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TOPOLOGICAL COMPLEXITY OF BASIS-CONJUGATING AUTOMORPHISM GROUPS

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We compute the topological complexity of Eilenberg–Mac Lane spaces associated to the group of automorphisms of a finitely generated free group that act by conjugation on a given basis, and to certain subgroups.

1. Introduction

Given a mechanical system, a motion planning algorithm is a function that assigns to any pair of states of the system, an initial state and a desired state, a continuous motion of the system starting at the initial state and ending at the desired state. Interest in such algorithms arises in robotics; see Latombe [1991] as a general reference. In a sequence of recent papers [2003; 2004; 2006], Farber develops a topological approach to the problem of motion planning, introducing a numerical invariant that gives a measure of the "navigational complexity" of the system.

Let X be a path-connected topological space, the space of all possible configurations of a mechanical system. In topological terms, the motion planning problem consists of finding an algorithm that takes pairs of configurations, that is, points $(x_0, x_1) \in X \times X$, and produces a continuous path $\gamma\colon [0, 1] \to X$ from the initial configuration $x_0 = \gamma(0)$ to the terminal configuration $x_1 = \gamma(1)$. Let PX be the space of all continuous paths in X, equipped with the compact-open topology. The map $\pi : PX \to X \times X$, $\gamma \mapsto (\gamma(0), \gamma(1))$, which sends a path to its endpoints, is a fibration. The motion planning problem then asks for a section of this fibration, a map $s\colon X \times X \to PX$ satisfying $\pi \circ s = \mathrm{id}_{X\times X}$. It would be desirable for a motion planning algorithm to depend continuously on the input. However, one can show that there exists a globally continuous motion planning algorithm $s\colon X \times X \to PX$ if and only if X is contractible; see [Farber 2003, Theorem 1]. One is thus led to study the discontinuities of such algorithms.

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For the space X, define the *topological complexity* $\mathsf{TC}(X)$ to be the Schwarz genus, or sectional category, of the path-space fibration:

$$\mathsf{TC}(X) := \operatorname{secat}(\pi\colon PX \to X \times X).$$

In other words, $\mathsf{TC}(X)$ is the smallest number k for which there exists an open cover $X \times X = U_1 \cup \cdots \cup U_k$ such that the map π admits a continuous section $s_j\colon U_j \to PX$ over each U_j satisfying $\pi \circ s_j = \mathrm{id}_{U_j}$. One can show that $\mathsf{TC}(X)$ is an invariant of the homotopy type of X; see [Farber 2003, Theorem 3].

Let X be an aspherical space, that is, a space whose higher homotopy groups vanish: $\pi_i(X) = 0$ for $i \geq 2$. Farber [2006, Section 31] poses the problem of computing the topological complexity of such a space in terms of algebraic properties of the fundamental group $G = \pi_1(X)$. In other words, given a discrete group G, define the topological complexity of G to be $\mathsf{TC}(G) := \mathsf{TC}(K(G, 1))$, the topological complexity of an Eilenberg–Mac Lane space of type $K(G, 1)$, and express $\mathsf{TC}(G)$ in terms of invariants such as the cohomological or geometric dimension of G if possible.

A number of results in the literature may be interpreted in the context of this problem. For a right-angled Artin group G, the topological complexity of an associated $K(G, 1)$-complex was computed in [Cohen and Pruidze 2008]. For the Artin pure braid group $G = P_n$, the configuration space $F(\mathbb{C}, n)$ of n ordered points in $\mathbb{C}$ is an associated Eilenberg–Mac Lane space. Similarly, the configuration space $F(\mathbb{C}_m, n)$ of n ordered points in $\mathbb{C}_m = \mathbb{C} \setminus \{m \text{ points}\}$ is an Eilenberg–Mac Lane space for the group $P_{n,m} = \ker(P_n \to P_m)$, the kernel of the homomorphism that forgets the last $n - m$ strands of a pure braid. In [Farber and Yuzvinsky 2004] and [Farber et al. 2007], Farber, Grant, and Yuzvinsky determine the topological complexity of these configuration spaces. All these results may be expressed in terms of the cohomological dimension, $\operatorname{cd}(G)$, of the underlying group G. For instance, one has $\mathsf{TC}(P_n) = \mathsf{TC}(F(\mathbb{C}, n)) = 2n - 2 = 2\operatorname{cd}(P_n)$.

