Pacific Journal of Mathematics

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Volume 238 No. 1 November 2008

PACIFIC JOURNAL OF MATHEMATICS
Vol. 238, No. 1, 2008

METABELIAN $\mathrm{SL}(n,\mathbb{C})$ REPRESENTATIONS OF KNOT GROUPS

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Dedicated to the memory of Jerry Levine

We give a classification of irreducible metabelian representations from a knot group into $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{C})$. If the homology of the n-fold branched cover of the knot is finite, we show that every irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representation is conjugate to a unitary representation and that the set of conjugacy classes of such representations is finite. In that case, we give a formula for this number in terms of the Alexander polynomial of the knot. These results are the higher rank generalizations of a result of Nagasato, who recently studied irreducible, metabelian $\mathrm{SL}(2,\mathbb{C})$ representations of knot groups. Finally we deduce the existence of irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representations of the knot group for any knot with nontrivial Alexander polynomial.

1. Introduction and statement of results

Given a knot $K \subset S^3$, we let $X_K = S^3 \setminus \nu(K)$ denote its complement. Gordon and Luecke [1989] showed that knots are determined by their complements by proving that, if K_1 and K_2 are knots and $\varphi : X_{K_1} \to X_{K_2}$ a homeomorphism, then K_1 and K_2 are equivalent. In the case of prime knots, Whitten [1987] proved that the homeomorphism type of the complement is determined by its fundamental group. Taken together, these two results reduce the classification of prime knots to that of knot groups.

Abstract groups are often better understood through their representations. For example, the knot invariants coming from finite representations of the knot group are an effective tool for distinguishing knots and constructing knot tables. More delicate information can be obtained from studying their character varieties; see [Culler and Shalen 1983] and [Klassen 1991]. For instance, independent work of Dunfield and Garoufalidis [2004] and Boyer and Zhang [2005] established the

MSC2000: primary 57M25; secondary 20C15.

Keywords: metabelian representation, knot group, Alexander polynomial, branched cover.

Boden was supported by a grant from NSERC. Friedl was supported by a CRM-ISM Fellowship and by CIRGET.

important result that the A-polynomial detects the unknot, and they prove this by exhibiting an arc of irreducible $\mathrm{SL}(2,\mathbb{C})$ characters on $\pi_1(X_K)$ for any nontrivial knot.

Since abelian representations of the knot group factor through $H_1(X_K)$, they provide little information beyond that known in classical knot theory. Among the nonabelian representations, the simplest are those that are trivial on the second commutator subgroup $\pi_1(X_K)^{(2)}$. Such representations are called *metabelian*. Various aspects of metabelian representations of knot groups have been studied by several authors, and we refer the reader to [Hartley 1979; Abdelghani et al. 2007; Jebali 2007].

In this paper, we study metabelian $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{C})$ representations of knot groups. We begin by recalling a result of Nagasato on irreducible metabelian $\mathrm{SL}(2,\mathbb{C})$ representations of knot groups. Here and throughout this paper, for any knot K, we denote its meridian and longitude by μ and λ, respectively.

Theorem 1.1 [Nagasato 2007, Proposition 1.1 and Theorem 1.2]. *If $K \subset S^3$ is a knot, then any irreducible metabelian representation $\alpha : \pi_1(X_K) \to \mathrm{SL}(2,\mathbb{C})$ satisfies*

$$\mathrm{tr}(\alpha(\mu)) = 0 \text{ and } \mathrm{tr}(\alpha(\lambda)) = 2.$$

Further, there exist only finitely many conjugacy classes of irreducible metabelian representations of $\pi_1(X_K)$ into $\mathrm{SL}(2,\mathbb{C})$, and their number equals

$$\frac{|\Delta_K(-1)|-1}{2}.$$

Xiao–Song Lin [2001, Proposition 4.2] had obtained the same count for the number of conjugacy classes of irreducible metabelian $SU(2)$ representations. (He attributed his result to Fox, and Eric Klassen [1991, Theorem 10] proved the same formula when counting binary dihedral representations.) In particular, Nagasato's result implies that every irreducible metabelian $\mathrm{SL}(2,\mathbb{C})$ representation is conjugate to a unitary one. This property is what one would expect from representations of finite groups. Although the quotient $\pi_1(X_K)/\pi_1(X_K)^{(2)}$ is not finite, we will show that for many n (for instance, whenever n is prime), every irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representation of $\pi_1(X_K)$ factors through a finite group.

Unitary metabelian representations of knot groups have been classified in [Friedl 2003; 2004], and here we give the corresponding classification result for $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{C})$. As a consequence, we derive the following higher rank analogue of Nagasato's result. In order to state it precisely, we introduce some notation. Given a knot $K \subset S^3$ and a positive integer n, we let L_n denote the n-fold cyclic branched cover of S^3 branched along K.

Theorem 1.2. *Suppose $K \subset S^3$ is a knot and n is a positive integer.*

(i) *Up to conjugation, any irreducible metabelian representation* $\alpha : \pi_1(X_K) \to \mathrm{SL}(n,\mathbb{C})$ *satisfies* $\alpha(\lambda) = I$ *and*

$$\alpha(\mu) = \begin{pmatrix} 0 & \dots & 0 & (-1)^{n+1} \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

In particular,

$$\mathrm{tr}(\alpha(\mu)) = 0 \text{ and } \mathrm{tr}(\alpha(\lambda)) = n.$$

(ii) *If* $H_1(L_n)$ *is finite, the number of conjugacy classes of irreducible metabelian* $\mathrm{SL}(n,\mathbb{C})$ *representations is finite. This number is given by*

$$\frac{1}{n}\#\big\{\,\chi : H_1(L_n) \to S^1 \mid \chi \text{ does not factor through } H_1(L_n) \to H_1(L_\ell) \text{ for any } \ell \mid n\big\}.$$

(iii) *If* $H_1(L_n)$ *is finite, every irreducible metabelian* $\mathrm{SL}(n,\mathbb{C})$ *representation of* $\pi_1(X_K)$ *is conjugate to a unitary representation.*

We know from [Gordon 1978, p. 17] or [Burde and Zieschang 1985, 8.21], for example, that the n-fold cyclic branched cover L_n has finite first homology group if and only if no root of the Alexander polynomial $\Delta_K(t)$ is an n-th root of unity. In that case, the order of the homology group is determined by the Alexander polynomial by the formula

$$|H_1(L_n)| = \left| \prod_{j=1}^{n-1} \Delta_K(e^{2\pi i j/n}) \right|. \tag{1}$$

From $\Delta_K(1) = 1$ and some basic algebra, it follows that $\Delta_K(z) \neq 0$ for any prime power root of unity z, we therefore deduce the well known fact that $H_1(L_n)$ is finite whenever n is a prime power. The following is now a straightforward corollary to Theorem 1.2. Note that when $n = 2$, this recovers Nagasato's formula.

