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**FINITE REPRESENTATION TYPE OF INFINITESIMAL
 q -SCHUR ALGEBRAS**

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We give a classification of the infinitesimal q -Schur algebras that have finite representation type.

1. Introduction

R. Dipper and G. James [1989; 1991] introduced q -analogues of Schur algebras, called q -Schur algebras. The q -Schur algebras are related to Hecke algebras of type A in precisely the same way that classical Schur algebras are related to group algebras of symmetric groups. The q -Schur algebras play an important role in the nondefining representation and cohomology theories of the finite general linear groups. The representations of q -Schur algebras $S_q(n, d)$ are equivalent to the polynomial representations of the quantum linear group $G := G_q(n)$ of a given degree d ; see [Parshall and Wang 1991, 11.2], [Donkin 1996, Section 4], and [Cox 1997, 1.6]. Infinitesimal Schur/ q -Schur algebras were introduced in [Doty et al. 1996; Cox 1997; 2000] as the dual algebras of the homogeneous components of the infinitesimal thickening (by the torus) of the Frobenius kernel. It turns out that infinitesimal q -Schur algebras control the polynomial representations of $G_r T$. Here $G_r T$ is the q -analogue of Jantzen subgroups, which can be regarded as infinitesimal thickenings of the Frobenius kernels G_r by the torus T .

It is important to classify the representation type of a finite-dimensional algebra. In the classical case, the representation type of Schur algebras and infinitesimal Schur algebras has been classified; see [Erdmann 1993; Doty et al. 1997; Doty et al. 1999]. In the quantum case, the classification of the representation type of the q -Schur algebras was given in [Erdmann and Nakano 2001]. The representation type of Hecke algebras has also been classified; see [Uno 1992; Erdmann and Nakano 2002; Ariki and Mathas 2004; Ariki 2005]. Here, we will classify the infinitesimal q -Schur algebras of finite representation type.

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Little q -Schur algebras were introduced as homomorphic images of infinitesimal quantum \mathfrak{gl}_n in [Du et al. 2005]; see also [Fu 2007]. The relationship between infinitesimal q -Schur algebras and little q -Schur algebras is similar to that of $G_r T$ and G_r ; see [Fu 2005]. We expect that the result of this paper can be used to study the representation type of little q -Schur algebras.

2. Main result

Let k be an algebraically closed field of characteristic $p \geq 0$. Let $q \in k$ and $q \neq 0, 1$. Let l be the multiplicative order of q in k^* ; that is, either q is a primitive l -th root of unity or $l = \infty$.

The q -Schur algebra $S_q(n, d)$ is a deformation of the classical Schur algebra $S(n, d)$. Let us recall the construction of the q -Schur algebra from the coordinated algebra of the quantum matrix monoid $M_q(n)$ as follows. Following [Dipper and Donkin 1991], let $A_q(n)$ be the k -algebra generated by the n^2 indeterminates c_{ij} , with $1 \leq i, j \leq n$, subject to the relations

$$\begin{aligned} c_{ij}c_{it} &= c_{it}c_{ij} && \text{for all } i, j, t, \\ c_{ij}c_{st} &= qc_{st}c_{ij} && \text{for } i > s \text{ and } j \leq t, \\ c_{ij}c_{st} &= c_{st}c_{ij} + (q-1)c_{sj}c_{it} && \text{for } i > s \text{ and } j > t. \end{aligned}$$

This is a bialgebra, with comultiplication and counit given by

$$\Delta(c_{ij}) = \sum_{t=1}^n c_{it} \otimes c_{tj} \quad \text{and} \quad \varepsilon(c_{ij}) = \delta_{ij}.$$

We denote by $M = M_q(n)$ the quantum matrix monoid with the coordinate algebra $k[M] = A_q(n)$. Let $A_q(n, d)$ denote the subspace of elements in $A_q(n)$ of degree d in the c_{ij} . Then the $A_q(n, d)$ are in fact subcoalgebras of $A_q(n)$ for all d . Hence $A_q(n, d)^*$ is an algebra, which is isomorphic to the q -Schur algebra $S_q(n, d)$ by [Dipper and Donkin 1991, 3.2.6].

Theorem 2.1 [Erdmann and Nakano 2001, Theorem 1.4(B)]. *The algebra $S_q(n, d)$ has infinite representation type if and only if q is a primitive l -th root of unity and one of the following holds:*

- (1) $n \geq 3$ and $d \geq 2l$;
- (2) $n = 2$, $p \neq 0$, $l \geq 3$ and $d \geq lp$;
- (3) $n = 2$, $p \geq 3$, $l = 2$ and d is even with $d \geq 2p$, or d is odd with $d \geq 2p^2 + 1$.

From now on, we assume $p > 0$ and q is a primitive l -th root of unity. Note that l and p must be coprime.

We recall the definition of infinitesimal q -Schur algebras of [Cox 1997; Cox 2000]. Let J_r be the ideal in $A_q(n)$ generated by $c_{ij}^{lp^{r-1}}$ for $1 \leq i \neq j \leq n$. This is a coideal; thus $A_q(n)/J_r$ is a bialgebra and gives rise to a quantum monoid, which we denote by $M_r D$. Let $A_q(n, d)_r$ be the image of $A_q(n, d)$ under the quotient map $k[M] = A_q(n) \rightarrow k[M_r D] = A_q(n)/J_r$. This subspace is also a subcoalgebra of $k[M_r D]$ for all d . The algebra $s_q(n, d)_r = A_q(n, d)_r^*$ is called the infinitesimal q -Schur algebra.

The main result of this paper is as follows.

Theorem 2.2. *Assume k is an algebraically closed field of characteristic $p > 0$ and $q \in k$ is a primitive l -th root of unity. Then the infinitesimal q -Schur algebra $s_q(n, d)_r$ has finite representation type if and only if one of the following holds:*

- (1) $n \geq 3$, $r \geq 2$ and $d < 2l$;
- (2) $n \geq 3$, $r = 1$ and $d < l$;
- (3) $n = 3$, $l = 3$, $r = 1$ and $d = 4, 5$;
- (4) $n = 3$, $l = 2$, $r = 1$ and $d = 2, 3$;
- (5) $n = 2$, $r \geq 2$ and $d < lp$;
- (6) $n = 2$, $l = 2$, $r \geq 3$ and d is odd with $2p + 1 \leq d < 2p^2 + 1$;
- (7) $n = 2$, $l = 2$, $r = 2$ and d is odd with $d \geq 2p + 1$;
- (8) $n = 2$ and $r = 1$.

For convenience, it will be useful to reformulate Theorem 2.2 as follows.

Theorem 2.3. *Assume k is an algebraically closed field of characteristic $p > 0$ and $q \in k$ is a primitive l -th root of unity. Then the infinitesimal q -Schur algebra $s_q(n, d)_r$ has infinite representation type if and only if one of the following holds:*

- (1) $n \geq 3$, $r \geq 2$ and $d \geq 2l$;
- (2) $n \geq 4$, $r = 1$ and $d \geq l$;
- (3) $n = 3$, $l \geq 4$, $r = 1$ and $d \geq l$;
- (4) $n = 3$, $l = 3$, $r = 1$ and either $d = 3$ or $d \geq 6$;
- (5) $n = 3$, $l = 2$, $r = 1$ and $d \geq 4$;
- (6) $n = 2$, $l \geq 3$, $r \geq 2$ and $d \geq lp$;
- (7) $n = 2$, $l = 2$, $r \geq 3$ and either d is even with $d \geq 2p$ or d is odd with $d \geq 2p^2 + 1$;
- (8) $n = 2$, $l = 2$, $r = 2$ and d is even with $d \geq 2p$.

3. Preliminaries

In this section, we shall prove some general results of infinitesimal q -Schur algebras, which will be used in Sections 4 and 5 to prove our main result.

