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We give a multimensional version of the p-adic heat equation, and show that its fundamental solution is the transition density of a Markov process.

1. Introduction

In recent years p-adic analysis has received a lot of attention due to its applications in mathematical physics; see for example [Albeverio and Karwowski 1991; 1994; Avetisov et al. 2002; Avetisov et al. 2003; Khrennikov 1994; 1997; Kochubeĭ 2001; Rammal et al. 1986; Vladimirov et al. 1994] and references therein. One motivation comes from statistical physics, in particular, in connection with models describing relaxation in glasses, macromolecules, and proteins. It has been proposed that the nonexponential nature of those relaxations is a consequence of a hierarchical structure of the state space which can in turn be connected to p-adic structures; see [Avetisov et al. 2002; Avetisov et al. 2003; Rammal et al. 1986]. In [Avetisov et al. 2002], it was demonstrated that p-adic analysis is a natural basis for the construction of a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes. To each of these models is associated a stochastic equation (the master equation). In several cases this equation is a p-adic parabolic equation of type

$$\left.\begin{aligned} \frac{\partial u(x,t)}{\partial t} + a(Au)(x,t) &= f(x,t) \\ u(x,0) &= \varphi(x) \end{aligned}\right\} \quad \text{for } x \in \mathbb{Q}_p^n \text{ and } t \in (0,T], \tag{1}$$

where a is a positive constant, A is pseudodifferential operator, and $\mathbb{Q}_p$ is the field of p-adic numbers. The simplest case occurs when $n = 1$ and A is the Vladimirov operator:

$$(D^{\alpha}\varphi)(x) = \mathcal{F}^{-1}_{\xi \to x}(|\xi|_p^{\alpha} \mathcal{F}_{x \to \xi}\varphi(x)) \quad \text{for } \alpha > 0,$$

where $\mathcal{F}$ is the Fourier transform. The fundamental solution of (1) is a density transition of a time- and space-homogeneous Markov process that is considered

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the p-adic counterpart of Brownian motion; see [Kochubeĭ 2001; Vladimirov et al. 1994].

It is relevant to mention that in the case $n = 1$, the fundamental solution of (1) when $A = D^\alpha$ (also called the p-adic heat kernel) has been studied extensively; see for example [Blair 1995; Haran 1990; 1993; Ismagilov 1991; Kochubeĭ 2001; Vladimirov et al. 1994].

A natural problem is to study the initial value problem (1) in the n-dimensional case. In [Zúñiga-Galindo 2008], the second author considered Cauchy's problem (1) when

$$(A\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}(|f(\xi)|_p^\alpha \mathcal{F}_{x\to\xi}\varphi(x)) \quad \text{for } \alpha > 0.$$

Here $f(\xi)$ is an elliptic homogeneous polynomial in n variables, and the datum φ is a locally constant and integrable function. Under these hypotheses, Zúñiga-Galindo established the existence of a unique solution to Cauchy's problem (1). In addition, the fundamental solution is a transition density of a Markov process with space state $\mathbb{Q}_p^n$.

In this paper we study Cauchy's problem (1) when A is the Taibleson pseudodifferential operator, which is defined through

$$(D_T^\beta\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}((\max_{1\le i\le n}|\xi_i|_p)^\beta \mathcal{F}_{x\to\xi}\varphi(x)) \quad \text{for } \beta > 0. \tag{2}$$

Recently Albeverio, Khrennikov, and Shelkovich [2006] studied D_T^β in the context of the Lizorkin spaces.

We prove existence and uniqueness of the Cauchy problem (1), (2) in spaces of increasing functions introduced in [Kochubeĭ 1991]; see Theorem 1. We also associate to the fundamental solution a transition density of a Markov process; see Theorem 2. These results constitute an extension of the corresponding results in [Kochubeĭ 2001; Vladimirov et al. 1994].

We want to mention here a relevant comment due to the referee. There exists a procedure, developed in [Kochubeĭ 2001] for elliptic equations, for reducing multidimensional problems over $\mathbb{Q}_p$ to one-dimensional problems over appropriate field extensions. In particular, the Taibleson operator is connected with the unramified extension of $\mathbb{Q}_p$ of degree n; see [Lemma 2.1]. The fundamental solutions corresponding to the multidimensional Cauchy problem and the problem over the unramified extension should be obtained from each other, up to a linear change of variables, as in the formula [(2.38)] for the elliptic case. Then many properties of the fundamental solution would follow directly from those known in the one-dimensional case. In this paper we use an elementary and independent method that has obvious advantages.

Let us explain the connection between the results of this paper and those of [Zúñiga-Galindo 2008]. There are infinitely many homogeneous polynomial functions satisfying

$$|f(\xi)|_p = (\max_{1\le i\le n}|\xi_i|_p)^d \quad \text{for any } \xi \in \mathbb{Q}_p^n,$$

where d denotes the degree of f (see Lemmas 14 and 15). Hence the pseudodifferential operators considered here are a subclass of the ones considered in that paper. However, the function spaces for the solutions and initial data are completely different. In this paper the initial datum and the solution to Cauchy problem (1), (2) are not necessarily bounded or integrable, whereas they are in the other paper.

Finally, our results can be extended to operators of the form

$$(A\varphi)(x) = a_0(x,t)(D_T^{\alpha}\varphi)(x) + \sum_{k=1}^{n} a_k(x,t)(D_T^{\alpha_k}\varphi)(x) + b(x,t)\varphi(x) \tag{3}$$

for $\alpha > 1$ and $0 < \alpha_1 < \cdots < \alpha_n < \alpha$, where $a_k(x,t)$ and $b(x,t)$ are bounded continuous functions. This is done using techniques presented in [Kochubeĭ 2001; 1991; 1988]. These results will appear later elsewhere.

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2. Preliminary results

As a general reference for p-adic analysis we refer the reader to [Taibleson 1975] and [Vladimirov et al. 1994]. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$, which is defined as follows: $|0|_p = 0$; if $x \in \mathbb{Q}^\times$ and $x = p^\gamma a/b$, where a and b are integers coprime to p, then $|x|_p = p^{-\gamma}$. The integer $\gamma = \gamma(x)$, called the *p-adic order of x*, will be denoted $\operatorname{ord}(x)$. We use the same symbol, $|\cdot|_p$, for the p-adic norm on $\mathbb{Q}_p$. We extend the p-adic norm to $\mathbb{Q}_p^n$ as follows:

$$\|x\|_p := \max_{1\le i\le n}|x_i|_p \quad \text{for } x = (x_1,\dots,x_n) \in \mathbb{Q}_p^n.$$

Note that $\|x\|_p = p^{-\min_{1\le i\le n}\{\operatorname{ord}(x_i)\}}$.

Any p-adic number $x \neq 0$ has a unique expansion $x = p^\gamma \sum_{j=0}^{\infty} x_j p^j$, where $\gamma = \operatorname{ord}(x) \in \mathbb{Z}$ and $x_j \in \{0, 1, \dots, p-1\}$. By using this expansion, we define the *fractional part of* $x \in \mathbb{Q}_p$, denoted as $\{x\}_p$, as the rational number

$$\{x\}_p := \begin{cases} 0 & \text{if } x = 0 \text{ or } \gamma \ge 0, \\ p^\gamma \sum_{j=0}^{|\gamma|-1} x_j p^j & \text{if } \gamma < 0. \end{cases}$$

For $\gamma \in \mathbb{Z}$, denote by $B_\gamma^n(a) = \{x \in \mathbb{Q}_p^n \mid \|x-a\|_p \le p^\gamma\}$ the ball of radius p^γ with center at $a = (a_1,\dots,a_n) \in \mathbb{Q}_p^n$, and let $B_\gamma^n := B_\gamma^n(0)$. Note that $B_\gamma^n(a) =$

$B_\gamma(a_1) \times \cdots \times B_\gamma(a_n)$, where $B_\gamma(a_j) = \{x_j \in \mathbb{Q}_p \mid |x_j - a_j|_p \le p^\gamma\}$ is the one-dimensional ball of radius p^γ with center at $a_j \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0(0) = \mathbb{Z}_p$, the ring of p-adic integers.