Corollary 1.3. *If* $n = p^k$ *is a prime power, then* $H_1(L_n)$ *is finite for any knot* K *and the number of conjugacy classes of irreducible metabelian* $\mathrm{SL}(n,\mathbb{C})$ *representations of the knot group is given by*

$$\frac{1}{p^k}\left(\left|H_1(L_{p^k})\right| - \left|H_1(L_{p^{k-1}})\right|\right).$$

If n *is prime, then* $L_1 = S^3$ *and Equation* (1) *implies that this number equals*

$$\frac{1}{n}\left(\left|H_1(L_n)\right| - \left|H_1(L_1)\right|\right) = \frac{1}{n}\left(\left|\prod_{j=1}^{n-1} \Delta_K(e^{2\pi i j/n})\right| - 1\right).$$

If K is a knot with trivial Alexander polynomial, then Equation (1) shows that L_n is a homology 3-sphere for each $n \geq 1$, and Theorem 1.2 implies there are no irreducible metabelian $\mathrm{SL}(n, \mathbb{C})$ representations of $\pi_1(X_K)$ for any $n \geq 2$. In fact, it is known (see [Livingston 2002, Theorem 1.2], for example) that L_n is a homology sphere for all $n \geq 1$ if and only if K has Alexander polynomial $\Delta_K(t) = 1$. Using this, we get the following conclusion.

Corollary 1.4. *Any knot with nontrivial Alexander polynomial admits infinitely many irreducible, pairwise nonconjugate metabelian representations.*

This is a consequence of Theorems 3.10 and 3.12, which give more precise information on conjugacy classes of irreducible metabelian representations for knots with nontrivial Alexander polynomial. Our proofs make use of the extensive work on homology groups of n-fold branched covers of knots that began with [Gordon 1972] and was continued in [Riley 1990; González-Acuña and Short 1991; Silver and Williams 2002], and our existence results of representations are strengthenings of previous ones (compare [Klassen 1991, Corollary 11]).

It is a straightforward exercise to extend all of these results to knots in homology spheres. In particular, we see that given a knot in a homology sphere with nontrivial Alexander polynomial, the knot group admits an irreducible, metabelian $\mathrm{SL}(n, \mathbb{C})$ representation for some $n \geq 2$. Of course, any metabelian representation of the knot group must send the longitude to the identity. It is interesting to compare this statement to Theorem 1.7 of [Frohman 1993], which shows that if K is a fibered knot of genus g in a rational homology sphere, then there exists an irreducible $SU(n)$ representation of the knot group sending the longitude to $e^{2\pi i/n} I$ for some $2 \leq n \leq g + 1$.

2. Metabelian representations into $\mathrm{GL}(n, \mathbb{C})$

Preliminaries. A group G is called metabelian if $G^{(2)} = \{e\}$, where $G^{(n)}$ is the n-th term of the derived series of G which is inductively defined by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. We say a representation $\varphi : G \to \mathrm{GL}(n, \mathbb{C})$ is metabelian if it factors through $G/G^{(2)}$.

A representation $\rho : G \to \mathrm{Aut}(\mathbb{C}^n) \cong \mathrm{GL}(n, \mathbb{C})$ is called *reducible* if there exists a proper subspace $V \subset \mathbb{C}^n$ such that ρ restricts to a representation $\rho : G \to \mathrm{Aut}(V)$. Otherwise ρ is called *irreducible* or *simple*. If α is the direct sum of simple representations, then α is called *semisimple*.

Given a representation $\alpha : G \to \mathrm{Aut}(V)$ we say that $\chi : G \to \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is a *weight* if there exists a nontrivial $v \in V$ such that $\alpha(g)(v) = \chi(g)(v)$ for all $g \in G$. For any weight χ, we set

$$V_\chi = \{v \in V \mid \alpha(g)(v) = \chi(g)(v) \text{ for all } g \in G\}.$$

Clearly V_χ is a nontrivial subspace of V. We refer to V_χ as the *weight space* of χ. If χ_1 and χ_2 are distinct weights, then it is not hard to show that $V_{\chi_1} \cap V_{\chi_2} = \{0\}$. Any abelian group has at least one weight, this follows easily from the fact that every irreducible representation of an abelian group is one-dimensional [Fässler and Stiefel 1992, p. 36].

Metabelian quotients of knot groups. Let $K \subset S^3$ be a knot. In the following we denote by $\widetilde{X_K}$ the infinite cyclic cover of X_K corresponding to the abelianization $\pi_1(X_K) \to \mathbb{Z}$. Therefore $\pi_1(\widetilde{X_K}) = \pi_1(X_K)^{(1)}$ and

$$H_1(X_K; \mathbb{Z}[t^{\pm 1}]) = H_1(\widetilde{X_K}) \cong \pi_1(X_K)^{(1)}/\pi_1(X_K)^{(2)}.$$

The $\mathbb{Z}[t^{\pm 1}]$-module structure is given on the right hand side by $t^n \cdot g := \mu^{-n} g \mu^n$, where μ is the meridian of K.

Given groups G and H together with a homomorphism $\varphi : G \to \mathrm{Aut}(H)$, the semidirect product $G \ltimes H$ is the group whose underlying set is just $G \times H$ and where the group structure is given by

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot g_2, \varphi(g_2)(h_1) \cdot h_2).$$

For a knot K, we set $\pi := \pi_1(X_K)$ and consider the short exact sequence

$$1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1.$$

Since $\pi/\pi^{(1)} = H_1(X_K) \cong \mathbb{Z}$, this sequence splits and we get isomorphisms

$$\begin{array}{ccccccc} \pi/\pi^{(2)} & \cong & \pi/\pi^{(1)} \ltimes \pi^{(1)}/\pi^{(2)} & \cong & \mathbb{Z} \ltimes \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \ltimes H_1(X_K; \mathbb{Z}[t^{\pm 1}]) \\ g & \mapsto & (g, \mu^{-\epsilon(g)} g) & \mapsto & (\epsilon(g), \mu^{-\epsilon(g)} g), \end{array}$$

where $n \in \mathbb{Z}$ acts by conjugation by μ^n on $\pi^{(1)}/\pi^{(2)}$ and by multiplication by t^n on $H_1(X_K; \mathbb{Z}[t^{\pm 1}])$. Thus, metabelian representations of $\pi_1(X_K)$ can be viewed as representations of $\mathbb{Z} \ltimes H_1(X_K; \mathbb{Z}[t^{\pm 1}])$ and vice versa.

Irreducible* $\mathbf{GL}(n, \mathbb{C})$ *representations of certain semidirect products. In this section we present a classification of the irreducible $\mathrm{GL}(n, \mathbb{C})$ representations of metabelian groups of the form $\mathbb{Z} \ltimes H$, where H is a finitely generated $\mathbb{Z}[t^{\pm 1}]$-module and where $n \in \mathbb{Z}$ acts on H by multiplication by t^n.

We begin with the prototypical example of a $\mathrm{GL}(n, \mathbb{C})$ representation of $\mathbb{Z} \ltimes H$. Fix $n \in \mathbb{N}$. Let $\chi : H \to \mathbb{C}^*$ be a character that factors through $H/(t^n - 1)$ and let

$z \in \mathbb{C}^*$. Then it is easy to verify that

$$\alpha = \alpha_{(z,\chi)} : \mathbb{Z} \ltimes H \to \mathrm{GL}(n, \mathbb{C})$$
$$(j,h) \mapsto \begin{pmatrix} 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}^j \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{n-1}h) \end{pmatrix}$$

defines a representation. Note that $\alpha(n,0)$ is a diagonal matrix with each diagonal entry equal to z. Also note that α factors through $\mathbb{Z} \ltimes H/(t^n - 1)$.

