LIPSCHITZ SOLUTIONS TO THE ISOMETRY RELATION FOR PAIRS OF RIEMANNIAN METRICS

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Let $M$ be a smooth manifold of dimension $n$ with two Riemannian metrics $g_1, g_2$ which are related by $a^2 g_1 < g_2 < b^2 g_1$. Let $\mathbb{R}^q$ be the Euclidean space with two Euclidean metrics $h_1, h_2$ such that $h_1 - h_2$ has distinct eigenvalues. Further, suppose that $c^2 h_1 - h_2$ is nondegenerate for each $c \in \langle a, b \rangle$, and $r_+(a^2 h_1 - h_2) \geq 2n$, where $r_+$ and $r_-$ denote respectively the positive and the negative ranks of an indefinite metric. Under these conditions we show that there exists an almost everywhere differentiable (Lipschitz) map $f : M \to \mathbb{R}^q$ satisfying $(df_x)^* h_i = g_i$ for $i = 1, 2$ for almost all $x \in M$.

1. Introduction

It is a classical result due to Nash and Kuiper that a Riemannian manifold $(M, g)$ admitting a $C^\infty$-immersion in $\mathbb{R}^q$ also admits a $C^1$-immersion $f : M \to \mathbb{R}^q$ such that $f^* h = g$ provided $q > n$, where $h$ is the canonical metric on $\mathbb{R}^q$. Gromov generalised this result via the method of convex integration by showing that if there exists a strictly short immersion of $(M, g)$ into another Riemannian manifold $(N, h)$ then there exists an isometric $C^1$-immersion $f : M \to N$, when $\dim N > \dim M$. He further proved that in the equidimensional case, there are almost everywhere differentiable (Lipschitz) maps whose derivatives $df$ are isometric almost everywhere on $M$. By an abuse of language, such maps will be referred as the Lipschitz isometric maps; classically, the Lipschitz maps which preserve the lengths of all rectifiable curves relative to the given metrics are referred as isometric maps. Our notion of Lipschitz isometric maps satisfy a much weaker condition; in fact, such an $f$ may collapse a submanifold of positive codimension in $M$ to a single point.

In this paper we generalise the above mentioned result of Gromov when both the manifolds $M$ and $N$ come with a pair of Riemannian metrics.

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Let $M$ be a smooth manifold of dimension $n$. Let $\mathbb{R}^q$ be the $q$-dimensional Euclidean space with two Euclidean metrics $h_1$ and $h_2$ which satisfy the following conditions: There exist two numbers $0 < a < b$ such that

1. $c^2h_1 - h_2$ is a nondegenerate indefinite form for each real number $c$ lying in $[a, b]$;
2. $r_+(a^2h_1 - h_2) \geq 2n$ and $r_-(b^2h_1 - h_2) \geq 2n$, where $r_+$ and $r_-$ denote respectively the positive and the negative ranks of an indefinite metric; and
3. if $A : \mathbb{R}^q \to \mathbb{R}^q$ is the unique linear isomorphism given by $h_2(v, w) = h_1(Av, w)$ for all $v, w \in \mathbb{R}^q$, then $A$ has distinct eigenvalues.

**Theorem 1.1.** Let $g_1, g_2$ be two Riemannian metrics which are related by $a^2g_1 < g_2 < b^2g_1$. Then under assumptions (1)–(3) mentioned above, there exists an almost everywhere differentiable (Lipschitz) map $f : M \to \mathbb{R}^q$ satisfying $(df_x)^*h_i = g_i$ for $i = 1, 2$ for almost all $x \in M$. Moreover, such maps are $C^0$-dense in the space of strictly $(g_1, g_2)$-short maps (see Definition 5.1).

We further observe that if $M$ is a one-dimensional manifold, then (under the same hypothesis) there exists a $C^1$-map $f : M \to \mathbb{R}^q$ such that $f^*h_i = g_i$ for $i = 1, 2$.

The maps $f$ obtained in Theorem 1.1 will be referred as Lipschitz isometric maps for pairs of metrics. If $\mathbb{R}^q$ is replaced by a general manifold $N$ in Theorem 1.1 we may have to presuppose the existence of strictly $(g_1, g_2)$-short maps in order to conclude the existence of Lipschitz isometric maps [Gromov 1986, 2.4.9 (A)]. It may be observed that $(g_1, g_2)$-short maps always exist for $N = \mathbb{R}^q$ (see Proposition 5.2).

In our earlier paper [D’Ambra and Datta 2002] we proved the existence of isometric $C^1$-immersions $M \to \mathbb{R}^q$ for pairs of Riemannian metrics when

$$r_\pm(c^2h_1 - h_2) \geq 3n + 2$$

for all $c \in [a, b]$, generalizing the Nash–Kuiper $C^1$-immersion theorem. The proof was based on Nash’s technique for obtaining isometric $C^1$-immersions.

In the present paper, we have substantially relaxed the restrictions on $r_\pm$, however, at the cost of $C^1$-regularity of solutions. Our study of Lipschitz isometric maps $f : (M, g_1, g_2) \to (\mathbb{R}^q, h_1, h_2)$ relies extensively on the convex integration theory which incorporates the essence of the approach of Kuiper [1955]. The key idea of the method of convex integration can be stated as follows: If $A$ is a connected subset of $\mathbb{R}^q$ such that the interior of the convex hull of $A$ contains the origin then there is a $C^1$-map $f : S^1 \to \mathbb{R}^q$ whose derivative maps $S^1$ into $A$. This can be viewed as the convex integration over a circle. However, in this paper we obtain only Lipschitz solutions in contrast with $C^1$-solutions in
The reason behind this is that we are unable to solve the connectivity problem for the subsets of the form $S_1 \cap S_2 \cap T$, where $S_1$ and $S_2$ are two spheres in $\mathbb{R}^q$ relative to the metrics $h_1$ and $h_2$ respectively and $T$ is an affine subspace in $\mathbb{R}^q$.

We organize the paper as follows. We devote Section 2 to review the basic language of $h$-principle theory and convex integration techniques to deal with open first order partial differential relations. In Section 3 we introduce the notion of $(h_1, h_2)$-regularity for $C^1$-maps $f : M \to \mathbb{R}^q$ and study the geometry underlying the regularity condition which plays a crucial role in our treatment. In Section 4 we prove the Main Lemma (Lemma 4.1) leading to Theorem 1.1 and in Section 5 we prove the existence of an approximate solution to our problem. The proof of the Main Theorem (Theorem 1.1) is given in Section 6. The one-dimensional case is separately studied in Section 7 where we show that there exists, in fact, a $C^1$-solution.

2. Review of convex integration techniques

In this section we recall the terminology of the theory of $h$-principle and discuss in brief the main result of convex integration technique following [Eliashberg and Mishachev 2002].

Let $f$ be the germ of some local $C^r$-map at $x \in M$. The $r$-jet of $f$ at $x$ is by definition the ordered tuple

$$j_f^r(x) = (x, f(x), Df(x), \ldots, D^r f(x)),$$

where $D^k f$ denotes the derivative map of $f$ of order $k$. The collection of all such $r$-jets constitutes the total space of a fibre bundle over $M$ which is denoted by $p^r : J^r(M, N) \to M$. The bundle is referred as the $r$-jet bundle associated with the space of $C^r$-maps from $M$ to $N$.