Let $d^n x$ be the Haar measure on $\mathbb{Q}_p^n$ normalized by the condition that the measure of B_0^n is 1.

A complex-valued function φ defined on $\mathbb{Q}_p^n$ is called *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that $\varphi(x + x') = \varphi(x)$ for $x' \in B_{l(x)}^n$.

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Schwartz–Bruhat function*, or a *test function*, if it is locally constant with compact support. The $\mathbb{C}$-vector space of Schwartz–Bruhat functions is denoted by $S(\mathbb{Q}_p^n)$. If $\varphi \in S(\mathbb{Q}_p^n)$, there exists an integer $l \ge 0$ such that $\varphi(x + x') = \varphi(x)$ for $x' \in B_{-l}^n$ and $x \in \mathbb{Q}_p^n$; see for example [Vladimirov et al. 1994, Lemma 1, page 79]. The largest of such numbers $l = l(\varphi)$ is called *the exponent of local constancy of* φ.

Let $S'(\mathbb{Q}_p^n)$ denote the set of all functionals (distributions) on $S(\mathbb{Q}_p^n)$. All the functionals on $S(\mathbb{Q}_p^n)$ are continuous; see for example [Vladimirov et al. 1994, Section VI.3].

Given $\xi = (\xi_1, \dots, \xi_n)$ and $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{i=1}^n \xi_i x_i$. The Fourier transform of $\varphi \in S(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \Psi(-\xi \cdot x)\varphi(\xi)\, d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where $\Psi(-\xi \cdot x) = \prod_{i=1}^n \Psi(-\xi_i x_i) = \exp(2\pi i \sum_{i=1}^n \{-\xi_i x_i\}_p)$. The function $\Psi(\alpha x_j) = \exp(2\pi i \sum_{i=1}^n \{\alpha x_j\}_p)$ is called *the standard additive character* of $\mathbb{Q}_p$. The Fourier transform is a linear isomorphism from $S(\mathbb{Q}_p^n)$ onto itself.

2.1. ***The Taibleson operator.*** We set

$$\Gamma_p^{(n)}(\alpha) := \frac{1 - p^{\alpha - n}}{1 - p^{-\alpha}} \quad \text{for } \alpha \ne 0.$$

This function is called the *p-adic gamma function*. The function

$$k_\alpha(x) = \frac{\|x\|_p^{\alpha - n}}{\Gamma_p^{(n)}(\alpha)} \quad \text{for } \alpha \in \mathbb{R} \setminus \{0, n\} \text{ and } x \in \mathbb{Q}_p^n,$$

is called *the multidimensional Riesz kernel*; it determines a distribution on $S(\mathbb{Q}_p^n)$ as follows. If $\alpha \ne 0, n$ and $\varphi \in S(\mathbb{Q}_p^n)$, then

$$\begin{aligned} (4)\quad \langle k_\alpha(x), \varphi(x)\rangle = \frac{1 - p^{-n}}{1 - p^{\alpha - n}}\varphi(0) + \frac{1 - p^{-\alpha}}{1 - p^{\alpha - n}} \int_{\|x\|_p > 1} \|x\|_p^{\alpha - n}\varphi(x)\, d^n x \\ + \frac{1 - p^{-\alpha}}{1 - p^{\alpha - n}} \int_{\|x\|_p \le 1} \|x\|_p^{\alpha - n}(\varphi(x) - \varphi(0))\, d^n x. \end{aligned}$$

Then $k_\alpha \in S'(\mathbb{Q}_p^n)$ for $\alpha \in \mathbb{R} \setminus \{0, n\}$. In the case $\alpha = 0$, by passing to the limit in (4), we obtain

$$\langle k_0(x), \varphi(x)\rangle := \lim_{\alpha\to 0}\langle k_\alpha(x), \varphi(x)\rangle = \varphi(0),$$

that is, $k_0(x) = \delta(x)$, the Dirac delta function, and then $k_\alpha \in S'(\mathbb{Q}_p^n)$ for $\alpha \in \mathbb{R}\setminus\{n\}$.

It follows from (4) that for $\alpha > 0$,

$$\langle k_{-\alpha}(x), \varphi(x)\rangle = \frac{1-p^\alpha}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \|x\|_p^{-\alpha-n}(\varphi(x)-\varphi(0))\, d^n x. \tag{5}$$

Lemma 1 [Taibleson 1975, Theorem III.4.5]. *As an element of* $S'(\mathbb{Q}_p^n)$, $(\mathcal{F}k_\alpha)(x)$ *equals* $\|x\|_p^{-\alpha}$ *for* $\alpha \neq n$.

Definition 1. The Taibleson pseudodifferential operator D_T^α for $\alpha > 0$ is defined as

$$(D_T^\alpha\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}(\|\xi\|_p^\alpha \mathcal{F}_{x\to\xi}\varphi) \quad \text{for } \varphi \in S(\mathbb{Q}_p^n).$$

As a consequence of the previous lemma and (5), we have

$$(D_T^\alpha\varphi)(x) = (k_{-\alpha} * \varphi)(x) = \frac{1-p^\alpha}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\alpha-n}(\varphi(x-y)-\varphi(x))\, d^n y. \tag{6}$$

The right side of (6) makes sense for a wider class of functions, for example, for locally constant functions $\varphi(x)$ satisfying

$$\int_{\|x\|_p\geq 1} \|x\|_p^{-\alpha-n}|\varphi(x)|\, d^n x < \infty.$$

3. The p-adic heat equation and the Taibleson operator

In this paper we consider the Cauchy problem

$$\left.\begin{aligned} \frac{\partial u(x,t)}{\partial t} + a(D_T^\alpha u)(x,t) &= f(x,t) \\ u(x,0) &= \varphi(x) \end{aligned}\right\} \quad \text{for } x \in \mathbb{Q}_p^n \text{ and } t \in (0, T], \tag{7}$$

where $a > 0$, $\alpha > 0$, and D_T^α is the Taibleson operator. In this section we show that (7) is a multidimensional analogue of the p-adic heat equation introduced in [Vladimirov et al. 1994].

3.1. ***The fundamental solution.*** The *fundamental solution* for the Cauchy problem (7) is defined as

$$Z(x,t) := \int_{\mathbb{Q}_p^n} \Psi(x\cdot\xi) e^{-at\|\xi\|_p^\alpha}\, d^n\xi. \tag{8}$$

Lemma 2. *The fundamental solution has the properties*

(i) $Z(x,t) = (1-p^{-n})\|x\|_p^{-n}\sum_{k=0}^\infty q^{-kn} e^{-at(q^{-k}\|x\|_p^{-1})^\alpha} - \|x\|_p^{-n} e^{-at(p\|x\|_p^{-1})^\alpha}$;

(ii) $Z(x,t)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\,\frac{1-p^{\alpha m}}{1-p^{-\alpha m-n}}\,(at)^m\|x\|_p^{-\alpha m-n}$ *for* $x\neq 0$;

(iii) $Z(x,t)\geq 0$ *for all* $x\in\mathbb{Q}_p^n$ *and* $t\in(0,T]$.