Our first goal is to determine which representations $\alpha_{(z,\chi)}$ are irreducible. We say that a character $\chi : H \to \mathbb{C}^*$ has *order* k if

$$k = \min\left\{\ell \in \mathbb{N} \mid \chi \text{ factors through } H/(t^\ell - 1)\right\}.$$

Given a character $\chi : H \to \mathbb{C}^*$, let $t^i\chi$ be the character defined by $(t^i\chi)(h) = \chi(t^i h)$. Obviously, if χ has order n, then $t^n\chi = \chi$. Conversely, the next lemma shows that the order of any character $\chi : H \to \mathbb{C}^*$ which factors through $H/(t^n - 1)$ divides n.

Lemma 2.1. *If $\chi : H \to \mathbb{C}^*$ is a character which factors through both $H/(t^n - 1)$ and $H/(t^\ell - 1)$, then χ also factors through $H/(t^{\gcd(n,\ell)} - 1)$.*

Proof. An easy exercise shows that $\gcd(t^n - 1, t^\ell - 1) = t^{\gcd(n,\ell)} - 1$ in the polynomial ring $\mathbb{Q}[t]$, and using the Euclidean algorithm, we find polynomials $p(t), q(t) \in \mathbb{Q}[t]$ with $(t^n - 1)p(t) + (t^\ell - 1)q(t) = t^{\gcd(n,\ell)} - 1$. Since the leading coefficients of $t^n - 1$ and $t^\ell - 1$ are units in $\mathbb{Z}$, we can arrange that $p(t), q(t)$ lie in $\mathbb{Z}[t]$.

For any $h \in H$, we have

$$\begin{aligned}\chi\left((t^{\gcd(n,\ell)} - 1)h\right) &= \chi\big[((t^n - 1)p(t) + (t^\ell - 1)q(t))h\big] \\ &= \chi\big[(t^n - 1)p(t)h\big] + \chi\big[(t^\ell - 1)q(t)h\big] = 0. \qquad \square\end{aligned}$$

We can now determine which representations $\alpha_{(z,\chi)}$ are irreducible.

Lemma 2.2. *Suppose $\chi : H \to \mathbb{C}^*$ is a character that factors through $H/(t^n - 1)$ and $z \in \mathbb{C}^*$. Then $\alpha_{(z,\chi)} : \mathbb{Z} \ltimes H \to \mathrm{GL}(n, \mathbb{C})$ is irreducible if and only if the character χ has order n.*

Proof. Throughout this proof, let $\alpha = \alpha_{(z,\chi)}$ and suppose χ has order n. Denote by $\gamma : H \to \mathrm{GL}(n, \mathbb{C})$ the restriction of α to $H = 0 \times H \subset \mathbb{Z} \ltimes H$, and let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{C}^n$. Then γ restricts to a representation on $\mathbb{C}e_i$ that is given by $t^{i-1}\chi$ for $i = 1, \dots, n$. Since χ has order n it follows that the characters $\chi, t\chi, \dots, t^{n-1}\chi : H \to \mathbb{C}^*$ are pairwise distinct. Clearly any H-invariant subspace of $\mathbb{C}^n$ must be of the form $\oplus_{i=1}^r \mathbb{C}e_{n_i}$ for some $\{n_1, \dots, n_r\} \subset \{1, \dots, n\}$. Let

$T = \alpha(1, 0)$ and notice that $T(e_i) = e_{i+1}$ (with indices taken modulo n). It follows that $\alpha(\mathbb{Z}, 0)$ acts transitively on the subspaces $\mathbb{C}e_i$. In particular we see that the only proper subrepresentation of α is the zero space. This concludes the proof that α is irreducible.

Now suppose χ has order $\ell < n$. It is a straightforward exercise to show that α is reducible. We will skip this part of the proof since it is also an immediate consequence of the proof of Theorem 2.3. □

The next result presents a classification of all irreducible $\mathrm{GL}(n, \mathbb{C})$ representations of semidirect products of the form $\mathbb{Z} \ltimes H$.

Theorem 2.3. (i) *Any irreducible representation $\alpha : \mathbb{Z} \ltimes H \to \mathrm{GL}(n, \mathbb{C})$ is conjugate to $\alpha_{(z,\chi)}$ for a character $\chi : H \to \mathbb{C}^*$ of order n and some $z \in \mathbb{C}^*$.*

(ii) *If $\chi_1, \chi_2 : H \to \mathbb{C}^*$ are characters of order n and $z_1, z_2 \in \mathbb{C}^*$, then $\alpha_{(z_1,\chi_1)}$ is conjugate to $\alpha_{(z_2,\chi_2)}$ if and only if $z_1 = z_2$ and $\chi_1 = t^k \chi_2$ for some k.*

Proof. We first prove (ii). For convenience, we set $\alpha_1 = \alpha_{(z_1,\chi_1)}$ and $\alpha_2 = \alpha_{(z_2,\chi_2)}$. First assume that α_1 and α_2 are conjugate. Note that

$$z_1(-1)^{n+1} = \det(\alpha_1(1, 0)) = \det(\alpha_2(1, 0)) = z_2(-1)^{n+1}.$$

Now denote by γ_i the restriction of α_i to H. Clearly the weights of γ_i are given by

$$\chi_i, t\chi_i, \ldots, t^{n-1}\chi_i.$$

Also recall that $t^{n+k}\chi_i = t^k\chi_i$. Since the sets of weights of γ_1 and γ_2 have to agree it now follows immediately that $\chi_1 = t^k\chi_2$ for some k.

Now assume $z_1 = z_2$ and $\chi_1 = t^k\chi_2$ for some integer k. Then it is easy to check that, for any $(j, h) \in \mathbb{Z} \ltimes H$, we have

$$\begin{pmatrix} 0 & \ldots & 0 & z_1 \\ 1 & \ldots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix}^{-k} \alpha_1(j, h) \begin{pmatrix} 0 & \ldots & 0 & z_1 \\ 1 & \ldots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix}^{k} = \alpha_2(j, h).$$

We now prove (i). Let $\alpha : \mathbb{Z} \ltimes H \to \mathrm{GL}(n, \mathbb{C})$ be an irreducible representation. We denote by γ the resulting representation $H \to 0 \times H \to \mathbb{Z} \ltimes H \xrightarrow{\alpha} \mathrm{GL}(n, \mathbb{C})$.

Since H is abelian there exists at least one weight $\chi : H \to \mathbb{C}^*$. Let ℓ be the order of χ, we write $\ell = \infty$ if χ does not factor through $H/(t^\ell - 1)$ for any ℓ. For $i = 0, \ldots, \ell - 1$ we consider the weight spaces

$$V_i := \{v \in \mathbb{C}^n \mid \gamma(h)(v) = \chi(t^i h)(v).\}$$

Since $\chi, t\chi, \ldots, t^{\ell-1}\chi$ are different we obtain that $V_0 \oplus V_1 \oplus \cdots \oplus V_{\ell-1}$ embeds in $\mathbb{C}^n$.