If $r = 1$ then

$$j_f^1(x) = (x, f(x), Df(x))$$

and $J^1(M, N)$ can be identified with the total space of the bundle $\text{Hom}(TM, TN)$.

A continuous map $\sigma : M \to J^1(M, N)$ is said to be a section if $p^1 \circ \sigma = \text{id}_M$.

If $f : M \to N$ is a $C^r$-map then its $r$-jet map $j_f^r$ defined by

$$j_f^r(x) = (x, f(x), Df(x), \ldots, D^r f(x))$$

is a section of $p^r$.

**Definition 2.1.** An $r$-th order partial differential relation is a subset $\mathcal{R}$ of $J^r(M, N)$. A $C^r$-map $f : M \to N$ is said to be a solution of $\mathcal{R}$ if its $r$-jet map $j_f^r$ maps $M$ into $\mathcal{R}$.
A section of \( p' \) whose image is contained in \( \mathcal{R} \) is called a formal solution of the differential relation. A formal solution of \( \mathcal{R} \) is said to be holonomic if it is the \( r \)-jet map of some \( C' \)-map \( f : M \to N \).

A differential relation \( \mathcal{R} \) is said to satisfy the \textit{h-principle} if every formal solution \( \sigma \) can be homotoped to a holonomic section in the space of all formal solutions.

**Definition 2.2.** Let \( \Omega \) be an open subset of a manifold \( M \). A continuous map \( f \) from \( \Omega \) into a manifold \( N \) is said to be piecewise \( C' \) if there exists a countable system of mutually disjoint open sets \( \Omega_j \subset \Omega \) which cover \( \Omega \) up to a set of measure zero and the restriction of \( f \) to each \( \Omega_j \) is \( C' \).

Let \( \mathcal{R} \subset J' (M, N) \) be an \( r \)-th order differential relation. A piecewise \( C' \)-map \( f : M \to N \) is said to be a piecewise \( C' \)-solution of \( \mathcal{R} \) if \( j_f^r(x) \in \mathcal{R} \) for all \( x \in M \) where the \( r \)-th derivative of \( f \) exists.

The convex integration technique gives solutions to h-principle for some differential relations which satisfy certain convexity condition. The key idea of the convex integration technique is stated in the following lemma.

**Lemma 2.3 [Gromov 1986, 2.4.1].** Let \( A \) be a connected subset of \( \mathbb{R}^q \) and let 0 belong to the interior of the convex hull of \( A \). Then there exists a \( C^1 \)-map \( f : [0, 1] \to \mathbb{R}^q \) such that \( f'(t) \in A \) for all \( t \in [0, 1] \). Moreover, \( f \) can be made to lie in an arbitrary small neighbourhood of 0.

If the connectivity condition on \( A \) is dropped in the above lemma then it delivers a piecewise linear map \( f \) such that \( f(0) = f(1) = 0 \) and \( f'(t) \in A \) whenever \( f \) is differentiable [Eliashberg and Mishachev 2002, §17.4(D)]. More generally we obtain:

**Proposition 2.4.** Let \( \mathcal{R} \) be an open subset of \( J^1 (\mathbb{R}, \mathbb{R}^q) \) and let \( f : [0, 1] \to \mathbb{R}^q \) be a continuous function which is \( C^1 \) on \( (0, 1) \). Suppose that \( j_1^f(x) \) lies in the convex hull of \( \mathcal{R}_{b(x)} \) for all \( x \in (0, 1) \), where

\[
b(x) = (x, f(x)) \in J^0 (\mathbb{R}, \mathbb{R}^q).
\]

Then \( f \) can be homotoped to a piecewise \( C^1 \)-solution \( f_1 \) of \( \mathcal{R} \) in any \( C^0 \)-neighbourhood of \( f \) such that \( f_1(0) = f(0) \) and \( f_1(1) = f(1) \).

**Proof:** Consider any \( \varepsilon > 0 \). Appealing to one-dimensional convex integration [Eliashberg and Mishachev 2002, §17.3] we can construct a piecewise linear map \( f^1 \) on the interval \([\varepsilon, 1 - \varepsilon]\) which coincides with \( f \) at the boundary points and is a piecewise \( C^1 \)-solution of \( \mathcal{R} \) on \((\varepsilon, 1 - \varepsilon)\). Next consider, for each \( n \geq 1 \), a pair of disjoint intervals \( I_n = [\varepsilon/2^n, \varepsilon/2^{n-1}] \) and \( J_n = [1 - \varepsilon/2^{n-1}, 1 - \varepsilon/2^n] \). The interior of these sets cover \([0, 1]\) up to a set of measure zero. Now, applying [Eliashberg and Mishachev 2002, §17.3] again to the restriction of \( f \) to \( I_n \cup J_n \) we obtain a piecewise linear map \( f^n \) on \( I_n \cup J_n \) which coincides with \( f \) at the endpoints and
satisfies the differential relation except at the points where the derivative does not exist. Further, we can choose $f^n$ to be $\varepsilon/2^n$-close to $f$ on the set. Now all these maps patch together to give a piecewise linear map $f_1$ on $(0, 1)$. Further, this map extends continuously to the closed interval $[0, 1]$ and the extended map satisfies the desired conditions. □

**Remark.** If $f$ is a solution of $\mathcal{R}$ on a neighbourhood of some closed subset $K$, then the homotopy remains constant on some (possibly smaller) open neighbourhood of $K$.

The result above may be generalised to a parametric version following [Eliashberg and Mishachev 2002, §17.5.1].

**Proposition 2.5.** Let $\mathcal{R}$ be an open subset of $I^l \times J^1(\mathbb{R}, \mathbb{R}^q)$ and

$$\mathcal{R}_p = p \times J^1(\mathbb{R}, \mathbb{R}^q) \cap \mathcal{R}.$$ Let $f : I^l \times I \to \mathbb{R}^q$ be a continuous function which is $C^1$ in the interior of $I^l \times I$. Let $f_p$ denote the restriction of $f$ to $p \times I$ and suppose that for each $p$, the pair $(f_p, \mathcal{R}_p)$ satisfies the hypothesis of Proposition 2.4. Then $f$ can be homotoped to a piecewise $C^1$-map $f_1$ in any $C^0$-neighbourhood of $f$ such that

1. $(f_1)_p$ is a piecewise $C^1$-solution of $\mathcal{R}_p$;
2. $f_1 = f$ on $I^l \times \{0, 1\}$;
3. the first order derivatives of $f_1(p, t)$ with respect to $p$ are arbitrarily $C^0$ close to the respective derivatives of $f(p, t)$.

Further, if $f_p$ is a genuine solution of $\mathcal{R}_p$ for $p \in \mathcal{C}p \partial I^l$ then the homotopy can be kept constant for $p \in \mathcal{C}p \partial I^l$. (The notation $\mathcal{C}p \partial I^l$ is used to denote a nonspecified open neighbourhood of $\partial I^l$ which may become smaller in the course of the argument.)

We shall now state the main result on convex integration which yields piecewise $C^1$-solutions to certain open relations. Before stating it we need to recall the basic language of $\perp$-jets.