Proof. (i) By expanding $Z(x,t)$ as

$$Z(x,t)=\sum_{k=-\infty}^{\infty}\int_{\|\xi\|_p=p^k}\Psi(x\cdot\xi)e^{-at\|\xi\|_p^{\alpha}}\,d^n\xi,$$

and applying

$$\int_{\|\xi\|_p=p^k}\Psi(x\cdot\xi)\,d^n\xi=\begin{cases}p^{kn}(1-p^{-n}) & \text{if } \|x\|_p\leq p^{-k},\\ -p^{kn}p^{-n} & \text{if } \|x\|_p=p^{-k+1},\\ 0 & \text{if } \|x\|_p>p^{-k+1},\end{cases}$$

(see [Taibleson 1975, Lemma III.4.1]), we obtain

$$Z(x,t)=(1-p^{-n})\|x\|_p^{-n}\sum_{k=0}^{\infty}p^{-kn}e^{-at(p^{-k}\|x\|_p^{-1})^{\alpha}}-\|x\|_p^{-n}e^{-at(p\|x\|_p^{-1})^{\alpha}}. \tag{9}$$

Note that by the previous expansion $Z(x,t)$ is a real-valued function.

(ii) By using the Taylor expansion of e^x in (9), and by exchanging the order of summation and summing the geometric progression, we find that

$$Z(x,t)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\,\frac{1-p^{\alpha m}}{1-p^{-\alpha m-n}}\,(at)^m\|x\|_p^{-\alpha m-n}\quad\text{for } x\neq 0.$$

(iii) Let $\Omega_l(x)$ denote the characteristic function of the ball $B_{-l}^n(0)$. Then $\mathcal{F}\Omega_l=p^{-nl}\Omega_{-l}$. The last part follows from this observation by means of the calculation

$$\begin{aligned}Z(x,t)&=\sum_{l=-\infty}^{\infty}e^{-atp^{l\alpha}}\int_{\|\xi\|_p=p^l}\Psi(x\cdot\xi)\,d^n\xi\\&=\sum_{l=-\infty}^{\infty}e^{-atp^{l\alpha}}(p^{n(l)}\Omega_{-l}(x)-p^{n(l-1)}\Omega_{-l+1}(x))\\&=\sum_{l=-\infty}^{\infty}p^{nl}(e^{-atp^{l\alpha}}-e^{-atp^{(l+1)\alpha}})\Omega_{-l}(x)\geq 0.\end{aligned}$$ □

Lemma 3. $Z(x,t)\leq Ct(t^{1/\alpha}+\|x\|_p)^{-\alpha-n}$ *for* $t>0$ *and* $x\in\mathbb{Q}_p^n$.

Proof. Let l an integer such that $p^{l-1} \le t^{1/\alpha} \le p^l$. Then

$$Z(x,t) \le \int_{\mathbb{Q}_p^n} e^{-at\|\xi\|_p^\alpha}\, d^n\xi \le \int_{\mathbb{Q}_p^n} e^{-ap^{\alpha(l-1)}\|\xi\|_p^\alpha}\, d^n\xi = \int_{\mathbb{Q}_p^n} e^{-a\|p^{-(l-1)}\xi\|_p^\alpha}\, d^n\xi \tag{10}$$

$$= p^{-(l-1)n} \int_{\mathbb{Q}_p^n} e^{-a\|\eta\|_p^\alpha}\, d^n\eta = C_0(\alpha) p^{-n} p^{-ln} \le C_1 t^{-n/\alpha}.$$

On the other hand, if $\|x\|_p \ge t^{1/\alpha}$, by applying Lemma 2(ii), we have

$$Z(x,t) \le \|x\|_p^{-n} \sum_{m=1}^{\infty} \frac{C_2^m}{m!} (t\|x\|_p^{-\alpha})^m \le C_3 t \|x\|_p^{-\alpha-n}. \tag{11}$$

These inequalities imply the result as follows. If $\|x\|_p \ge t^{1/\alpha}$, by (11),

$$Z(x,t) \le C_3 t \|x\|_p^{-\alpha-n} \le 2^{\alpha+n} C_3 t (t^{1/\alpha} + \|x\|_p)^{-\alpha-n}.$$

If $\|x\|_p < t^{1/\alpha}$, by (10),

$$Z(x,t) \le C_1 t^{-n/\alpha} \le 2^{\alpha+n} C_1 t (t^{1/\alpha} + \|x\|_p)^{-\alpha-n}. \qquad \square$$

Lemma 3 shows in particular that the function $Z(x,t)$ belongs, with respect to x, to $L_1(\mathbb{Q}_p^n) \cap L_2(\mathbb{Q}_p^n)$.

Corollary 1. $\displaystyle\int_{\mathbb{Q}_p^n} Z(x,t)\, d^n x = 1.$

3.2. *The spaces $\mathfrak{M}_\lambda$ and pseudodifferentiability of the fundamental solution.*

Definition 2. Denote by $\mathfrak{M}_\lambda$ for $\lambda > 0$ the set of complex-valued locally constant functions $\varphi(x)$ on $\mathbb{Q}_p^n$ such that $|\varphi(x)| \le C(\varphi)(1+\|x\|_p^\lambda)$. If the function φ depends also on a parameter t, we shall say that $\varphi \in \mathfrak{M}_\lambda$ *uniformly with respect to* t if its constant C and its exponent of local constancy $l(\varphi)$ do not depend on t.

Lemma 4. *If $\varphi \in \mathfrak{M}_\lambda$ for $\lambda < \alpha$, with α as in* (7), *then*

$$\lim_{t\to 0^+} \int_{\mathbb{Q}_p^n} Z(x-\xi,t)\varphi(\xi)\, d^n\xi = \varphi(x). \tag{12}$$

Proof. By Corollary 1, Lemma 2(iii), and Lemma 3, we have

$$\Big|\int_{\mathbb{Q}_p^n} Z(x-\xi,t)\varphi(\xi)\, d^n\xi - \varphi(x)\Big| = \Big|\int_{\mathbb{Q}_p^n} Z(x-\xi,t)(\varphi(\xi)-\varphi(x))\, d^n\xi\Big|$$

$$\le \int_{\mathbb{Q}_p^n} Z(x-\xi,t)|\varphi(\xi)-\varphi(x)|\, d^n\xi$$

$$\le C \int_{\mathbb{Q}_p^n} t(t^{1/\alpha} + \|x-\xi\|_p)^{-\alpha-n} |\varphi(\xi)-\varphi(x)|\, d^n\xi =: I(x,t).$$

Let η be the exponent of locally constancy of φ. Since $\varphi \in \mathfrak{M}_\lambda$ for $\lambda < \alpha$, we can rewrite $I(x,t)$ as

$$\begin{aligned} I(x,t) &= C \int_{\|\xi - x\|_p > p^\eta} t(t^{1/\alpha} + \|\xi - x\|_p)^{-\alpha-n} |\varphi(\xi) - \varphi(x)| \, d^n\xi \\ &\le I_1(x,t) + I_2(x,t), \end{aligned}$$

with

$$\begin{aligned} I_1(x,t) &:= C_1 t \int_{\|\xi - x\|_p > p^\eta} \frac{1 + \|\xi\|_p^\lambda}{(t^{1/\alpha} + \|x - \xi\|_p)^{\alpha+n}} \, d^n\xi, \\ I_2(x,t) &:= C t |\varphi(x)| \int_{\|\xi - x\|_p > p^\eta} (t^{1/\alpha} + \|\xi - x\|_p)^{-\alpha-n} \, d^n\xi. \end{aligned}$$

