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We derive an explicit lower bound for the volume of a hyperbolic orbifold, dependent on the dimension and the maximal order of torsion in the orbifold's fundamental group.

1. Introduction

A complete orientable *hyperbolic n-orbifold* is an orbit space $\mathbb{H}^n/\Gamma$, where Γ is a discrete group of orientation-preserving isometries of $\mathbb{H}^n$. An orientable hyperbolic n-manifold is the quotient of $\mathbb{H}^n$ by a discrete *torsion-free* subgroup of $\mathrm{Isom}_+(\mathbb{H}^n)$. Explicit lower bounds for the volume of a hyperbolic 3-manifold, as well as for the volume of a hyperbolic 3-orbifold, were given by Meyerhoff [1986]. Later, explicit bounds for manifolds in all dimensions were constructed by Martin [1989a] and Friedland and Hersonsky [1993]. Wang's finiteness theorem [1972] asserts that, for n greater than three, the set of volumes of hyperbolic n-orbifolds is discrete in the real numbers. Hence, a lower bound for the volume of a hyperbolic orbifold exists in all dimensions. In this paper we prove the following result.

Theorem 1 (Main Theorem). *Let Γ be a discrete group of orientation-preserving isometries of $\mathbb{H}^n$. Assume that Γ has no torsion element of order greater than k. Then*

$$\mathrm{Vol}(\mathbb{H}^n/\Gamma) \geq \mathscr{A}(n,k)$$

where $\mathscr{A}(n,k)$ is an explicit constant depending only on n and k, given by

$$\mathscr{A}(n,k) = \sup_{r>0}\left(1 + \left(\frac{e(n+1)(1+\cosh r)}{\sinh r}\right)^2 \cosh 6r \sin^{-2}\left(\frac{\pi}{k}\right)\right)^{-(n+1)^2} \omega_{n-1}\int_0^r \sinh^{n-1} u\, du.$$

Here $\omega_{n-1} = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the standard sphere in euclidean n-space.

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As a corollary, we obtain the following analogue of Hurwitz's formula for groups acting on surfaces.

Corollary 2. *If M is an orientable hyperbolic n-manifold and G is a group of orientation-preserving isometries of M containing no torsion elements of order greater than k, then*

$$|G| \leq \frac{\mathrm{Vol}(M)}{\mathscr{A}(n,k)}.$$

2. Preliminaries

We denote *hyperbolic n-space* by $\mathbb{H}^n$ and define it as

$$\mathbb{H}^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : -x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = -1, x_1 > 0\right\}$$

together with the Riemannian metric induced on $\mathbb{H}^n$ by the quadratic form

$$ds^2 = -dx_1^2 + dx_2^2 \cdots + dx_{n+1}^2.$$

The Riemannian metric gives rise to a distance function. Given two vectors

$$x, y \in \mathbb{H}^n$$

the *hyperbolic distance* between x and y is denoted by $d_{\mathbb{H}}(x, y)$ and defined by the equation

$$\cosh d_{\mathbb{H}}(x, y) = x_1 y_1 - \cdots - x_{n+1} y_{n+1}.$$

Let e_i denote the standard basis element. We will make particular use of

$$e_1 = (1, 0, \dots, 0)$$

which is an element of $\mathbb{H}^n$. The group of isometries of hyperbolic space will be identified with the Lie group $O^+(1, n)$ [Beardon 1983]. The subgroup $SO^+(1, n)$, consisting of all elements of $O^+(1, n)$ with determinant 1, corresponds to orientation-preserving isometries of $\mathbb{H}^n$. The symbol I_n will denote the $n \times n$ identity matrix. The *torsion* elements of a discrete group of isometries of hyperbolic space, that is isometries of finite order, are called *elliptic*. We will use the two terms interchangeably.

For an element A of $O^+(1, n)$ we define its operator norm to be

$$\|A\| = \max\left\{|Av| : v \in \mathbb{R}^{n+1} \text{ and } |v| = 1\right\}.$$

A subgroup Γ of $O^+(1, n)$ is *discrete* if for each $c > 0$, the set

$$\{A \in \Gamma : \|A\| < c\}$$

is finite. The group Γ is *elementary* if and only if it has a finite orbit in $\overline{\mathbb{H}}^n$. Otherwise Γ is *nonelementary*. A finitely generated nonelementary subgroup of

$O^+(1,n)$ is discrete if and only if every two-generator subgroup is discrete [Martin 1989b].

Let A be any $n \times n$ matrix. The *spectrum*, denoted by $\sigma(A)$, is the set of all eigenvalues of A. The *spectral radius*, denoted by $r_\sigma(A)$, is defined by the equation

$$r_\sigma(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

An important alternative definition for the operator norm is

$$\|A\| = \sqrt{r_\sigma(A^t A)}.$$

The *conformal ball model* of hyperbolic n-space consists of $\mathbb{B}^n$, the open unit ball in $\mathbb{R}^n$, together with the metric

$$ds^2_{\mathbb{B}} = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1-|x|^2)^2}.$$

By [Gehring and Martin 1987, Theorem 3.7], if Γ is a discrete group of isometries of $\mathbb{B}^n$, then Γ has the following

Convergence property. *For each infinite sequence of elements in Γ there exists a subsequence $\{\gamma_j\}$ and points x_0, y_0 in $\overline{\mathbb{B}}^n$ such that*

$$\lim_{j\to\infty} \gamma_j = y_0 \quad \text{and} \quad \lim_{j\to\infty} \gamma_j^{-1} = x_0$$

uniformly on compact subsets in $\overline{\mathbb{B}}^n \setminus \{x_0\}$ and $\overline{\mathbb{B}}^n \setminus \{y_0\}$, respectively.

