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The aim of this work is to prove partial regularity of the p-harmonic maps with potential. The main difficultly caused by the potential is in finding the equation satisfied by the scaling function, which breaks down the blow-up processing. We directly estimate the Morrey energy to avoid the difficulties caused by blowing up.

1. Introduction

Suppose (M, g) and (N, h) are two smooth compact Riemannian manifolds. Let $u : M \to N$ be a map, and let $H(u, x)$ be a function on $N \times M$. We consider the functional

$$E(u) = \int_M (e(u) - H(u, x)), \tag{1-1}$$

where $e(u) = (1/p)(\operatorname{trace} u^*h)^{p/2}$ is the p-energy density of u. Here and below (usually), we leave implicit the integration measure. By Nash's embedding theorem, we can think of N as a submanifold that is isomorphically embedded in some Euclidean space $\mathbb{R}^k$. Since N is a submanifold of $\mathbb{R}^k$, we may consider the tangent space T_yN at point $y \in N$ as a subspace of $\mathbb{R}^k$. Let $P(y) : \mathbb{R}^k \to T_yN$ be the orthogonal projection. A critical point of the functional $E(u)$ is called a p-harmonic map with potential (for $p = 2$, see [Fardoun and Ratto 1997]). The Euler–Lagrange equation of $E(u)$ is

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}A(u)(\nabla u, \nabla u) + P(u)(\nabla^N H(u, x)), \tag{1-2}$$

where $\nabla^N H(u, x)$ is the gradient of H on N and $A(u)(\cdot\,,\cdot)$ is the second fundamental form on N. For u belonging to $W^{1,p}(M; N)$, as defined in (1-4) below, we

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say u is a weakly p-harmonic map if, for any $\phi \in C_0^\infty(M; \mathbb{R}^k)$,

$$\int_M |\nabla u|^{p-2} \nabla u \nabla \phi = \int_M (|\nabla u|^{p-2} A(u)(\nabla u, \nabla u) + P(u)(\nabla^N H(u, x)))\phi.$$

The Euler–Lagrange equation can be seen as requiring vanishing variation of the target manifold for energy (1-1). On the other hand, we can consider the vanishing variation of the domain. For simplicity, we assume M is a subset Ω in $\mathbb{R}^m$ and set $u_\tau(x) = u(x+\tau\varphi)$, where $\varphi \in C_0^\infty(\Omega, \mathbb{R}^m)$ and τ is small enough. Then computing $\frac{d}{d\tau} E(u(x+\tau\varphi))|_{\tau=0} = 0$ leads to

$$\partial_\alpha |\nabla u|^p - p\partial_\beta(|\nabla u|^{p-2}\partial_\beta u \cdot \partial_\alpha u) + p\partial_\alpha H(u, x) = 0 \tag{1-3}$$

in the distributional sense.

The weakly p-harmonic maps with potential H in $W^{1,p}(\Omega, N)$ that satisfy (1-3) are called stationary p-harmonic maps with the potential. Here

$$W^{1,p}(\Omega, N) = \{u \in W^{1,p}(\Omega, \mathbb{R}^k) \mid u(x) \in N \text{ for almost all } x \in \Omega\}. \tag{1-4}$$

Stationary harmonic maps were first introduced in [Schoen 1984]. Rivière's example [1995] shows that a weakly harmonic map may be discontinuous everywhere.

The partial regularity for stationary harmonic maps was proved by Evans [1991] in the case N is a sphere and by Bethuel [1993] in the general case. S. A. Chang, L. Wang, and P. C. Yang gave an elementary argument in their interesting paper [Chang et al. 1999]. In two-dimensional case, Hélein [1991a] proved that weak harmonic maps are smooth, and J. Qing [1993] proved the boundary regularity. For p-harmonic maps, there are some interesting results by T. Toro and C. Wang [1995] and C. Wang [1998]. There are some papers dealing with the existence of harmonic maps and their heat flows with potential [Fardoun and Ratto 1997; Fardoun et al. 2000; Fardoun and Regbaoui 2002, 2003], but there seem to be very few results about the regularity of such maps and their heat flows. In [Chu and Liu 2006], we proved the partial regularity of p-harmonic maps with potential under strongly restrictive conditions on the potential, when $1 < p \le 2$. The main difficulty caused by the potential H is in finding the equation satisfied by the scaling function, which breaks down the blow-up processing.