Now, since $\alpha > 0$ and $t > 0$, we have $I_2(x,t) \le C_2 t|\varphi(x)|$, and since $\lambda < \alpha$,

$$\begin{aligned} I_1(x,t) &\le C_1 t \Big(C_3 + \int_{\|\tau\|_p > p^\eta} \frac{\|x - \tau\|_p^\lambda}{\|\tau\|_p{}^{\alpha+n}} \, d^n\xi \Big) \\ &\le C_1 t \Big(C_3 + \int_{p^\eta < \|\tau\|_p^\eta \le \|x\|_p} \frac{\|x - \tau\|_p^\lambda}{\|\tau\|_p{}^{\alpha+n}} \, d^n\xi + \int_{\|\tau\|_p > \|x\|_p} \frac{\|x - \tau\|_p^\lambda}{\|\tau\|_p{}^{\alpha+n}} \, d^n\xi \Big) \\ &= C_1 t \Big(C_4(x) + \int_{\|\tau\|_p > \|x\|_p} \frac{1}{\|\tau\|_p{}^{\alpha-\lambda+n}} \, d^n\xi \Big) = C_5(x) t. \end{aligned}$$

Therefore

$$\lim_{t \to 0^+} \left| \int_{\mathbb{Q}_p^n} Z(x - \xi, t) \varphi(\xi) \, d^n\xi - \varphi(x) \right| \le \lim_{t \to 0^+} C_6(x) t = 0. \qquad \square$$

For reference, we summarize the properties of the fundamental solution:

Proposition 1. *The fundamental solution has the properties*

(i) $Z(x,t) \ge 0$ *for all* $x \in \mathbb{Q}_p^n$ *and* $t \in (0,T]$;

(ii) $\int_{\mathbb{Q}_p^n} Z(x,t) \, d^n x = 1$ *for any* $t > 0$;

(iii) *if* $\varphi \in S(\mathbb{Q}_p^n)$, *then* $\lim_{(x,t)\to(x_0,0)} \int_{\mathbb{Q}_p^n} Z(x - \eta, t)\varphi(\eta) d^n\eta = \varphi(x_0)$;

(iv) $Z(x, t + t') = \int_{\mathbb{Q}_p^n} Z(x - y, t) Z(y, t') d^n y$ *for* $t, t' > 0$.

Proof. (i), (ii), and (iii) are already established (see Lemma 2(iii), Corollary 1, and Lemma 4). The last assertion is proved as follows: since $e^{-at\|\xi\|_p^\alpha} \in L^1(\mathbb{Q}_p^n)$,

$$\begin{aligned} \int_{\mathbb{Q}_p^n} Z(x - y, t_1) Z(y, t_2) \, d^n y &= \mathcal{F}^{-1}(\mathcal{F}(Z(y, t_1) * Z(y, t_2))) \\ &= \mathcal{F}^{-1}(e^{-at_1\|\xi\|_p^\alpha} e^{-at_2\|\xi\|_p^\alpha}) \\ &= Z(x, t_1 + t_2). \end{aligned} \qquad \square$$

Proposition 2. *If* $b > 0,\ 0 \leq \lambda < \alpha$, *and* $x \in \mathbb{Q}_p^n$, *then*

$$I(b, x) = \int_{\mathbb{Q}_p^n} (b + \|x - \xi\|_p)^{-\alpha-n} \|\xi\|_p^{\lambda}\, d^n\xi \leq Cb^{-\alpha}(1 + \|x\|_p^{\lambda}),$$

where the constant C *does not depend on* b *or* x.

Proof. Let m be an integer such that $p^{m-1} \leq b \leq p^m$. Then

$$(b + \|x - \xi\|_p)^{-\alpha-n} \leq (p^{m-1} + \|x - \xi\|_p)^{-\alpha-n},$$

and

$$\begin{aligned} I(b, x) \leq I(p^{m-1}, x) &= \int_{\mathbb{Q}_p^n} (p^{m-1}|_p + \|x - \xi\|_p)^{-\alpha-n} \|\xi\|_p^{\lambda}\, d^n\xi \\ &= p^{(m-1)(-\alpha-n)} \int_{\mathbb{Q}_p^n} (1 + \|p^{m-1}x - p^{m-1}\xi\|_p)^{-\alpha-n} \|\xi\|_p^{\lambda}\, d^n\xi \\ &= p^{(m-1)(\lambda-\alpha)} \int_{\mathbb{Q}_p^n} (1 + \|p^{m-1}x - \eta\|_p)^{-\alpha-n} \|\eta\|_p^{\lambda}\, d^n\eta \\ &= p^{(m-1)(\lambda-\alpha)} I(1, p^{m-1}x). \end{aligned}$$

Let $p^{m-1}x = y$ and $\|y\|_p = p^l$. We have $I(1, y) = I_1(y) + I_2(y) + I_3(y)$, where

$$\begin{aligned} I_1(y) &= \sum_{k=-\infty}^{l-1} \int_{\|\eta\|_p = p^k} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^{\lambda}\, d^n\eta, \\ I_2(y) &= \int_{\|\eta\|_p = p^l} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^{\lambda}\, d^n\eta, \\ I_3(y) &= \sum_{k=l+1}^{\infty} \int_{\|\eta\|_p = p^k} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^{\lambda}\, d^n\eta. \end{aligned}$$

The result follows from the following estimates:

Claim A. $I_1(y) \leq C_0(1 + \|y\|_p)^{-\alpha-n} \|y\|_p^{\lambda+n}$.

Claim B. $I_2(y) \leq C_1 \|y\|_p^{\lambda}$.

Claim C. $I_2(y) \leq C_2$.

Indeed, from the claims we have $I(1, y) \leq C_3(1 + \|y\|_p^{\lambda})$, and by (13),

$$\begin{aligned} I(b, x) &\leq C_3 p^{(m-1)(\lambda-\alpha)}(1 + p^{(1-m)\lambda}\|x\|_p^{\lambda}) \\ &\leq C_3 p^{-m\alpha}(1 + \|x\|_p^{\lambda}) \leq Cb^{-\alpha}(1 + \|x\|_p^{\lambda}). \end{aligned}$$

Proof of Claim A.

$$I_1(y) = \sum_{k=-\infty}^{l-1} \int_{\|\eta\|_p = p^k} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^{\lambda} \, d^n\eta$$

$$= (1 - p^{-n})(1 + \|y\|_p)^{-\alpha-n} \sum_{k=-\infty}^{l-1} p^{(\lambda+n)k} \leq C_0(1 + \|y\|_p)^{-\alpha-n} \|y\|_p^{\lambda+n},$$

where $C_0 = (1 - p^{-n})p^{-\lambda-n}/1 - p^{-\lambda-n}$. □

Proof of Claim B. Let $\widetilde{y} \in \mathbb{Q}_p$ such that $|\widetilde{y}|_p = p^l = \|y\|_p$. Then

$$I_2(y) = \int_{\|\eta\|_p = p^l} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^{\lambda} \, d^n\eta$$

$$= \|y\|_p^{\lambda} \int_{\|\eta\|_p = p^l} (1 + |\widetilde{y}|_p \|\widetilde{y}^{-1} y - \widetilde{y}^{-1}\eta\|_p)^{-\alpha-n} \, d^n\eta$$

$$= \|y\|_p^{\lambda-\alpha} \int_{\|\eta\|_p = 1} (\|y\|_p^{-1} + \|u - \eta\|_p)^{-\alpha-n} \, d^n\eta, \quad \text{with } u = \widetilde{y}^{-1} y.$$

We set $A_m = \{\eta \in \mathbb{Q}_p^n \mid \|\eta\|_p = 1 \text{ and } \|u - \eta\|_p = p^{-m}\}$ for $m \in \mathbb{N}$, and for I a nonempty subset of $\{1, 2, \ldots, n\}$, we have

$$A_{m,I} = \{\eta \in A_m \mid |u_i - \eta_i|_p = p^{-m} \text{ if } i \in I \text{ and } |u_i - \eta_i|_p < p^{-m} \text{ if } i \notin I\},$$

where $u = (u_1, \ldots, u_n)$ and $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{Q}_p^n$, with $\|\eta\|_p = \|u\|_p = 1$.