Outline. The proof of the Main Theorem will come in three steps. In Section 3 we prove that an upper bound on the order of an elliptic isometry A of $\mathbb{H}^n$ gives a lower bound on the operator norm of $A - I_{n+1}$. In Section 4 we show that, up to conjugation, an upper bound on the maximal order of torsion of a discrete group of isometries of $\mathbb{H}^n$ leads to a uniform lower bound on $\|A - I_{n+1}\|$ for all $A \neq I_{n+1}$. Finally, in Section 5, we establish an upper bound on the number of elements of a discrete group of isometries of $\mathbb{H}^n$ that fail to move a ball of radius r off itself. This allows us to bound the volume of the image of such a ball in the orbit space.

3. Norm bound for low-order torsion elements

Proposition 3. *Let $A \in O^+(1,n)$ be an elliptic element of order at most k. Then*

$$\|A - I_{n+1}\| \geq c_k$$

where $c_k := 2\sin^2(\frac{\pi}{k})e^{-2}$.

The first step is to prove a version of Proposition 3 for the elements of the subgroup of $O^+(1,n)$ that fix e_1. Next, we will consider the remaining elliptic elements, which are all conjugate to elements which fix e_1. Two different bounds on $\|A - I_{n+1}\|$ that depend on the distance between the fixed point set of A and e_1 will be developed. Finally, all results will be combined to prove the proposition.

Define $E(n)$ to be the subgroup of $O^+(1, n)$ that stabilizes the vector e_1. We identify $E(n)$ with the *orthogonal group* $O(n)$ by noting that for each $A \in E(n)$, there exists $A^* \in O(n)$ such that

$$A = \begin{pmatrix} 1 & \\ & A^* \end{pmatrix}$$

Using this identification we may carry over properties of $O(n)$ to $E(n)$. In particular

$$A \in E(n) \Rightarrow A^{-1} = A^t. \tag{3-1}$$

A basic result of linear algebra, the proof of which, for instance, can be found in [Gruenberg and Weir 1967, section 6.4], gives us the following

Lemma 4. *Given $A \in E(n)$ there exists $B \in E(n)$ such that*

$$BAB^{-1} = \begin{pmatrix} 1 & & & & & \\ & A_1 & & & & \\ & & \ddots & & & \\ & & & A_l & & \\ & & & & -I_s & \\ & & & & & I_t \end{pmatrix} \tag{3-2}$$

where l, s, t are nonnegative integers and

$$A_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

where $0 < \theta_i < \pi$.

Remark 5. If in Lemma 4 the order of A is at most k, then clearly

$$\frac{2\pi}{k} \leq \theta_i < \pi.$$

Since the set of eigenvalues of a matrix is conjugacy invariant, so is the operator norm.

Lemma 6. *Let A be a real $(n+1) \times (n+1)$ matrix and let $B \in E(n)$, then*

$$\|BAB^{-1}\| = \|A\|.$$

The following proposition gives our result for elements of $E(n)$.

Proposition 7. *If A is a nonidentity element of $E(n)$ of order at most k, then*

$$\|A - I_{n+1}\| \geq 2\sin\left(\frac{\pi}{k}\right).$$

Proof. Write $A = BA'B^{-1}$, where A' has the form of the right-hand side of (3-2) and B is the appropriate element of $E(n)$. Then

$$\begin{aligned}\|A - I_{n+1}\| &= \|BA'B^{-1} - I_{n+1}\| \\ &= \|B(A' - I_{n+1})B^{-1}\| \\ &= \|A' - I_{n+1}\| \qquad \text{(by Lemma 6)} \\ &\geq |(A' - I_{n+1})e_2| \\ &= |A'e_2 - e_2|.\end{aligned}$$

Assume $l \neq 0$, then

$$|A'e_2 - e_2| = \sqrt{(\cos\theta_1 - 1)^2 + \sin^2\theta_1} = 2\sin\left(\frac{\theta_1}{2}\right) \geq 2\sin\left(\frac{\pi}{k}\right)$$

by Remark 5. If $l = 0$, then $k = 2$ and A' has the form

$$\begin{pmatrix} 1 & & \\ & -I_s & \\ & & I_t \end{pmatrix},$$

and therefore

$$|A'e_2 - e_2| = |-e_2 - e_2| = 2 = 2\sin\frac{\pi}{2}. \qquad \square$$

Now consider the more general case where A is an elliptic element of $O^+(1, n)$ which does not fix e_1. Our first approach will give us a bound in the case where the fixed point set of A is "close" to e_1. We will show that A is conjugate, by an isometry whose norm we can explicitly calculate, to an elliptic element of the same order which fixes e_1. Proposition 7 can then be used to obtain a bound on $\|A - I_{n+1}\|$.

Proposition 8. *Let $A \in O^+(1, n)$ be an elliptic element of order at most k, which does not fix e_1. Let δ be the hyperbolic distance from e_1 to the fixed point set of A. Then*

$$\|A - I_{n+1}\| \geq 2\sin\left(\frac{\pi}{k}\right)e^{-2\delta}.$$

Proof. Let b be the closest point in the fixed point set of A to e_1. Let

$$\hat{b} = (\cosh\delta, \sinh\delta, 0, \ldots, 0).$$

Then $d_{\mathbb{H}}(e_1, b) = d_{\mathbb{H}}(e_1, \hat{b}) = \delta$. Therefore, there exists $\hat{A} \in E(n)$ such that $\hat{A}\hat{b} = b$.