In this note, we shall consider the regularity of stationary p-harmonic maps into the sphere $\mathbb{S}^{n-1}$. We prove the partial regularity for general potential H when $1 < p < \dim M$. Our method is to estimate directly the Morrey energy.

We prove the following theorem in this paper.

Theorem 1.1. *Let M be a compact Riemannian manifold with* $\dim M \ge 3$. *Let N be the sphere $\mathbb{S}^{n-1}$, and let $u : M \to \mathbb{S}^{n-1}$ be a stationary p-harmonic map with* *smooth potential $H(u,x)$, where $1 < p < \dim M$. There exists a closed subset Σ of M such that $\mathcal{H}^{\dim M - p}(\Sigma) = 0$ and $u \in C^{1,\alpha}(M \setminus \Sigma, N)$.*

This theorem also holds when $p = \dim M$, in which case the singular set is empty. For simplicity, we shall also assume throughout that M is $\mathbb{R}^m$ with the standard metric.

2. Decay lemma

First we shall prove a monotonicity formula for stationary p-harmonic maps.

Proposition 2.1. *There exists a constant C^* such that*

$$C^* r_1^{p+1} + \frac{1}{r_1^{m-p}} \int_{B_{r_1}(x)} |\nabla u|^p \leq C^* r_2^{p+1} + \frac{e}{r_2^{m-p}} \int_{B_{r_2}(x)} |\nabla u|^p$$

for any $x \in \mathbb{R}^m$ and $0 < r_1 < r_2 \leq 1$.

Proof. Let $B_r(x) = B_r(0) = B_r$, and set $\zeta = \phi(|y|)y$, where

$$\phi(s) = \begin{cases} 1 & \text{if } s \leq r, \\ 1 + (r-s)/h & \text{if } r < s < r+h, \\ 0 & \text{if } s > r+h. \end{cases}$$

Then $\partial_j \zeta^j = \phi' y^j y^i / |y| + \phi \delta_{ij}$, and stationary condition (1-3) becomes

$$\int \bigl(|\nabla u|^p (\phi' |y| + \phi m) - p |\nabla u|^{p-2} \partial_i u \partial_j u (\phi' y^i y^j / |y| + \phi \delta_{ij}) \bigr) + \int p (\partial_{u^i} H(u,x) \partial_j u^i + \partial_{x_j} H(u,x)) y^j \phi = 0.$$

Letting $h \to 0^+$, we have $\int_\Omega |\nabla u|^p \phi' |y| \to -r \int_{\partial B(r)} |\nabla u|^p$ and

$$\int_\Omega |\nabla u|^{p-2} \partial_i u \partial_j u \phi' y^i y^j |y|^{-1} \to -r^{-1} \int_{\partial B(r)} |\nabla u|^{p-2} |r u_r|^2,$$

where $r u_r = y \nabla_y u$. Thus we obtain

$$(m-p) \int_{B_r} |\nabla u|^p - r \int_{\partial B_r} |\nabla u|^p + p r^{-1} \int_{\partial B_r} |\nabla u|^{p-2} |r u_r|^2 + p \int_{B_r} (\partial_{u^i} H(u,x) \partial_j u^i + \partial_{x_j} H(u,x)) y^j = 0,$$

that is,

$$\begin{aligned}\frac{d}{dr}\Big(\frac{1}{r^{m-p}}\int_{B_r}|\nabla u|^p\Big) &= \frac{1}{r^{m-p+1}}\Big((p-m)\int_{B_r}|\nabla u|^p + r\int_{\partial B_r}|\nabla u|^p\Big)\\ &= \frac{1}{r^{m-p+1}}\Big(pr^{-1}\int_{\partial B_r}|\nabla u|^{p-2}|ru_r|^2 + p\int_{B_r}(\partial_{u^i}H(u)\partial_j u^i + \partial_{x_j}H(u,x))y^j\Big)\\ &\geq \frac{1}{r^{m-p+1}}\Big(-cr\int_{B_r}(|\nabla u|+1)\Big) \geq \frac{1}{r^{m-p+1}}\Big(-Cr^{m+1} - r\int_{B_r}|\nabla u|^p\Big).\end{aligned}$$