With this notation we have $A_m \subseteq \bigcup_I A_{m,I}$ and

$$\operatorname{vol}(A_{m,I}) \leq (p^{-m}(1 - p^{-1}))^{|I|}(p^{-m-1})^{n-|I|},$$

where $|I|$ denotes the cardinality of I. Then

$$\operatorname{vol}(A_m) \leq \sum_{|I|=0}^{n} \binom{n}{|I|} (p^{-m}(1 - p^{-1}))^{|I|}(p^{-m-1})^{n-|I|} = p^{-mn},$$

and

$$I_2(y) = \|y\|_p^{\lambda-\alpha} \sum_{m=0}^{\infty} \int_{A_m} (\|y\|_p^{-1} + \|u - \eta\|_p)^{-\alpha-n} \, d^n\eta$$

$$\leq \|y\|_p^{\lambda-\alpha} \sum_{m=0}^{\infty} (\|y\|_p^{-1} + p^{-m})^{-\alpha-n} p^{-mn}$$

$$= \frac{\|y\|_p^{\lambda-\alpha}}{1 - p^{-n}} \sum_{m=0}^{\infty} \int_{\|\eta\|_p = p^{-m}} (\|y\|_p^{-1} + \|\eta\|_p)^{-\alpha-n} \, d^n\eta$$

$$
\begin{aligned}
&= \frac{\|y\|_p^{\lambda-\alpha}}{1-p^{-n}} \int_{\|\eta\|_p\le 1} (\|y\|_p^{-1} + \|\eta\|_p)^{-\alpha-n}\, d^n\eta \\
&\le C_1' \|y\|_p^{\lambda-\alpha} \int_{\mathbb{Q}_p^n} (\|y\|_p^{-1} + \|\eta\|_p)^{-\alpha-n}\, d^n\eta \\
&= C_1' \|y\|_p^{\lambda-\alpha} \int_{\mathbb{Q}_p^n} (\|y\|_p^{-1} + \|y\|_p^{-1}\|\widetilde{y}\eta\|_p)^{-\alpha-n}\, d^n\eta \\
&= C_1' \|y\|_p^{\lambda+n} \int_{\mathbb{Q}_p^n} (1 + \|\widetilde{y}\eta\|_p)^{-\alpha-n}\, d^n\eta \\
&= C_1' \|y\|_p^{\lambda} \int_{\mathbb{Q}_p^n} (1 + \|\tau\|_p)^{-\alpha-n}\, d^n\tau \le C_1 \|y\|_p^{\gamma}. \qquad \square
\end{aligned}
$$

Proof of Claim C.

$$
\begin{aligned}
I_3(y) &= \sum_{k=l+1}^{\infty} \int_{\|\eta\|_p = p^k} (1 + \|\eta\|_p)^{-\alpha-n} \|\eta\|_p^{\lambda}\, d^n\eta \\
&\le \int_{\mathbb{Q}_p^n} (1 + \|\eta\|_p)^{-\alpha-n} \|\eta\|_p^{\lambda}\, d^n\eta = C. \qquad \square
\end{aligned}
$$

This completes the proof of Proposition 2. □

Lemma 5. *If* $\alpha > 0$, *then*

$$
\|x\|_p^{\alpha} = \frac{1}{\Gamma_p^{(n)}(-\alpha)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\alpha-n} (\Psi(-x\cdot y) - 1)\, d^n y \quad \textit{for all } x \in \mathbb{Q}_p^n. \tag{13}
$$

Proof. This is a slight change of the proof of [Kochubeĭ 2001, Proposition 2.3]. □

Lemma 6. *If* $0 < \gamma \le \alpha$, *then*

$$
(D_T^{\gamma} Z)(x,t) = \int_{\mathbb{Q}_p^n} \Psi(x\cdot\eta) \|\eta\|_p^{\gamma} e^{-at\|\eta\|_p^{\alpha}}\, d^n\eta.
$$

Proof. By Lemma 2(ii), $Z(x-y,t) = Z(x,t)$ for $\|y\| < \|x\|$. Then we can use (6) to calculate $(D_T^{\gamma} Z)(x,t)$:

$$
\begin{aligned}
(D_T^{\gamma} Z)(x,t) &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\gamma-n} (Z(x-y,t) - Z(x,t))\, d^n y \\
&= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p \ge \|x\|_p} \|y\|_p^{-\gamma-n} (Z(x-y,t) - Z(x,t))\, d^n y.
\end{aligned}
$$

We now use Lemma 3 to obtain

$$|(D_T^{\gamma} Z)(x,t)| \leq \left|\frac{1}{\Gamma_p^{(n)}(-\gamma)}\right| \int_{\|y\|_p \geq \|x\|_p} (Ct\|y\|_p^{-\gamma-\alpha-2n} + Z(x,t)\|y\|_p^{-\gamma-n})\, d^n y$$
$$< \infty. \tag{14}$$

This shows that $(D_T^{\gamma} Z)(x,t)$ exists. We now compute this function explicitly.

We set

$$Z^{(m)}(x,t) := \int_{\|\xi\|_p \leq p^m} \Psi(x \cdot \xi) e^{-at\|\xi\|_p^{\alpha}}\, d^n \xi.$$

Then $Z^{(m)}(x,t)$ is bounded and locally constant as function of x, and the exponent of local constancy is m. From these observations and by using Lemma 2(ii) and (6), we calculate

$$\begin{aligned}
(D_T^{\gamma} Z^{(m)})(x,t) &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\gamma-n} (Z^{(m)}(x-y,t) - Z^{(m)}(x,t))\, d^n y \\
&= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-m}} \|y\|_p^{-\gamma-n} (Z^{(m)}(x-y,t) - Z^{(m)}(x,t))\, d^n y \\
&= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-m}} \|y\|_p^{-\gamma-n} \int_{\|\eta\|_p \leq p^m} e^{-at\|\eta\|_p^{\alpha}} \Psi(x \cdot \eta)(\Psi(-y \cdot \eta) - 1)\, d^n \eta\, d^n y \\
&= \int_{\|\eta\|_p \leq p^m} e^{-at\|\eta\|^{\alpha}} \Psi(x \cdot \eta) \\
&\qquad \times \left(\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-m}} \|y\|_p^{-\gamma-n} (\Psi(-y \cdot \eta) - 1) d^n y \right) d^n \eta.
\end{aligned}$$

Note that if $\|y\|_p \leq p^{-m}$, then $\Psi(-y \cdot \eta) = 1$ for all η such that $\|\eta\|_p \leq p^m$. Using this observation and Lemma 5, $(D_T^{\gamma} Z^{(m)})(x,t)$ becomes

$$\begin{aligned}
\int_{\|\eta\|_p \leq p^m} e^{-at\|\eta\|_p^{\alpha}} \Psi(x \cdot \eta) \left(\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\gamma-n} (\Psi(-y \cdot \eta) - 1)\, d^n y \right) d^n \eta \\
= \int_{\|\eta\|_p \leq p^m} e^{-at\|\eta\|^{\alpha}} \Psi(x \cdot \eta) \|\eta\|_p^{\gamma}\, d^n \eta.
\end{aligned}$$

By the dominated convergence theorem and (14) we have

$$(D_T^{\gamma} Z)(x,t) = \int_{\mathbb{Q}_p^n} e^{-at\|\eta\|_p^{\alpha}} \Psi(x \cdot \eta) \|\eta\|_p^{\gamma}\, d^n \eta. \qquad \square$$

Lemma 7. *We have*

$$\frac{\partial Z}{\partial t}(x,t) = -a \int_{\mathbb{Q}_p^n} \Psi(x \cdot \xi) \|\xi\|_p^{\alpha} e^{-at\|\xi\|_p^{\alpha}}\, d^n \xi,$$

$$\frac{\partial Z}{\partial t}(x,t) = -a (D_T^{\gamma} Z)(x,t) \quad \textit{for } 0 < \gamma \leq \alpha.$$