The pure braid group P_n and the group $P_{n,m}$ may be realized as subgroups of $\operatorname{Aut}(F_n)$, the automorphism group of the finitely generated free group $F_n = \langle x_1, \ldots, x_n \rangle$. The purpose of this note is to determine the topological complexity of several other subgroups of $\operatorname{Aut}(F_n)$.

Let $G = P\Sigma_n$ be the "group of loops", the group of motions of a collection of $n \geq 2$ unknotted, unlinked circles in 3-space, where each (oriented) circle returns to its original position. This group may be realized as the basis-conjugating automorphism group, or pure symmetric automorphism group, of F_n, consisting of all automorphisms that, for the fixed basis $\{x_1, \ldots, x_n\}$ for F_n, send each generator to a conjugate of itself. A presentation for $P\Sigma_n$ was found by McCool [1986]. In particular, this group is generated by automorphisms $\alpha_{i,j} \in \operatorname{Aut}(F_n)$ for $1 \leq i \neq j \leq n$, defined by $\alpha_{i,j}(x_i) = x_j x_i x_j^{-1}$ and $\alpha_{i,j}(x_k) = x_k$ for $k \neq i$. Also of interest is the "upper triangular McCool group", the subgroup $P\Sigma_n^+$ of $P\Sigma_n$ generated by $\alpha_{i,j}$ for $i < j$. The main results of this note may be summarized as follows.

Theorem. *The topological complexity of the basis-conjugating automorphism group is*

$$\mathsf{TC}(P\Sigma_n) = 2n - 1.$$

The topological complexity of the upper triangular McCool group is

$$\mathsf{TC}(P\Sigma_n^+) = 2n - 2.$$

Let X be an Eilenberg–Mac Lane complex of type $K(G, 1)$ for either $G = P\Sigma_n$ or $G = P\Sigma_n^+$. Since the topological complexity $\mathsf{TC}(X) = \mathsf{TC}(G)$ of X is the Schwarz genus of the path-space fibration, it admits several useful bounds. For instance, one has

$$\mathsf{TC}(X) = \mathrm{secat}(\pi : PX \to X \times X) \leq \mathrm{cat}(X \times X) \leq 2\,\mathrm{cat}(X) - 1 \leq 2\dim(X) + 1,$$

where $\mathrm{cat}(X)$ denotes the Lusternik–Schnirelmann category of X; see Schwarz [1961; 1962] and James [1978] as classical references. One also has a cohomological lower bound

$$\mathsf{TC}(X) \geq 1 + \mathsf{cl}(\ker(\pi^* \colon H^*(X \times X; \mathbb{Q}) \to H^*(PX; \mathbb{Q}))),$$

where $\mathsf{cl}(A)$ denotes the cup length of a graded ring A, the largest integer q for which there are homogeneous elements $a_1, \ldots, a_q$ of positive degree in A such that $a_1 \cdots a_q \neq 0$. Using the Künneth formula, the fact that $PX \simeq X$, and the equality $H^*(X; \mathbb{Q}) = H^*(G; \mathbb{Q})$, the kernel of $\pi^* \colon H^*(X \times X; \mathbb{Q}) \to H^*(PX; \mathbb{Q})$ may be identified with the kernel $Z = Z(H^*(G; \mathbb{Q}))$ of the cup-product map

$$H^*(G; \mathbb{Q}) \otimes H^*(G; \mathbb{Q}) \xrightarrow{\cup} H^*(G; \mathbb{Q});$$