Let

$$T = \begin{pmatrix} \cosh\delta & \sinh\delta & & & \\ \sinh\delta & \cosh\delta & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

define $B := \hat{A}T$ and let

$$\tilde{A} = B^{-1}AB.$$

Since $\tilde{A}$ fixes e_1, $\tilde{A} \in E(n)$ and

$$\begin{aligned} \|\tilde{A} - I_{n+1}\| &= \|B^{-1}AB - I_{n+1}\| \\ &= \|B^{-1}(A - I_{n+1})B\| \\ &\le \|B^{-1}\|\|A - I_{n+1}\|\|B\| \\ &= \|B\|^2\|A - I_{n+1}\|, \end{aligned}$$

since $\|B\| = \|B^{-1}\|$ for all $B \in O^+(1,n)$.

Furthermore

$$\begin{aligned} B^tB &= (\hat{A}T)^t(\hat{A}T) \\ &= T^t\hat{A}^t\hat{A}T \\ &= T^t\hat{A}^t\hat{A}T \qquad \text{(by (3-1))} \\ &= T^tT \\ &= \begin{pmatrix} \cosh 2\delta & \sinh 2\delta & & & \\ \sinh 2\delta & \cosh 2\delta & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}. \end{aligned}$$

The eigenvalues of B^tB are 1, $e^{-2\delta}$ and $e^{2\delta}$, and thus $\|B\| = e^{\delta}$. Therefore,

$$\begin{aligned} \|A - I_{n+1}\| &\ge \|\tilde{A} - I_{n+1}\|\|B\|^{-2} \\ &= \|\tilde{A} - I_{n+1}\|e^{-2\delta} \\ &\ge 2\sin\left(\frac{\pi}{k}\right)e^{-2\delta} \qquad \text{(by Proposition 7).} \end{aligned}$$

□

In Proposition 8 our bound goes to zero as $\delta \to \infty$. This is our estimate for "small" values of δ. The following proposition uses a different method to address the case where δ is "large". If the fixed point set of an elliptic element A is far from e_1 then A must move e_1 a significant amount. This ensures, by the definition of the operator norm, that $\|A - I_{n+1}\|$ can be bounded away from zero.

Proposition 9. *Let $A \in O^+(1,n)$ be an elliptic element of order at most k, which does not fix e_1. Let δ be the hyperbolic distance from e_1 to the fixed point set of A. Then*

$$\|A - I_{n+1}\| \geq 2\sinh^2\delta \sin^2\Big(\frac{\pi}{k}\Big).$$

The proof of the following lemma is straight-forward.

Lemma 10. *Let A be an elliptic isometry of $\mathbb{H}^n$ of order at most k. Let b be a fixed point of A and let v be an element of $\mathbb{H}^n$ such that the geodesic g_1 containing v and b is perpendicular to the fixed point set of A. Suppose g_2 is the geodesic containing b and $A(v)$, then the angle of intersection θ between g_1 and g_2 is greater than or equal to $2\pi/k$.*

Proof of Proposition 9. Let $A = (a_{ij})$. By the definition of the operator norm we have the following

$$\begin{aligned}\|A - I_{n+1}\| &\geq |(A - I_{n+1})e_1| \\ &= |Ae_1 - e_1| \\ &= \big|(a_{11} - 1, a_{21}, \ldots, a_{(n+1)1})\big| \\ &\geq |a_{11} - 1|.\end{aligned}$$

On the other hand,

$$\cosh d_{\mathbb{H}}(e_1, Ae_1) = 1 \cdot a_{11} = a_{11}.$$

Therefore,

$$\|A - I_{n+1}\| \geq |\cosh d_{\mathbb{H}}(e_1, Ae_1) - 1|.$$

Let b be the point in the fixed point set of A closest to e_1. We can apply hyperbolic cosine rule to the triangle with vertices e_1, Ae_1 and b. If θ is the angle of the triangle at the point b, then

$$\begin{aligned}\cosh d_{\mathbb{H}}(e_1, Ae_1) = \cosh d_{\mathbb{H}}(b, e_1) \cosh d_{\mathbb{H}}(b, Ae_1)& \\ - \sinh d_{\mathbb{H}}(b, e_1) \sinh d_{\mathbb{H}}(b, Ae_1) \cos\theta&.\end{aligned}$$

Therefore,

$$\begin{aligned}\cosh d_{\mathbb{H}}(e_1, Ae_1) &= \cosh^2\delta - \sinh^2\delta\cos\theta \\ &= \cosh^2\delta - \sinh^2\delta\Big(1 - 2\sin^2\Big(\frac{\theta}{2}\Big)\Big) \\ &= \cosh^2\delta - \sinh^2\delta + 2\sinh^2\delta\sin^2\Big(\frac{\theta}{2}\Big) \\ &= 1 + 2\sinh^2\delta\sin^2\Big(\frac{\theta}{2}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\|A - I_{n+1}\| &\geq \left|1 + 2\sinh^2\delta\sin^2\left(\frac{\theta}{2}\right) - 1\right| \\ &= 2\sinh^2\delta\sin^2\left(\frac{\theta}{2}\right) \\ &\geq 2\sinh^2\delta\sin^2\left(\frac{\pi}{k}\right) \quad \text{(by Lemma 10).}\end{aligned}$$

□

The following lemma follows immediately from Propositions 8 and 9.