Proof. The first part follows by applying the dominated convergence theorem. The second part follows from the first one by Lemma 6. □

Lemma 8. *We have*

$$\left|\frac{\partial Z}{\partial t}(x,t)\right| \le C(t^{1/\alpha}+\|x\|_p)^{-\alpha-n},$$
$$|(D_T^{\gamma} Z)(x,t)| \le C(t^{1/\alpha}+\|x\|_p)^{-\gamma-n}.$$

Proof. The proof uses the same reasoning as the proof of Lemma 3. □

Corollary 2. $\displaystyle\int_{\mathbb{Q}_p^n} (D_T^{\gamma} Z)(x,t)\, d^n x = 0.$

3.3. *The Cauchy problem for the multidimensional p-adic heat equation.*

Theorem 1. *Let* $\varphi(x),\ f(x,t) \in \mathfrak{M}_\lambda$, $0 \le \lambda < \alpha$ *be continuous functions. Then the Cauchy problem*

$$\left.\begin{aligned} \frac{\partial u(x,t)}{\partial t} + a(D_T^{\alpha} u)(x,t) &= f(x,t) \\ u(x,0) &= \varphi(x) \end{aligned}\right\} \quad \text{for } x \in \mathbb{Q}_p^n \text{ and } t \in (0,T], \tag{15}$$

with $a > 0$ *and* $\alpha > 0$, *has a continuous solution in* $\mathfrak{M}_\lambda$ *given by*

$$u(x,t) = \int_{\mathbb{Q}_p^n} Z(x-\xi,t)\varphi(\xi)\, d^n\xi + \int_0^t \left(\int_{\mathbb{Q}_p^n} Z(x-\xi,t-\tau) f(\xi,\tau)\, d^n\xi\right) d\tau. \tag{16}$$

Let $u_1(x,t)$ be the first summand in (16), and let $u_2(x,t)$ be the second.

The theorem will proved through the following lemmas.

Lemma 9. *The solution* $u(x,t)$ *belongs to* $\mathfrak{M}_\lambda$ *uniformly with respect to* t *and satisfies the initial conditions of Theorem 1.*

Proof. We first show that $u_1(x,t) \in \mathfrak{M}_\lambda$ uniformly with respect to t. Since φ is locally constant, there exists an $l \in \mathbb{N}$ such that $\varphi(\xi+y) = \varphi(\xi)$ for any $\|y\|_p \le p^{-l}$. By the change of variables $y - \xi = -\eta$ in $u_1(x,t)$, we have that $u_1(x,t)$ is locally constant. Now using Lemma 3 and Proposition 2, we have $|u_1(x,t)| \le C(1+\|x\|)^{\lambda}$, and thus $u_1(x,t) \in \mathfrak{M}_\lambda$ uniformly with respect to t.

By similar reasoning, one shows that $u_2(x,t)$ is locally constant in x and that $|u_2(x,t)| \le CT(1+\|x\|)^{\lambda}$. Therefore $u(x,t) = u_1(x,t)+u_2(x,t) \in \mathfrak{M}_\lambda$ uniformly with respect to t.

We now show $\lim_{t\to 0^+} u(x,t) = \varphi(x)$. By Lemma 4, $\lim_{t\to 0^+} u_1(x,t) = \varphi(x)$, and $\lim_{t\to 0^+} u_2(x,t) = 0$, since $|u_2(x,t)| \le Ct(1+\|x\|)^{\lambda}$ for $t \le T$. □

We now compute the partial derivatives of $u_1(x,t)$, $u_2(x,t)$ with respect to t.

Lemma 10. $\displaystyle\frac{\partial u_1}{\partial t}(x,t) = \int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x-\xi,t)\varphi(\xi)\, d^n\xi.$

Proof. The result follows by the dominated convergence theorem. □

Lemma 11.

$$\frac{\partial u_2}{\partial t}(x,t) = \int_0^t \Big(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x-\xi, t-\tau)(f(\xi,\tau) - f(x,\tau))\, d^n\xi \Big)\, d\tau + f(x,t).$$

Proof. Let

$$u_{2,h}(x,t) := \int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} Z(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau,$$

where h is a small positive number. Then $(u_h(x,t+t') - u_h(x,t))/t'$ equals

$$(17) \quad \int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} \frac{Z(x-\xi, t+t'-\tau) - Z(x-\xi, t-\tau)}{t'} f(\xi,\tau)\, d^n\xi \Big)\, d\tau$$
$$+ \int_{t-h}^{t-h+t'} \Big(\int_{\mathbb{Q}_p^n} \frac{Z(x-\xi, t+t'-\tau) - Z(x-\xi, t-\tau)}{t'} f(\xi,\tau)\, d^n\xi \Big)\, d\tau$$
$$+ \frac{1}{t'} \int_{t-h}^{t-h+t'} \Big(\int_{\mathbb{Q}_p^n} Z(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau.$$

By taking $t' \to 0^+$, the first integral in (17) tends to

$$\int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau.$$

By using the continuity of the functions

$$\int_{\mathbb{Q}_p^n} (Z(x-\xi, t+t'-\tau) - Z(x-\xi, t-\tau)) f(\xi,\tau)\, d^n\xi,$$
$$\int_{\mathbb{Q}_p^n} Z(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi$$

with respect to τ, the second integral tends to zero. The third integral tends to $\int_{\mathbb{Q}_p^n} Z(x-\xi, h) f(\xi, t-h)\, d^n\xi$. Hence

$$\frac{\partial u_{2,h}}{\partial t}(x,t) = \int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau + \int_{\mathbb{Q}_p^n} Z(x-\xi, h) f(\xi, t-h)\, d^n\xi.$$

This expression can be rewritten as

$$\begin{aligned}(18)\qquad \frac{\partial u_{2,h}}{\partial t}(x,t) &= \int_0^{t-h}\Big(\int_{\mathbb{Q}_p^n}\frac{\partial Z}{\partial t}(x-\xi,t-\tau)(f(\xi,\tau)-f(x,\tau))\,d^n\xi\Big)d\tau\\ &\quad+\int_0^{t-h}\Big(\int_{\mathbb{Q}_p^n}\frac{\partial Z}{\partial t}(x-\xi,t-\tau)f(x,\tau)\,d^n\xi\Big)d\tau\\ &\quad+\int_{\mathbb{Q}_p^n} Z(x-\xi,h)(f(\xi,t-h)-f(\xi,t))\,d^n\xi\\ &\quad+\int_{\mathbb{Q}_p^n} Z(x-\xi,h)f(\xi,t)\,d^n\xi.\end{aligned}$$

The first integral contains no singularity at $t=\tau$ due to Lemma 8 and the local constancy of f. By Corollary 2, the second integral in (18) is equal to zero. The third can be written as the sum of the integrals over $\{\xi\in\mathbb{Q}_p^n \mid \|\xi\|_p\le p^m\}$ and its complement; one integral is estimated using the uniform continuity of f, while the other contains no singularity. Hence this integral tends to zero as h approaches zero from the right. By Lemma 4, the fourth integral tends to $f(x,t)$ as $h\to 0^+$; therefore

$$\frac{\partial u_2}{\partial t}(x,t)=\int_0^t\Big(\int_{\mathbb{Q}_p^n}\frac{\partial Z}{\partial t}(x-\xi,t-\tau)(f(\xi,\tau)-f(x,t))\,d^n\xi\Big)\,d\tau+f(x,t).\quad\square$$

As a consequence of Lemmas 10–11, we obtain a proposition:

Proposition 3.