see [Farber 2003, Theorem 7]. We call the cup length of the ideal Z of zero-divisors the *zero-divisor cup length* of $H^*(G; \mathbb{Q})$ and denote it by $\mathsf{zcl}(H^*(G; \mathbb{Q})) = \mathsf{cl}(Z)$. In this notation, the cohomological lower bound reads

$$\mathsf{TC}(G) \geq 1 + \mathsf{zcl}(H^*(G; \mathbb{Q})).$$

This note is organized as follows. After a discussion of basis-conjugating automorphism groups in Section 2, including the determination of their geometric dimensions, we use the (known) structure of the cohomology rings of these groups to compute the zero-divisor cup lengths of these rings in Section 3. These results are used in Section 4 to find the topological complexity of these groups. We conclude with some remarks concerning formality in Section 5.

2. Basis-conjugating automorphism groups

Let N be a compact set contained in the interior of a manifold M. Generalizing the familiar interpretation of a braid as the motion of $N = \{n \text{ distinct points}\}$ in $M = \mathbb{R}^2$, Dahm [1962] defines a motion of N in M as a path h_t in $\mathscr{H}_c(M)$, the space of homeomorphisms of M with compact support, satisfying $h_0 = \mathrm{id}_M$ and $h_1(N) = N$. With an appropriate notion of equivalence, the set of equivalence classes of motions of N in M is a group, and, furthermore, there is a homomorphism from this group to the automorphism group of the fundamental group $\pi_1(M \setminus N)$.

Goldsmith [1981] gives an exposition of Dahm's (unpublished) work, with particular attention paid to the case where $N = \mathscr{L}_n$ is a collection of n unknotted, unlinked circles in $M = \mathbb{R}^3$. Let $\mathscr{G}_n$ denote the corresponding motion group. Goldsmith shows that $\mathscr{G}_n$ is generated by three types of motions — flipping a single circle, interchanging two (adjacent) circles, and pulling one circle through another — and that the Dahm homomorphism $\phi\colon \mathscr{G}_n \to \mathrm{Aut}(\pi_1(\mathbb{R}^3 \setminus \mathscr{L}_n))$ is an embedding.

Choose a basepoint $e \in \mathbb{R}^3$ that is disjoint from $\mathscr{L}_n = C_1 \cup \cdots \cup C_n$, and for each i, let x_i be (the homotopy class of) a loop based at e linking C_i once. This identifies $\pi_1(\mathbb{R}^3 \setminus \mathscr{L}_n, e) = F_n$ with the free group generated by $x_1, \ldots, x_n$. With this identification, the generators of the motion group $\mathscr{G}_n \hookrightarrow \mathrm{Aut}(F_n)$ correspond to automorphisms ρ_i (flip C_i), τ_i (switch C_i and C_{i+1}), and $\alpha_{i,j}$ (pull C_i through C_j) defined by

$$\rho_i(x_k) = \begin{cases} x_k^{-1} & \text{if } k = i, \\ x_k & \text{if } k \neq i, \end{cases} \qquad \tau_i(x_k) = \begin{cases} x_{k+1} & \text{if } k = i, \\ x_{k-1} & \text{if } k = i+1, \\ x_k & \text{if } k \neq i, i+1, \end{cases}$$

and

$$\alpha_{i,j}(x_k) = \begin{cases} x_j x_k x_j^{-1} & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases} \tag{2-1}$$

Let $\varphi : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/[F_n, F_n]) \cong \mathrm{GL}(n, \mathbb{Z})$ denote the epimorphism induced by the abelianization homomorphism $F_n \to F_n/[F_n, F_n] \cong \mathbb{Z}^n$. There is a corresponding short exact sequence

$$1 \longrightarrow \mathrm{IA}_n \longrightarrow \mathrm{Aut}(F_n) \xrightarrow{\ \varphi\ } \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1\ ,$$

