| H_i + \frac{F_i^\perp}{2(t_0 - t)} \right|^2 d\mu_s^i \right. \\ &\quad + \delta \int_{\Sigma_s^i} \frac{|A_i|^2}{*\omega^2} \rho(F_i, 0, s, 0) d\mu_s^i \\ &\quad \left. + 2 \int_{\Sigma_s^i} \frac{|\nabla *\omega|^2}{*\omega^3} \rho(F_i, 0, s, 0) d\mu_s^i \right). \end{aligned}$$

Note $t_0 + \lambda_i^{-2}s \rightarrow t_0$ for any fixed s as $i \rightarrow \infty$ and $\lim_{t \rightarrow t_0} \int_{\Sigma_t} (1/*\omega) \rho d\mu_t$ exists. By the above monotonicity formula, this implies that, for any fixed s_1 and s_2 ,

$$\begin{aligned} 0 &\leftarrow \int_{\Sigma_{s_1}^i} \frac{1}{*\omega} \rho(F_i, 0, s_1, 0) d\mu_{s_1}^i - \int_{\Sigma_{s_2}^i} \frac{1}{*\omega} \rho(F_i, 0, s_2, 0) d\mu_{s_2}^i \\ &\geq \int_{s_1}^{s_2} \int_{\Sigma_s^i} \frac{1}{*\omega} \rho(F_i, 0, s, 0) \left| H_i + \frac{F_i^\perp}{2(t_0 - t)} \right|^2 d\mu_s^i \\ &\quad + \delta \int_{s_1}^{s_2} \int_{\Sigma_s^i} \frac{|A_i|^2}{*\omega^2} \rho(F_i, 0, s, 0) d\mu_s^i \\ &\quad + 2 \int_{s_1}^{s_2} \int_{\Sigma_s^i} \frac{|\nabla *\omega|^2}{*\omega^3} \rho(F_i, 0, s, 0) d\mu_s^i. \end{aligned}$$

Since $*\omega$ is bounded below, we have

$$\int_{s_1}^{s_2} \int_{\Sigma_s^i} |A_i|^2 \rho(F_i, 0, s, 0) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

This implies that for any compact $K \subset R^{2n}$,

$$(3-3) \quad \int_{\Sigma_{s_i}^i \cap K} |A_i|^2 \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Now we claim that for the graphic mean curvature flow the fact that $*\omega$ has a positive lower bound implies that (X_0, t_0) is a regular point. Without loss of

generality, we assume the origin 0 is on the boundary of $\Sigma_{s_i}^i$. Since

$$*\omega = \frac{1}{\prod_{i=1}^n (1+\lambda_i^2)} \leq \frac{1}{1+\lambda_i^2} = \frac{1}{1+|df_t|^2}$$

and because $*\omega$ has a positive lower bound, Σ_t can be written as the graph of a map $f_t : \Sigma_1 \rightarrow \Sigma_2$ with uniformly bounded $|df_t|$. We consider the diverging sequence of f in \mathbb{R}^{2n} by λ_i given by

$$\tilde{f}_i(y) = \lambda_i f_{t_0+\lambda_i^{-2}s_i}(y),$$

where $y \in \mathbb{R}^n$. It is clear that $|d\tilde{f}_i|$ is also uniformly bounded and $\lim_{i \rightarrow \infty} \tilde{f}_i(0) = 0$. By the Arzela theorem, $\tilde{f}_i \rightarrow \tilde{f}_\infty$ in C^α on any compact set. Note that by [Ilmanen 1995, inequality (29)], we have

$$|A_i| \leq |\nabla d\tilde{f}_i| \leq C(1 + |d\tilde{f}_i|^3)|A_i|,$$

where $\nabla d\tilde{f}_i$ is measured with respect to the induced metric on $\Sigma_{s_i}^i$. From Equation (3-3) we know that

$$\int_{\Sigma_{s_i}^i \cap K} |\nabla d\tilde{f}_i|^2 \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

which implies that $\tilde{f}_i \rightarrow \tilde{f}_\infty$ in $C^\alpha \cap W_{loc}^{1,2}$ and that the second derivative of \tilde{f}_∞ is 0. By [Chen and Li 2004, Main Theorem], we know that $\Sigma_{s_i}^i \rightarrow \Sigma^\infty$ and that Σ^∞ is independent of s . Therefore Σ^∞ is the graph of a linear function. Therefore

$$\lim_{i \rightarrow \infty} \int \rho(F_i, 0, s_i, 0) d\mu_{s_i}^i = \int \rho(F_\infty, 0, -1, 0) d\mu^\infty = 1,$$