Let $\tau$ be an integrable hyperplane field on $\mathbb{R}^n$. With respect to this $\tau$ we define an equivalence relation $\sim$ on $J^1(\mathbb{R}^n, \mathbb{R}^q)$ as follows: If $(x, y, \alpha), (x, y, \beta)$ lie in the same fibre over $(x, y) \in J^0(\mathbb{R}^n, \mathbb{R}^q)$, then

$$\alpha \sim \beta \quad \text{if and only if} \quad \alpha|_{\tau} = \beta|_{\tau}.$$ The equivalence class of $(x, y, \alpha)$, denoted as $P_\alpha$, is an affine subspace of dimension $q$ in the jet space. Indeed, if we fix a vector field $v$ on $\mathbb{R}^n$ transversal to $\tau$, then $(x, y, \beta) \in P_\alpha$ is completely determined by $\beta(v) \in \mathbb{R}^q$. Thus relative to $\tau$ we can slice the 1-jet space into $q$-dimensional affine subspaces $P_\alpha$. $P_\alpha$ is
called the principal subspace through \((x, y, \alpha)\) corresponding to \(\tau\). The set of equivalence classes is denoted by \(J^1(\mathbb{R}^n, \mathbb{R}^q)\) and there is a canonical projection \(p : J^1(\mathbb{R}^n, \mathbb{R}^q) \to J^1(\mathbb{R}^n, \mathbb{R}^q)\) which takes a 1-jet onto its equivalence class. Let \(j^1_f = p \circ j^1_f\).

Identifying \(j^1_f(x)\) with \((j^1_f(x), df(x))\) we can write
\[
J^1(\mathbb{R}^n, \mathbb{R}^q) = J^1(\mathbb{R}^n, \mathbb{R}^q) \times \mathbb{R}^q.
\]

Note that when \(n = 1\), \(J^1(\mathbb{R}^1, \mathbb{R}^q) = J^0(\mathbb{R}^1, \mathbb{R}^q) = \mathbb{R} \times \mathbb{R}^q\).

**Theorem 2.6.** Let \(\mathcal{R}\) be an open subset of \(J^1(\mathbb{R}^n, \mathbb{R}^q)\). Let \(f_0 : I^n \to \mathbb{R}^q\) be a piecewise \(C^1\)-function such that \(j^1_f(x)\) lies in the convex hull of \(\mathcal{R}_{b(\cdot)}\) whenever the derivative exists, where \(b(x) = j^1_f(x)\). Then there exists a piecewise \(C^1\)-solution of \(\mathcal{R}\), \(f_1 : I^n \to \mathbb{R}^q\), which is homotopic to \(f_0\). Moreover, the homotopy can be made to lie in an arbitrary \(C^0\)-neighbourhood of \(f_0\).

Further, if \(f_0\) is a piecewise \(C^1\)-solution of \(\mathcal{R}\) on some open neighbourhood of a compact set \(K \subset I^n\), then the homotopy remains constant on some (possibly smaller) neighbourhood of \(K\).

For the sake of completeness we include the proof from [Eliashberg and Mishachev 2002].

**Proof.** Consider the splitting of the cube \(I^n\) as \(I^{n-1} \times I\). Form a relation
\[
\mathcal{R}^1 \subset I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)
\]
fibred over \(I^{n-1}\) as follows:

For each \(x \in I^n\) let \(P(j^1_f(x))\) denote the principal subspace through \(j^1_f(x)\) corresponding to the splitting \(I^{n-1} \times I\). Let \(\Omega(f(p, t))\) be the subset defined by
\[
\{j^1_f(p, t)\} \times \Omega(f(p, t)) = P(j^1_f(p, t)) \cap \mathcal{R}.
\]

By the given hypothesis, \(\partial_t f(p, t)\) belongs to the convex hull of \(\Omega(f(p, t))\) in \(P(j^1_f(x))\). Since \(\mathcal{R}\) is open there is an open neighbourhood \(D^q(f(p, t))\) of \(f(p, t)\) in \(\mathbb{R}^q\) and an open subset \(\Omega'(f(p, t))\) contained in \(\Omega(f(p, t))\) such that

1. \(\Omega'(f(p, t))\) contains \(\partial_t f(p, t)\) in its convex hull and
2. \(\{(p, t) \times D^q(f(p, t)) \times \partial_p f(p, t) \times \Omega'(f(p, t))\} \subset \mathcal{R}\) for all \((p, t) \in I^{n-1} \times I\).

In the above, \(\partial_t\) and \(\partial_p\) respectively denote the derivatives of the function with respect to the coordinates \(t\) and \(p\).

For each \(p \in I^{n-1}\) define a relation \(\mathcal{R}^1_p \subset J^1(\mathbb{R}, \mathbb{R}^q)\) as
\[
\mathcal{R}^1_p = \{(t, y, v) \in I \times \mathbb{R}^q \times \mathbb{R}^q : y \in D^q(f(p, t)), v \in \Omega'(f(p, t))\}.
\]

Then \(\mathcal{R}^1 = \bigcup_p \{p\} \times \mathcal{R}^1_p\) is a fibred relation in \(I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)\) which is defined over an open neighbourhood of the graph of the section \(f\) in \(I^n \times \mathbb{R}^q\).
Further for an appropriate choice of $\Omega'(f(p, t))$ we may assume that $\mathcal{R}^1$ is an open fibred relation in $I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)$.

Also note that for a fixed $p \in I^{n-1}$, $t \mapsto f(p, t)$ is a short solution of $\mathcal{R}^1_p$.

We now apply the parametric one-dimensional convex integration to obtain a piecewise $C^1$-homotopy $f_t$ of fibrewise “short” (see [Eliashberg and Mishachev 2002] for the definition) solutions of $\mathcal{R}^1$ which is $C^0$ close to $f$ and satisfies

$$f_t(p, 0) = f(p, 0) \quad \text{and} \quad f_t(p, 1) = f(p, 1)$$

for all $p \in I^{n-1}$. Furthermore, the first order derivatives of $f_1(p, t)$ with respect to the parameter $p$ (wherever exist) are arbitrarily $C^0$ close to the respective derivatives of $f(p, t)$. Hence,

$$\left( f(p, t), \partial_p f(p, t), \partial_t f_1(p, t) \right) \in \mathcal{R}.$$ 

Since $\mathcal{R}$ is open and since the derivatives of $f_1$ with respect to $p$ are arbitrarily close to the respective derivatives of $f$ it follows that

$$\left( f_1(p, t), \partial_p f_1(p, t), \partial_t f_1(p, t) \right) \in \mathcal{R}.$$ 

Thus $f_1$ is a solution of $\mathcal{R}$ with the desired properties. \hfill \square

**Remark 2.7.** We refer the reader to [Gromov 1986, p. 218] for a general result on the existence of (almost everywhere differentiable) Lipschitz solutions to some differential relations.

### 3. $(h_1, h_2)$ regularity and underlying geometry

Throughout this section $h_1$ and $h_2$ will denote two positive definite symmetric bilinear forms on $\mathbb{R}^q$. For any subspace $V$ of $\mathbb{R}^q$, we shall denote its orthogonal complement with respect to $h_i$ by $V^\perp_i$ for $i = 1, 2$.

**Definition 3.1.** A subspace $V$ of $\mathbb{R}^q$ is said to be $(h_1, h_2)$-regular if $V^\perp_1$ is transversal to $V^\perp_2$.

Observe that if $A : \mathbb{R}^q \rightarrow \mathbb{R}^q$ is the (unique) linear transformation defined by $h_2(v, w) = h_1(Av, w)$ for all $v, w \in \mathbb{R}^q$, then a subspace $V$ in $\mathbb{R}^q$ is regular if and only if $V + A(V)$ has the maximum dimension.