Set $z(r) = (1/r^{m-p})\int_{B_r}|\nabla u|^p$. Then for any $0 < r \leq 1$,

$$\frac{d}{dr}(\exp(r)z(r)) \geq -Cr^p\exp(r) \geq -C^*(p+1)r^p,$$

that is, $\frac{d}{dr}(\exp(r)z(r) + C^*r^{p+1}) \geq 0$, which proves the proposition. □

For $B_r(x) \subset \mathbb{R}^m$, we define the renormalized energy

$$U(x,r) = r^p + \frac{1}{r^{m-p}}\int_{B_r(x)}|\nabla u|^p.$$

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. *Suppose u is a stationary p-harmonic map into the sphere $\mathbb{S}^{n-1}$. Then*

$$U(x_0, r/2) \leq C_0\Big(U(x_0,r)^{1/p}U(x_0,r) + r^{-m}\int_{B_r(x_0)}|u-u_r|^p\Big),$$

where C_0 is a universal constant.

Proof. When $N = \mathbb{S}^{n-1}$, Equation (1-2) becomes

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p u + P(u)(\nabla^N H(u,x)). \tag{2-1}$$

Let $\phi \in C_0^\infty(B_r(x_0))$ be a standard cutoff function, with $\phi = 1$ on $B_{r/2}(x_0)$ and $|\nabla\phi^{p-1}| \leq c/r$. By multiplying Equation (2-1) by $\phi^p(u-u_r)$, where $u_r = u_{x_0,r} = ⨍_{B_r(x_0)} u$, and then integrating, we obtain

$$\begin{aligned}\int|\nabla u|^p\phi^p &= \int|\nabla u|^p u\cdot\phi^p(u-u_r) + \int P(u)(\nabla^N H(u,x))\cdot\phi^p(u-u_r)\\ &\qquad - \int|\nabla u|^{p-2}\nabla u\nabla\phi^p(u-u_r)\\ &\leq Cr^m + \int|\nabla u|^p u\cdot\phi^p(u-u_r)\\ &\qquad + \frac14\int|\nabla u|^p\phi^p + \frac{C}{r^p}\int_{B_r(x_0)}|u-u_r|^p,\end{aligned} \tag{2-2}$$

where we used $|P(u)\nabla^N H| \le C$. We estimate the term $\int |\nabla u|^p u \cdot \phi^p (u-u_r)$ as follows. Taking wedge product with respect to u in (2-1), we have

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u \wedge u) = P(u)\nabla^N (H(u,x)) \wedge u.$$

In particular,

$$-\nabla \cdot (\phi^{p-1}|\nabla u|^{p-2}\nabla u \wedge u) = -\nabla \phi^{p-1}|\nabla u|^{p-2}\nabla u \wedge u \\ + \phi^{p-1}P(u)\nabla^N (H(u,x)) \wedge u.$$

For $p' = p/(p-1)$, we have

$$\int |\nabla \cdot (\phi^{p-1}|\nabla u|^{p-2}\nabla u \wedge u)|^{p'} \le Cr^m + \frac{C}{r^{p'}} \int_{B_r(x_0)} |\nabla u|^p. \tag{2-3}$$

On the other hand, by Hélein's trick [1991a; 1991b],

$$|\nabla u|^p u = |\nabla u|^{p-2} \sum_{\alpha, j} (u^i \partial_\alpha u^j - u^j \partial_\alpha u^i) \frac{\partial u^j}{\partial x_\alpha} = |\nabla u|^{p-2} (\nabla u \wedge u)\nabla u,$$

and we have

$$\begin{aligned} \int |\nabla u|^p u \cdot \phi^p (u-u_r) &= \int |\nabla u|^{p-2}(\nabla u \wedge u)\nabla u \cdot \phi^p (u-u_r) \\ &= \int \phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)\nabla(\phi(u-u_r)) \cdot (u-u_r) \\ &\quad - \int \phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)(\nabla \phi)(u-u_r)\cdot(u-u_r) \\ &=: I_1 + I_2. \end{aligned}$$