where $\mathrm{IA}_n = \ker \varphi$ is the well-known group of automorphisms of F_n that induce the identity on $H_1(F_n; \mathbb{Z})$. Brownstein and Lee [1993] considered the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker(\varphi \circ \phi) & \longrightarrow & \mathscr{G}_n & \xrightarrow{\varphi\circ\phi} & \mathbb{Z}/2 \wr \Sigma_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{IA}_n & \longrightarrow & \mathrm{Aut}(F_n) & \xrightarrow{\ \varphi\ } & \mathrm{GL}(n, \mathbb{Z}) & \longrightarrow & 1, \end{array}$$

where the vertical maps are embeddings. They showed that the image of $\mathscr{G}_n$ under $\varphi \circ \phi$ is the wreath product $\mathbb{Z}/2 \wr \Sigma_n$, the reflection group of type D_n. The kernel of $\varphi \circ \phi$ corresponds to the group $\mathscr{C}_n$ of "pure motions" of $\mathscr{L}_n$, motions that bring each oriented circle back to its original position. The isomorphic image of $\ker(\varphi \circ \phi)$ in $\mathrm{Aut}(F_n)$, that is, the intersection $\mathrm{IA}_n \cap \phi(\mathscr{G}_n)$, is the basis-conjugating automorphism group of the free group.

Definition 2.1. The *basis-conjugating automorphism group* of the free group F_n is the subgroup of $\mathrm{Aut}(F_n)$ generated by the elements $\alpha_{i,j}$ from (2-1) with $1 \le i, j \le n$, and $i \ne j$. Following [Jensen et al. 2006], we denote this group by $P\Sigma_n$.

McCool [1986] showed that $P\Sigma_n$ admits a presentation with the aforementioned generators and defining relations

$$\left\{ \begin{array}{ll} [\alpha_{i,j}, \alpha_{k,l}] & \text{for } i, j, k, l \text{ distinct}, \\ [\alpha_{i,j}, \alpha_{k,j}] & \text{for } i, j, k \text{ distinct}, \\ [\alpha_{i,j}, \alpha_{i,k}\alpha_{j,k}] & \text{for } i, j, k \text{ distinct}, \end{array} \right\} \tag{2-2}$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ denotes the commutator.

An "upper triangular" version of the basis-conjugating automorphism group has been studied in a number of recent works; see [Bardakov and Mikhailov 2008; Cohen et al. 2007; Cohen et al. 2008].

Definition 2.2. The *upper triangular McCool group* $P\Sigma_n^+$ is the subgroup of $P\Sigma_n$ generated by the elements $\alpha_{i,j}$ with $i < j$, subject to the relevant relations (2-2).

The upper triangular McCool group $P\Sigma_n^+$ shares a number of features with the Artin pure braid group P_n. For instance, both groups may be realized as iterated semidirect products of free groups:

$$P_n = F_{n-1} \rtimes_{\eta_{n-1}} \cdots \rtimes_{\eta_2} \rtimes F_1 \quad \text{and} \quad P\Sigma_n^+ = F_{n-1} \rtimes_{\mu_{n-1}} \cdots \rtimes_{\mu_2} \rtimes F_1.$$

For the pure braid group, the action of the free group F_k on F_m with $1 \le k < m \le n-1$ is given by the restriction of the Artin representation $\eta_m \colon P_m \to \mathrm{Aut}(F_m)$; see for instance [Birman 1974]. For the upper triangular McCool group, the action of $F_k = \langle \alpha_{n-k,j} \mid n-k+1 \le j \le n \rangle$ on $F_m = \langle \alpha_{n-m,j} \mid n-m+1 \le j \le n \rangle$, that is, the homomorphism $\mu_m \colon \rtimes_{j=1}^{m-1} F_j \to \mathrm{Aut}(F_m)$, was determined in [Cohen et al. 2008] (with different notation). Using the relations (2-2), one can check that

$$\mu_m(\alpha_{j,p})(\alpha_{i,q}) = \alpha_{j,p}^{-1}\alpha_{i,q}\alpha_{j,p} = \begin{cases} \alpha_{i,p}\alpha_{i,q}\alpha_{i,p}^{-1} & \text{if } q = j, \\ \alpha_{i,q} & \text{otherwise}, \end{cases}$$

where $i = n-m,\ \ j = n-k,\ \ 1 \le i < j < p \le n$, and $i+1 \le q \le n$.