Now recall that the group structure of $\mathbb{Z} \ltimes H$ is given by

$$(j_1, h_1)(j_2, h_2) = (j_1 + j_2, t^{j_2} h_1 + h_2)$$

In particular for any $h \in H$, we have

$$(j, 0)(0, t^j h) = (j, t^j h) = (0, h)(j, 0).$$

Therefore, setting $T = \alpha(1, 0)$, we see that

$$T^j \alpha(0, t^j h) = \alpha(j, 0)\alpha(0, t^j h) = \alpha(j, t^j h) = \alpha(0, h)\alpha(j, 0) = \alpha(0, h)T^j.$$

It follows that $T^j \gamma(t^j h) = \gamma(h) T^j$, and for $v \in V_0$, we see that

$$\gamma(h) T^j v = T^j \gamma(t^j h) v = T^j \chi(t^j h) v = \chi(t^j h) T^j v.$$

Hence T^j induces a map $V_0 \to V_j$ (where we take indices modulo ℓ) which is an isomorphism with inverse T^{-j}. This shows $\dim(V_j) = \dim(V_0) \geq 1$ for $j = 1, \ldots, \ell - 1$, which together with the fact that $V_0 \oplus V_1 \oplus \cdots \oplus V_{\ell-1}$ embeds in $\mathbb{C}^n$, implies that $\ell \leq n$, in particular ℓ is finite.

T^ℓ induces an isomorphism of V_0. Let v be an eigenvector of $T^\ell : V_0 \to V_0$ and let W be the subspace of $\mathbb{C}^n$ spanned by $\{v, Tv, \ldots, T^{\ell-1} v\}$. Clearly α restricts to a representation of W, and so irreducibility of α implies $\ell = \dim(W) = n$. It is straightforward to see that, in terms of the basis $v, Tv, \ldots, T^{n-1} v$, the representation α is given by $\alpha_{(z,\chi)}$, where z is the eigenvalue corresponding to the eigenvector v of T^ℓ. □

3. Metabelian representations into SL(n, $\mathbb{C}$)

Metabelian* SL(n, $\mathbb{C}$)*-representations. In this section we apply the previous results to give a classification of irreducible metabelian $\mathrm{SL}(n, \mathbb{C})$ representations of knot groups. We begin with an elementary observation.

Lemma 3.1. *Two representations $\alpha, \beta : G \to \mathrm{SL}(n, \mathbb{C})$ that are conjugate over $\mathrm{GL}(n, \mathbb{C})$ are also conjugate over $\mathrm{SL}(n, \mathbb{C})$.*

Proof. Assume that there exists a matrix $P \in \mathrm{GL}(n, \mathbb{C})$ such that $P\alpha(g)P^{-1} = \beta(g)$ for all $g \in G$. Then let z be an n-th root of $\det(P)$. Clearly $\det(z^{-1}P) = 1$ and $(z^{-1}P)\alpha(g)(z^{-1}P)^{-1} = \beta(g)$ for all $g \in G$. □

As before, we suppose H is a finitely generated $\mathbb{Z}[t^{\pm 1}]$-module and we consider $\mathrm{SL}(n, \mathbb{C})$ representations of the semidirect product $\mathbb{Z} \ltimes H$, where $n \in \mathbb{Z}$ acts on H by multiplication by t^n. Throughout this section, we make the additional assumption on H that multiplication by $t - 1$ is an isomorphism. Notice first that this holds for the principle application we have in mind. Indeed, the long exact sequence in

homology

$$\cdots \to H_{i+1}(X_K;\mathbb{Z}) \to H_i(\widetilde{X_K};\mathbb{Z}) \xrightarrow{t-1} H_i(\widetilde{X_K};\mathbb{Z}) \to H_i(X_K;\mathbb{Z}) \to \cdots$$

shows that $H = H_1(X_K;\mathbb{Z}[t^{\pm 1}])$ has this property.

Lemma 3.2. *Let H be a $\mathbb{Z}[t^{\pm 1}]$-module such that multiplication by $t-1$ is an isomorphism, $\chi : H \to \mathbb{C}^*$ a character that factors through $H/(t^n-1)$, and $z \in \mathbb{C}^*$. Set $\alpha = \alpha_{(z,\gamma)}$. Then for any $(j,h) \in \mathbb{Z} \ltimes H$ we have*

$$\det(\alpha(j,h)) = (-1)^{(n+1)j} z^j.$$

Proof. Let χ be a character that factors through $H/(t^n-1)$ and let $(j,h) \in \mathbb{Z} \ltimes H$. It is straightforward to see that

$$\det(\alpha(j,h)) = (-1)^{(n+1)j} z^j \prod_{i=0}^{n-1} \chi(t^i h) = (-1)^{(n+1)j} z^j \chi\left(\sum_{i=0}^{n-1} t^i h\right).$$

Since multiplication by $t-1$ is an isomorphism on H, we have $h' \in H$ with $(t-1)h' = h$. Thus

$$\chi\left(\sum_{i=0}^{n-1} t^i h\right) = \chi\left(\sum_{i=0}^{n-1} t^i (t-1)h'\right) = \chi\left((t^n-1)h'\right) = 1,$$

since χ factors through $H/(t^n-1)$. □

If H is a $\mathbb{Z}[t^{\pm 1}]$-module such that multiplication by $t-1$ is an isomorphism, and $\chi : H \to \mathbb{C}^*$ is a character of order n, then it follows from Lemma 3.2 that, for $(j,h) \in \mathbb{Z} \ltimes H$, setting

$$\alpha_\chi(j,h) = \begin{pmatrix} 0 & \dots & 0 & (-1)^{n+1} \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}^j \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{n-1}h) \end{pmatrix}$$

defines an SL$(n,\mathbb{C})$ representation.

Theorem 3.3. *Let H be a $\mathbb{Z}[t^{\pm 1}]$-module such that multiplication by $t-1$ is an isomorphism. Then the following hold*:

(i) *If $\chi : H \to \mathbb{C}^*$ is a character of order n, then α_χ defines an irreducible* SL$(n,\mathbb{C})$ *representation.*

(ii) *Given two characters $\chi_1, \chi_2 : H \to \mathbb{C}^*$ of order n, the representations α_{χ_1} and α_{χ_2} are conjugate if and only if $\chi_1 = t^k \chi_2$ for some k.*

(iii) *For any irreducible representation $\alpha : \mathbb{Z} \ltimes H \to$ SL$(n,\mathbb{C})$ there exists a character $\chi : H \to \mathbb{C}^*$ of order n such that α is conjugate to α_χ.*

The theorem is an immediate consequence of Theorem 2.3 and Lemmas 2.2, 3.1, and 3.2.

Proofs of Theorem 1.2 and Corollary 1.3. We need two more basic lemmas before we are in a position to prove Theorem 1.2 and Corollary 1.3. The following result is easy to prove.