Noting that $|u| = 1$, we have by Hölder's inequality that

$$|I_2| \le \frac{1}{4}\int |\nabla u|^p \phi^p + Cr^{-p}\int_{B_r(x_0)} |u-u_r|^p. \tag{2-4}$$

In order to estimate I_1, we recall that the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$ is the set of functions $g \in L^1$ such that the maximal function

$$g^*(x) = \sup_{r>0} \left| \frac{1}{r^m} \int g(y)\psi((x-y)/r)dy \right|$$

is also in L^1. Here ψ represents any smooth function with support in the unit ball and with $\int \psi = 1$. A norm on $\mathcal{H}^1$ is $\|g\|_{\mathcal{H}^1} = \|g\|_{L^1} + |g^*|_{L^1}$.

A fundamental theorem of Fefferman asserts that the dual space of $\mathcal{H}^1$ is BMO (see [Strömberg and Torchinsky 1989]), and

$$\left| \int gf \right| \le c\|g\|_{\mathcal{H}^1} \|f\|_{BMO}.$$

We have the following lemmas.

Lemma 2.2. *$V(x) := \phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)\nabla(\phi(u-u_r))$ belongs to $\mathcal{H}^1(\mathbb{R}^m)$. Also*

$$\|V\|_{\mathcal{H}^1(\mathbb{R}^m)} \le C\Big(r^{m+1} + \int_{B_r(x_0)} |\nabla u|^p + r^{-p}\int_{B_r(x_0)} |u-u_r|^p\Big),$$

where C is an absolute constant.

Proof. Set $\psi_\rho(\cdot) = (1/\rho^m)\psi(\cdot/\rho)$ and

$$(\psi_\rho * V)(x) = \frac{1}{\rho^m}\int V(y)\psi((x-y)/\rho)dy.$$

Making use of integration by parts, we have

$$\begin{aligned}
(\psi_\rho * V)(x) &= \rho^{-m}\int \phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)\nabla(\phi(u-u_r))\psi\Big(\frac{x-y}{\rho}\Big)dy\\
&= \rho^{-m}\int \nabla\left(\phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)\,(\phi(u-u_r)-c)\right)\psi\Big(\frac{x-y}{\rho}\Big)dy\\
&\quad - \rho^{-m}\int \nabla\left(\phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)\right)(\phi(u-u_r)-c)\psi\Big(\frac{x-y}{\rho}\Big)dy\\
&= \rho^{-(m+1)}\int \phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)(\phi(u-u_r)-c)\nabla\psi\Big(\frac{x-y}{\rho}\Big)dy\\
&\quad - \rho^{-m}\int \nabla\left(\phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)\right)(\phi(u-u_r)-c)\psi\Big(\frac{x-y}{\rho}\Big)dy\\
&=: I_{11} + I_{12},
\end{aligned}$$

where $c = (\phi(u-u_r))_\rho$. Using Hölder's inequality and the Sobolev–Poincaré inequality, we obtain

$$\begin{aligned}
|I_{11}| &\le \frac{C}{\rho}\Big(⨍_{B_\rho(x)} |\phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u)|^{q'}\Big)^{1/q'}\Big(⨍_{B_\rho(x)} |\phi(u-u_r)-c|^q\Big)^{1/q}\\
&\le C\Big(⨍_{B_\rho(x)} |\phi\nabla u|^{(p-1)q'}\Big)^{1/q'}\Big(⨍_{B_\rho(x)} |\nabla(\phi(u-u_r))|^{q^*}\Big)^{1/q^*}\\
&\le CM(|\phi\nabla u|^{(p-1)q'})^{1/q'}(x)\, M(|\nabla(\phi(u-u_r))|^{q^*})^{1/q^*}(x),
\end{aligned}$$

where $1 < q' < p'$, $1/q' + 1/q = 1$, $q^* = mq/(m+q)$ and $p < q < pm/(m-p)$.