**Definition 3.2.** A vector $v \in \mathbb{R}^q$ is said to be $(h_1, h_2)$-regular provided the one-dimensional subspace $\langle v \rangle$ spanned by $v$ is a $(h_1, h_2)$-regular subspace of $\mathbb{R}^q$.

**Observation 1.** A vector $v$ is $(h_1, h_2)$-regular if and only if $v$ and $Av$ are linearly independent, $A : \mathbb{R}^q \rightarrow \mathbb{R}^q$ being the unique linear map defined above. Consequently, the set of nonregular vectors precisely consists of the eigen-vectors of $A$. 


The following observation brings out the underlying geometry of the \((h_1, h_2)\-regular\ vectors.

**Observation 2.** Let \((\mathbb{R}^q, h_1, h_2)\) be as in the above. We shall denote the norms of a vector \(w \in \mathbb{R}^q\) relative to \(h_1\) and \(h_2\) by \(\|w\|_1\) and \(\|w\|_2\) respectively. Let

\[
S_r = \{ w \in \mathbb{R}^q \mid \|w\|_1 = r \} \quad \text{and} \quad E_r = \{ w \in \mathbb{R}^q \mid \|w\|_2 = r \}
\]

denote the spheres of radius \(r\) in \(\mathbb{R}^q\) relative to the two metrics. Observe that, a vector \(v \in S_r \cap E_r\) is \((h_1, h_2)\-regular\ if and only if \(S_r\) and \(E_r\) intersect transversally at \(v\). Indeed, \(v\) is a regular vector if and only if \(v^\perp_1\) is transversal to \(v^\perp_2\). If \(v \in S_r \cap E_r\), then \(v^\perp_1\) is tangent to \(S_r\) at \(v\) and \(v^\perp_2\) is tangent to \(E_r\) at \(v\). Therefore it follows that \(S_r\) is transversal to \(E_r\) at \(v\).

**Observation 3.** Let \(V\) be a \((h_1, h_2)\-regular\ subspace of \(\mathbb{R}^q\) of dimension \((n - 1)\) and let

\[
X = V^\perp_1 \cap V^\perp_2 = (V \oplus A(V))^\perp_1.
\]

Then \(X\) has codimension \(2(n - 1)\) in \(\mathbb{R}^q\). For any vector \(w \in \mathbb{R}^q\), \(\tau \oplus \langle w \rangle\) is an \((h_1, h_2)\-regular\ subspace if and only if \(w^\perp_1 \cap X\) is transversal to \(w^\perp_2 \cap X\) in \(X\). Indeed, \(V \oplus \langle w \rangle\) is a \((h_1, h_2)\-regular\ subspace if and only if \((V \oplus \langle w \rangle)^\perp_1\) is transversal to \((V \oplus \langle w \rangle)^\perp_2\), that is, if and only if

\[
\text{codim}\left((V \oplus \langle w \rangle)^\perp_1 \cap (V \oplus \langle w \rangle)^\perp_2\right) = 2n.
\]

This is equivalent to saying \(X \cap w^\perp_1 \cap w^\perp_1\) has codimension \(2\) in \(X\). Thus \(w^\perp_1 \cap X\) is transversal to \(w^\perp_2 \cap X\).

Let \(T\) be a translate of \(X\) through \(w\). Suppose that \(r = \|w\|_1\) and \(r' = \|w\|_2\). Since \(w^\perp_1 \cap X\) is the tangent space of \(S_r \cap T\) at \(w\) and \(w^\perp_2 \cap X\) is the tangent space of \(E_r \cap T\) at \(w\), it follows from the above that the sets \(S_r \cap T\) and \(E_r \cap T\) intersect transversally in \(T\) at \(w\).

In particular, we can show that if \(w\) is in \(X\), then \(V \oplus \langle w \rangle\) is \((h_1, h_2)\-regular\ if and only if \(w\) is \((\bar{h}_1, \bar{h}_2)\-regular\, where \(\bar{h}_1\) and \(\bar{h}_2\) denote the restrictions of \(h_1\) and \(h_2\) respectively to \(X\).

Let \(\bar{A}\) denote the unique linear transformation \(X \rightarrow X\) such that

\[
\bar{h}_2(v, w) = \bar{h}_1(\bar{A}v, w) \quad \text{for} \quad v, w \in V.
\]

If \(w \in X\) is \((\bar{h}_1, \bar{h}_2)\-regular\ then \(w\) and \(\bar{A}(w)\) are linearly independent. Let \(A(w) = x + x^\perp\), where \(x \in X\) and \(x^\perp \in X^\perp_1\). Then

\[
h_2(w, v) = h_1(Aw, v) = h_1(x + x^\perp, v) = h_1(x, v)
\]

for all \(v \in X\). Hence \(x = \bar{A}(w)\). This proves that \(Aw = \bar{A}w + x^\perp\). Since \(w, \bar{A}w\) are linearly independent in \(X\) and \(x^\perp \notin X\) it follows that \(w\) and \(Aw\) are linearly independent and consequently, \(V \oplus \langle w \rangle\) is \((h_1, h_2)\-regular.\)
Definition 3.3. Let $N$ be a smooth manifold with two Riemannian metrics $h_1$ and $h_2$. A smooth map $f : M \to N$ will be called $(h_1, h_2)$-regular if for each $x \in M$, $df_x(T_xM)$ is a $(h_1, h_2)$-regular subspace of $T_{f(x)}N$. 

Proposition 3.4 [D’Ambra and Datta 2002]. Let $h_1, h_2$ be two positive definite symmetric bilinear forms on $\mathbb{R}^q$ such that the eigen-values of $A$ (as defined above) are all distinct. Then a generic map $f : M \to \mathbb{R}^q$ is $(h_1, h_2)$-regular if $q$ exceeds $3 \dim M - 1$.

4. The Main Lemma

Let $M$ be a smooth manifold of dimension $n$. Let $\mathbb{R}^q$ be the $q$-dimensional Euclidean space. In what follows $h_1$ and $h_2$ will denote two Euclidean metrics on $\mathbb{R}^q$ which satisfy the following conditions:

There exist two numbers $0 < a < b$, such that

(1) $c^2h_1 - h_2$ is a nondegenerate indefinite form for each real number $c$ lying in $[a, b]$;

(2) $r_+(a^2h_1 - h_2) \geq 2n$ and $r_-(b^2h_1 - h_2) \geq 2n$, where $r_+$ and $r_-$ denote respectively the positive and the negative ranks of an indefinite metric; and

(3) if $A : \mathbb{R}^q \to \mathbb{R}^q$ is the unique linear isomorphism given by $h_2(v, w) = h_1(Av, w)$ for all $v, w \in \mathbb{R}^q$, then $A$ has distinct eigenvalues.

Lemma 4.1. Let $g_1$ and $g_2$ be two Riemannian metrics on $M$ which are related by $a^2 g_1 < g_2 < b^2 g_2$. Let $f : M \to \mathbb{R}^q$ be an $(h_1, h_2)$-regular immersion such that

$$g_1 - f^* h_1 = \phi^2 d\psi^2 \quad \text{and} \quad g_2 - f^* h_2 = c^2 \phi^2 d\psi^2,$$

where $\phi, \psi$ are smooth functions on $M$, $\phi$ has compact support contained in an open set $U$ of $M$ and $a < c < b$.