Let $\bar{G} = GL(n, k)$ be the general linear group with coordinate algebra

$$k[\bar{G}] = k[x_{ij} \text{ for } 1 \leq i, j \leq n; \delta^{-1}], \quad \text{where } \delta = \det(x_{ij})_{n \times n}.$$

There are several different quantum deformations of \bar{G} ; see [Parshall and Wang 1991; Dipper and Donkin 1991; Du et al. 1991]. We will use the version introduced by Dipper and Donkin: let $\delta_q = \sum_{\pi \in \mathcal{S}_n} (-1)^{\ell(\pi)} c_{1,1\pi} c_{2,2\pi} \cdots c_{n,n\pi}$ be the q -determinant in $A_q(n)$, where \mathcal{S}_n is the symmetric group and $\ell(\pi)$ is the length of π . Since $c_{ij} \delta_q = q^{i-j} \delta_q c_{ij}$ for $1 \leq i, j \leq n$, we may localize the bialgebra $A_q(n)$ at δ_q . They proved that the localization $A_q(n)_{\delta_q}$ is a Hopf algebra. Let $G = G_q(n)$ be the quantum linear group whose coordinate algebra is $k[G] = A_q(n)_{\delta_q}$.

The torus $T = T_q(n)$ is defined to be the subgroup of G with defining ideal of $k[G]$ generated by all c_{ij} with $i \neq j$. Similarly, we can define $D = D_q(n)$ to be the submonoid of M with defining ideal of $k[M]$ generated by all c_{ij} with $i \neq j$. Following [Du et al. 1991, (3.1)], let F be the quantum Frobenius morphism $F: G \rightarrow \bar{G}$ with comorphism $F^\#: k[\bar{G}] \rightarrow k[G]$ defined by $F^\#(x_{ij}) = c_{ij}^l$ for all i, j . We also have the usual Frobenius map F for \bar{G} taking x_{ij} to x_{ij}^p . Let $F^r = F^{r-1}F$, and let G_r be the kernel of F^r . Then G_r is the subgroup of G with defining ideal of $k[G]$ generated by the elements $c_{ij}^{lp^{r-1}} - \delta_{ij}$ for $1 \leq i, j \leq n$ and $\delta_q^{lp^{r-1}} - 1$. Similarly, we may define M_r to be the submonoid of M with defining ideal of $k[M]$ generated by the elements $c_{ij}^{lp^{r-1}} - \delta_{ij}$ for $1 \leq i, j \leq n$. Let $G_r T$ be the subgroup of G with defining ideal generated by the elements $c_{ij}^{lp^{r-1}}$ for $1 \leq i \neq j \leq n$. Note that $k[M_r]$ is isomorphic to $k[G_r]$ and $k[G_r T]$ is the localization of $k[M_r D]$ at the quantum determinant.

Let \mathbb{G}_m be the multiplicative group with coordinate algebra $k[t, t^{-1}]$, and let \mathbb{M}_m be the multiplicative monoid with coordinate algebra $k[t]$. Let

$$X(T) = \text{Hom}(T, \mathbb{G}_m) \quad \text{and} \quad P(D) = \text{Hom}(D, \mathbb{M}_m).$$

As usual, we identify $X(T)$ with \mathbb{Z}^n and $P(D)$ with \mathbb{N}^n . Let

$$\begin{aligned} X^+(T) &= \{\lambda \in X(T) \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}, & P^+(D) &= P(D) \cap X^+(T), \\ \Lambda(n, d) &= \{\lambda \in \mathbb{N}^n \mid \sum_{1 \leq i \leq n} \lambda_i = d\}, & \Lambda^+(n, d) &= \Lambda(n, d) \cap X^+(T). \end{aligned}$$

For $\lambda \in \Lambda^+(n, d)$, let $\nabla(\lambda)$ and $\bar{\nabla}(\lambda)$ be the costandard modules for $S_q(n, d)$ and $S(n, d)$, respectively, with highest weight λ ; see [Donkin 1998, Appendix]. Let $L(\lambda) = \text{soc}_{S_q(n,d)} \nabla(\lambda)$ and $\bar{L}(\lambda) = \text{soc}_{S(n,d)} \bar{\nabla}(\lambda)$. The sets $\{L(\lambda) \mid \lambda \in \Lambda^+(n, d)\}$

and $\{\bar{L}(\lambda) \mid \lambda \in \Lambda^+(n, d)\}$ form complete sets of inequivalent irreducible $S_q(n, d)$ -modules and $S(n, d)$ -modules, respectively. For $\lambda \in X(T)$, let $\widehat{L}_r(\lambda)$ be the corresponding irreducible $G_r T$ -module; see [Donkin 1998; Cox 1997; Cox 2000].

Let $X_r(T) = P_r(D) = \{\lambda \in X(T) \mid 0 \leq \lambda_i - \lambda_{i+1} \leq lp^{r-1} - 1, 1 \leq i \leq n\}$, where $\lambda_{n+1} = 0$. By [Cox 1997; Cox 2000] the set $\{\widehat{L}_r(\lambda) \mid \lambda \in \Gamma_r^d(D)\}$ forms a complete set of nonisomorphic simple $s_q(n, d)_r$ -modules, where

$$\Gamma_r(D) = P_r(D) + lp^{r-1}P(D) \quad \text{and} \quad \Gamma_r^d(D) = \{\lambda \in \Gamma_r(D) \mid \sum_{i=1}^n \lambda_i = d\}.$$

By [Donkin 1998, 3.2] and [Cox 1997, 1.7], for $\alpha = \lambda + lp^{r-1}\mu \in \Lambda^+(n, d)$ with $\lambda \in P_r(D)$ and $\mu \in P^+(D)$, we have

$$(3-1) \quad L(\alpha) \Big|_{s_q(n, d)_r} \cong \bigoplus_{j=1}^s \widehat{L}_r(\alpha^{(j)}).$$

where $\alpha^{(j)} = \lambda + lp^{r-1}\mu^{(j)}$ and $\{\mu^{(j)} : 1 \leq j \leq s\}$ is some enumeration of the weights of $\bar{L}(\mu)$.

Let $\Xi(n, d)$ be the set of $n \times n$ matrices with nonnegative integer entries summing to d . For $A \in \Xi(n, d)$, let $c^A = c_{11}^{a_{11}} c_{12}^{a_{12}} \cdots c_{1n}^{a_{1n}} c_{21}^{a_{21}} c_{22}^{a_{22}} \cdots c_{2n}^{a_{2n}} c_{n1}^{a_{n1}} c_{n2}^{a_{n2}} \cdots c_{nn}^{a_{nn}} \in A_q(n, d)$. By [Dipper and Donkin 1991, 1.1.8] the set $\{c^A \mid A \in \Xi(n, d)\}$ forms a k -basis for $A_q(n, d)$. For $A \in \Xi(n, d)$, we write ϕ_A for the element for $S_q(n, d) = A_q(n, d)^*$ dual to c^A . For $A \in \Xi(n, d)$, let

$$(3-2) \quad [A] = v^{-d_A} \phi_A \quad \text{with} \quad d_A = \sum_{i \geq k, j < l} a_{ij} a_{kl}.$$

Then the set $\{[A] \mid A \in \Xi(n, d)\}$ forms a k -basis for $s_q(n, d)_r$. By [Cox 1997, 5.3.1], the set

$$(3-3) \quad \{[A] \mid A \in \Xi(n, d), a_{ij} < lp^{r-1} \text{ for } i \neq j\}$$

forms a k -basis for $s_q(n, d)_r$.

Lemma 3.1. *For any $\lambda, \mu \in \Gamma_r^d(D)$, we have*

$$\text{Ext}_{s_q(n, d)_r}^1(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)) = \text{Ext}_{s_q(n, d)_r}^1(\widehat{L}_r(\mu), \widehat{L}_r(\lambda)).$$

Proof. By [Beřlinson et al. 1990, 3.10], [Du 1992, A.1] and [Du et al. 1991, 5.7], there is an antiautomorphism Ψ on the q -Schur algebra $S_q(n, d)$ defined by sending $[A]$ to $[{}^t A]$ for all $A \in \Xi(n, d)$, where ${}^t A$ is the transpose of A . By (3-3), we have $\Psi(s_q(n, d)_r) = s_q(n, d)_r$. Using the antiautomorphism Ψ on the infinitesimal q -Schur algebra $s_q(n, d)_r$, we may construct from any $s_q(n, d)_r$ -module M the contravariant dual module M^0 . It is easy to see that $(\widehat{L}_r(\lambda))^0 \cong \widehat{L}_r(\lambda)$ for any $\lambda \in \Gamma_r^d(D)$. We can now imitate the proof of [Jantzen 1987, II, 2.12(4)] to get the result. □

For generalizing [Doty et al. 1997, 2.3] to the quantum case, we have to prove the following two lemmas.