Lemma 11. *Let $A \in O^+(1,n)$ be an elliptic element of order at most k. Let δ be the hyperbolic distance from e_1 to the fixed point set of A. Then*

$$\|A - I_{n+1}\| \geq \max\left\{2\sinh^2\delta\sin^2\left(\frac{\pi}{k}\right), 2\sin\left(\frac{\pi}{k}\right)e^{-2\delta}\right\}.$$

Hence, we have

Lemma 12. *Let $A \in O^+(1,n)$ be an elliptic element of order at most k. Then*

$$\|A - I_{n+1}\| \geq \inf_{\delta>0}\max\left\{2\sinh^2\delta\sin^2\left(\frac{\pi}{k}\right), 2\sin\left(\frac{\pi}{k}\right)e^{-2\delta}\right\}.$$

Proof of Proposition 3. We must show that the bound of Lemma 12 agrees with the uniform bound c_k. We divide the proof into two cases. First, assume $\delta \geq 1$. Since $\sinh^2\delta$ is an increasing function, we have

$$2\sinh^2\delta\sin^2\left(\frac{\pi}{k}\right) \geq 2\sinh^2(1)\sin^2\left(\frac{\pi}{k}\right) \geq 2\sin^2\left(\frac{\pi}{k}\right)e^{-2}.$$

Now assume $\delta \leq 1$. Note that $e^{-2\delta}$ is a decreasing function. Also note that $k \geq 1$ since A is nontrivial. Therefore,

$$\sin^2\left(\frac{\pi}{k}\right) \leq \sin\left(\frac{\pi}{k}\right).$$

Hence,

$$2\sin\left(\frac{\pi}{k}\right)e^{-2\delta} \geq 2\sin^2\left(\frac{\pi}{k}\right)e^{-2\delta} \geq 2\sin^2\left(\frac{\pi}{k}\right)e^{-2}.$$

Therefore, by Lemma 12,

$$\begin{aligned}\|A - I_{n+1}\| &\geq \inf_{\delta>0}\max\left\{2\sinh^2\delta\sin^2\left(\frac{\pi}{k}\right), 2\sin\left(\frac{\pi}{k}\right)e^{-2\delta}\right\} \\ &\geq 2\sin^2\left(\frac{\pi}{k}\right)e^{-2}.\end{aligned}$$

□

4. Norm bound for low-order torsion groups

On his way to providing lower bounds for the volumes of hyperbolic manifolds, Martin proved the following theorem.

Theorem 13 [Martin 1989a]. *Let Γ be a discrete nonelementary torsion free subgroup of $O^+(1,n)$. Then there is an $\alpha \in O^+(1,n)$ such that*

$$\|A\| \, \|A - I_{n+1}\| \geq \frac{1}{2\sqrt{2}}$$

for all $A \in \alpha\Gamma\alpha^{-1}$, $A \neq I_{n+1}$.

In this section we prove an orbifold version of this result. That is we drop the condition that Γ is torsion free and replace it with a bound on the order of torsion. Our proof is similar in outline to Martin's, with Proposition 3 allowing us to control elliptic elements.

Recall from the beginning of Section 3,

$$c_k := 2\sin^2\left(\frac{\pi}{k}\right)e^{-2}.$$

The following proposition is our orbifold version of Theorem 13.

Proposition 14. *Let Γ be a discrete nonelementary subgroup of $SO^+(1,n)$ which contains no torsion elements of order greater than k. Then there is an $\alpha \in O^+(1,n)$ such that*

$$\|A - I_{n+1}\| \geq c_k$$

for all $A \in \alpha\Gamma\alpha^{-1}$, $A \neq I_{n+1}$.

The proof will be by contradiction. In proving Proposition 14, we will pass between the ball and hyperboloid model of hyperbolic n-space.

As Γ is discrete, it is countable. Therefore Γ has a countable number of parabolic and hyperbolic fixed points on the boundary of $\mathbb{B}^n$. Furthermore the set of fixed points of each elliptic element on the boundary of $\mathbb{B}^n$ forms at most a codimension 2 subspace of $\mathbb{S}^{n-1}$. Therefore, we may assume (by conjugation) that no element of the group Γ fixes the north or south poles (N, S respectively) of $\mathbb{S}^{n-1}$. For each $t > 0$ let α_t represent, in the ball or hyperboloid model, the hyperbolic isometry that corresponds to a pure translation by t in the geodesic from S to N.

Lemma 15. *For each fixed $A \in \Gamma - \{I_{n+1}\}$,*

$$\lim_{t\to\infty} \|\alpha_t A \alpha_t^{-1}\| = \infty. \tag{4-1}$$

Proof. Let $\vec{0}$ be the origin in $\mathbb{B}^n$. Note that

$$\lim_{t\to\infty} \alpha_t^{-1}(\vec{0}) = S.$$

Hence, for $A \in \Gamma - \{I_{n+1}\}$,

$$\lim_{t\to\infty} A\alpha_t^{-1}(\vec{0}) = A(S).$$

Since $A(S) \neq S$, there exists a neighborhood V of S in $\overline{\mathbb{B}}^n$ such that $A(S) \notin V$. Thus,

$$\lim_{t\to\infty} \alpha_t A \alpha_t^{-1}(\vec{0}) = N.$$

Transferring from the ball model to the hyperboloid model, we have

$$\lim_{t\to\infty} |\alpha_t A \alpha_t^{-1}(e_1)| = \infty.$$

Therefore, (4-1) follows. □

For the remainder of this section we will work under the assumption that Proposition 14 fails. That is,

Assumption 16. *Let Γ be a discrete nonelementary subgroup of $SO^+(1,n)$ which contains no torsion elements of order greater than k. We assume that for all $\alpha \in O^+(1,n)$ there exists*

$$A \in \alpha\Gamma\alpha^{-1}, \quad A \neq I_{n+1},$$

such that

$$\|A - I_{n+1}\| < c_k.$$

Under this assumption, we will construct an infinite sequence $\{A_i\}$ of elements of Γ and a diverging sequence $\{t(i)\}$ of positive real numbers, so that

$$\| \alpha_{t(i)} A_i \alpha_{t(i)}^{-1} - I_{n+1} \| < c_k,$$
$$\| \alpha_{t(i)} A_{i+1} \alpha_{t(i)}^{-1} - I_{n+1} \| < c_k$$

for all i. We then use Martin and Friedland–Hersonsky's generalization of Jørgensen's inequality to show that $\{\|\alpha_{t(i)} A_i \alpha_{t(i)}^{-1}\|\}$ is unbounded and obtain a contradiction.