Lemma 3.4. *Let A be a finite abelian group. Then*

$$|\mathrm{Hom}(A, \mathbb{C}^*)| = |\mathrm{Hom}(A, S^1)| = |\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})| = |A|.$$

Let K be a knot and $H = H_1(\widetilde{X_K}; \mathbb{Z}) = H_1(X_K; \mathbb{Z}[t^{\pm 1}])$. Denote by W_n the n-fold cover of X_K, which we can view as a subset of L_n. Then the projection from $\widetilde{X_K}$ to W_n induces a map

$$\pi_n : H_1(\widetilde{X_K}; \mathbb{Z}) \to H_1(W_n) \to H_1(L_n).$$

The main properties of this map are found, for example, in [Friedl 2003]:

Lemma 3.5. *For any n the map π_n factors through $H/(t^n - 1)$. Given $\ell \,|\, n$ we have a commutative diagram*

$$\begin{array}{ccc} H/(t^n-1) & \xrightarrow{\cong} & H_1(L_n) \\ \downarrow & & \downarrow \\ H/(t^\ell-1) & \xrightarrow{\cong} & H_1(L_\ell), \end{array}$$

where the horizontal maps are isomorphisms and the vertical maps are surjections.

Using Theorem 3.3 we can now give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let $K \subset S^3$ and $n \in \mathbb{N}$ and let $\alpha : \pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$ be an irreducible metabelian representation. It is well known that the longitude λ lies in $\pi_1(X_K)^{(2)}$, hence $\alpha(\lambda) = I$ and $\mathrm{tr}(\alpha(\lambda)) = n$. This, together with Theorem 3.3 (iii) completes the proof of part (i).

We now turn to the proof of (ii). In the following we write $H = H_1(\widetilde{X_K}; \mathbb{Z})$. Recall that multiplication by $t - 1$ is an isomorphism on H. By Theorem 3.3, the number of conjugacy classes of irreducible metabelian $\mathrm{SL}(n, \mathbb{C})$ representations of $\pi_1(S^3 \backslash K)$ is given by

$$N = \#\left\{\chi : H \to \mathbb{C}^* \mid \chi \text{ of order } n\right\} / \sim,$$

where $\chi_1 \sim \chi_2$ if and only if $\chi_1 = t^k \chi_2$ for some k.

Any character $\chi : H \to \mathbb{C}^*$ of order n factors through $H/(t^n - 1)$, which is by Lemma 3.5 isomorphic to $H_1(L_n)$. Now assume that $H_1(L_n)$ is finite. It follows

immediately from Lemma 3.4 that N is finite. Also note that the group $\mathbb{Z}/n = \langle t \mid t^n = 1\rangle$ acts freely on the set of characters of order n, hence

$$N = \frac{1}{n}\#\left\{\chi : H \to S^1 \mid \chi \text{ of order } n\right\}.$$

By applying Lemmas 2.1 and 3.5 we see that

$$N = \frac{1}{n}\#\left\{\chi : H_1(L_n) \to S^1 \mid \chi \text{ does not factor through } H_1(L_\ell) \text{ for any } \ell \,|\, n\right\},$$

as claimed.

Finally we prove (iii). Suppose $\alpha : \mathbb{Z} \ltimes H \to \mathrm{SL}(n,\mathbb{C})$ is an irreducible representation. Then by Theorem 3.3 the representation α is conjugate to α_χ for some character $\chi : H \to \mathbb{C}^*$ of order n. Note that χ factors through $H/(t^n-1) \cong H_1(L_n)$ which is finite by hypothesis. In particular $\chi(h)$ has finite order for each $h \in H$, i.e. χ is a unitary character $\chi : H \to S^1 \subset \mathbb{C}^*$. It is now clear from the definition that α_χ is a unitary representation. □

Remark 3.6. This proof shows that every irreducible representation $\alpha : \mathbb{Z} \ltimes H \to \mathrm{SL}(n,\mathbb{C})$ factors through $\mathbb{Z}/n \ltimes H/(t^n - 1)$, which is a finite group whenever $H_1(L_n)$ is finite.

Finally, Corollary 1.3 is an immediate consequence of the following more general result.

Theorem 3.7. *Given a knot K and $n \in \mathbb{N}$ with $H_1(L_n)$ finite, the number of conjugacy classes of irreducible metabelian* $\mathrm{SL}(n,\mathbb{C})$ *representations of the knot group is given by*

$$\frac{1}{n}\sum_{k|n} \mu(k)\left|H_1(L_{n/k})\right|,$$

where μ is the Möbius function.

Recall that given n with prime decomposition $n = p_1^{n_1} \cdot \cdots \cdot p_s^{n_s}$ with $n_i \geq 1$ and distinct primes $p_1, \ldots, p_s$ the Möbius function is defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n_i \geq 2 \text{ for some } i, \\ (-1)^s & \text{otherwise.} \end{cases}$$

Proof. Throughout the proof, we make repeated use of the general fact that, for any $m|n$, the projection $H_1(L_n) \to H_1(L_m)$ is surjective.

First consider the case $n = p^k$ is a prime power. Then $|H_1(L_{p^k})|$ is automatically finite, and Theorem 1.2 implies that the number of conjugacy classes of irreducible metabelian representations is finite and is given by

$$\frac{1}{p^k}\#\left\{\chi : H_1(L_{p^k}) \to S^1 \mid \chi \text{ does not factor through } H_1(L_\ell) \text{ for any } \ell \,|\, p^k\right\}.$$

If $\ell \,|\, p^k$, then $\ell = p^j$ for some $j < k$. It follows that the projection $H_1(L_{p^{k-1}}) \to H_1(L_{p^j})$ is surjective, thus any character $\chi : H_1(L_{p^k}) \to S^1$ which factors through $H_1(L_\ell)$ must also factor through $H_1(L_{p^{k-1}})$. Lemma 3.4 and Equation (1) now show that the number of conjugacy classes of irreducible metabelian representations is given by

$$\frac{1}{p^k}\left(\left|H_1(L_{p^k})\right| - \left|H_1(L_{p^{k-1}})\right|\right) = \prod_{j=1}^{p^k} \Delta_K(e^{2\pi i j/p^k}) - \prod_{j=1}^{p^{k-1}} \Delta_K(e^{2\pi i j/p^{k-1}}).$$

This agrees with the formula given by the theorem in the case $n = p^k$ is a prime power. This also proves Corollary 1.3.

Next consider the case $n = p^k q^\ell$, where p and q are distinct primes. If $H_1(L_{p^k q^\ell})$ is finite, Theorem 1.2 implies that the number of conjugacy classes of irreducible metabelian representations is finite and is given by

$$\frac{1}{p^k q^\ell} \#\left\{\chi : H_1(L_{p^k q^\ell}) \to S^1 \mid \chi \text{ does not factor through } H_1(L_m) \text{ for any } m | p^k q^\ell\right\}.$$

In this case, any character $\chi : H_1(L_{p^k q^\ell}) \to S^1$ which factors through $H_1(L_m)$ for some $m | p^k q^\ell$ must also factor through either $H_1(L_{p^k q^{\ell-1}})$ or $H_1(L_{p^{k-1} q^\ell})$. Further, by Lemma 2.1, any character $\chi : H_1(L_{p^k q^\ell}) \to S^1$ which factors through both $H_1(L_{p^k q^{\ell-1}})$ and $H_1(L_{p^{k-1} q^\ell})$ must also factor through $H_1(L_{p^{k-1} q^{\ell-1}})$. Lemma 3.4 now shows that the number of conjugacy classes of irreducible metabelian representations is given by

$$\left|H_1(L_{p^k q^\ell})\right| - \left|H_1(L_{p^k q^{\ell-1}})\right| - \left|H_1(L_{p^{k-1} q^\ell})\right| + \left|H_1(L_{p^{k-1} q^{\ell-1}})\right|.$$

This agrees with the formula given by the theorem in the case $n = p^k q^\ell$.