This implies that

$$\begin{aligned} \lim_{t \rightarrow t_0} \int \rho(F, X_0, t, t_0) &= \lim_{i \rightarrow \infty} \int \rho(F, X_0, t_0 + \lambda_i^{-2}s_i, t_0) \\ (3-4) \qquad \qquad \qquad &= \lim_{i \rightarrow \infty} \int \rho(F_i, 0, s_i, 0) d\mu_{s_i}^i = 1. \end{aligned}$$

White's regularity theorem [2002] tells us that if $\lim_{t \rightarrow t_0} \rho(F, X_0, t, t_0) \leq 1$, then (X_0, t_0) is a regular point. Thus (3-4) tells us that (X_0, t_0) is a regular point. We have proved the theorem. □

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References

- [Bretscher 1997] O. Bretscher, *Linear algebra with applications*, Prentice Hall, Upper Saddle River, NJ, 1997. MR 1429616 Zbl 0896.15001
- [Chen and Li 2004] J. Chen and J. Li, “Singularity of mean curvature flow of Lagrangian submanifolds”, *Invent. Math.* **156**:1 (2004), 25–51. MR 2004m:53116 Zbl 1059.53052
- [Chen et al. 2002] J. Y. Chen, J. Y. Li, and G. Tian, “Two-dimensional graphs moving by mean curvature flow”, *Acta Math. Sin. (Engl. Ser.)* **18**:2 (2002), 209–224. MR 2003c:53094 Zbl 1028.53069
- [Ecker and Huisken 1989] K. Ecker and G. Huisken, “Mean curvature evolution of entire graphs”, *Ann. of Math. (2)* **130**:3 (1989), 453–471. MR 91c:53006 Zbl 0696.53036
- [Ilmanen 1995] T. Ilmanen, “Singularities of mean curvature flow of surfaces”, Preprint, 1995, Available at <http://www.math.ethz.ch/~ilmanen/papers/pub.html/sing.dvi>.
- [Li and Li 2003] J. Li and Y. Li, “Mean curvature flow of graphs in $\Sigma_1 \times \Sigma_2$ ”, *J. Partial Differential Equations* **16**:3 (2003), 255–265. MR 2004e:53102 Zbl 1038.53065
- [Tsui and Wang 2004] M.-P. Tsui and M.-T. Wang, “Mean curvature flows and isotopy of maps between spheres”, *Comm. Pure Appl. Math.* **57**:8 (2004), 1110–1126. MR 2005b:53110 Zbl 1067.53056
- [Wang 2001] M.-T. Wang, “Deforming area preserving diffeomorphism of surfaces by mean curvature flow”, *Math. Res. Lett.* **8**:5-6 (2001), 651–661. MR 2003f:53122 Zbl 1081.53056
- [Wang 2002] M.-T. Wang, “Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension”, *Invent. Math.* **148**:3 (2002), 525–543. MR 2003b:53073 Zbl 1039.53072
- [White 2002] B. White, “A local regularity theorem for classical mean curvature flow”, Preprint, 1999, revised 2002, Available at <http://math.stanford.edu/~white/localreg.pdf>.
- [Yuan 2002] Y. Yuan, “A Bernstein problem for special Lagrangian equations”, *Invent. Math.* **150**:1 (2002), 117–125. MR 2003k:53060 Zbl 1027.53055

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XIAOLI HAN
THE ABDUS SALAM ICTP
STRADA COSTIERA 11
34014 TRIESTE
ITALY
xhan@ictp.it