$$\begin{aligned}
|I_{12}| &\le C\Big(⨍_{B_\rho(x)} |\nabla(\phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u))|^{q'}\Big)^{1/q'}\Big(⨍_{B_\rho(x)} |\phi(u-u_r)-c|^q\Big)^{1/q}\\
&\le C\Big(⨍_{B_\rho(x)} |\nabla(\phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u))|^{q'}\Big)^{1/q'}\Big(⨍_{B_\rho(x)} |\nabla(\phi(u-u_r))|^{q^*}\Big)^{1/q^*}\rho\\
&\le CM(|r\nabla(\phi^{p-1}|\nabla u|^{p-2}(\nabla u \wedge u))|^{q'})^{1/q'}(x)\, M(|\nabla(\phi(u-u_r))|^{q^*})^{1/q^*}(x),
\end{aligned}$$

where $M(f)(x)$ is the maximum function of f. Thus by (2-3),

$$\begin{aligned}
&\int \sup_\rho |(\psi_\rho * V)(x)| \\
&\le C\int M(|\phi\nabla u|^{(p-1)q'})^{1/q'}(x)\, M(|\nabla(\phi(u-u_r))|^{q^*})^{1/q^*}(x) \\
&\quad + C\int M(|r\nabla(\phi^{p-1}|\nabla u|^{p-2}(\nabla u\wedge u))|^{q'})^{1/q'}(x)\, M(|\nabla(\phi(u-u_r))|^{q^*})^{1/q^*}(x) \\
&\le C\int M(|\phi\nabla u|^{(p-1)q'})^{p'/q'}(x) + C\int M(|\nabla(\phi(u-u_r))|^{q^*})^{p/q^*}(x) \\
&\quad + C\int M(|r\nabla(\phi^{p-1}|\nabla u|^{p-2}(\nabla u\wedge u))|^{q'})^{p'/q'}(x) \\
&\quad + C\int M(|\nabla(\phi(u-u_r))|^{q^*})^{p/q^*}(x) \\
&\le C\Big(\int |\nabla u|^p\phi^p + |r\nabla(\phi^{p-1}|\nabla u|^{p-2}(\nabla u\wedge u))|^{p'} + |\nabla(\phi(u-u_r))|^p\Big) \\
&\le C\Big(r^{m+1} + \int_{B_r(x_0)} |\nabla u|^p + r^{-p}\int_{B_r(x_0)} |u-u_r|^p\Big) \\
&\le Cr^m + C\int_{B_r(x_0)} |\nabla u|^p. \qquad \square
\end{aligned}$$

From the definition of I_1, we may assume $u-u_r=0$ on $\mathbb{R}^m\backslash B_r(x_0)$. Hence for $x\in B_r(x_0)$ and $t\le r$, the monotonicity inequality implies

$$\begin{aligned}
\fint_{B_t(x)} |u-u_r-(u-u_r)_{x,t}| &= \fint_{B_t(x)} |u-u_{x,t}| \\
&\le C\Big(t^{p-m}\int_{B_t(x)} |\nabla u|^p\Big)^{1/p} \le CU(x_0,r)^{1/p},
\end{aligned}$$

that is,

$$\|u-u_r\|_{BMO(\mathbb{R}^m)} \le CU(x_0,r)^{1/p}. \tag{2-5}$$

From Lemma 2.2 and (2-5) we have

$$|I_1| \le C\Big(r^m + \int_{B_r(x_0)} |\nabla u|^p\Big)U(x_0,r)^{1/p}. \tag{2-6}$$

Combining (2-2) and (2-4) with (2-6), we obtain

$$\begin{aligned}
\int_{B_{r/2}(x_0)} |\nabla u|^p \le CU(x_0,r)^{1/p}&\Big(r^m + \int_{B_r(x_0)} |\nabla u|^p\Big) \\
&+ C\Big(r^m + r^{-p}\int_{B_r(x_0)} |u-u_r|^p\Big). \qquad \square
\end{aligned}$$