Now consider the general case $n = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1, \ldots, p_r$ are distinct primes. We will show that the number of conjugacy classes of irreducible metabelian representations is given by

$$\frac{1}{n} \sum_{s=0}^{r} \sum_{i_1 < \cdots < i_s} (-1)^s \left|H_1\left(L_{n/(p_{i_1} \ldots p_{i_s})}\right)\right|.$$

This formula agrees with the one given in the theorem.

Assume $H_1(L_n)$ is finite, and apply Theorem 1.2 to see that the number of conjugacy classes of irreducible metabelian representations is finite and equal to

$$\frac{1}{n} \#\left\{\chi : H_1(L_n) \to S^1 \mid \chi \text{ does not factor through } H_1(L_m) \text{ for any } m | n\right\}.$$

In this case, since $n = p_1^{k_1} \cdots p_r^{k_r}$, any character $\chi : H_1(L_n) \to S^1$ which factors through $H_1(L_m)$ for some $m | n$ must factor through $H_1(L_{n/p_i})$ for some $1 \le i \le r$.

Further, by Lemma 2.1, for $1 \le i < j \le r$, any character $\chi : H_1(L_n) \to S^1$ which factors through $H_1(L_{n/p_i})$ and $H_1(L_{n/p_j})$ must also factor through $H_1(L_{n/(p_i p_j)})$. Repeated application of Lemma 2.1 gives the general statement that, for $1 \le i_1 < \cdots < i_s \le r$, any character $\chi : H_1(L_n) \to S^1$ which factors through each of $H_1(L_{n/p_{i_1}}), H_1(L_{n/p_{i_2}}), \ldots, H_1(L_{n/p_{i_s}})$ must also factor through $H_1(L_{n/(p_{i_1}\cdots p_{i_s})})$.

Using this fact, Lemma 3.4, and the principle of inclusion-exclusion, we obtain the desired result. □

Existence results. In this section, we prove several results on existence of irreducible metabelian SL($n, \mathbb{C}$) representations of knot groups and on existence of faithful metabelian SL($n, \mathbb{C}$) representations of knot groups. Notice that Theorem 1.2 implies that any irreducible metabelian representation $\alpha : \pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$ of a knot group sends the meridian to a matrix of order n and as such is not faithful.

We begin with the problem of existence of irreducible representations. If $K \subset S^3$ is a knot with trivial Alexander polynomial, then $\pi_1(X_K)^{(1)} = \pi_1(X_K)^{(2)}$, i.e. any metabelian representation is already abelian. This shows there are no irreducible metabelian representations $\alpha : \pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$ for any $n \ge 2$ for knots with trivial Alexander polynomial.

On the other hand, if the Alexander polynomial is not trivial, then we will see that there always exist irreducible metabelian SL($n, \mathbb{C}$) representations of $\pi_1(X_K)$. In fact, by using information about the homology groups $H_1(L_n)$ of the n-fold branched covers of K, we prove the existence of infinitely many conjugacy classes of irreducible metabelian representations in many cases. Fortunately for us, the homology groups $H_1(L_n)$ have been extensively studied, and we shall make frequent use of the ideas and results from many of the excellent papers on the subject, including [Gordon 1972; Riley 1990; González-Acuña and Short 1991; Silver and Williams 2002; Livingston 2002].

In what follows, we denote by X_n the character variety of metabelian representations $\alpha : \pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$, and by $X_n^* \subset X_n$ the subvariety of characters of irreducible representations. Since X_n^* is a variety it either consists of finitely many points or it contains positive dimensional components. It follows from Theorem 2.3 that X_n^* consists of finitely many points if and only if there exist only finitely many conjugacy classes of irreducible metabelian representations $\pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$.

Lemma 3.8. *Let $n \in \mathbb{N}$ such that $H_1(L_n)$ is finite. Then*

(i) *the variety X_n^* consists of finitely many points or is empty, and*

(ii) *any irreducible representation $\pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$ is conjugate to a unitary representation.*

Proof. This is a reformulation of Theorem 1.2. □

The next lemma treats the case when $H_1(L_n)$ is infinite and shows that X_n^* is either empty or contains positive dimensional components.

Lemma 3.9. *Let n such that $H_1(L_n)$ is infinite. Then the following are equivalent*:

(a) *there exists an irreducible representation $\pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$,*

(b) *there exist infinitely many conjugacy classes of irreducible representations $\pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$, none of which is conjugate to a unitary representation,*

(c) *there exist infinitely many conjugacy classes of irreducible unitary representation $\pi_1(X_K) \to SU(n)$,*

(d) *there exists a character $\chi : \mathrm{Tor}(H_1(L_n)) \to S^1$ that does not factor through $\mathrm{Tor}(H_1(L_n)) \to \mathrm{Tor}(H_1(L_\ell))$ for any $\ell \,|\, n$ with $b_1(L_\ell) = b_1(L_n)$.*

Proof. Given an abelian group A we write $R(A) = \mathrm{Hom}(A, \mathbb{C}^*)$. Note that $R(A)$ is a complex variety of dimension $\mathrm{rk}(A)$. If $\ell \,|\, n$, the epimorphism $\pi : H_1(L_n) \to H_1(L_\ell)$ induces an injective map $\pi^* : R(H_1(L_\ell)) \to R(H_1(L_n))$. We say that $\chi \in R(A)$ is unitary if χ lies in $U(A) := \mathrm{Hom}(A, S^1) \subset R(A)$, otherwise we say χ is nonunitary. It follows from Theorem 2.3 that given a character $\chi \in R(H_1(L_n))$ of order n, the representation α_χ is conjugate to a unitary representation if and only if χ is unitary, i.e. if and only if $\chi \in U(H_1(L_n))$.

For any unitary character $\chi : \mathrm{Tor}(H_1(L_n)) \to S^1$, we define $R_\chi(H_1(L_n)) \subset R(H_1(L_n))$ by setting

$$R_\chi(H_1(L_n)) = \{\alpha : H_1(L_n) \to \mathbb{C}^* \mid \alpha|_{\mathrm{Tor}(H_1(L_n))} = \chi\}.$$

Fixing a splitting $H_1(L_n) = F_n \oplus \mathrm{Tor}(H_1(L_n))$, where F_n is torsion free, we see that any character $\beta : F_n \to \mathbb{C}^*$ determines a character

$$\alpha = \beta \oplus \chi : F_n \oplus \mathrm{Tor}(H_1(L_n)) \to \mathbb{C}^*$$

with $\alpha \in R_\chi(H_1(L_n))$. Thus $R_\chi(H_1(L_n))$ is a complex variety of the same dimension as $R(F_n)$, indeed $\dim R_\chi(H_1(L_n)) = \mathrm{rk}(H_1(L_n)) > 0$

We first show (d) implies (b), so let $\chi : \mathrm{Tor}(H_1(L_n)) \to S^1$ be the character whose existence is guaranteed by (d). To conclude (b), it suffices by Theorem 3.3 to show that the set

$$R_\chi(H_1(L_n)) \;\setminus\; \bigcup_{\ell \,|\, n} \pi^* R(H_1(L_\ell)) \tag{2}$$

contains infinitely many nonunitary characters. If $\ell \,|\, n$ with $b_1(L_n) = b_1(L_\ell)$, then $R_\chi(H_1(L_n)) \cap \pi^* R(H_1(L_\ell)) = \varnothing$ by hypothesis. On the other hand, if $\ell \,|\, n$ with $b_1(L_\ell) < b_1(L_n)$, then $\pi^* R(H_1(L_\ell)))$ has codimension at least one in $R(H_1(L_n))$. It follows that the set (2) is open and dense in $R_\chi(H_1(L_n))$ and therefore contains infinitely many nonunitary characters.