Consideration of centers provides another similarity between these groups. For a group G, let $Z(G)$ denote the center of G, and let $\overline{G} = G/Z(G)$. It is well known that the center of the pure braid group is infinite cyclic and that $P_n \cong \overline{P}_n \times Z(P_n) = \overline{P}_n \times \mathbb{Z}$. The analogous result holds for the upper triangular McCool group.

Proposition 2.3. *The center of the upper triangular McCool group $P\Sigma_n^+$ is infinite cyclic, the quotient $\overline{P\Sigma}_n^+ = F_{n-1} \rtimes_{\mu_{n-1}} \cdots \rtimes_{\mu_3} F_2$ is an iterated semidirect product of free groups, and $P\Sigma_n^+ \cong \overline{P\Sigma}_n^+ \times Z(P\Sigma_n^+) = \overline{P\Sigma}_n^+ \times \mathbb{Z}$.*

Proof. Consider the element $c = \alpha_{1,n}\alpha_{2,n} \cdots \alpha_{n-1,n}$ of the group $P\Sigma_n^+$. Using (2-2), it is readily checked that c commutes with all the generators of $P\Sigma_n^+$, and so $c \in Z(P\Sigma_n^+)$. Also it is clear that $c \in \mathrm{Aut}(F_n)$ has infinite order. Consequently, the infinite cyclic subgroup $C = \langle c \rangle$ is contained in the center $Z(P\Sigma_n^+)$.

Since $\alpha_{n-1,n} = (\alpha_{1,n}\alpha_{2,n} \cdots \alpha_{n-2,n})^{-1} \cdot c$, the group $P\Sigma_n^+$ admits a presentation with generators c and $\alpha_{i,j}$ for $1 \le i < j \le n$ and $(i, j) \ne (n-1, n)$, relations $[c, \alpha_{i,j}]$ for all $i < j$, and the relations (2-2) (not involving $\alpha_{n-1,n}$). Thus, $P\Sigma_n^+ \cong C \times (P\Sigma_n^+/C)$. Since the free group F_1 in the iterated semidirect product decomposition $P\Sigma_n^+ = \rtimes_{j=1}^{n-1} F_j$ is generated by $\alpha_{n-1,n}$, it is clear from the above discussion that $P\Sigma_n^+/C = F_{n-1} \rtimes_{\mu_{n-1}} \cdots \rtimes_{\mu_3} F_2$. An easy inductive argument reveals that the center of this quotient is trivial. It follows that $C = Z(P\Sigma_n^+)$, which completes the proof. □

Despite the aforementioned similarities, the groups P_n and $P\Sigma_n^+$ are not isomorphic; see Bardakov and Mikhailov [2008].

Definition 2.4. Let G be a group. The *cohomological dimension* $\mathrm{cd}(G)$ of G is the smallest integer n such that $H^q(G; M) = 0$ for any G-module M and all $q > n$. The *geometric dimension* $\mathrm{geom\,dim}(G)$ of the group G is the smallest dimension of an Eilenberg–Mac Lane complex of type $K(G, 1)$.

Proposition 2.5. *Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then*

$$\mathrm{geom\,dim}(P\Sigma_n) = \mathrm{cd}(P\Sigma_n) = n - 1.$$

Proof. Collins [1989] showed that, for each n, the cohomological dimension of $P\Sigma_n$ is as asserted: $\mathrm{cd}(P\Sigma_n) = n - 1$. A classical result of Eilenberg and Ganea [1957] states that, for groups of cohomological dimension at least 3, the geometric dimension is equal to the cohomological dimension. Thus, the assertion holds for $P\Sigma_n$ with $n > 3$.