Lemma 3.2. *For any $\lambda, \mu \in X_r(T)$, the restriction map*

$$\text{res}_{G, G_r} : \text{Ext}_G^1(L(\lambda), L(\mu)) \rightarrow \text{Ext}_{G_r}^1(L(\lambda), L(\mu)) \quad \text{is injective.}$$

Proof. Let \bar{G}^r be the factor group of G whose coordinate algebra is the sub-Hopf algebra of $k[G]$ generated by the elements $c_{ij}^{lp^{r-1}}$ for $1 \leq i, j \leq n$ and $\delta_q^{-lp^{r-1}}$. Note that the factor group \bar{G}^r is isomorphic to \bar{G} via F^r .

By [Parshall and Wang 1991, (2.11.1) and (2.8.2)(3)], we have the five term exact sequence

$$\begin{aligned} 0 \rightarrow H^1(\bar{G}^r, \text{Hom}_{G_r}(L(\lambda), L(\mu))) \\ \rightarrow \text{Ext}_G^1(L(\lambda), L(\mu)) \rightarrow \text{Ext}_{G_r}^1(L(\lambda), L(\mu))^{\bar{G}^r} \\ \rightarrow H^2(\bar{G}^r, \text{Hom}_{G_r}(L(\lambda), L(\mu))) \rightarrow \text{Ext}_G^2(L(\lambda), L(\mu)). \end{aligned}$$

Since \bar{G}^r is isomorphic to \bar{G} , by [Jantzen 1987, II, 4.11] we have $H^i(\bar{G}^r, k) = 0$ for all $i > 0$. Hence we have $\text{Ext}_G^1(L(\lambda), L(\mu)) \cong \text{Ext}_{G_r}^1(L(\lambda), L(\mu))^{\bar{G}^r}$. \square

Lemma 3.3. *For any G -module N , we have*

$$\text{soc}_G N \cong \bigoplus_{\lambda \in X_r(T)} \text{soc}_G \text{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda).$$

Proof. By [Donkin 1998, 3.2(4)], the natural map $f : \text{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda) \rightarrow N$ for $\lambda \in X_r(T)$ is a morphism of G -modules. In fact, the map f is injective, since $\text{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda)$ is isomorphic to $(\text{soc}_{G_r} N)_{L(\lambda)}$ via f . So we can view $\text{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda)$ as a submodule of N for $\lambda \in X_r(T)$. By [Parshall and Wang 1991, 2.8.2(3) and 2.10.2], for $\lambda \in X_r(T)$ there exists a \bar{G} -module V such that

$$\text{soc}_G(\text{Hom}_{G_r}(L(\lambda), N)) \cong \text{soc}_G((N \otimes L(\lambda)^*)^{G_r}) \cong \text{soc}_G(V^{F^r}) \cong (\text{soc}_{\bar{G}} V)^{F^r}.$$

It follows from [Donkin 1998, 3.2(5)] that $\text{soc}_G \text{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda)$ is a semisimple G -module for each $\lambda \in X_r(T)$. On the other hand, each simple G -submodule W of N is isomorphic to $\bar{L}(\mu)^{F^r} \otimes L(\lambda)$ for some $\lambda \in X_r(T)$ and $\mu \in X^+(T)$. By the proof of [Donkin 1998, 3.2(5)], $\bar{L}(\mu)^{F^r} \cong \text{Hom}_{G_r}(L(\lambda), W) \subseteq \text{Hom}_{G_r}(L(\lambda), N)$. Hence $W \subseteq \text{soc}_G \text{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda)$. The assertion follows. \square

Now using the above two lemmas we can prove the following result, which gives information about the restriction of extensions of simple M -modules to $M_r D$.

Proposition 3.4. (1) *If $\lambda, \mu \in X_r(T)$, then the restriction map*

$$\text{res}_{M, M_r D} : \text{Ext}_M^1(L(\lambda), L(\mu)) \rightarrow \text{Ext}_{M_r D}^1(L(\lambda), L(\mu)) \quad \text{is injective.}$$

- (2) Let N be an M -module with two composition factors $L(\lambda)$ and $L(\mu)$, where $\lambda \in X_r(T)$ and $\mu \in P^+(D)$ with $\text{soc}_M N \cong L(\lambda)$. Assume that $L(\mu) = \bigoplus_{j=1}^s \widehat{L}_r(\mu_j)$ is the decomposition of $L(\mu)$ as $M_r D$ -modules. If $\widehat{L}_r(\lambda) \not\cong \widehat{L}_r(\mu_j)$ as G_r -modules for all j , then $\text{soc}_{M_r D} N \cong L(\lambda)$.

Proof. The proof is almost the same as [Doty et al. 1997, 2.3]. For $\lambda, \mu \in X_r(T)$, we have the commutative diagram

$$\begin{array}{ccc} \text{Ext}_M^1(L(\lambda), L(\mu)) & \xrightarrow{\text{res}_{M, M_r D}} & \text{Ext}_{M_r D}^1(L(\lambda), L(\mu)) \\ \text{res}_{M, G} \downarrow & & \downarrow \text{res}_{M_r D, G_r T} \\ \text{Ext}_G^1(L(\lambda), L(\mu)) & \xrightarrow{\text{res}_{G, G_r T}} & \text{Ext}_{G_r T}^1(L(\lambda), L(\mu)). \end{array}$$

By Lemma 3.2 we know that $\text{res}_{G, G_r T}$ is injective, and by [Donkin 1996, 4(5)] the map $\text{res}_{M, G}$ is an isomorphism. Hence the assertion (1) follows.

Now we consider part (2). If $\text{soc}_{M_r D} N \not\cong L(\lambda)$, then $\widehat{L}_r(\mu_j)$ is a simple factor of $\text{soc}_{M_r D} N$ for some $1 \leq j \leq s$. By [Cox 1997, bottom of page 76], [Cox 2000, §4] and [Donkin 1998, 3.1(18)], we have $\text{soc}_{M_r D} N \cong \text{soc}_{G_r T} N \cong \text{soc}_{G_r} N$. Hence $\widehat{L}_r(\mu_j)$ is a factor of $\text{soc}_{G_r} N$. It follows from Lemma 3.3 that $\text{soc}_G N$ is not simple. This is a contradiction. \square

Now we shall describe some results which will be used to reduce the general question of representation type of infinitesimal q -Schur algebras to that of finding the representation type of $s_q(n, d)_r$ for small n and small d . The first result relates the representation type of $s_q(n, d)_r$ with $s_q(n', d)_r$ where $n' \geq n$.

Theorem 3.5. *Assume $n' \geq n$. If $s_q(n, d)_r$ has infinite representation type, then so does $s_q(n', d)_r$.*

Proof. Let $e = \sum_{\lambda \in \Lambda(n, d)} [\text{diag}(\lambda)] \in S_q(n', d)_r$. (See (3-2) for the definition of $[\text{diag}(\lambda)]$.) Then we have $e S_q(n', d)_r e \cong S_q(n, d)_r$. Hence the assertion follows by [Erdmann 1990, I 4.7]. \square

Lemma 3.6. *There is a surjective homomorphism φ_d from $s_q(n, d+n)_r$ to $s_q(n, d)_r$ for any d .*

Proof. By [Donkin 1998, 4.2(18)], there is a surjective homomorphism φ_d from $S_q(n, d+n)$ to $S_q(n, d)$. It is easy to check that restriction induces a surjective homomorphism φ_d from $s_q(n, d+n)_r$ to $s_q(n, d)_r$. \square

By the above lemma, we get the following corollary which relates the representation type of $s_q(n, d)_r$ to that of $s_q(n, d+n)_r$.