Then there exists a piecewise $C^1$-map $\tilde{f}$ which is a fine $C^0$-approximation of $f$ and has the following properties:

(1) $\tilde{f}$ coincides with $f$ outside $U$;

(2) $\tilde{f}^* h_i$ is arbitrarily close to $g_i$ ($\tilde{f}^* h_i \approx g_i$) for $i = 1, 2$ relative to the fine $C^0$-topology on each component where $\tilde{f}$ is $C^1$.

Proof. Let $\mathcal{F}$ denote the subset of $J^1(M, \mathbb{R}^q)$ consisting of all 1-jets $(x, y, \alpha)$ such that $\alpha^* h_1 = g_1$ and $\alpha^* h_2 = g_2$. Let $\tau$ be the hyperplane field over $U$ defined by $\ker d\psi$. Then $\tau$ is integrable and its integral submanifolds are precisely the level sets of the function $\psi$.

Consider the bundle $p_\perp : J^1(U, \mathbb{R}^q) \to J^1(U, \mathbb{R}^q)$
relative to the hyperplane distribution $\tau$ on $U$. An element $b$ of $J_{1}^\perp(M, \mathbb{R}^q)$ is of the form $b = (x, y, \beta)$, where $x \in U$, $y \in \mathbb{R}^q$ and $\beta : \tau_x \to \mathbb{R}^q$ is a linear map. The fibre over $b$ consists of all linear maps $\alpha : T_x M \to \mathbb{R}^q$ which restricts to $\beta$ on $\tau_x$.

To describe the intersection of the relation $\mathcal{J}$ with the principal subspaces of the fibration $p_{1}^\perp$, we choose a vector field $v_0$ on $TU$ such that

$$\|v_0\|_1 = \sqrt{g_1(v_0, v_0)} = 1 \quad \text{and} \quad g_1(v_0, \tau) = 0$$
on $U \supset \text{supp } \phi$. Let $\|v_0\|_2 = \sqrt{g_2(v_0, v_0)} = r$; then $r$ is a smooth function on $U$ satisfying the inequality $a < r(x) < b$ for all $x \in U$. Let $p' : \mathcal{J} \to J_{1}^\perp(M, \mathbb{R}^q)$ denote the restriction of $p_{1}^\perp$ to $\mathcal{J}$. Recall that a 1-jet $(x, y, \alpha)$ in a principal subspace $J_{1}^b(U, \mathbb{R}^q)$ is completely determined by its value at $v_0$. Moreover, if

$$(x, y, \alpha) \in \mathcal{J}_b = J_{1}^b(U, \mathbb{R}^q) \cap \mathcal{J},$$

then $\alpha(v_0)$ is contained in the unique affine space

$$T_b = \{ w \in \mathbb{R}^q \mid h_1(w, \beta(\tau)) = 0 \text{ and } h_2(w, \beta(v)) = g_2(v_0, v) \text{ for all } v \in \tau \},$$

where $b = (x, y, \beta) \in J_{1}^\perp(U, \mathbb{R}^q)$. Note that the equation $h_2(w, \beta(v)) = g_2(v_0, v)$ defines an affine subspace of $\mathbb{R}^q$ which is a translate of $\beta(\tau)^{\perp_2}$. If $\alpha$ is $(h_1, h_2)$-regular then, in particular, $\beta(\tau_x)^{\perp_1}$ is transversal to $\beta(\tau_x)^{\perp_2}$ and the same is true for any translates of these spaces. Thus $T_b$ is an affine plane of codimension $(n - 1)$. Moreover, this is the translate of the vector subspace $X_b = \beta(\tau_x)^{\perp_1} \cap \beta(\tau_x)^{\perp_2}$ in $\mathbb{R}^q$.

Thus $J_{1}^b(U, \mathbb{R}^q) \cap \mathcal{J}$ is contained in an affine subbundle of codimension $(n - 1)$ (over some open subset of $J_{1}^\perp(U, \mathbb{R}^q)$). Further, it follows that if

$\alpha \in J_{1}^b(U, \mathbb{R}^q) \cap \mathcal{J}$

then $\|\alpha(v_0)\|_1 = 1$ and $\|\alpha(v_0)\|_2 = r$. Therefore we can characterize $J_{1}^b(U, \mathbb{R}^q) \cap \mathcal{J}$ as

$$J_{1}^b(U, \mathbb{R}^q) \cap \mathcal{J} = \{ w \in T_b : \|w\|_1 = 1, \|w\|_2 = r \}.$$
Proof of Sublemma 4.2. Observe that

1. \( df_x(v_0) \) lies in \( T_{b(x)} \), and
2. \( df_x(v_0) \) satisfies the equation
   \[
   c^2(1 - \|w\|_1^2) = r^2 - \|w\|_2^2
   \]
since \( g_2 - f^*h_2 = c^2(g_1 - f^*h_1) \).

The above equation can be equivalently expressed as \((c^2h_1 - h_2)(w, w) = c^2 - r^2\). This represents a generalised hyperboloid \( H \) since \( r_\pm(c^2h_1 - h_2) \geq 2n \). It may be seen easily that \( H \cap S = E \cap S = H \cap E \).

Since \( r_\pm(c^2h_1 - h_2) \geq 2n \), \( H \) is generated by affine subspaces of dimension \( 2n - 1 \). To see this, let \( h \) be a nondegenerate symmetric bilinear form on \( \mathbb{R}^q \) of signature \((q_+, q_-)\). Let \( v \in H \) be such that \( h(v, v) = d \neq 0 \) and let \( V \) denote the \( h \)-orthogonal complement of the subspace generated by \( v \). Then \( V \) has dimension \( n - 1 \) and

\[
    r_+(h|_V) \geq q_+ - 1, \quad r_-(h|_V) \geq q_- - 1.
\]

Consequently, \( V \) admits a regular \( h \)-isotropic subspace \( I \) of dimension

\[
    \min(q_+ - 1, q_- - 1).
\]

Here regularity means that \( I \) does not intersect the kernel of \( h|_V \). Consider the affine subspace \( W = I + v \). It is easy to see that \( h(w, w) = d \) for every \( w \in W \). This proves the above assertion.

Let \( A_x \) be an affine subspace in \( H \) which passes through \( df_x(v_0) \). Since

\[
    \text{codim } T_x = 2(n - 1) < 2n - 1 = \dim A_x,
\]

the intersection \( T_x \cap A_x \) is an affine subspace of dimension at least 1. Since \( df_x(v_0) \in T_x \cap A_x \) and \( \|df_x(v_0)\|_1 < 1 \), \( T_x \cap A_x \cap S \) contains at least two points and \( df_x(v_0) \) lies in the convex hull of this intersection. Noting that \( T_x \cap A_x \cap S \subset T_x \cap E \cap S \),

we conclude that \( df_x(v_0) \) lies in the convex hull of \( T_x \cap E \cap S \). This completes the proof of Sublemma 4.2.