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t)&=\int_{\mathbb{Q}_p^n}\frac{\partial Z}{\partial t}(x-\xi,t)\varphi(\xi)\,d^n\xi\\ &\quad+\int_0^t\Big(\int_{\mathbb{Q}_p^n}\frac{\partial Z}{\partial t}(x-\xi,t-\tau)(f(\xi,\tau)-f(x,t))\,d^n\xi\Big)\,d\tau+f(x,t).\end{aligned}$$

We now consider the action of the operator D_T^γ for $0<\gamma\le\alpha$ on $u(x,t)$. We first note that $(D_T^\gamma u)(x,t)$ is defined if $\gamma>\lambda$. This follows from (6) using $u(x,t)\in\mathfrak{M}_\lambda$.

Lemma 12. *Let $\lambda<\gamma\le\alpha$. Then*

$$(D_T^\gamma u_1)(x,t)=\int_{\mathbb{Q}_p^n}(D_T^\gamma Z)(x-\xi,t)\varphi(\xi)\,d^n\xi.$$

Proof. Let $Z_\gamma(x,t):=(D_T^\gamma Z)(x,t)$ and

$$(19)\qquad Z_{\gamma,l}(x,t):=\frac{1}{\Gamma_p^{(n)}(-\gamma)}\int_{\|y\|_p>p^{-l}}\|y\|_p^{-\gamma-n}(Z(x-y,t)-Z(x,t))\,d^n y.$$

By the Fubini theorem

$$\begin{aligned}
&\frac{1}{\Gamma_p^{(n)}(-\gamma)}\int_{\|y\|_p>p^{-l}}\|y\|_p^{-\gamma-n}(u_1(x-y,t)-u_1(x,t))\,d^ny\\
&=\frac{1}{\Gamma_p^{(n)}(-\gamma)}\int_{\|y\|_p>p^{-l}}\|y\|_p^{-\gamma-n}\Big(\int_{\mathbb{Q}_p^n}(Z(x-y-\xi,t)-Z(x-\xi,t))\varphi(\xi)\,d^n\xi\Big)\,d^ny\\
&=\int_{\mathbb{Q}_p^n}\varphi(\xi)\Big(\frac{1}{\Gamma_p^{(n)}(-\gamma)}\int_{\|y\|_p>p^{-l}}\|y\|_p^{-\gamma-n}(Z(x-y-\xi,t)-Z(x-\xi,t))\,d^ny\Big)\,d^n\xi\\
&=\int_{\mathbb{Q}_p^n}Z_{\gamma,l}(x-\xi,t)\varphi(\xi)\,d^n\xi.
\end{aligned}$$

Let m be a fixed positive integer. Then the last integral can be expressed as

$$\int_{\|x-\xi\|_p\ge p^{-m}}Z_{\gamma,l}(x-\xi,t)\varphi(\xi)\,d^n\xi+\int_{\|x-\xi\|_p<p^{-m}}Z_{\gamma,l}(x-\xi,t)\varphi(\xi)\,d^n\xi.$$

Now if $\|x\|_p\ge p^{-m}$ and $l>m$, then $Z_{\gamma,l}(x,t)=Z_\gamma(x,t)$, and

$$\begin{aligned}
(20)\quad &\frac{1}{\Gamma_p^{(n)}(-\gamma)}\int_{\|y\|_p>p^{-l}}\|y\|_p^{-\gamma-n}(u_1(x-y,t)-u_1(x,t))\,d^ny\\
&=\int_{\|x-\xi\|_p\ge p^{-m}}Z_{\gamma}(x-\xi,t)\varphi(\xi)\,d^n\xi+\int_{\|x-\xi\|_p<p^{-m}}Z_{\gamma,l}(x-\xi,t)\varphi(\xi)\,d^n\xi
\end{aligned}$$

for $l>m$. Now using Fubini's theorem and taking $\lim_{l\to\infty}$, we obtain that

$$\begin{aligned}
(21)\quad &\lim_{l\to\infty}\int_{\|x-\xi\|_p<p^{-m}}Z_{\gamma,l}(x-\xi,t)\varphi(\xi)\,d^n\xi\\
&=\int_{\mathbb{Q}_p^n}\|y\|_p^{-\gamma-n}\frac{1}{\Gamma_p^{(n)}(-\gamma)}\\
&\qquad\times\Big(\int_{\|x-\xi\|_p<p^{-m}}(Z(x-\xi-y,t)-Z(x-\xi,t))\varphi(\xi)\,d^n\xi\Big)\,d^ny\\
&=\int_{\|x-\xi\|_p<p^{-m}}\frac{1}{\Gamma_p^{(n)}(-\gamma)}\\
&\qquad\times\Big(\int_{\mathbb{Q}_p^n}\|y\|_p^{-\gamma-n}(Z(x-\xi-y,t)-Z(x-\xi,t))\,d^ny\Big)\varphi(\xi)\,d^n\xi\\
&=\int_{\|x-\xi\|_p<p^{-m}}Z_{\gamma}(x-\xi,t)\varphi(\xi)\,d^n\xi.
\end{aligned}$$

Since $\|y\|_p^{-\gamma-n}(u_1(x-y,t)-u_1(x,t))$ is integrable as function of y (because $u_1(x,t)\in\mathfrak{M}_\lambda$ for $\gamma>\lambda$ by Lemma 9), the result follows by taking $\lim_{l\to\infty}$ in (20) and using (21). $\square$

Lemma 13. *Let $\lambda < \gamma \le \alpha$. Then*

$$(D_T^\gamma u_2)(x,t) = \int_0^t \Big(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau.$$

Proof. We set $u_{2,h}(x,t) := \int_0^{t-h} (\int_{\mathbb{Q}_p^n} Z(x-y, t-\theta) f(y,\theta)\, d^n y)\, d\theta$. Then

$$\begin{aligned}
&\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} (u_{2,h}(x-y,t) - u_{2,h}(x,t))\, d^n y \\
&= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} \\
&\qquad \times \Big(\int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} (Z(x-y-\xi, t-\tau) - Z(x-\xi, t-\tau)) f(\xi,\tau)\, d^n\xi \Big)\, d\tau \Big)\, d^n y \\
&= \int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} Z_{\gamma,l}(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau,
\end{aligned}$$

with $Z_{\gamma,l}(x,t)$ as in (19). We now note that

$$Z_{\gamma,l}(x,t) = \int_{\mathbb{Q}_p^n} \psi(x\cdot\xi) P_l(\xi) e^{-at\|\xi\|_p^\alpha}\, d^n\xi,$$

where

$$P_l(\xi) = \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} (\psi(-y\cdot\xi) - 1)\, d^n y.$$

By reasoning similar to that in [Kochubeĭ 2001, page 142], we have

$$|P_l(\xi)| \le \frac{2\|\xi\|_p^\gamma}{|\Gamma_p^{(n)}(-\gamma)|} \int_{\|u\|_p > 1} \|u\|_p^{-\gamma-n}\, d^n u = C\|\xi\|_p^\gamma,$$

whence $|Z_{\gamma,l}(x,t)| \le C'$. Also, if $\|x-\xi\|_p \ge p^{-(l-1)}$ then $Z_{\gamma,l}(x-\xi, t-\tau) = Z_\gamma(x-\xi, t-\tau)$. Therefore

$$\begin{aligned}
&\int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} Z_{\gamma,l}(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau \\
&\qquad = \int_0^{t-h} \Big(\int_{\|x-\xi\|_p \ge p^{-(l-1)}} Z_\gamma(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau \\
&\qquad\qquad + \int_0^{t-h} \Big(\int_{\|x-\xi\|_p < p^{-(l-1)}} Z_{\gamma,l}(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau.
\end{aligned}$$

By taking $l \to \infty$, we obtain that

$$
\begin{aligned}
(D_T^\gamma u_{2,h})(x,t) &= \int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x-\xi, t-\tau) f(\xi,\tau)\, d^n\xi \Big)\, d\tau \\
&= \int_0^{t-h} \Big(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x-\xi, t-\tau)(f(\xi,\tau) - f(x,\tau))\, d^n\xi \Big)\, d\tau \\
&= \int_0^{t-h} \Big(\int_{\|x-\xi\|>p^{-l}} (D_T^\gamma Z)(x-\xi, t-\tau)(f(\xi,\tau) - f(x,\tau))\, d^n\xi \Big)\, d\tau,
\end{aligned}
$$

where l is the exponent of local constancy of $f(\xi,\tau)$ (see Corollary 2). Finally, since $u_{2,h} \in \mathcal{M}_\lambda$ uniformly in h (see Lemma 3), by taking $h \to 0^+$ and using the dominated convergence theorem, we have the announced formula. □

As a consequence of Lemmas 7, 12, and 13, we obtain the following result.