Definition 17. For each $A \in \Gamma - \{I_{n+1}\}$, let

$$U_A := \{\, t > 0 : \|\alpha_t A \alpha_t^{-1} - I_{n+1}\| < c_k \,\}.$$

It is clear from Lemma 15 and Assumption 16 that

$$\mathcal{U} = \{U_A : A \in \Gamma - \{I_{n+1}\}\}$$

forms an open cover of the positive real line by bounded sets.

Lemma 18. *There exists a sequence $t(i) \to \infty$ and a sequence $U_{A_i} \in \mathcal{U}$ such that*

$$t(i) \in U_i \cap U_{i+1}.$$

Proof. For each $A \in \Gamma - \{I_{n+1}\}$, let

$$\mathcal{U}_{[1,2]} := \{U_A \cap [1,2]\}.$$

Then, $\mathcal{U}_{[1,2]}$ is an open cover of $[1, 2]$. For $x, y \in [1, 2]$, we say

$$x \sim y$$

if there exist $r, s \in \mathbb{Z}^+$ and a sequence

$$\{U^1_{A_i}\}_{i=r}^{s} \subset \mathcal{U}_{[1,2]}$$

such that $x \in U^1_{A_r}$, $y \in U^1_{A_s}$ and $U^1_{A_i} \cap U^1_{A_i+1} \neq \varnothing$ for all i with $r \leq i \leq s-1$.

Reflexivity, symmetry and transitivity are immediate from the fact that $\mathcal{U}_{[1,2]}$ is an open cover. Therefore $\sim$ is an equivalence relation on $[1, 2]$. Now, let $E \subset [1, 2]$ be an equivalence class and let $x \in E$. There exists an open set $U^1_{A_i} \in \mathcal{U}_{[1,2]}$ such that $x \in U^1_{A_i}$. By definition of our equivalence relation, if $y \in U^1_{A_i}$ then $y \in E$. Therefore we have that $U^1_{A_i} \subset E$. Thus E is an open set. Hence, $[1, 2]$ can be divided into disjoint, open equivalence classes. Since $[1, 2]$ is connected, there is only one equivalence class.

Since $1 \sim 2$ there exists an $m_1 \in \mathbb{Z}^+$ and a sequence

$$\{U^1_{A_i}\}_{i=1}^{m_1} \subset \mathcal{U}_{[1,2]}$$

such that

$$1 \in U^1_{A_1}, \quad 2 \in U^1_{A_{m_1}} \quad \text{and} \quad U^1_{A_i} \cap U^1_{A_{i+1}} \neq \varnothing$$

for all i with $1 \leq i \leq m_1 - 1$. Define $t(1) := 1$, $t(m_1) := 2$ and select $t(i)$ from $U^1_{A_{i-1}} \cap U^1_{A_i}$ for $2 \leq i \leq m_1 - 1$.

Now consider $\mathcal{U}_{[2,3]} = \{U^2_A\}$, where $U^2_A := U_A \cap [2, 3]$. By repeating the program above, we can define $t(i)$ for $m_1 + 1 \leq i \leq m_2$, where m_2 is an integer larger than m_1 and $t(m_2) = 3$. We can then define the corresponding U_{A_i} for $m_1 + 1 \leq i \leq m_2$.

Continuing in this way, we define the required sequences. □

The next lemma will be the key in what follows.

Lemma 19. *For all i,*

$$\|\alpha_{t(i)} A_i \alpha^{-1}_{t(i)} - I_{n+1}\| < c_k,$$
$$\|\alpha_{t(i)} A_{i+1} \alpha^{-1}_{t(i)} - I_{n+1}\| < c_k.$$

Proof. As $t(i) \in U_{A_i} \cap U_{A_{i+1}}$ the lemma follows immediately from Definition 17. □

Next, we prove that the set $\{\|\alpha_{t(i)} A_i \alpha^{-1}_{t(i)}\|\}$ is unbounded. This will be shown to contradict Lemma 19 and thus establish Proposition 14. The following is a special case of [Friedland and Hersonsky 1993, Theorem 2.11] and [Martin 1989b, Theorem 4.5].

Theorem 20. *Let $\Gamma \subset O^+(1,n)$ be a discrete group. Let τ be the unique positive solution of the cubic equation $2\tau(1+\tau)^2 = 1$. If $A, B \in \Gamma$ such that $\langle A, B\rangle$ is a discrete group and*

$$\|A - I_{n+1}\| < \tau, \qquad \|B - I_{n+1}\| < \tau,$$

then $\langle A, B\rangle$ is a nilpotent group.

Remark 21. We note here that

$$\begin{aligned} \tau &> 0.2971 \\ &> 2e^{-2} \\ &\geq 2\sin^2\left(\frac{\pi}{k}\right)e^{-2} \quad \text{for all } k \\ &= c_k. \end{aligned}$$

We now prove the following claim.