Now we conclude the proof of the Main Lemma (Lemma 4.1). Since \( \mathcal{J} \) is not an open relation we cannot directly apply Theorem 2.6 to the pair \((f, \mathcal{J})\). We take an arbitrary small open neighbourhood \( \tilde{\mathcal{J}} \) of \( \mathcal{J} \) and apply Theorem 2.6 to the pair \((\tilde{f}, \tilde{\mathcal{J}})\). Thus we obtain a fine \( C^0 \)-approximation of \( f \) by a piecewise \( C^1 \)-solution \( \tilde{f} \) of \( \tilde{\mathcal{J}} \). Choosing \( \tilde{\mathcal{J}} \) sufficiently small, we can make \( \tilde{f}^*h_1 \) and \( \tilde{f}^*h_2 \) arbitrarily \( C^0 \) close to the pair \((g_1, g_2)\) as desired. This completes the proof. \( \square \)
5. Approximate solution

We recall the definition of short maps from [D’Ambra and Datta 2002].

**Definition 5.1.** Let $M$ be a manifold with two Riemannian metrics $g_1$ and $g_2$. A $C^1$-map $f_0 : M \to (\mathbb{R}^q, h_1, h_2)$ is $(g_1, g_2)$-short if the metrics $g_1 - f_0^*(h_1)$ and $g_2 - f_0^*(h_2)$ on $M$ are positive definite. This will be expressed by $g_i - f_0^*(h_i) > 0$ or $g_i > f_0^*(h_i)$, for $i = 1, 2$.

**Proposition 5.2.** Let $M$ be a $C^\infty$-manifold with two Riemannian metrics $g_1$ and $g_2$ which are related by $a^2 g_1 < g_2 < b^2 g_1$. Then there exists a $(g_1, g_2)$-short $C^\infty$-immersion $f_0 : M \to (\mathbb{R}^q, h_1, h_2)$ which also satisfies the inequalities

\[
\begin{align*}
& a^2(g_1 - f_0^*h_1) < (g_2 - f_0^*h_2) < b^2(g_1 - f_0^*h_1), \\
& a^2 f_0^*h_1 < f_0^*h_2 < b^2 f_0^*h_1.
\end{align*}
\]

**Proof.** For any number $c$ with $a < c < b$, consider the nondegenerate form $\tilde{h} = c^2 h_1 - h_2$. By the hypothesis of Theorem 1.1, $r_+(\tilde{h}) \geq 2n$ and $r_-(\tilde{h}) \geq 2n$. Therefore, there exists a $C^1$-immersion $f : M \to \mathbb{R}^q$ such that $f^*(\tilde{h}) = 0$. (This follows from an exercise in [Gromov 1986, 2.4.9, Corollary (2’)].) Such an $f$ clearly satisfies the relation $a^2 f^*h_1 < f^*h_2 < b^2 f^*h_1$. Moreover, without any loss of generality we may assume that the map $f$ satisfying the above inequality is smooth, because if that is not the case we replace $f$ by a $C^\infty$-immersion which is sufficiently $C^1$ close to $f$.

Now, if $M$ is a closed manifold, then starting with an $f$ as above we can obtain the required $f_0$ by scaling the map $f$ with a suitable scalar (see the corresponding result in [D’Ambra and Datta 2002]). To obtain such an $f_0$ in the case of open manifolds we have to employ the partition of unity techniques. \hfill $\boxempty$

Let $\mathcal{F}$ denote the set of all piecewise $C^1$-maps $f : M \to \mathbb{R}^q$ which satisfy the following conditions at each point $x \in M$ where $f$ is differentiable:

F1. $f$ is $(h_1, h_2)$-regular;
F2. $f$ is $(g_1, g_2)$-short;
F3. $a^2(g_1 - f^*h_1) < g_2 - f^*h_2 < b^2(g_1 - f^*h_1)$;
F4. $a^2 f^*h_1 < f^*h_2 < b^2 f^*h_1$.

**Proposition 5.3.** Let $f_0 : M \to \mathbb{R}^q$ belong to $\mathcal{F}$ and let $0 < \varepsilon < 1$ be any positive number. Then there exists a piecewise $C^1$-map $f_1 \in \mathcal{F}$ such that the following conditions are satisfied:

(1) $\varepsilon g_1 < f_1^*h_1 < g_1$ on the set of points where $f$ is differentiable;
(2) $f_1$ is arbitrarily close to $f_0$ in the fine $C^0$-topology.
Remark 5.4. Condition (1) in the above proposition implies that $f_1$ is strictly $g_1$-short and the induced metric $f_1^*h_1$ is sufficiently close to $g_1$ when $\varepsilon$ is close to 1.

Proof. Fix a locally finite open covering \{U_i\} of $M$ by coordinate neighbourhoods. Since the metrics $g_1 - f^*h_1$ and $g_2 - f^*h_2$ are related by the inequalities (5-1) we can get simultaneous decomposition of $g_1 - f^*h_1$ and $g_2 - f^*h_2$ as

$$\varepsilon(g_1 - f^*h_1) = \sum_i \phi_i^2 d\psi_i^2$$ and $$\varepsilon(g_2 - f^*h_2) = \sum_i \psi_i^2 d\phi_i^2$$

where $c_i$'s are constants which lie between $a$ and $b$, and $\phi_i$’s and $\psi_i$’s are smooth real valued functions. Further, for each $i$, the function $\phi_i$ has compact support contained in $U_i$ [D’Ambra and Datta 2002, Decomposition Lemma]. Let us define two sequences of Riemannian metrics \{g_1^i\} and \{g_2^i\} as

$$g_1^i = g_1^{i-1} + \phi_i^2 d\psi_i^2$$ and $$g_2^i = g_2^{i-1} + \psi_i^2 d\phi_i^2$$

where $g_1^0 = f^*h_1$ and $g_2^0 = f^*h_2$. Clearly, $g_1^1 < g_1$ and $g_2^1 < g_2$ for each $i$. Further, since $a^2 f^*h_1 < f^*h_2 < b^2 f^*h_1$ and $a < c_1 < b$ for each $i$, $a^2 g_1^i < g_2^1 < b^2 g_1^i$ for each $i$.

By applying the Main Lemma (Lemma 4.1) successively (with an appropriate choice of $\tilde{\phi}$ for each $i$) we obtain a sequence of piecewise $C^1$-maps such that

$$\tilde{f}_i^*h_{\alpha} \approx g_i^\alpha,$$

for $\alpha = 1, 2, i = 1, 2, \ldots$ and $\tilde{f}_i$ lies in a given neighbourhood of $f$ in the fine $C^0$-topology. Note that each $\tilde{f}_i$ satisfies conditions F2 and F4. Since supp $\phi_i \subset U_i$ for each $i$, where \{U_i\} is a locally finite open covering of $M$, the sequence $\tilde{f}_i$ is eventually constant near any point $x \in M$. Therefore the sequence converges to a piecewise $C^1$-map on $V$. Let

$$f_1 = \lim_{i \to \infty} \tilde{f}_i.$$

If $\tilde{f}_i^*h_{\alpha}$ are sufficiently close to $g_i^\alpha$ for $\alpha = 1, 2$ and for all $i$, then $f_1$ can be made to satisfy F2, F3 and F4. Further,

$$g_1 - f_1^*h_1 \approx g_1 - (f_1^*h_1 + \varepsilon(g_1 - f_1^*h_1)) = (1 - \varepsilon)(g_1 - f_1^*h_1) < (1 - \varepsilon)g_1.$$

Hence $f_1$ satisfies $\varepsilon g_1 < f_1^*h_1 < g_1$. \qed

6. Proof of the Main Theorem

We begin this section with some preliminaries on Lipschitz maps.