Lemma 2.3. *There is a universal constant C_1 such that for each $\tau \in (0, 1/2]$, there exists a constant $\epsilon_1 > 0$ such that, for $U(x_0, r) \le \epsilon_1^p$,*

$$\frac{1}{(\tau r)^m} \int_{B_{\tau r}(x_0)} |u - u_{\tau r}|^p \le C_1 \tau^p U(x_0, r).$$

Proof. The proof is by contradiction. For fixed positive τ and $C_1 > 0$, if the lemma were false, then there would exist positive x_k and r_k such that $U(x_k, r_k) = \lambda_k^p \to 0$. However

$$\frac{1}{(\tau r_k)^m} \int_{B_{\tau r_k}(x_k)} |u - u_{\tau r_k}|^p > C_1 \tau^p \lambda_k^p. \tag{2-7}$$

Transform variables into the unit ball $B_1(0)$, and set $v_k(x) \equiv (u_k(x) - (u_k)_{0,1})/\lambda_k$, where $u_k = u(x_k + r_k x)$ and $(u_k)_{0,1} = ⨍_{B_1(0)} u_k$. We have

$$\int_{B_1(0)} |\nabla v_k|^p \le 1 \qquad \text{and} \qquad \int_{B_1(0)} v_k = 0.$$

Therefore $\{v_k\}$ is bounded in $W^{1,p}(B_1(0))$. Suppose

$$\nabla v_k \rightharpoonup \nabla v \text{ in } L^p(B_1(0)) \text{ as } v_k \to v \text{ almost everywhere in } B_1(0). \tag{2-8}$$

It is easy to show that v_k satisfies the equation

$$-\operatorname{div}(|\nabla v_k|^{p-2} \nabla v_k) = \lambda_k |\nabla v_k|^p u_k + (r_k^p / \lambda_k^{p-1}) P(u_k) \nabla^N H(u_k, x_k + r_k x)$$

in $B_1(0)$. From the definition of $U(x_k, r_k)$, we know that $r_k^p \le \lambda_k^p$. This yields $r_k^p / \lambda_k^{p-1} \le \lambda_k \to 0$. Hence v satisfies $-\operatorname{div}(|\nabla v|^{p-1} \nabla v) = 0$ in $B_1(0)$. Then, for $\tau \in (0, 1/2]$, the regularity of p-Laplace equations gives

$$\frac{1}{\tau^m} \int_{B_\tau(0)} |v - (v)_{0,\tau}|^p \le C_2 \tau^p, \tag{2-9}$$

where C_2 is a constant that is independent of k. Now scaling (2-7) to v_k, letting $k \to \infty$ in (2-7), and using (2-8), we obtain $(1/\tau^m) \int_{B_\tau(0)} |v - (v)_{0,\tau}|^p \ge C_1 \tau^p$, which contradicts (2-9) if we choose $C_1 > C_2$. □

Now we use the lemmas above to prove the following decay result.

Proposition 2.2. *For any $\gamma \in (0, 1)$, there exist a constant $\theta \in (0, 1/2]$ and an $\varepsilon_0 > 0$ such that $U(x_0, \theta r) \le \theta^{p\gamma} U(x_0, r)$ for $U(x_0, r) < \varepsilon_0^p$.*

Proof. Let $\gamma \in (0, 1)$. For every $\theta \in (0, 1/2]$ to be determined later, there exists an integer $k \ge 0$ with $1/2^{k+2} < \theta \le 1/2^{k+1}$. Write

$$G(x_0, r) \equiv \frac{1}{r^m} \int_{B_r(x_0)} |u - (u)_{x_0,r}|^p.$$

Take $\tau = 1/2$ in Lemma 2.3. If $\varepsilon_0 \le \varepsilon_1$, then $G(x_0, 2^{-k}r) \le C_1 2^{-kp} U(x_0, r)$. By this and Lemma 2.1,

$$\begin{aligned} U(x_0, 2^{-(k+1)}r) &\le C_0(U(x_0, 2^{-k}r)^{1/p} U(x_0, 2^{-k}r) + G(x_0, 2^{-k}r)) \\ &\le C_0(\varepsilon_0 U(x_0, 2^{-k}r) + G(x_0, 2^{-k}r)) \\ &\le C_0\varepsilon_0 U(x_0, 2^{-k}r) + C_0 C_1 2^{-kp} U(x_0, r), \end{aligned}$$

that is,

$$U(x_0, 2^{-(k+1)}r) \le C_0\varepsilon_0 U(x_0, 2^{-k}r) + C_0 C_1 2^{-kp} U(x_0, r). \tag{2-10}$$