Claim 22. *The set $\{\|\alpha_{t(i)}A_i\alpha_{t(i)}^{-1}\|\}$ is unbounded.*

Proof. Since $\alpha_{t(i)}A_i\alpha_{t(i)}^{-1}$ and $\alpha_{t(i)}A_{i+1}\alpha_{t(i)}^{-1}$ are elements of the discrete group $\alpha_{t(i)}\Gamma\alpha_{t(i)}^{-1}$,

$$\left\langle \alpha_{t(i)}A_i\alpha_{t(i)}^{-1}, \alpha_{t(i)}A_{i+1}\alpha_{t(i)}^{-1}\right\rangle$$

is a discrete group. By Lemma 19, Theorem 20 and Remark 21,

$$\left\langle \alpha_{t(i)}A_i\alpha_{t(i)}^{-1}, \alpha_{t(i)}A_{i+1}\alpha_{t(i)}^{-1}\right\rangle$$

is nilpotent and thus elementary. Therefore $\langle A_i, A_{i+1}\rangle$ is discrete and elementary.

By Assumption 16, if A_i is elliptic it has order at most k. This implies

$$\|\alpha_{t(i)}A_i\alpha_{t(i)}^{-1} - I_{n+1}\| \geq c_k$$

by Proposition 3. However that directly contradicts the definition of U_{A_i}. We conclude that no element of $\{A_i\}$ is elliptic. So for all i, A_i and A_{i+1} are either both parabolic sharing a common fixed point or both hyperbolic sharing a common axis. Therefore, either each A_i is hyperbolic or each A_i is parabolic and the A_i all have a common fixed point set.

Let Δ be the subgroup of Γ generated by all A_i. Since the fixed points of any element of Δ is fixed by the entire group, Δ is elementary. The set $\{A_i\}$ is an infinite sequence in Δ, since each U_A is bounded and $t_i \to \infty$. Therefore there exists a subsequence $\{A_{i_j}\}$ and points $x_0, y_0 \in \overline{\mathbb{B}}^n$ such that

$$\lim_{j\to\infty} A_{i_j} = y_0 \quad \text{and} \quad \lim_{j\to\infty} A_{i_j}^{-1} = x_0$$

uniformly on compact subsets in $\overline{\mathbb{B}}^n \setminus \{x_0\}$ and $\overline{\mathbb{B}}^n \setminus \{y_0\}$, respectively.

Since x_0 and y_0 are accumulation points of a discrete elementary group with no elliptic elements, they are fixed points of elements in Δ. Now

$$\{x_0, y_0\} \cap \{N, S\} = \varnothing$$

since N and S are not fixed points. Therefore, since

$$\lim_{j\to\infty} \alpha^{-1}_{t(i_j)}(\vec{0}) = S \neq x_0, \quad \text{and} \quad \lim_{j\to\infty} A_{i_j} = y_0$$

uniformly on compact subsets of $\overline{\mathbb{B}}^n \setminus \{x_0\}$, we have that $\lim_{j\to\infty} A_{i_j}\alpha^{-1}_{t(i_j)}(\vec{0}) = y_0$. Hence,

$$\lim_{j\to\infty} \alpha_{t(i_j)} A_{i_j} \alpha^{-1}_{t(i_j)}(\vec{0}) = N.$$

Thus, transferring from the ball model to the hyperboloid model, we have

$$\lim_{j\to\infty} |\alpha_{t(i_j)} A_{i_j} \alpha^{-1}_{t(i_j)}(e_1)| = \infty.$$

Therefore,

$$\lim_{j\to\infty} \|\alpha_{t(i_j)} A_{i_j} \alpha^{-1}_{t(i_j)}\| = \infty. \qquad \square$$

Proof Proposition 14. To complete the proof of Proposition 14 we observe that by Lemma 19,

$$\begin{aligned} \big|\|\alpha_{t(i)} A_i \alpha^{-1}_{t(i)}\| - 1\big| &= \big|\|\alpha_{t(i)} A_i \alpha^{-1}_{t(i)}\| - \|I_{n+1}\|\big| \\ &\leq \|\alpha_{t(i)} A_i \alpha^{-1}_{t(i)} - I_{n+1}\| \\ &< c_k < 1, \end{aligned}$$

which implies that

$$\|\alpha_{t(i)} A_i \alpha^{-1}_{t(i)}\| < 2,$$

contradicting Claim 22. $\square$

5. Proof of main result

In this section we prove several technical lemmas. These lemmas will be used to show that if the conclusion of Proposition 14 holds we can establish an upper bound on the number of elements of Γ that fail to move a ball of radius r off itself. This result will allow us to bound the volume of the image of such a ball in the orbit space, thereby establishing Theorem 1.

First, we see that a bound on the translation distance of a hyperbolic isometry implies a bound on the entries of the associated matrix.

Lemma 23. *Given $r > 0$, if $A \in O^+(1, n)$ is such that*

$$A\big(\overline{B(e_1, r)}\big) \cap \overline{B(e_1, r)} \neq \varnothing, \tag{5-1}$$

then for all i, j,

$$|a_{ij}| \leq \kappa(r),$$

where $\kappa(r) := \frac{1+\cosh r}{\sinh r}\sqrt{\cosh 6r}$.