Definition 6.1. Let $(X, d)$ and $(Y, d')$ be two metric spaces and let $f : X \to Y$ be a continuous map. The map $f$ is said to be Lipschitz if there is a constant $K > 0$
such that \( d'(f(x), f(x')) < K d(x, x') \) for all \( x, x' \in X \). \( K \) is called the Lipschitz constant for \( f \).

A Riemannian metric \( g \) on a \( C^\infty \)-manifold \( M \) induces a canonical metric space structure on \( M \). If we denote this metric by \( d_g \), then the distance \( d_g(x, x') \) between two points \( x, x' \in M \) is defined to be the infimum of the lengths of all piecewise \( C^1 \)-paths in \( M \) joining \( x \) and \( x' \).

**Definition 6.2.** A continuous map \( f : (M, g) \to (N, h) \) from a Riemannian manifold \( (M, g) \) into another Riemannian manifold \( (N, h) \) will be called Lipschitz if it is a Lipschitz map relative to the metrics \( d_g \) and \( d_h \) on \( M \) and \( N \) respectively.

**Example 6.3.** A \( C^1 \)-isometric map \( f : (M, g) \to (N, h) \) between Riemannian manifolds is a Lipschitz map with a Lipschitz constant equal to 1. Hence, every \( g \)-short map is also a Lipschitz map.

A Riemannian metric \( g \) on a manifold \( M \) induces a canonical volume measure which we denote by \( \mu_g \). Measurability on \( (M, g) \) is therefore to be understood in terms of this \( \mu_g \). Observe that if \( g' \) is another Riemannian metric on \( M \) then a set \( A \) in \( M \) has measure zero relative to \( \mu_g \) if and only if it has measure zero relative to \( \mu_{g'} \).

We recall the following facts about Lipschitz maps between Riemannian manifolds from [Weaver 1999].

- Every Lipschitz map between Riemannian manifolds is almost everywhere differentiable, since a Lipschitz map \( f : \Omega \to \mathbb{R}^q \) defined on some open subset of \( \mathbb{R}^n \) is almost everywhere differentiable.
- The Lipschitz functions on a Riemannian manifold are precisely those which have bounded measurable exterior derivative \( df \).

**Definition 6.4.** A Lipschitz map \( f : (M, g) \to (N, h) \) from a Riemannian manifold \( (M, g) \) into another Riemannian manifold \( (N, h) \) will be called Lipschitz isometric if \( df_x : T_x M \to T_{f(x)} N \) is isometric for almost all \( x \in M \).

- If \( g_1 \) and \( g_2 \) are two Riemannian metrics on a manifold \( M \) satisfying \( a^2 g_1 < g_2 < b^2 g_1 \) then a map \( f : M \to \mathbb{R}^q \) is Lipschitz with respect to the pair \( (g_1, h_1) \) if and only if it is Lipschitz with respect to the pair \( (g_2, h_2) \), where \( h_1, h_2 \) are two linear metrics on \( \mathbb{R}^q \). Therefore, there is no ambiguity when we speak of almost everywhere differentiable Lipschitz maps in the context of Theorem 1.1.

**Proof of Theorem 1.1.** Since \((h_1, h_2)\)-regular immersions are generic for \( q \geq 3 \dim M \), it follows from Proposition 5.2 that there is a \((h_1, h_2)\)-regular immersion \( f_0 : M \to \mathbb{R}^q \) which satisfies the inequalities in (5-1).

Let \( \mathcal{R} \) denote the set of all 1-jets \((x, y, \alpha)\) which satisfy the following properties:
Let \( \alpha \) be a sequence of positive numbers with 0
such that
\[ a_2^2(1 - \alpha h_1) < g_2 - \alpha h_2 < b^2(1 - \alpha h_1); \]
\[ a_2^2 < \alpha h_2 < b^2 \alpha h_1. \]

For every \( \eta > 0 \) define relations \( \mathcal{R}_\eta \) by
\[ \mathcal{R}_\eta = \mathcal{R} \cap \{ (x, y, \alpha) : (1 - \eta)g_1 < \alpha h_1 < g_1 \}. \]

Let \( \mathcal{J} \) denote the isometry relation
\[ \mathcal{J} = \{ (x, y, \alpha) \in J^1(M, \mathbb{R}^q) : \alpha h_1 = g_1, \alpha h_2 = g_2 \}, \]
then:

- Each \( \mathcal{R}_\eta \) is an open relation.
- The fibres of \( \mathcal{J} \) over \( J^0(M, \mathbb{R}^q) \) are compact sets. Hence, the relations \( \mathcal{R}_\eta \) are
uniformly bounded over compact sets in \( M \).
- Let \( \eta_i \) be a sequence of positive numbers such that \( \eta_i \to 0 \). If \( \alpha_i \in \mathcal{R}_{\eta_i} \) and
\( \alpha_i \to \alpha \), then \( \alpha \in \mathcal{J} \). (Compare with [Gromov 1986, p. 218].)

Let \( \eta_i \) be a sequence of constants converging to zero and \( \delta_i \) be a sequence of
positive continuous functions on \( M \) such that the series \( \sum \delta_i \) converges pointwise
on \( M \). By applying Proposition 5.3 we obtain a sequence of piecewise \( C^1 \)-maps
\( f_i : M \to \mathbb{R}^q \) for \( i = 1, 2, \ldots \) such that \( f_i \) is a piecewise \( C^1 \)-solution of the
relation \( \mathcal{R}_{\eta_i} \) and the distance between \( f_i(x) \) and \( f_{i+1}(x) \) is less than \( \delta_i(x) \) for all
\( x \in M \). Thus the sequence \( \{f_i\} \) converges (in the \( C^0 \) compact open topology) to
a continuous function \( f \) on \( M \). Since \( f_i \) is a piecewise \( C^1 \)-solution of the relation
\( \mathcal{R}_{\eta_i} \), it is Lipschitz (relative to \( (g_1, h_1) \)) and the Lipschitz constants of \( f_i \) are
uniformly bounded. Hence the limit function \( f \) is also a Lipschitz map [Weaver 1999]. Consequently, \( f \) is almost everywhere differentiable and the \( L^\infty \) norm of
\( df \) is finite on any coordinate neighbourhood of \( M \).

We would further like to show that the sequence \( df_i, i = 1, 2, \ldots \), converges to
\( df \) in \( L^1(\Omega) \) for any compact coordinate neighbourhood \( \Omega \). Since \( L^1 \) convergence
of a sequence of functions guarantees the almost everywhere convergence of a
subsequence of the original sequence to \( df \), this would imply that \( f \) is a Lipschitz
solution of \( \mathcal{J} \) on all of \( M \) (by a property of \( \mathcal{R}_\eta \) discussed above).