Corollary 3.7. *If $s_q(n, d)_r$ has infinite representation type, then $s_q(n, d+n)_r$ does as well.*

Using the translation functor for G defined in [Erdmann and Nakano 2001, 2.4], we can define the translation functor for $G_r T$ as follows. Let Φ^+ be a set of positive roots for the root system of type A_{n-1} , and let $\rho = (n-1, n-2, \dots, 0)$ and

$$\bar{C}_{\mathbb{Z}} = \{\lambda \in X(T) \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq l \text{ for all } \alpha \in \Phi^+\}.$$

Let W_l be the affine Weyl group. For any $G_r T$ -module V and $\lambda, \mu \in X(T)$, let $pr_\lambda V$ be the largest $G_r T$ -submodule of V such that $\mu \in W_l \cdot \lambda$ for every composition factor $\widehat{L}_r(\mu)$ of V . For $\lambda, \mu \in \bar{C}_{\mathbb{Z}}$, let v be the unique weight in $X^+(T) \cap W(\mu - \lambda)$. Then the translation functor $T_\lambda^\mu : \text{mod}(G_r T) \rightarrow \text{mod}(G_r T)$ is defined by

$$T_\lambda^\mu(V) = pr_\mu(\widehat{L}_r(v) \otimes pr_\lambda V).$$

For $V \in \text{mod}(G_r T)$, let $\mathcal{F}_{M_r D}(V)$ be the unique maximal $G_r T$ -submodule of V that lifts to an $M_r D$ -module. For any $\lambda, \mu \in \bar{C}_{\mathbb{Z}}$, define a truncated translation functor \bar{T}_λ^μ to be the composite $\mathcal{F}_{M_r D} \circ T_\lambda^\mu$. By restriction, \bar{T}_λ^μ induces a functor from $\text{mod}(M_r D)$ into itself.

For $\lambda \in \Gamma_r^d(D)$, let $\mathcal{B}_r^d(\lambda)$ be the block of $s_q(n, d)_r$ containing λ . Since a simple $s_q(n, d)_r$ -module appears as a composition factor of exactly one block and the simple $s_q(n, d)_r$ -modules are indexed by elements of $\Gamma_r^d(D) \subseteq \mathbb{Z}^n$, we may identify blocks for $s_q(n, d)_r$ with subsets of \mathbb{Z}^n .

Theorem 3.8. *Assume the block $\mathcal{B}_r^d(\lambda)$ of $s_q(n, d)_r$ has infinite representation type. Suppose that $\mu \in \Gamma_r^d(D)$ is a weight in the same facet as λ with $\mu - \lambda \in P(D)$. Then $s_q(n, d)_r$ has infinite representation type.*

Proof. Since λ and μ lie in the same facet, there exist unique elements $\lambda', \mu' \in \bar{C}_{\mathbb{Z}}$ in the same facet and a unique $w \in W_l$ with $w \cdot \lambda' = \lambda$ and $w \cdot \mu' = \mu$. Since $\mu' - \lambda' \in W(\mu - \lambda)$ and $\mu - \lambda \in P(D)$, we have $\mu' - \lambda' \in P(D)$. View $\bar{T}_{\lambda'}^{\mu'}$ as a functor from $\{V \in \text{mod}(s_q(n, d)_r) \mid pr_{\lambda'} V = V\}$ to $\{V \in \text{mod}(s_q(n, d')_r) \mid pr_{\mu'} V = V\}$. Then one can prove $\bar{T}_{\mu'}^{\lambda'} \circ \bar{T}_{\lambda'}^{\mu'}$ is equivalent to identity functor as in the proof of [Doty et al. 1997, 4.2]. It follows that the functor $\bar{T}_{\lambda'}^{\mu'}$ preserves indecomposable modules and isomorphism classes. The assertion follows. \square

4. Infinite representation type

In this section, we will prove that the infinitesimal q -Schur algebra has infinite representation type for the cases listed in Theorem 2.3.

Proposition 4.1. *The algebra $s_q(n, d)_r$ has infinitesimal representation type if one of the following holds:*

- (1) $n \geq 3$, $d \geq 2l$ and either $r \geq 3$ or both $r = 2$ and $p \geq 5$;
- (2) $n = 2$, $l \geq 3$, $r \geq 3$ and $d \geq lp$.
- (3) $n = 2$, $l = 2$, $r \geq 3$ and d is even with $d \geq 2p$

(4) $n = 2, l = 2, r \geq 4$ and d is odd with $d \geq 2p^2 + 1$.

Proof. (1) Suppose either $r \geq 3$ or both $r = 2$ and $p \geq 5$. Then $lp^{r-1} \geq 4l > d$ for $d = 2l, 2l + 1, 2l + 2$. By (3-3) and Theorem 2.1, the algebra $s_q(3, d)_r = S_q(3, d)$ has infinite representation type for $d = 2l, 2l + 1, 2l + 2$. So by Corollary 3.7 and Theorem 3.5, we have $s_q(n, d)_r$ has infinite representation type for $n \geq 3$ and $d \geq 2l$.

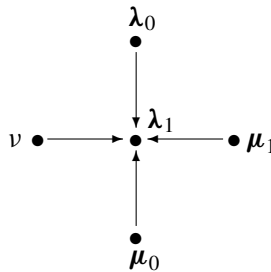
(2) Suppose $l \geq 3$ and $r \geq 3$. Then $lp^{r-1} \geq lp^2 > d$ for $d = lp, lp + 1$. By (3-3) and Theorem 2.1, the algebra $s_q(2, d)_r = S_q(2, d)$ has infinite representation type for $d = lp, lp + 1$. So by Corollary 3.7, we have $s_q(2, d)_r$ has infinite representation type for $d \geq lp$.

(3) Suppose $l = 2$ and $r \geq 3$. Then $lp^{r-1} \geq 2p^2 > 2p$. By (3-3) and Theorem 2.1, the algebra $s_q(2, 2p)_r = S_q(2, 2p)$ has infinite representation type. So by Corollary 3.7, we have $s_q(2, d)_r$ has infinite representation type for d even with $d \geq 2p$.

(4) Suppose $l = 2$ and $r \geq 4$. Then $lp^{r-1} \geq 2p^3 > 2p^2 + 1$. By (3-3) and Theorem 2.1, the algebra $s_q(2, 2p^2 + 1)_r = S_q(2, 2p^2 + 1)$ has infinite representation type. So by Corollary 3.7, we have $s_q(2, d)_r$ has infinite representation type for d odd with $d \geq 2p^2 + 1$. □

Proposition 4.2. *Assume $l = 2$. Then the algebra $s_q(2, d)_3$ has infinite representation type for d odd with $d \geq 2p^2 + 1$.*

Proof. Let $\lambda_0 = (2p^2 + 1, 0), \lambda_1 = (2p^2 + 1 - 2p, 2p), \mu_0 = (2p^2 - 1, 2)$ and $\mu_1 = (2p^2 - 2p - 1, 2p + 2)$. By [Erdmann and Nakano 2001, 3.2], the classical Schur algebra $S(2, p^2)$ is Morita equivalent to the principal block component of $S_q(2, 2p^2 + 1)$. It follows from [Erdmann 1993, 5.2] that $\text{Ext}_M^1(L(\lambda_1), L(\sigma)) \neq 0$ for $\sigma = \lambda_0, \mu_0, \mu_1$. By (3-1) we have $L(\lambda_0)|_{M_3D} \cong \widehat{L}_3(\lambda_0) \oplus \widehat{L}_3(\nu)$, where $\nu = (1, 2p^2)$. Hence by Proposition 3.4, we have $\text{Ext}_{M_3D}^1(\widehat{L}_3(\lambda_1), \widehat{L}_3(\sigma)) \neq 0$ for $\sigma = \lambda_0, \mu_0, \mu_1, \nu$. The Ext^1 quiver of $s_q(2, 2p^2 + 1)_3$ has a four subspace quiver as a subquiver, as illustrated.