Proof. For all $j \neq 1$ and for all i,

$$\begin{aligned}
|a_{ij}| &\leq |Ae_j| \\
&= \left|A\left(\frac{\cosh r}{\sinh r}e_1 + e_j - \frac{\cosh r}{\sinh r}e_1\right)\right| \\
&\leq \frac{1}{\sinh r}\big|A\big((\cosh r)e_1 + (\sinh r)e_j\big)\big| + \frac{\cosh r}{\sinh r}|Ae_1|.
\end{aligned}$$

The assumption (5-1) implies that

$$A\big(\overline{B(e_1, r)}\big) \subset \overline{B(e_1, 3r)}.$$

Now, let

$$v = (\cosh r)e_1 + (\sinh r)e_j.$$

Then

$$\cosh d_{\mathbb{H}}(v, e_1) = \cosh r.$$

Since r and $d_{\mathbb{H}}(v, e_1)$ are nonnegative we have that $d_{\mathbb{H}}(v, e_1) = r$. Therefore

$$v \in \overline{B(e_1, r)}.$$

Clearly $e_1 \in \overline{B(e_1, r)}$. Furthermore, as any element of $\overline{B(e_1, 3r)}$ has euclidean length at most

$$\sqrt{\sinh^2 3r + \cosh^2 3r},$$

we have

$$|a_{ij}| \leq \left(\frac{1}{\sinh r} + \frac{\cosh r}{\sinh r}\right)\sqrt{\sinh^2 3r + \cosh^2 3r} = \frac{1+\cosh r}{\sinh r}\sqrt{\cosh 6r}$$

for all $j \neq 1$ and for all i. For $j = 1$,

$$|a_{i1}| \leq |Ae_1| \leq \sqrt{\sinh^2 3r + \cosh^2 3r} = \sqrt{\cosh 6r}.$$

Since

$$\frac{1+\cosh r}{\sinh r} > 1,$$

the result follows. □

A bound on the entries of a matrix A implies a bound on its operator norm.

Lemma 24. *Let* A *be any* $(n+1) \times (n+1)$ *matrix. Given* $d > 0$, *if* $|a_{ij}| \leq d$ *for all* i *and* j, *then*

$$\|A\| \leq d(n+1).$$

Proof. Let v be a vector of unit length. Let A_i denote the i-th row of the matrix A. Then

$$|A_i \cdot v| \leq |A_i||v| = |A_i| = \sqrt{\sum_{j=1}^{n+1} (a_{ij})^2} \leq \sqrt{\sum_{j=1}^{n+1} d^2} = d\sqrt{n+1}.$$

Hence,

$$\begin{aligned} |Av| &= \left|(A_1 \cdot v, \ldots, A_{n+1} \cdot v)\right| \\ &= \sqrt{(A_1 \cdot v)^2 + \cdots + (A_{n+1} \cdot v)^2} \\ &\leq \sqrt{(n+1)(n+1)d^2} \\ &= d(n+1). \end{aligned}$$

Since v was chosen arbitrarily, by our definition of the operator norm,

$$\|A\| \leq d(n+1). \qquad \square$$

Given two matrices A and B, a bound on the distance between corresponding entries gives a bound on $\|AB^{-1} - I_{n+1}\|$.

Lemma 25. *Given $K, L > 0$, let*

$$\delta := \frac{L}{(n+1)^2 K}.$$

For $A, B \in O^+(1,n)$, if $|b_{ij}| \leq K$ and $|a_{ij} - b_{ij}| \leq \delta$ for all i and j, then

$$\|AB^{-1} - I_{n+1}\| \leq L.$$

Proof.

$$\|AB^{-1} - I_{n+1}\| = \|(A-B)B^{-1}\| \leq \|(A-B)\|\|B^{-1}\| = \|(A-B)\|\|B\|.$$

Therefore, by Lemma 24,

$$\|AB^{-1} - I_{n+1}\| \leq (n+1)\delta(n+1)K = (n+1)^2\delta K. \qquad \square$$

The following lemma bounds the size of a bounded uniformly discrete subset of $\mathbb{R}^p$.

Lemma 26. *Let $p \in \mathbb{Z}^+$ and $q, s \in \mathbb{R}^+$ be given. Let $M \subset \mathbb{R}^p$ be such that*

(i) *$(a_i) \in M$ implies that $|a_i| \leq q$ for all i, and*

(ii) *$(a_i), (b_i) \in M$ implies that $|a_i - b_i| > s$ for some i.*

Then

$$|M| \leq \left(\frac{2q}{s} + 1\right)^p.$$

Proof. We divide the interval $[-q, q]$ as follows

$$
\begin{aligned}
E_1 &= [-q, -q+s), \\
E_2 &= [-q+s, -q+2s), \\
&\ \vdots \\
E_{[\frac{2q}{s}]+1} &= \Big[-q + \Big[\frac{2q}{s}\Big]s, q\Big].
\end{aligned}
$$

Now each $a = (a_i) \in M$ is an element of a p-cylinder

$$E_a = E_{j_1} \times E_{j_2} \times \cdots \times E_{j_p},$$

that is, $a_i \in E_{j_i}$, for all i. We note that by condition (ii) above, if

$$(a_i) \in E_a, \quad \text{and} \quad (a_i) \neq (b_i)$$

then $(b_i) \notin E_a$.

Hence we need only count the number of possible cylinders. As there are a maximum of $[\frac{2q}{s}] + 1$ choices for p different positions, our conclusion follows. $\square$

We now use our series of lemmas to bound the set of elements of a discrete group of hyperbolic isometries that fail to move a ball of radius r off itself.