To see that (d) implies (c), let $U_\chi(H_1(L_n)) \subset U(H_1(L_n))$ be the corresponding subset of unitary characters defined by setting

$$U_\chi(H_1(L_n)) = \{\alpha : H_1(L_n) \to S^1 \mid \alpha|_{\mathrm{Tor}(H_1(L_n))} = \chi\}.$$

The same argument as above now shows that

$$U_\chi(H_1(L_n)) \setminus \bigcup_{\ell|n} \pi^* U(H_1(L_\ell))$$

contains infinitely many unitary characters, and Theorem 3.3 applies to show (c).

Now we show that if (d) does not hold, then every character $\alpha : H_1(L_n) \to \mathbb{C}^*$ factors through $H_1(L_n) \to H_1(L_\ell)$ for some ℓ with $\ell|n$. Consequently, there are no characters of order n, and Theorem 3.3 shows that none of (a), (b) or (c) hold.

Suppose $\alpha : H_1(L_n) \to \mathbb{C}^*$ and fix a splitting $H_1(L_n) = F_n \oplus \mathrm{Tor}(H_1(L_n))$, where F_n is torsion-free. Writing $\alpha = \beta \oplus \chi : F_n \oplus \mathrm{Tor}(H_1(L_n)) \to \mathbb{C}^*$, by assumption, we see χ must factor through $\mathrm{Tor}(H_1(L_n)) \to \mathrm{Tor}(H_1(L_\ell))$ for some $\ell|n$ with $b_1(L_\ell) = b_1(L_n)$. Let $\chi' : \mathrm{Tor}(H_1(L_\ell)) \to S^1$ be the corresponding character. Since $b_1(L_n) = b_1(L_\ell)$, the restriction of π_* gives an isomorphism from F_n to some torsion-free submodule of $H_1(L_\ell)$, and since $\pi_* : H_1(L_n) \to H_1(L_\ell)$ is surjective, the image $F_\ell = \pi_*(F_n)$ determines a splitting

$$H_1(L_\ell) = F_\ell \oplus \mathrm{Tor}(H_1(L_\ell)).$$

Letting $\beta' : F_\ell \to \mathbb{C}^*$ be the character defined by the condition $\beta = \beta' \circ \pi_*$, it follows easily that α is the pullback of $\alpha' = \beta' \oplus \chi' : F_\ell \oplus \mathrm{Tor}(H_1(L_\ell)) \to \mathbb{C}^*$. Thus α factors through $H_1(L_n) \to H_1(L_\ell)$, and this completes the proof. □

Theorem 3.10. *Suppose $K \subset S^3$ is a knot with Alexander polynomial such that a zero of $\Delta_K(t)$ is a root of unity. Let m be the minimal number such that every root of unity that is a zero of $\Delta_K(t)$ is in fact an m-th root of unity. Let $\lambda_1(t)$ be the first Alexander invariant of K.*

(i) *If $\lambda_1(t)|(t^m - 1)$, then there exist irreducible metabelian* SL($n, \mathbb{C}$) *representations of the knot group $\pi_1(X_K)$ only for $2 \le n \le m$. Furthermore the variety X_m^* contains positive dimensional components.*

(ii) *If $\lambda_1(t) \not| (t^m - 1)$, then the variety X_{km}^* contains positive dimensional components for infinitely many $k \ge 2$.*

(iii) *If $(n, m) = 1$, then X_n^* is finite or empty. If all zeroes of $\Delta_K(t)$ are roots of unity, then $X_n^* = \varnothing$ for all $(n, m) = 1$.*

(iv) *If not all zeroes of $\Delta_K(t)$ are roots of unity, then there exist infinitely many n with $(n, m) = 1$ for which X_n^* is nonempty.*

For the proof we need the following lemma.

Lemma 3.11. *Let m be the minimal number such that every root of unity which is a zero of $\Delta_K(t)$ is in fact an m-th root of unity. Then for any k we have $b_1(L_{km}) = b_1(L_m)$, and for any n such that $m \nmid n$ we have $b_1(L_n) < b_1(L_m)$.*

Proof of Lemma 3.11. We write $\Delta_K(t) = \prod_{i=1}^{d}(t - z_i)$ for $z_1, \dots, z_d \in \mathbb{C}$. It is well known that

$$b_1(L_n) = \#\{z_i \mid 1 \le i \le d,\ z_i^n = 1\}. \tag{3}$$

(This formula can be deduced by studying

$$H_1(X_K; \mathbb{C}[t, t^{-1}]) = H_1(X_K; \mathbb{Z}[t, t^{-1}]) \otimes \mathbb{C}$$

and using the fact that $\mathbb{C}[t, t^{-1}]$ is a PID.) It is clear from the formula and the definition of m that $b_1(L_n)$ is maximal if and only if $m|n$. □

Proof of Theorem 3.10. We first prove (i). If $\lambda_1(t)|(t^m - 1)$, the main theorem of [Gordon 1972] shows that the first homology groups of L_n are periodic and moreover $H_1(L_n) = H_1(L_{(n,m)})$. This implies that every character $\chi : H_1(L_n) \to \mathbb{C}^*$ factors through $H_1(L_n) \to H_1(L_{(m,n)})$. Thus if $n > m$, no character has order n. This shows that irreducible metabelian representations $\alpha : \pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$ can only occur for n with $2 \le n \le m$.

It is an immediate consequence of Lemmas 3.9 and 3.11 that the variety X_m^* contains positive dimensional components

We now prove (ii). By Lemmas 3.9 and 3.11 we only have to show that for any K there exists an $k > K$ and a character $\mathrm{Tor}(H_1(L_{km})) \to S^1$ which does not factor through a character $\mathrm{Tor}(H_1(L_{\ell m})) \to S^1$ for any $\ell \,|\, k$. Set $M = \sum_{\ell=2}^{K} \left|\mathrm{Tor}(H_1(L_{\ell m}))\right|$ and define

$$T_K = \{k \mid \text{ there exists a character } \chi : \mathrm{Tor}(H_1(L_{km})) \to \mathbb{C}^* \text{ which does not factor through } \mathrm{Tor}(H_1(L_{km})) \to \mathrm{Tor}(H_1(L_{\ell m})) \text{ for any } \ell \,|\, k \text{ with } \ell \le K\}.$$

Since $\lambda_1(t) \nmid (t^m - 1)$, it follows from the combination of [Gordon 1972, Theorem 4.7] and [Silver and Williams 2002, Theorem 2.1] that $\left|\mathrm{Tor}(H_1(L_{km}))\right| \to \infty$ as $k \to \infty$. In particular, there exists k with $k > K$ such that $\left|\mathrm{Tor}(H_1(L_{km}))\right| > M$. Clearly this implies that $T_K \neq \varnothing$.