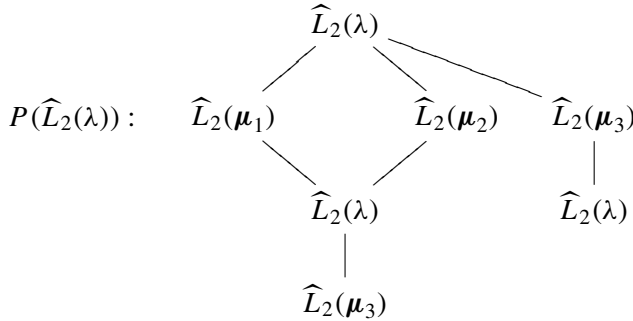


By [Gel'fand and Ponomarev 1972], $s_q(2, 2p^2 + 1)_3$ is of infinite type. Now the assertion follows from Corollary 3.7. □

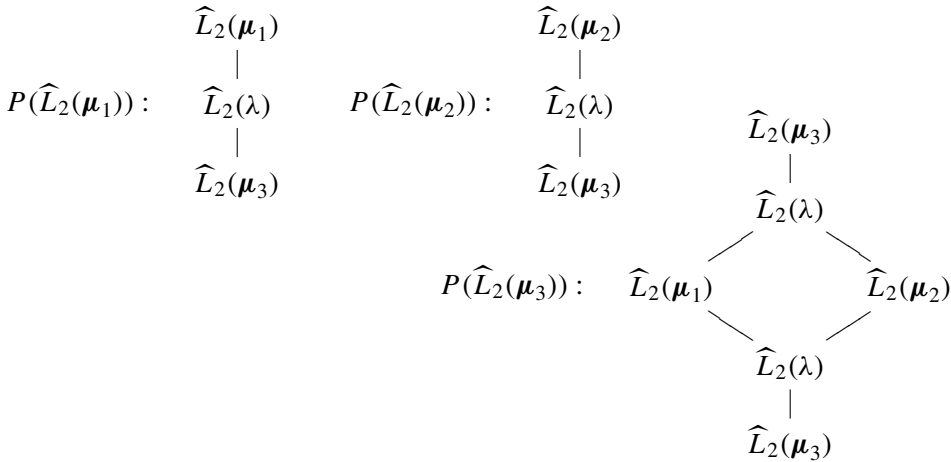
By Propositions 4.1 and 4.2, we know that the algebras listed in Theorem 2.3 for $r \geq 3$ have infinite representation type. It remains to check the algebras listed there have infinite representation type for $r = 1, 2$.

Proposition 4.3. *Assume $p = 2$. Then the algebra $s_q(n, d)_2$ has infinite representation type for $n \geq 2$ and $d \geq 2l$.*

Proof. Let $\mu_1 = (2l, 0)$, $\mu_2 = (0, 2l)$, $\mu_3 = (2l - 1, 1)$ and $\lambda = (l, l)$. By [Cox 1997, 6.2.13] and [Cox 2000, 5.12], the block $B_2^{2l}(\lambda)$ of $s_q(2, 2l)_2$ is equal to $\{\mu_i, \lambda \mid 1 \leq i \leq 3\}$. By [Erdmann and Nakano 2001, Proposition 3.3(B)] and Proposition 3.4, the projective cover of $\widehat{L}_2(\lambda)$ has the following structure.



The vertices in this diagram correspond with composition factors, and the edges indicate a nonsplit extension. The structure of the projective covers for the other simple modules in the block $B_2^{2l}(\mu)$ are given below.



By [Doty et al. 1997, 5.2], the basic algebra of $B_2^{2l}(\lambda)$ is isomorphic to the basic algebra of the infinitesimal Schur algebra $s(2, 4)_2$. So $B_2^{2l}(\lambda)$ has infinite representation type. It follows that the algebra $s_q(2, 2l)_2$ has infinite representation type.

Since $p = 2$ and l and p are coprime, we have $l \geq 3$. So the weight $\mu = (l + 1, l)$ belongs to $\Gamma_2^{2l+1}(D)$, lies in the same facet as λ , and $\mu - \lambda \in P(D)$. It follows from Theorem 3.8 that $s_q(2, 2l + 1)_2$ has infinite representation type. Now the assertion follows by Corollary 3.7 and Theorem 3.5. □

Proposition 4.4. *Assume $p \geq 3$. Then the algebra $s_q(2, lp + 2j)_2$ has infinite representation type for $j \geq 0$.*

Proof. Let $\lambda = (lp - l, l)$, $\gamma = (lp, 0)$, $\beta = (lp - 1, 1)$, $\tau = (0, lp)$ and $\eta = (lp - l - 1, l + 1)$. By [Thams 1994], the M -modules $\nabla(\lambda)$ and $\nabla(\gamma)$ have the following structure.

$$\nabla(\lambda) : \begin{array}{c} L(\eta) \\ | \\ L(\lambda) \end{array} \qquad \nabla(\gamma) : \begin{array}{c} L(\beta) \\ | \\ L(\lambda) \\ | \\ L(\gamma) \end{array}$$

By (3-1) we have $L(\gamma)|_{M_2D} \cong \widehat{L}_2(\gamma) \oplus \widehat{L}_2(\tau)$. By Proposition 3.4, it follows that $\text{Ext}_{M_2D}^1(\widehat{L}_2(\lambda), \widehat{L}_2(\sigma)) \neq 0$ for $\sigma = \gamma, \beta, \tau, \eta$. The Ext^1 -quiver for $s_q(2, lp)_2$ has a four subspace quiver as subquiver. Hence $s_q(2, lp)_2$ is of infinite type. Hence the assertion follows by Corollary 3.7. □

Corollary 4.5. *Assume $l \geq 3$. Then the algebra $s_q(2, d)_2$ has infinite representation type for $d \geq lp$.*

Proof. If $p = 2$, then the assertion follows from Proposition 4.3. Now we assume $p \geq 3$. Let $\lambda = (lp - l, l)$ and $\mu = (lp - l + 1, l)$. Then the weight μ belongs to $\Gamma_2^{lp+1}(D)$, it lies in the same facet as λ since $l \geq 3$, and $\mu - \lambda \in P(D)$. By the proof of Proposition 4.4, the block $B_2^{lp}(\lambda)$ of $s_q(2, lp)_2$ has infinite representation type. It follows from Theorem 3.8 that the algebra $s_q(2, lp + 1)_2$ has infinite representation type. Hence the assertion follows by Corollary 3.7. □

Proposition 4.6. *Assume $p = 3$. Then the algebra $s_q(n, d)_2$ has infinite representation type for $n \geq 3$ and $d \geq 2l$.*

Proof. There are two cases.

(1) Suppose $l > 2$. Then $3l > d$ for $d = 2l, 2l + 1, 2l + 2$. By (3-3) and Theorem 2.1, the algebra $s_q(3, d)_2 = S_q(3, d)$ has infinite representation type for $d = 2l, 2l + 1, 2l + 2$. So by Corollary 3.7 and Theorem 3.5, $s_q(n, d)_2$ has infinite representation type for $n \geq 3$ and $d \geq 2l$.

(2) Suppose $l = 2$. By (3-3) and Theorem 2.1, the algebra $s_q(3, d)_2 = S_q(3, d)$ has infinite representation type for $d = 4, 5$. But by Proposition 4.4 and Theorem 3.5, the algebra $s_q(3, 6)_2$ has infinite representation type. So by Corollary 3.7 and Theorem 3.5, $s_q(n, d)_2$ has infinite representation type for $n \geq 3$ and $d \geq 4$. □

By Proposition 4.1(1) and Propositions 4.3–4.6, we know that the algebras listed in Theorem 2.3 for $r = 2$ have infinite representation type. We can now concentrate on the situation when $r = 1$.