Definition 27.

$$\mathcal{H}(r, \Gamma) := \Big\{A \in \Gamma : A\big(\overline{B(e_1, r)}\big) \cap \overline{B(e_1, r)} \neq \varnothing\Big\}.$$

Lemma 28. *Let Γ be a subgroup of $O^+(1, n)$ such that $\|A - I_{n+1}\| \geq c_k$ for all $A \in \Gamma - I_{n+1}$. For $r > 0$,*

$$|\mathcal{H}(r, \Gamma)| \leq \Big(\frac{2\kappa(r)^2(n+1)^2}{c_k} + 1\Big)^{(n+1)^2},$$

where c_k and $\kappa(r)$ are the constants of Proposition 3 and Lemma 23 respectively.

Proof. Let $r > 0$ be given. Let A and B be distinct elements of $\mathcal{H}(r, \Gamma)$. Then A and B are distinct elements of Γ and, therefore, AB^{-1} is a nonidentity element of Γ. So by assumption, we have

$$\|AB^{-1} - I_{n+1}\| \geq c_k.$$

By Lemma 23, $|b_{ij}| \leq \kappa(r)$ for all i, j. Thus by Lemma 25, there exists i_0, j_0, such that

$$|a_{i_0 j_0} - b_{i_0 j_0}| > \frac{c_k}{(n+1)^2\kappa(r)}.$$

We now apply Lemma 26, with

$$p = (n+1)^2, \quad q = \kappa(r) \quad \text{and} \quad s = \frac{c_k}{(n+1)^2\kappa(r)}.$$

In this case,
$$\Big(\frac{2q}{s}+1\Big)^p = \Big(\frac{2\kappa(r)^2(n+1)^2}{c_k}+1\Big)^{(n+1)^2}. \qquad \square$$

We are now prepared to prove our main result, which for convenience is restated below.

Theorem 29 (Main Theorem). *Let Γ be a discrete group of orientation-preserving isometries of $\mathbb{H}^n$. Assume that Γ has no torsion element of order greater than k. Then*
$$\mathrm{Vol}(\mathbb{H}^n/\Gamma) \geq \mathcal{A}(n,k)$$
where $\mathcal{A}(n,k)$ is the constant depending only on n and k defined in the introduction.

Proof. Let Γ be a discrete subgroup of $SO^+(1,n)$ which contains no torsion elements of order greater than k. If Γ is an elementary group, its covolume would be infinite. Therefore, we may assume that Γ is nonelementary. Hence by Proposition 14, there exists a group Γ', conjugate to Γ, such that
$$\|A - I_{n+1}\| \geq c_k$$
for all $A \in \Gamma'$ and $A \neq I_{n+1}$.

Let π be the covering projection from hyperbolic n-space onto $Q = \mathbb{H}^n/\Gamma'$. For $r > 0$, let
$$\mathcal{H} := \big\{A \in \Gamma' : A\big(\overline{B(e_1,r)}\big) \cap \overline{B(e_1,r)} \neq \varnothing\big\}.$$

The map π restricted to $B(e_1,r)$ is a local isometry away from the singular locus of Q. Notice that the singular locus has volume zero. If $x \in \pi(B(e_1,r))$ then
$$|\pi^{-1}(x) \cap B(e_1,r)| \leq |\mathcal{H}|.$$

Therefore,
$$\mathrm{Vol}(\mathbb{H}^n/\Gamma) = \mathrm{Vol}(\mathbb{H}^n/\Gamma') \geq \mathrm{Vol}(\pi(B(e_1,r)) \geq \frac{\mathrm{Vol}(B(e_1,r))}{|\mathcal{H}|}.$$

The numerator of the above fraction is the volume of a hyperbolic n-ball of radius r. This can be computed explicitly in terms of ω_{n-1}, the volume of a ball of radius 1 in euclidean n-space. The denominator is bounded by Lemma 28. Hence,
$$\mathrm{Vol}(\mathbb{H}^n/\Gamma) \geq \omega_{n-1}\int_0^r \sinh^{n-1}(u)du\,\Big(\frac{2\kappa(r)^2(n+1)^2}{c_k}+1\Big)^{-(n+1)^2}. \qquad \square$$

Remark 30. Explicit covolume estimates for Kleinian groups containing elliptics with order greater than or equal to 6 have been calculated in [Gehring and Martin 1999]. Volume bounds for arithmetic hyperbolic orbifolds in even dimensions have been produced in [Belolipetsky 2004].

6. Corollaries

Corollary 2 follows readily from Theorem 1. We recall that the quotient Q of a hyperbolic manifold M by its group of orientation-preserving isometries G is an orientable hyperbolic orbifold, as long as $\pi_1(M)$ is not virtually abelian, in which case $\mathrm{Vol}(M)$ is infinite and Corollary 2 is vacuous. Note that

$$\mathrm{Vol}(Q) = \frac{\mathrm{Vol}(M)}{|G|}.$$

Since, under the assumptions of Corollary 2, $\mathrm{Vol}(Q) \geq \mathscr{A}(n,k)$, we obtain our result.

The Mostow–Prasad rigidity theorem [Mostow 1968; Prasad 1973] implies that if M has finite volume then one can identify the group of isometries of M with $\mathrm{Out}(\pi_1(M))$. With this in mind, we can give the following more topological version of Corollary 2.

Corollary 31. *If M is a finite volume orientable hyperbolic n-manifold and G is a subgroup of* $\mathrm{Out}(\pi_1(M))$ *containing no torsion elements of order greater than k, then*

$$|G| \leq \frac{2\,\mathrm{Vol}(M)}{\mathscr{A}(n,k)}.$$

Remark 32. The corollaries to Theorem 1 are related to a volume bound for a closed hyperbolic n-manifold with a symmetry of prime order obtained in [Gehring et al. 1998]. This bound depends on the order of the symmetry and the dimension and volume of its fixed set.

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