Since $P\Sigma_2 = F_2$ is the free group generated by $\alpha_{2,1}$ and $\alpha_{1,2}$, the case $n = 2$ is immediate.

It remains to consider the case $n = 3$. The group $P\Sigma_3$ is generated by six elements $\alpha_{i,j}$ with $1 \le i \ne j \le 3$. Let $\beta_1 = \alpha_{2,1}\alpha_{3,1}$, $\beta_2 = \alpha_{1,2}\alpha_{3,2}$, and $\beta_3 = \alpha_{1,3}\alpha_{2,3}$, and observe that these elements generate the inner automorphism group $\mathrm{Inn}(F_3)$ of F_3, which is isomorphic to F_3. As noted in [Brownstein and Lee 1993], the group $P\Sigma_3 = \mathrm{Inn}(F_3) \rtimes F$ is a semidirect product, where $F = \langle \alpha_{1,2}, \alpha_{2,1}, \alpha_{3,1} \rangle$ is also a free group on 3 generators. Thus, $P\Sigma_3 \cong F_3 \rtimes F_3$ is a semidirect product of two finitely generated free groups.

For an arbitrary iterated semidirect product of finitely generated free groups G, Cohen and Suciu [1998, Section 1.3] give an explicit construction of a $K(G,1)$-complex X_G. If $G = \rtimes_{i=1}^{\ell} F_{d_i}$, the complex X_G is ℓ-dimensional. In particular, for the group $G = P\Sigma_3$, this construction yields a 2-dimensional $K(G,1)$-complex. We therefore have $\operatorname{geom\,dim}(P\Sigma_3) = \operatorname{cd}(P\Sigma_3) = 2$. □

A similar result holds for the upper triangular McCool groups.

Proposition 2.6. *Suppose $P\Sigma_n^+$ is the upper triangular McCool group, and let $\overline{P\Sigma}_n^+ = P\Sigma_n^+/Z(P\Sigma_n^+)$. Then*

$$\operatorname{geom\,dim}(P\Sigma_n^+) = \operatorname{cd}(P\Sigma_n^+) = n-1 \text{ and } \operatorname{geom\,dim}(\overline{P\Sigma}_n^+) = \operatorname{cd}(\overline{P\Sigma}_n^+) = n-2.$$

Proof. Since $\overline{P\Sigma}_n^+ = F_{n-1} \rtimes_{\mu_{n-1}} \cdots \rtimes_{\mu_3} \rtimes F_2$ and $P\Sigma_n^+ = \overline{P\Sigma}_n^+ \times \mathbb{Z}$ are iterated semidirect products of finitely generated free groups, this follows immediately from the results of [Cohen and Suciu 1998]. □

3. Structure of the cohomology ring

As noted in Section 1, the zero-divisor cup length of the cohomology ring of a group provides a lower bound for the topological complexity. In this section, we determine this lower bound for the groups $P\Sigma_n$ and $P\Sigma_n^+$.

Let $A = \bigoplus_{k=0}^{\ell} A^k$ be a graded algebra over a field $\Bbbk$, and recall that the cup length $\mathsf{cl}(A)$ is the largest integer q for which there are homogeneous elements $a_1, \dots, a_q$ of positive degree in A such that $a_1 \cdots a_q \neq 0$. The tensor product $A \otimes A$ has a natural graded algebra structure, with multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1|\cdot|u_2|} u_1 u_2 \otimes v_1 v_2.$$

Let $\mu : A \otimes A \to A$ denote the multiplication homomorphism, and let $Z = \ker(\mu)$ be the ideal of zero-divisors. The zero-divisor cup length of A, denoted by $\mathsf{zcl}(A)$, is the cup length of this ideal: $\mathsf{zcl}(A) = \mathsf{cl}(Z)$. Observe that if $a \in A$, then the element $\bar{a} = a \otimes 1 - 1 \otimes a \in Z$ is a zero-divisor.