Proposition 4.7. *Assume $l \geq 3$. Then the algebra $s_q(3, l)_1$ has infinite representation type.*

Proof. Let $\lambda = (l-1, 1, 0)$, $\gamma = (l, 0, 0)$, $\beta = (0, l, 0)$, $\eta = (0, 0, l)$ and $\tau = (l-2, 1, 1)$. By [Thams 1994], the M -module $\nabla(\gamma)$ has only two composition factors $L(\lambda)$ and $L(\gamma)$. So $\text{Ext}_M^1(L(\lambda), L(\gamma)) \neq 0$. It is clear that $L(\gamma)|_{M_1 D} \cong \widehat{L}_1(\gamma) \oplus \widehat{L}_1(\beta) \oplus \widehat{L}_1(\eta)$. It follows from Proposition 3.4(2) that $\text{Ext}_{M_1 D}^1(\widehat{L}_1(\lambda), \widehat{L}_1(\sigma)) \neq 0$ for $\sigma = \gamma, \beta, \eta$. By [Xi 1999], $\text{Ext}_{M_1 D}^1(\widehat{L}_1(\lambda), \widehat{L}_1(\tau)) \neq 0$. The Ext^1 quiver for $s_q(3, l)_1$ has a four subspace quiver as subquiver. Hence the algebra $s_q(3, l)_1$ has infinite representation type. \square

Corollary 4.8. *Assume $l \geq 4$. Then the algebra $s_q(3, d)_1$ has infinite representation type for $d \geq l$.*

Proof. Let $\tau = (l-2, 1, 1)$, $\mu_1 = (l-2, 2, 1)$ and $\mu_2 = (l-2, 2, 2)$. For $i = 1, 2$, we have $\mu_i \in \Gamma_1^{l+i}(D)$, μ_i lies in the same facet as τ since $l \geq 4$, and $\mu_i - \tau \in P(D)$. From the proof of Proposition 4.7 we know that the block $B_1^l(\tau)$ of $s_q(3, l)_1$ has infinite representation type. It follows from Theorem 3.8 that $s_q(3, l+i)_1$ has infinite representation type for $i = 1, 2$. Hence by Proposition 4.7 and Corollary 3.7, the algebra $s_q(3, d)_1$ has infinite representation type for $d \geq l$. \square

Lemma 4.9. *Let s_1 be one of the algebras $s_q(3, 7)_1$ or $s_q(3, 8)_1$ for $l = 3$, or $s_q(3, 4)_1$ or $s_q(3, 5)_1$ for $l = 2$. Then the algebra s_1 has infinite representation type.*

Proof. Let X, Y_1, Y_2 and Z denote the following simple s_1 -modules.

Simple modules	$s_q(3, 7)_1$ $l = 3$	$s_q(3, 8)_1$ $l = 3$	$s_q(3, 4)_1$ $l = 2$	$s_q(3, 5)_1$ $l = 2$
X	$\widehat{L}_1(2, 5, 0)$	$\widehat{L}_1(3, 5, 0)$	$\widehat{L}_1(1, 3, 0)$	$\widehat{L}_1(3, 0, 2)$
Y_1	$\widehat{L}_1(4, 3, 0)$	$\widehat{L}_1(4, 4, 0)$	$\widehat{L}_1(2, 2, 0)$	$\widehat{L}_1(3, 1, 1)$
Y_2	$\widehat{L}_1(1, 3, 3)$	$\widehat{L}_1(1, 4, 3)$	$\widehat{L}_1(0, 2, 2)$	$\widehat{L}_1(1, 1, 3)$
Z	$\widehat{L}_1(4, 2, 1)$	$\widehat{L}_1(4, 2, 2)$	$\widehat{L}_1(2, 1, 1)$	$\widehat{L}_1(2, 2, 1)$

We consider the algebra $s_q(3, 7)_1$ for $l = 3$. By [Donkin 1998, 4.2(9) and 4.2(15)] and [Thams 1994], we know that the M -modules $\nabla(4, 3, 0)$ and $\nabla(4, 0, 0)$ have the following structure.

$$\nabla(4, 3, 0) : \begin{array}{c} L(4, 2, 1) \\ | \\ L(4, 3, 0) \end{array} \qquad \nabla(4, 0, 0) : \begin{array}{c} L(2, 2, 0) \\ | \\ L(4, 0, 0) \end{array}$$

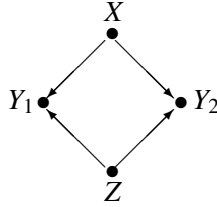
It is clear that $\bar{L}(1, 1, 0) = \bar{\nabla}(1, 1, 0)$. Hence by Weyl's character formula we know that the weights of $\bar{L}(1, 1, 0)$ are $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$. It follows from (3-1) that $L(4, 3, 0)|_{M_1D} \cong \widehat{L}_1(4, 3, 0) \oplus \widehat{L}_1(1, 3, 3) \oplus \widehat{L}_1(4, 0, 3)$. Hence by Proposition 3.4 and Lemma 3.1, $\text{Ext}_{M_1D}^1(\widehat{L}_1(\sigma), \widehat{L}_1(4, 2, 1)) \neq 0$ for $\sigma = (4, 3, 0)$ and $(1, 3, 3)$. Since

$$L(4, 0, 0)|_{M_1D} \cong \widehat{L}_1(4, 0, 0) \oplus \widehat{L}_1(1, 3, 0) \oplus \widehat{L}_1(1, 0, 3),$$

by Lemma 3.1, there exist nonsplit extensions of the form

$$\begin{aligned} 0 \rightarrow \widehat{L}_1(4, 0, 0) \rightarrow M_1 \rightarrow \widehat{L}_1(2, 2, 0) \rightarrow 0, \\ 0 \rightarrow \widehat{L}_1(1, 0, 3) \rightarrow M_2 \rightarrow \widehat{L}_1(2, 2, 0) \rightarrow 0. \end{aligned}$$

Now, by tensoring these short exact sequences by the one-dimensional module $\widehat{L}_1(0, 3, 0)$, we have that $\text{Ext}_{M_1D}^1(\widehat{L}_1(\sigma), \widehat{L}_1(2, 5, 0)) \neq 0$ for $\sigma = (4, 3, 0)$ and $(1, 3, 3)$. For the other algebras, we can prove the existence of the extensions in a similar manner. So one of the components of the separated quiver of the Ext^1 -quiver of s_1 contains the subquiver given below.



The algebra s_1/J^2 has the same Ext^1 quiver as s_1 , where $J = \text{Rad } s_1$. By [Pierce 1982, 11.8], s_1/J^2 has infinite representation type. Hence s_1 has infinite representation type. □

Lemma 4.10. *The algebra $s_q(3, 6)_1$ has infinite representation type for $l = 2$.*

Proof. Let $\lambda = (2, 2, 2)$, $\gamma = (4, 1, 1)$, $\beta = (2, 3, 1)$, $\tau = (2, 1, 3)$ and $\eta = (3, 3, 0)$. By [Donkin 1998, 4.2(9) and 4.2 (15)] and [Thams 1994], we know that the M -module $\nabla(\gamma)$ has two composition factors with top $L(\lambda)$. By [Erdmann and Nakano 2001, 5.6], we know that the M -module $\nabla(\eta)$ has two composition factors with top $L(\lambda)$. By (3-1), we have $L(\gamma)|_{M_1D} \cong \widehat{L}_1(\gamma) \oplus \widehat{L}_1(\beta) \oplus \widehat{L}_1(\tau)$. Upon restriction to M_1D , we have $\text{Ext}_{M_1D}^1(\widehat{L}_1(\lambda), \widehat{L}_1(\sigma)) \neq 0$ for $\sigma = \gamma, \beta, \tau, \eta$ by Proposition 3.4. So the Ext^1 -quiver for $s_q(3, 6)_1$ has a four subspace quiver as a subquiver. The assertion follows. □

Proposition 4.11. *The algebra $s_q(3, d)_1$ has infinite representation type for*

- (1) $l = 3$ and $d \geq 6$;
- (2) $l = 2$ and $d \geq 4$.

Proof. By Proposition 4.7 and Corollary 3.7, the algebra $s_q(3, 6)_1$ has infinite representation type for $l = 3$. Hence the assertion follows by Lemmas 4.9 and 4.10 and Corollary 3.7. \square

Proposition 4.12. *The algebra $s_q(n, d)_1$ has infinite representation type for $n \geq 4$ and $d \geq l$.*

Proof. If $l \geq 4$, then the assertion follows from Corollary 4.8 and Theorem 3.5. Now we assume $l < 4$. Then $l = 2$ or $l = 3$. For $l = 3$, set $s_1 = s_q(4, 4)_1$ or $s_q(4, 5)_1$, and for $l = 2$, set $s_1 = s_q(4, 2)_1$ or $s_q(4, 3)_1$. For the algebra s_1 let $\alpha, \beta, \lambda, \eta$ and μ be the following weights.