Making use of (2-10) and taking ε_0 such that $2^p C_0 \epsilon_0 \le \min\{1/2, \varepsilon_1\}$, we get

$$\begin{aligned} U(x_0, \theta r) &\le 2^{m-2} U(x_0, 2^{-(k+1)}r) \\ &\le 2^{m-2} U(x_0, r)((C_0\varepsilon_0)^{k+1} + C_1 C_0 \textstyle\sum_{i=0}^{k} (C_0\varepsilon_0)^{k-i} 2^{-ip}) \\ &\le 2^{m-2} U(x_0, r)((C_0\varepsilon_0)^{k+1} + C_1 C_0 2^{-kp+1}). \end{aligned}$$

Since $(k+1)\log 2 \le -\log\theta < (k+2)\log 2$, we thus have

$$(C_0\varepsilon_0)^{k+1} = \exp\{(k+1)\log(C_0\varepsilon_0)\} \le \theta^{|\log(C_0\varepsilon_0)|/\log 2}$$

and

$$C_1 C_0 2^{-kp+1} = 2^{1+2p} C_1 C_0 \exp(-(k+2)p\log 2) \le 2^{1+2p} C_1 C_0 \theta^p.$$

Then, taking $\theta \in (0, 1/2]$ such that $2^{m-2}(1 + 2^{1+2p} C_1 C_0)\theta^{p(1-\gamma)} \le 1$, we obtain $U(x_0, \theta r) \le \theta^{p\gamma} U(x_0, r)$. □

3. The proof of Theorem 1.1

For any $\rho < r/4$, we can find an integer $k \ge 0$ such that $\theta^{k+1} r < 2\rho \le \theta^k r$. For any $x \in B_\rho(x_0)$, we have by the monotonicity formula in Proposition 2.1 and the decay formula in Proposition 2.2 that

$$U(x, \rho) \le CU(x_0, 2\rho) \le C\theta^{p-m} U(x_0, \theta^k r) \le C\theta^{p-m}\theta^{kp\gamma} U(x_0, r).$$

Since $(k+1)\log\theta + \log r < \log 2\rho \le k\log\theta + \log r$, we obtain

$$C\theta^{p-m}\theta^{kp\gamma} = C\theta^{p-m}\exp(kp\gamma\log\theta) \le C\theta^{p-m-p\gamma} 2^{p\gamma} (\rho/r)^{p\gamma}.$$

Thus

$$U(x, \rho) \le C(\theta, \gamma, p)(\rho/r)^{p\gamma} U(x_0, r).$$

Let $\Omega_0 = \{x \in \mathbb{R}^m \mid U(x, r) < \epsilon_0^p\}$. Then Ω_0 is open, and the standard covering argument implies $\mathcal{H}^{m-p}(\Omega \setminus \Omega_0) = 0$. Furthermore if $x \in \Omega_0$, then $U(y, \rho) \le C(\theta, \gamma, p)\rho^{p\gamma}$ for all y near x and for sufficiently small radii $\rho > 0$. Hence u

belongs to $C^{0,\gamma}$; see [Giaquinta 1983]. Now let $x \in \Omega_0$ and $R > 0$. We consider the Dirichlet boundary problem

$$\begin{aligned} -\operatorname{div}(|\nabla w|^{p-2}\nabla w) &= 0 \quad \text{for } y \in B_{2R}(x), \\ w &= u \quad \text{for } y \in \partial B_{2R}(x). \end{aligned}$$