In [1993], Brownstein and Lee determined the low-dimensional cohomology $H^{\leq 2}(P\Sigma_n; \mathbb{Z})$ of the basis-conjugating automorphism group, and conjectured the general ring structure in terms of generators and relations. This conjecture was recently proved by Jensen, McCammond, and Meier [2006, Theorem 6.7]. For our purposes, it suffices to work with coefficients in the field $\Bbbk = \mathbb{Q}$ of rational numbers. So we suppress coefficients and denote the rational cohomology of a group G by $H^*(G) = H^*(G; \mathbb{Q})$ throughout this section and the next.

Theorem 3.1 [Jensen et al. 2006]. *The rational cohomology algebra $H^*(P\Sigma_n)$ is isomorphic to E/I, where E is the exterior algebra over $\mathbb{Q}$ generated by degree* *one elements $a_{i,j}$ for $1 \leq i \neq j \leq n$, and I is the homogeneous ideal generated by the degree two elements*

$$\begin{aligned} &a_{i,j}a_{j,i} && \text{for } i,\ j \text{ distinct, and} \\ &a_{k,j}a_{j,i} - a_{k,j}a_{k,i} - a_{i,j}a_{k,i} && \text{for } i,\ j, k \text{ distinct.} \end{aligned}$$

This result may be used to exhibit an explicit basis for $H^q(P\Sigma_n)$ for each q with $0 \leq q \leq n-1$; see [Jensen et al. 2006, Section 6]. Call an element of the form $a_{i,j}a_{j,k} \cdots a_{s,t}a_{t,i}$ a cyclic product. Then $H^q(P\Sigma_n)$ has a basis consisting of those q-fold products $a_{i_1,j_1}a_{i_2,j_2} \cdots a_{i_q,j_q}$ of the one-dimensional generators that do not contain any cyclic products and have distinct first indices $i_1, \ldots, i_q$. It follows that the Poincaré polynomial of $P\Sigma_n$ is $\sum_{q \geq 0} \dim H^q(P\Sigma_n) \cdot t^q = (1+nt)^{n-1}$. In particular, $H^i(P\Sigma_n) = 0$ for $i \geq n$, and the cup length of $H^*(P\Sigma_n)$ is $n-1$.

We use these results to find the zero-divisor cup length of the ring $H^*(P\Sigma_n)$.

Theorem 3.2. *Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then the zero-divisor cup length of the rational cohomology algebra of $P\Sigma_n$ is*

$$\mathsf{zcl}(H^*(P\Sigma_n)) = 2n - 2.$$

Proof. In general, the zero-divisor cup length of an algebra A cannot exceed the cup length of the tensor product $A \otimes A$, which is twice the cup length of A itself: $\mathsf{zcl}(A) \leq \mathsf{cl}(A \otimes A) = 2\,\mathsf{cl}(A)$. Since $\mathsf{cl}(H^*(P\Sigma_n)) = n-1$ by Theorem 3.1, it follows that $\mathsf{zcl}(H^*(P\Sigma_n)) \leq 2n-2$.

For the reverse inequality, we work in the aforementioned basis for $H^*(P\Sigma_n)$ and the corresponding induced basis for the tensor product $H^*(P\Sigma_n) \otimes H^*(P\Sigma_n)$. Observe that any monomial in the generators of $H^*(P\Sigma_n)$ that contains a cyclic product must vanish, and that any finite expression in $H^*(P\Sigma_n)$ can be reduced to an expression in the basis elements after finitely many applications of the relation

$$a_{k,j}a_{k,i} = a_{k,j}a_{j,i} + a_{i,j}a_{k,i} \tag{3-1}$$

by eliminating, step-by-step, repetition in the first index.