	$s_q(4, 4)_1$ $l = 3$	$s_q(4, 5)_1$ $l = 3$	$s_q(4, 2)_1$ $l = 2$	$s_q(4, 3)_1$ $l = 2$
α	(4, 0, 0, 0)	(5, 0, 0, 0)	(2, 0, 0, 0)	(3, 0, 0, 0)
β	(1, 3, 0, 0)	(2, 3, 0, 0)	(0, 2, 0, 0)	(1, 2, 0, 0)
λ	(1, 0, 3, 0)	(2, 0, 3, 0)	(0, 0, 2, 0)	(1, 0, 2, 0)
η	(1, 0, 0, 3)	(2, 0, 0, 3)	(0, 0, 0, 2)	(1, 0, 0, 2)
μ	(2, 2, 0, 0)	(2, 2, 1, 0)	(1, 1, 0, 0)	(1, 1, 1, 0)

By [Thams 1994], we know that $\nabla(\alpha)$ has two composition factors with top $L(\mu)$. By (3-1), we have $L(\alpha)|_{M_1 D} \cong L(\alpha) \oplus L(\beta) \oplus L(\lambda) \oplus L(\eta)$. It follows from Proposition 3.4(2) that $\text{Ext}_{M_1 D}^1(\widehat{L}_1(\mu), \widehat{L}_1(\sigma)) \neq 0$ for $\sigma = \alpha, \beta, \lambda, \eta$. The Ext^1 quiver of s_1 has a four subspace quiver as a subquiver. Hence the algebra s_1 has infinite representation type. By Propositions 4.7 and 4.11 and Theorem 3.5, the algebra $s_q(4, d)_1$ has infinite representation type for $l = 3$ and $d = 3, 6$ or $l = 2$ and $d = 4, 5$. Hence the assertion follows by Corollary 3.7 and Theorem 3.5. \square

5. Finite representation type

In this section, we will prove the infinitesimal q -Schur algebra has finite representation type for the cases listed in Theorem 2.2.

Proposition 5.1. *The algebra $s_q(n, d)_r$ has finite representation type if one of the following holds:*

- (1) $n \geq 3$, $r \geq 2$ and $d < 2l$;
- (2) $n \geq 3$, $r = 1$ and $d < l$;
- (3) $n = 2$, $r \geq 2$ and $d < lp$;
- (4) $n = 2$, $l = 2$, $r \geq 3$ and d is odd with $2p + 1 \leq d < 2p^2 + 1$;
- (5) $n = 2$ and $r = 1$.

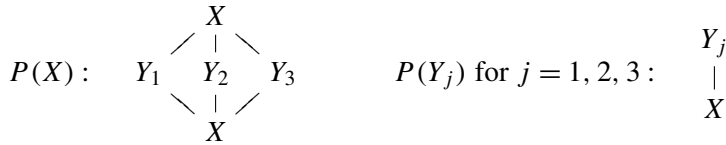
Proof. In cases (1)–(4), we have $s_q(n, d)_r = S_q(n, d)$. Hence the assertion in these cases follows from Theorem 2.1. In the last case, the assertion follows from [Erdmann and Fu 2008, 3.7]. □

Proposition 5.2. *Let s_1 be the algebra $s_q(3, 4)_1$ or $s_q(3, 5)_1$ for $l = 3$, or $s_q(3, 2)_1$ or $s_q(3, 3)_1$ for $l = 2$. Then the algebra s_1 has finite representation type.*

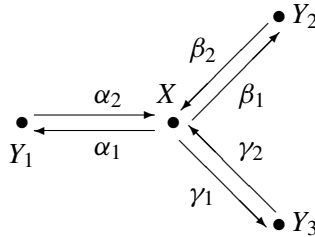
Proof. All nonsemisimple blocks of s_1 are Morita equivalent to a basic algebra B . The algebra B has four simple modules X, Y_1, Y_2, Y_3 . The following table provides the correspondence between the simple modules for B and the simple modules for the blocks of s_1 .

Simple modules	$l = 3$			$l = 2$	
	$s_q(3, 4)_1$	$s_q(3, 5)_1$ block 1	$s_q(3, 5)_1$ block 2	$s_q(3, 2)_1$	$s_q(3, 3)_1$
Y_1	$\widehat{L}_1(4, 0, 0)$	$\widehat{L}_1(5, 0, 0)$	$\widehat{L}_1(4, 1, 0)$	$\widehat{L}_1(2, 0, 0)$	$\widehat{L}_1(3, 0, 0)$
Y_2	$\widehat{L}_1(1, 3, 0)$	$\widehat{L}_1(2, 3, 0)$	$\widehat{L}_1(1, 4, 0)$	$\widehat{L}_1(0, 2, 0)$	$\widehat{L}_1(1, 2, 0)$
Y_3	$\widehat{L}_1(1, 0, 3)$	$\widehat{L}_1(2, 0, 3)$	$\widehat{L}_1(1, 1, 3)$	$\widehat{L}_1(0, 0, 2)$	$\widehat{L}_1(1, 0, 2)$
X	$\widehat{L}_1(2, 2, 0)$	$\widehat{L}_1(2, 2, 1)$	$\widehat{L}_1(3, 2, 0)$	$\widehat{L}_1(1, 1, 0)$	$\widehat{L}_1(1, 1, 1)$

The projective covers of these modules have the following structure.



The Ext^1 quiver for B is illustrated below with relations $\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2$ and all other products zero.



Hence by [Pierce 1982, 11.8], the algebra B/J^2 has finite representation type, where $J = \text{Rad}(B)$. Since $P(X)$ is injective and $P(X)$ is the only indecomposable projective B -module of radical length greater than two, the algebra B has also finite representation type. □

We have now proved that the algebras listed in Theorem 2.2(1)–(6) and (8) are of finite type. It remains to check (7). Recall that δ_q is the q -determinant in

$A_q(n)$. We shall denote the corresponding 1-dimensional $G_q(n)$ -module by the same symbol δ_q . For simplicity, we shall denote by δ_q the restriction to $G_r T$ of the q -determinant module. We need the following reduction lemma.

Lemma 5.3. *Assume \mathcal{B} is a block of $s_q(2, d)_r$ such that $\lambda - \mathbf{1} \in \Gamma_r(D)$ for any $\lambda \in \mathcal{B}$, where $\mathbf{1} = (1, 1)$. Then $\mathcal{B}' := \{\lambda - \mathbf{1} \mid \lambda \in \mathcal{B}\}$ is a block of $s_q(2, d-2)_r$, and \mathcal{B}' is Morita equivalent to \mathcal{B} .*

Proof. By [Cox 1997, 5.2] and [Cox 2000, Section 4], it is easy to check that \mathcal{B}' is a block of $s_q(2, d-2)_r$. Since $P(\widehat{L}_r(\lambda)) \cong P(\widehat{L}_r(\lambda - \mathbf{1})) \otimes \delta_q$ for any $\lambda \in \mathcal{B}$, the assertion follows. \square

By [Thams 1994], we have the following result.

Lemma 5.4. *Assume $l = 2$ and $0 \leq d < 2p$. Then*

$$\dim L(d, 0) = \begin{cases} d+1 & \text{if } d \text{ is odd,} \\ d/2+1 & \text{if } d \text{ is even.} \end{cases}$$

Let \mathcal{A}_s for $s \geq 2$ denote the quiver the figure below with relations $\alpha_1\beta_1 = 0 = \beta_s\alpha_s$, $\alpha_j\alpha_{j+1} = 0 = \beta_{j+1}\beta_j$ and $\beta_j\alpha_j = \alpha_{j+1}\beta_{j+1}$ for $j = 1, 2, \dots, s-1$.

$$\begin{array}{ccccccc} \bullet & \xrightarrow{\alpha_1} & \bullet & \xrightarrow{\alpha_2} & \bullet & \cdots & \bullet & \xrightarrow{\alpha_s} & \bullet \\ X_0 & & X_1 & & X_2 & & X_{s-1} & & X_s \\ & \xleftarrow{\beta_1} & & \xleftarrow{\beta_2} & & & & \xleftarrow{\beta_s} & \end{array}$$

For $k, t \geq 0$, let

$$\begin{aligned} \lambda_{0,k}(1) &= (2p + 2k + 1, 0) + (t - k)\mathbf{1}, \\ \lambda_{1,k}(1) &= (2p - 1, 2k + 2) + (t - k)\mathbf{1}, \\ \lambda_{2,k}(1) &= (2k + 1, 2p) + (t - k)\mathbf{1}. \end{aligned}$$

Let $\mathcal{B}_{k,t}(1) = \{\lambda_{0,k}(1), \lambda_{1,k}(1), \lambda_{2,k}(1)\}$.