However, to prove the desired \( L^1 \) convergence we need to choose the functions
\( \delta_i \) appropriately. First we fix a locally finite open covering of \( M \) by coordinate
neighbourhoods \( \{ \Omega_\alpha : \alpha = 1, 2, \ldots \} \). For our convenience we choose each \( \Omega_\alpha \) to
be compact. Suppose we have already constructed \( \delta_i \) and \( f_i \) for \( i = 1, 2, \ldots, k \).
Let \( \{ \varepsilon_\alpha \} \) be a sequence of positive numbers with 0 < \( \varepsilon_\alpha < 2^{-\alpha} \) such that
\[ \| df_i * \rho_{\varepsilon_\alpha} - df_i \|_{L^1(\Omega_\alpha)} \leq 2^{-\alpha}. \]
The functions $\rho_\varepsilon$ are defined as in [Müller and Šverák 2003] by $\rho_\varepsilon = \varepsilon^{-n} \rho(x/\varepsilon)$, where $\rho : \mathbb{R}^n \to \mathbb{R}$ is the mollifying kernel, that is, a smooth nonnegative function supported in the open unit disc in $\mathbb{R}^n$ with $\int \rho \, dx = 1$.

Observing that there exists a positive continuous function $\varepsilon$ on $M$ which is strictly less than $\varepsilon_\alpha$ on $\Omega_\alpha$ for each $\alpha = 1, 2, \ldots$, define

$$\delta_{i+1} = \varepsilon \delta_i.$$ 

Now we apply Proposition 5.3 to obtain a piecewise $C^1$-solution of $\mathcal{R}_{\delta_{i+1}}$ such that $|f_{i+1} - f_i| < \delta_{i+1}$. Proceeding this way we construct a sequence $\{f_i\}, i = 1, 2, \ldots$, which has all the desired property.

Now, arguing exactly as in [Müller and Šverák 2003, Theorem 3.2] we can prove that $df_i$ converges to the derivative map of $f$ in $L^1(\Omega_\alpha)$ for each $\alpha$. This completes the proof of the theorem. □

**Remark 6.5.** The proof of the main theorem begins with an immersion $f_0 : M \to \mathbb{R}^q$ satisfying the inequalities (5-1). If $\mathbb{R}^q$ is replaced by a general manifold $N$ then such maps are no longer guaranteed. This is the main obstruction to generalise the result for arbitrary manifold $N$ in the place of $\mathbb{R}^q$. However, assuming the existence of such maps we may possibly prove the existence of Lipschitz isometric maps for pairs of Riemannian metrics [Gromov 1986, 2.4.9 (A)].

### 7. One-dimensional case

In this section we discuss the one-dimensional case which is the motivation to the general problem.

Let $M = S^1$ be the unit circle and let $g_1 = d\theta^2$ be the canonical metric on $S^1$. Let $g_2 = c^2 g_1$. If $f : S^1 \to \mathbb{R}^q$ is a $C^1$-immersion such that $f^* h_1 = g_i$ for $i = 1, 2$ then

$$\| \frac{\partial f}{\partial \theta} \|_1 = 1 \quad \text{and} \quad \| \frac{\partial f}{\partial \theta} \|_2 = c,$$

where $\| \cdot \|_i$ denote the norms relative to the metric $h_i$ for $i = 1, 2$. In other words, $\frac{\partial f}{\partial \theta} \in A$, where $A$ is given by

$$A = \{ y = (y_1, \ldots, y_q) \in \mathbb{R}^q : \sum y_i^2 = 1 \quad \text{and} \quad \sum \lambda_i y_i^2 = c^2 \}.$$

**Lemma 7.1.** Let $h_1$ and $h_2$ be two inner products on $\mathbb{R}^q$ such that $h_1 - h_2$ is nondegenerate. Let $S_1$ and $S_2$ denote the unit spheres relative to the metrics $h_1$ and $h_2$ respectively. Then $S_1 \cap S_2$ has the same homotopy type as $S^{r_+ - 1} \times S^{r_- - 1}$, where $r_+$ and $r_-$ are respectively the positive and the negative ranks of $h_1 - h_2$. Consequently, if $r_+ \geq 2$ then $S_1 \cap S_2$ is connected. Further the interior of the convex hull of $S_1 \cap S_2$ contains the origin.
Proof. Let $h_1 - h_2$ be nondegenerate. Note that a nonzero vector $v$ satisfies

$$(h_1 - h_2)(v, v) = 0$$

if and only if $\lambda v$ satisfies the same equation for all $\lambda$. This means that the onedimensional subspace $\ell_v$ containing $v$ lies completely inside the solution space $C$ of $h_1 - h_2 = 0$. In other words, the solution space of this equation in $\mathbb{R}^q$ is a cone.

Now, if $h$ is an arbitrary positive definite quadratic form on $\mathbb{R}^q$, then $\ell_v$ intersects the unit sphere relative to $h$ in exactly two points. Thus we see that $S_1 \cap S_2$ has the same homotopy type as the space of nonzero solutions of the equation $h_1 - h_2 = 0$.

Choose basis vectors in $\mathbb{R}^q$ so that both $h_1$ and $h_2$ are in the diagonal form. The set $S_1 \cap S_2$ has the same homeomorphism type as the solution space of the system of equations

$$x_1^2 + \cdots + x_{r_+}^2 + y_1^2 + \cdots + y_{r_-}^2 = 1,$$

$$x_1^2 + \cdots + x_{r_+}^2 - y_1^2 - \cdots - y_{r_-}^2 = 0,$$

which is further equivalent to

$$x_1^2 + x_2^2 + \cdots + x_{r_+}^2 = \frac{1}{2},$$

$$y_1^2 + y_2^2 + \cdots + y_{r_-}^2 = \frac{1}{2}.$$ 

Therefore, $S_1 \cap S_2$ has the homeomorphism type of $S^{r+1} \times S^{r-1}$, which is $k$-connected for $k \leq \min(r_+ - 2, r_- - 2)$. Thus if $r_+ \geq 2$ then $S_1 \cap S_2$ is connected and nowhere flat. (Note that in the lowest admissible dimension the intersection is topologically equivalent to a torus embedded in $S^3$.) Also note that if $(\bar{x}_1, \ldots, \bar{x}_{r_+}, \bar{y}_1, \ldots, \bar{y}_{r_-}) \in S_1 \cap S_2$ then $(\pm \bar{x}_1, \ldots, \pm \bar{x}_{r_+}, \pm \bar{y}_1, \ldots, \pm \bar{y}_{r_-}) \in S_1 \cap S_2$, so that the convex hull of $S_1 \cap S_2$ has nonempty interior and $0$ belongs to the interior convex hull of $S_1 \cap S_2$. □

It follows from the above lemma that if $r_+(c^2h_1 - h_2) \geq 2$, then $A$ is connected and the interior of the convex hull of $A$ contains the origin. Thus, by Lemma 2.3 there exists a $C^1$-immersion $f : S^1 \to \mathbb{R}^q$ such that $f^*h_i = g_i$ for $i = 1, 2$ when $r_+(c^2h_1 - h_2) \geq 2$.

On the other hand there does not exist any such isometric immersion if $q \leq 3$ since it is observed in [Gromov 1986, 2.4.1(A) Example] that if $f : S^1 \to \mathbb{R}^q$ is a $C^1$-map whose derivative takes the unit circle $S^1$ into a (connected) subset $A$, then the convex hull of $A$ must contain the origin. Indeed, if $q = 3$ and $h_1 - h_2$ is a nondegenerate indefinite form, then $A$ is a disjoint union of two circles none of which contains the origin in its convex hull, thereby ruling out the existence of $C^1$-immersion with the desired isometry property.

We conclude the paper with a conjecture:
Conjecture. If \( r_\pm (c^2 h_1 - h_2) \geq 2n + 1 \) for all \( c \in [a, b] \), then it is possible to obtain a \( C^1 \)-solution of the general problem.

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