Proposition 4. *We have*

$$
\begin{aligned}
(D_T^\gamma u)(x,t) &= \int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x-\xi, t)\varphi(\xi)\, d^n\xi \\
&\qquad + \int_0^{t} \Big(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x-\xi, t-\tau)(f(\xi,\tau) - f(x,\tau))\, d^n\xi \Big)\, d\tau, \\
a(D_T^\gamma u)(x,t) &= -\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x-\xi, t)\varphi(\xi)\, d^n\xi \\
&\qquad - \int_0^{t} \Big(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x-\xi, t-\tau)(f(\xi,\tau) - f(x,\tau))\, d^n\xi \Big)\, d\tau,
\end{aligned}
$$

for $0 < \gamma \le \alpha$.

Proof of Theorem 1. By Lemma 9, $u(x,t) \in \mathfrak{M}_\lambda$ uniformly with respect to t, and $u(x,t)$ satisfies the initial condition of Theorem 1. By Propositions 3 and 4, $u(x,t)$ is a solution of Cauchy problem (15). □

3.4. ***Taibleson operator and elliptic pseudodifferential operators.*** For a polynomial $g(x) \in \mathbb{Z}_p[x_1,\dots,x_n]$, we denote by $\overline{g}(x) \in \mathbb{F}_p[x_1,\dots,x_n]$ its reduction modulo p, that is, the polynomial obtained by reducing the coefficients of $g(x)$ modulo p. Let $f(x) \in \mathbb{Z}_p[x_1,\dots,x_n]$ with $f(0)=0$ be a nonconstant homogeneous polynomial of degree d such that $\overline{f}(x) \neq 0$. We say *that* $f(x)$ *is elliptic modulo* p if $\{x \in \mathbb{F}_p^n \mid \overline{f}(x) = 0\} = \{0\}$, and that $f(x)$ *is elliptic over* $\mathbb{Q}_p$ if $\{x \in \mathbb{Q}_p^n \mid f(x) = 0\} = \{0\}$. Note that if f elliptic modulo p, then f is elliptic over $\mathbb{Q}_p$.

If I is a nonempty subset of $\{1,\dots,n\}$, we define $f_I(x)$ as the polynomial mapping obtained by restricting $f(x)$ to the set $T_I := \{x \in \mathbb{Z}_p^n \mid x_i \neq 0 \Leftrightarrow i \in I\}$. Likewise, we define $\overline{f}_I(x)$ by restricting $f(x)$ to $\overline{T}_I := \{x \in \mathbb{F}_p^n \mid x_i \neq 0 \Leftrightarrow i \in I\}$.

Definition 3. Let $f(x) \in \mathbb{Z}_p[x_1, \dots, x_n]$ with $f(0) = 0$ be a nonconstant homogeneous polynomial of degree d with coefficients in $\mathbb{Z}_p^\times$. We say that $f(x)$ is strongly elliptic modulo p if $\overline{f}_I(x)$ is *elliptic modulo* p for every nonempty subset I of $\{1, \dots, n\}$.

Example 1. Let $f(x) = x^2 - \upsilon y^2$, with $\upsilon \in \mathbb{Z}_p^\times \setminus (\mathbb{Z}_p^\times)^2$, where

$$(\mathbb{Z}_p^\times)^2 := \{x \in \mathbb{Z}_p^\times \mid x = y^2 \text{ for some } y \in \mathbb{Z}_p^\times\}.$$

Then $f(x)$ is strongly elliptic modulo p.

Lemma 14. *There are infinitely many strongly elliptic polynomials modulo* p.

Proof. The proof is by induction on n, the number of variables. The case $n = 1$ is clear. Assume that the result is true for $1 \le n \le k$ and $k \ge 2$. Let $g(x_1, \dots, x_k)$ be a strongly elliptic polynomial modulo p of degree d. Set any $\upsilon \in \mathbb{Z}_p^\times$ such that $\bar{\upsilon}$ does not have an l-th root in $\mathbb{F}_p^\times$ for some $l \ge 2$, and $f(x_1, \dots, x_{k+1}) = g(x_1, \dots, x_k)^l - \upsilon x_{k+1}^{ld}$. Then $f(x_1, \dots, x_{k+1})$ is strongly elliptic modulo p. □

Lemma 15. *Let $f(x) \in \mathbb{Z}_p[x_1, \dots, x_n]$ with $f(0) = 0$ be a nonconstant homogeneous polynomial of degree d with coefficients in $\mathbb{Z}_p^\times$. If $f(x)$ is strongly elliptic modulo p, then*

$$|f(x)|_p = \|x\|_p^d \quad \textit{for } x \in \mathbb{Q}_p^n. \tag{22}$$

Proof. We set $A := \{(z_1, \dots, z_n) \in \mathbb{Z}_p^n$ such that $|z_i|_p = 1$ for some $i\}$. Since $f(x)$ is elliptic over $\mathbb{Q}_p$,

$$(\sup_{z \in A} |f(z)|_p) \|x\|_p^d \le |f(x)|_p \le (\inf_{z \in A} |f(z)|_p) \|x\|_p^d,$$

(see [Zúñiga-Galindo 2008, Lemma 1]). Thus, to prove the result it is sufficient to show that $|f|_p\,|_A \equiv 1$. Given a nonempty subset I of $\{1, \dots, n\}$, we define

$$A_I = \{x \in A \text{ such that } |x_i| = 1 \Leftrightarrow i \in I\}.$$

Then $\bigcup_I A_I$ is a partition of A when I runs through all nonempty subsets of $\{1, \dots, n\}$, and to show (22), it sufficient to prove that $|f|_p\big|_{A_I} \equiv 1$ for every nonempty subset I.

Without loss of generality we may assume that $I = \{1, \dots, r\}$ for $1 \le r \le n$. Thus, if $x \in A_I$, then $x_i \in \mathbb{Z}_p^\times$ for $i = 1, \dots, r$, and $x_i \in p\mathbb{Z}_p$ for $i = r+1, \dots, n$, and $\overline{f}(x) = \overline{f}_I(x) \ne 0$, since f is strongly elliptic modulo p; therefore $|f|_p\big|_{A_I} \equiv 1$. □

4. Markov processes and fundamental solutions

Theorem 2. *The fundamental solution $Z(x, t)$ is a transition density of a time- and space-homogeneous nonexploding right-continuous strict Markov process without discontinuities of the second kind.*

Proof. By Proposition 1(iv), the family of operators

$$(\Theta(t) f)(x) = \int_{\mathbb{Q}_p^n} Z(x-\eta, t) f(\eta) d^n \eta$$

has the semigroup property. We know that $Z(x, t) > 0$ and that $\Theta(t)$ preserves the function $f(x) \equiv 1$ (see Proposition 1). Thus $\Theta(t)$ is a Markov semigroup. The required properties of the corresponding Markov process follow from Proposition 1 and general theorems of the theory of Markov processes; see [Dynkin 1961] and also [Vladimirov et al. 1994, Section XVI]. □

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