Proposition 5.5. *Assume $l = 2$. Then*

- (1) *For $0 \leq t \leq p-2$, the nonsemisimple blocks of $s_q(2, 2p+2t+1)_2$ are $\mathcal{B}_{k,t}(1)$ for $0 \leq k \leq t$ and are Morita equivalent to \mathcal{A}_2 .*
- (2) *The nonsemisimple blocks of $s_q(2, 4p-1)_2$ are $\mathcal{B}_{k,p-1}(1)$ for $0 \leq k \leq p-2$ and are Morita equivalent to \mathcal{A}_2 .*

Proof. We consider the algebra $s_q(2, 2p+1)_2$. Let

$$\lambda_0 = (2p + 1, 0), \quad \lambda_1 = (2p - 1, 2), \quad \lambda_2 = (1, 2p).$$

By [Cox 1997, 6.2.13] and [Cox 2000, 5.12], we have $\mathcal{B}_2^{2p+1}(\lambda_0) = \{\lambda_0, \lambda_1, \lambda_2\}$. Since the classical Schur algebra $S(2, p)$ is Morita equivalent to the principal block component of $S_q(2, 2p+1)$ by [Erdmann and Nakano 2001, 3.2], we get the

structure of projective covers for the simple modules in $\mathcal{B}_2^{2p+1}(\lambda_0)$ as follows by [Erdmann 1993, 5.1] and Proposition 3.4.

$$\begin{array}{l}
 P(\widehat{L}_2(\lambda_0)) : \begin{array}{c} \widehat{L}_2(\lambda_0) \\ | \\ \widehat{L}_2(\lambda_1) \end{array} \\
 \\
 P(\widehat{L}_2(\lambda_2)) : \begin{array}{c} \widehat{L}_2(\lambda_2) \\ | \\ \widehat{L}_2(\lambda_1) \end{array}
 \end{array}
 \quad
 P(\widehat{L}_2(\lambda_1)) : \begin{array}{ccc} & \widehat{L}_2(\lambda_1) & \\ & / \quad \backslash & \\ \widehat{L}_2(\lambda_0) & & \widehat{L}_2(\lambda_2) \\ & \backslash \quad / & \\ & \widehat{L}_2(\lambda_1) & \end{array}$$

Hence the basic algebra for $\mathcal{B}_2^{2p+1}(\lambda_0)$ is isomorphic to \mathcal{A}_2 . Since $s_q(2, 2p-1)_2 = S_q(2, 2p-1)$ is semisimple by [Erdmann and Nakano 2001, 1.3], the algebra $s_q(2, 2p+1)_2$ has only one nonsemisimple block $\mathcal{B}_2^{2p+1}(\lambda_0)$ by Lemma 5.3. Now by induction on t , the assertion follows. □

Assume $a \geq 2$ and $0 \leq t \leq p-1$. For $0 \leq k \leq p-t-3$, let

$$\mathcal{B}_k(a-1) = \{\lambda_{i,k}(a-1) \mid 0 \leq i \leq 2a-2\},$$

where

$$\lambda_{2s,k}(a-1) = (2p(a-s) - p + k + t + 1, 2ps + p - k + t),$$

$$\lambda_{2s+1,k}(a-1) = (2p(a-s-1) + p - k + t - 1, 2p(s+1) - p + k + t + 2).$$

For $p-t-2 \leq k \leq p-2$, let

$$\mathcal{B}_k(a) = \{\lambda_{i,k}(a) \mid 0 \leq i \leq 2a\},$$

where

$$\lambda_{2s,k}(a) = (2p(a-s) - p + 2t + 3 + k, 2ps + p - k - 2),$$

$$\lambda_{2s+1,k}(a) = (2p(a-s) + p - k - 3, 2ps - p + 2t + 4 + k).$$

Proposition 5.6. *Assume $l=2$. Then for $a \geq 2$ and $0 \leq t \leq p-1$, the nonsemisimple blocks of $s_q(2, 2pa + 2t + 1)_2$ are $\mathcal{B}_k(a-1)$ for $0 \leq k \leq p-t-3$ and $\mathcal{B}_k(a)$ for $p-t-2 \leq k \leq p-2$. Furthermore, the block $\mathcal{B}_k(a-1)$ is Morita equivalent to \mathcal{A}_{2a-2} for each $0 \leq k \leq p-t-3$, and the block $\mathcal{B}_k(a)$ is Morita equivalent to \mathcal{A}_{2a} for each $p-t-2 \leq k \leq p-2$.*

Proof. Let $d = 2pa + 2t + 1$. By induction on a and Proposition 5.5, one can prove for each $0 \leq k \leq p-t-3$ and $0 \leq i \leq 2a-2$ there exist indecomposable $s_q(2, d)_2$ -modules $P(\lambda_{i,k}(a-1))$ with the same structure as the projective cover $P(X_i)$ for the simple \mathcal{A}_{2a-2} -modules X_i , and for each $p-t-2 \leq k \leq p-2$

and $0 \leq i \leq 2a$, there exist indecomposable $s_q(2, d)_2$ -modules $P(\lambda_{i,k}(a))$ with the same structure as the projective cover $P(X_i)$ for the simple \mathcal{A}_{2a} -modules X_i . Let

$$\begin{aligned} A_1 &= \{2p(a-i-1, i) + (2p+t, t+1) \mid 0 \leq i \leq a-1\}, \\ A_2 &= \{2p(a-i-2, i) + (3p+t, p+t+1) \mid 0 \leq i \leq a-2\}. \end{aligned}$$

One can check

$$\Gamma_2^d(D) = \left(\bigcup_{k=0}^{p-t-3} \mathcal{B}_k(a-1) \right) \cup \left(\bigcup_{k=p-t-2}^{p-2} \mathcal{B}_k(a) \right) \cup A_1 \cup A_2.$$

By Lemma 5.4, for $\lambda_{i,k}(a-1) \in \mathcal{B}_k(a-1)$, we have

$$\dim(\widehat{L}_2(\lambda_{i,k}(a-1))) = \begin{cases} 2p-2k-2 & \text{if } i \text{ is odd,} \\ 2k+2 & \text{if } i \text{ is even.} \end{cases}$$

Similarly, for $\lambda_{i,k}(a) \in \mathcal{B}_k(a)$, we have

$$\dim(\widehat{L}_2(\lambda_{i,k}(a))) = \begin{cases} 4p-2k-2t-6 & \text{if } i \text{ is odd,} \\ -2p+2k+2t+6 & \text{if } i \text{ is even,} \end{cases}$$

and $\dim(\widehat{L}_2(\mu)) = 2p$ for each $\mu \in A_1 \cup A_2$. It follows from [Cox 1997, 5.3.2] that

$$\begin{aligned} \sum_{k=0}^{p-t-3} \sum_{\mu \in \mathcal{B}_k(a-1)} \dim P(\mu) \dim \widehat{L}_2(\mu) &+ \sum_{k=p-t-2}^{p-2} \sum_{\mu \in \mathcal{B}_k(a)} \dim P(\mu) \dim \widehat{L}_2(\mu) \\ &+ \sum_{\mu \in A_1 \cup A_2} (\dim \widehat{L}_2(\mu))^2 = 4p^2(2ap-2p+2t+3) \\ &= \dim s_q(2, d)_2. \end{aligned}$$

Hence for $\mu \in A_1 \cup A_2$, the $s_q(2, d)_2$ -module $\widehat{L}_2(\mu)$ is projective, and for other $\mu \in \Gamma_2^d(D)$, the projective cover of $\widehat{L}_2(\mu)$ is isomorphic to $P(\mu)$. The assertion follows. \square

Proposition 5.7. *Assume $l=2$. Then the algebra $s_q(2, d)_2$ has finite representation type for d odd with $d \geq 2p+1$.*

Proof. By [Doty et al. 1997, 6.3], the algebra \mathcal{A}_s has finite representation type for s even with $s \geq 2$. Hence the assertion follows from Propositions 5.5 and 5.6. \square

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