The theory of regularity of elliptic systems [Hamburger 1992; Tolksdorf 1984] implies

$$M^p \equiv \sup_{B_R(x)} |\nabla w|^p \le C \fint_{B_{2R}(x)} (1 + |\nabla w|^p)$$

and $\operatorname{osc}(\nabla w, B_\rho(x)) \le C(1+M)(\rho/R)^\mu$, where $\rho \le R$ and $0 < \mu < 1$. Consequently,

$$\int_{B_\rho(x)} |\nabla w - (\nabla w)_{x,\rho}|^p \le C(1+M)\rho^m \Big(\frac{\rho}{R}\Big)^{p\mu} \tag{3-1}$$

and

$$\int_{B_\rho(x)} |\nabla u - (\nabla u)_{x,\rho}|^p \le C(1+M)\rho^m \Big(\frac{\rho}{R}\Big)^{p\mu} + C\int_{B_\rho(x)} |\nabla u - \nabla w|^p. \tag{3-2}$$

Next we estimate

$$\int_{B_\rho(x)} |\nabla u - \nabla w|^p.$$

Extending $u - w = 0$ in $\Omega \setminus B_{2R}(x)$, multiplying (2-1) by $u - w$, integrating, and using the Hölder continuity of u, we get

$$\begin{aligned} &\int_{B_{2R}(x)} (|\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w)(\nabla u - \nabla w) \\ &\qquad = \int_{B_{2R}(x)} |\nabla u|^p u(u-w) + \int_{B_{2R}(x)} (\nabla_u H(u,x) - (\nabla_u H(u,x)\cdot u)u)(u-w) \\ &\qquad \le CR^\gamma \int_{B_{2R}(x)} (|\nabla u|^p + 1). \end{aligned}$$

For $p \ge 2$, we have

$$\int_{B_{2R}(x)} |\nabla u - \nabla w|^p \le CR^\gamma \int_{B_{2R}(x)} (|\nabla u|^p + 1). \tag{3-3}$$

For $1 < p < 2$, we have as in [Liu 1997] that

$$\begin{aligned} &\int_{B_{2R}(x)} |\nabla u - \nabla w|^p \\ &\quad \le \Big(\int_{B_{2R}(x)} (|\nabla u| + |\nabla w|)^p\Big)^{(2-p)/2} \Big(\int_{B_{2R}(x)} (|\nabla u| + |\nabla w|)^{p-2} |\nabla u - \nabla w|^2\Big)^{p/2} \end{aligned}$$

$$\leq \Big(\int_{B_{2R}(x)} (|\nabla u| + |\nabla w|)^p\Big)^{(2-p)/2}$$
$$\times\Big(\frac{1}{p-1}\int_{B_{2R}(x)} (|\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w)(\nabla u - \nabla w)\Big)^{p/2}$$
$$\leq C R^{\gamma p/2}\int_{B_{2R}(x)} (|\nabla u|^p + 1),$$

where we have used $\int_{B_{2R}(x)}|\nabla w|^p \leq C\int_{B_{2R}(x)}|\nabla u|^p$. Set

$$\Lambda(x, R) = \frac{1}{R^m}\int_{B_{2R}(x)} (|\nabla u|^p + 1).$$

Then from (3-1), (3-2), and (3-3), we get

$$\int_{B_\rho(x)} |\nabla u - (\nabla w)_{x,\rho}|^p \leq 2^{p-1}\int_{B_\rho(x)} \big(|\nabla u - \nabla w|^p + |\nabla w - (\nabla w)_{x,\rho}|^p\big)$$
$$\leq C(1+M)\rho^m(\rho/R)^{p\mu} + C\Lambda R^{\gamma p/2+m},$$

and thus

$$\int_{B_\rho(x)} |\nabla u - (\nabla u)_{x,\rho}|^p \leq C(\rho^m(\rho/R)^{p\mu} + R^{p\gamma/2+m}).$$

Take $\rho = \frac{1}{2}R^{1+\theta}$ and $\theta = (p\gamma/2)/(m+p\mu) > 0$. Then

$$\int_{B_\rho(x)} |\nabla u - (\nabla u)_{x,\rho}|^p \leq C R^{m+p\gamma/2} \leq C\rho^{m+\theta p\mu/(1+\theta)}.$$

Hence $\nabla u \in C^{0,\alpha}(\Omega_0)$. □

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