Let $k = \min(T_K)$ and pick a character $\chi : \mathrm{Tor}(H_1(L_{km})) \to \mathbb{C}^*$ that does not factor through $\mathrm{Tor}(H_1(L_{km})) \to \mathrm{Tor}(H_1(L_{\ell m}))$ for any $\ell \,|\, k$ with $\ell \le K$. But then χ does not factor through $\mathrm{Tor}(H_1(L_\ell))$ for any $\ell \,|\, k$ with $\ell > K$ by the minimality of k. The claim now follows from Lemma 3.9.

We now turn to the proof of (iii). If $(n, m) = 1$, it follows from (3) that $H_1(L_n)$ is finite. If all zeroes of $\Delta_K(t)$ are roots of unity, it follows from [Gordon 1972, Theorem 4.5] that in fact $H_1(L_n)$ is trivial. Claim (iii) now follows immediately from Theorem 1.2.

Finally we give a proof of (iv). Let $N \in \mathbb{N}$. We have to show that there exists $n > N$ with $(n, m) = 1$ for which there exists an irreducible metabelian representation $\alpha : \pi_1(X_K) \to \mathrm{SL}(n, \mathbb{C})$. Set $M = \sum_{\ell=2}^{N} |H_1(L_\ell)|$ and define

$$S_N = \{n \mid (n, m) = 1 \text{ and there exists a character } \chi : \mathrm{Tor}(H_1(L_n)) \to \mathbb{C}^* \text{ that does not factor through } \mathrm{Tor}(H_1(L_n)) \to \mathrm{Tor}(H_1(L_\ell)) \text{ for any } \ell \mid n \text{ with } \ell \leq N\}.$$

By [Riley 1990] or [González-Acuña and Short 1991] (or alternatively by [Silver and Williams 2002, Theorem 2.1]) there exists n with $(n, m) = 1$ and $n > N$ such that $\left|\mathrm{Tor}(H_1(L_n))\right| > M$. Clearly this implies that $S_N \neq \varnothing$.

Let $n = \min(S_N)$ and pick a character $\chi : \mathrm{Tor}(H_1(L_n)) \to \mathbb{C}^*$ that does not factor through $\mathrm{Tor}(H_1(L_n)) \to \mathrm{Tor}(H_1(L_\ell))$ for any $\ell \mid n$ with $\ell \leq N$. But then χ does not factor through $\mathrm{Tor}(H_1(L_\ell))$ for any $\ell \mid n$ with $\ell > N$ by the minimality of n. The claim now follows from Theorem 1.2. □

The proof of Theorem 3.10 (iii) and (iv) (setting $m = 1$) also immediately gives the following result.

Theorem 3.12. *Suppose $K \subset S^3$ is a knot with Alexander polynomial $\Delta_K(t) \neq 1$ and such that no zero of $\Delta_K(t)$ is a root of unity. Then for any n the set X_n^* is finite or empty. Further, X_n^* is nonempty for infinitely many n.*

Now we turn our attention to the problem of existence of faithful representations of the metabelian quotient of a knot group. First note that it follows from Theorem 1.2 that given a semisimple representation $\alpha : \pi_1(X_K)/\pi_1(X_K)^{(2)} \to \mathrm{SL}(n, \mathbb{C})$ we have $\alpha(\mu^n) = I$, i.e. α can not be faithful. Also note that a unitary representation is necessarily semisimple, hence by the above can not be faithful.

On the other hand if we study nonsemisimple nonunitary representations, then we can always find one that is faithful. The next result produces a faithful representation into $\mathrm{GL}(n, \mathbb{C})$, but it can easily be modified to give a faithful reducible representation into $\mathrm{SL}(n + 2, \mathbb{C})$.

Proposition 3.13. *Given any knot $K \subset S^3$ there exists a faithful reducible representation of $\pi_1(X_k)/\pi_1(X_K)^{(2)}$.*

Representations similar to the ones used in our proof appear also in [Jebali 2007].

Proof. It is well known that $H_1(X_K; \mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}$-torsion free, in particular we get an injection $H_1(X_K; \mathbb{Z}[t^{\pm 1}]) \to H_1(X_K; \mathbb{Z}[t^{\pm 1}]) \otimes \mathbb{C} = H_1(X_K; \mathbb{C}[t^{\pm 1}])$. Since $\mathbb{C}[t^{\pm 1}]$ is a PID, we have an isomorphism

$$H_1(X_K; \mathbb{C}[t^{\pm 1}]) \cong \bigoplus_{i=1}^{n} \mathbb{C}[t^{\pm 1}]/(t - z_i)^{r_i}$$

of $\mathbb{C}[t^{\pm 1}]$-modules for some $z_i \in \mathbb{C}$, $r_i \in \mathbb{N}$. Note that this gives rise to an injective group homomorphism

$$\mathbb{Z} \ltimes H_1(X_K; \mathbb{C}[t^{\pm 1}]) \to \prod_{i=1}^{n} \mathbb{Z} \ltimes \mathbb{C}[t^{\pm 1}]/(t - z_i)^{r_i},$$

here $n \in \mathbb{Z}$ acts on $\mathbb{C}[t^{\pm 1}]/(t - z_i)^{r_i}$ by multiplication by $z_i^n = t^n$. By taking direct sums of representations it therefore suffices to find a faithful representation of $\mathbb{Z} \ltimes \mathbb{C}[t^{\pm 1}]/(t-z)^r$ for some $z \in \mathbb{C}$, $r \in \mathbb{N}$. Given an element $p \in \mathbb{C}[t^{\pm 1}]/(t-z)^r$ there exist unique $a_0, \dots, a_{r-1} \in \mathbb{C}$ such that p is represented by $\sum_{i=0}^{r-1} a_i (t-z)^i$. Let $x \in S^1$ an element of infinite order. We then consider the representation $\alpha : \mathbb{Z} \ltimes \mathbb{C}[t^{\pm 1}]/(t-z)^r \to \mathrm{GL}(r+1, \mathbb{C})$ defined by

$$(0, p) \mapsto \begin{pmatrix} 1 & a_0 & a_1 & \cdots & a_{r-1} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{pmatrix} \quad \text{and} \quad (1, 0) \mapsto x \cdot \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & z & 1 & \cdots & 0 \\ \vdots & & z & \ddots & \vdots \\ & & & \ddots & 1 \\ 0 & \cdots & & 0 & z \end{pmatrix}.$$

The group structure on $\mathbb{Z} \ltimes \mathbb{C}[t^{\pm 1}]/(t-z)^r$ is given by $(n', h') + (n, h) = (n' + n, z^n h' + h)$. Using this, it is not difficult to check that α is indeed a representation. It is easy to verify that α is also a faithful representation. □

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Received March 29, 2008.

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