For each $i < n$, consider the elements $\mathbf{x}_i = a_{i,i+1}$ and $\mathbf{y}_i = a_{i+1,i}$ in $H^*(P\Sigma_n)$ and the corresponding zero divisors $\bar{\mathbf{x}}_i = \mathbf{x}_i \otimes 1 - 1 \otimes \mathbf{x}_i$ and $\bar{\mathbf{y}}_i = \mathbf{y}_i \otimes 1 - 1 \otimes \mathbf{y}_i$ in the tensor product $H^*(P\Sigma_n) \otimes H^*(P\Sigma_n)$. We claim that the product

$$M = \prod_{i=1}^{n-1} \bar{\mathbf{x}}_i \cdot \prod_{i=1}^{n-1} \bar{\mathbf{y}}_i = \bar{\mathbf{x}}_1\bar{\mathbf{x}}_2 \cdots \bar{\mathbf{x}}_{n-1}\bar{\mathbf{y}}_1\bar{\mathbf{y}}_2 \cdots \bar{\mathbf{y}}_{n-1}$$

of these $2n-2$ zero divisors is different from zero. To prove this, we use the relation (3-1) to express M in terms of the specified basis of the tensor product, and identify at least one monomial left unchanged by the reduction process.

If I is a subset of $[n-1]=\{1,2,\dots,n-1\}$, let $|I|$ denote the cardinality of I, and let $U_I = z_1\cdots z_{n-1}$ and $V_I = \hat{z}_1\cdots\hat{z}_{n-1}$, where

$$z_i = \begin{cases}\mathbf{y}_i, & \text{if } i\notin I,\\ \mathbf{x}_i, & \text{if } i\in I\end{cases} \quad\text{and}\quad \hat{z}_i = \begin{cases}\mathbf{y}_i, & \text{if } i\in I,\\ \mathbf{x}_i, & \text{if } i\notin I.\end{cases}$$

Then, using the fact that $\bar{\mathbf{x}}_i\bar{\mathbf{y}}_i = \mathbf{y}_i\otimes\mathbf{x}_i - \mathbf{x}_i\otimes\mathbf{y}_i$, we have

$$M = \sum_{I\subseteq[n-1]} (-1)^{|I|} U_I\otimes V_I. \tag{3-2}$$

When $I=\varnothing$ is the empty set, the summand $U_\varnothing\otimes V_\varnothing$ in (3-2) is

$$U_\varnothing\otimes V_\varnothing = \mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_{n-1}\otimes\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_{n-1} = a_{2,1}a_{3,2}\cdots a_{n,n-1}\otimes a_{1,2}a_{2,3}\cdots a_{n-1,n}.$$

This monomial is already a basis element of $H^{n-1}(P\Sigma_n)\otimes H^{n-1}(P\Sigma_n)$.

We claim that the expression of any other summand $(-1)^{|I|}U_I\otimes V_I$ of (3-2) in terms of our basis for $H^*(P\Sigma_n)\otimes H^*(P\Sigma_n)$ will avoid the specified basis element $U_\varnothing\otimes V_\varnothing$. Clearly, if the monomial U_I is already a basis element of $H^*(P\Sigma_n)$, there is nothing to prove. Otherwise, U_I contains a factor $a_{k,j}a_{k,i}$ for at least one k with $1<k<n$, and these are the only generators in the product U_I involving index k. Applying the relation (3-1) to the product $a_{k,j}a_{k,i}$, we obtain (up to sign)

$$U_I = (a_{k,j}a_{j,i} + a_{i,j}a_{k,i})\cdot(\text{other factors}) = a_{k,j}P + a_{k,i}Q,$$

where P and Q are monomials in the generators $a_{r,s}$ of $H^*(P\Sigma_n)$ with $r\neq k$ and $s\neq k$. Further application of reductive relation (3-1) to P and Q will result in no further appearance of k in the indices. Hence writing $U_I = a_{k,j}P + a_{k,i}Q$ in the specified basis for $H^*(P\Sigma_n)$ will yield a linear combination of basis elements, each with exactly one factor involving index k. On the other hand, our fixed monomial $U_\varnothing = a_{n,n-1}\cdots a_{k+1,k}a_{k,k-1}\cdots a_{2,1}$ contains two factors involving index k. Therefore the basis monomial $U_\varnothing\otimes V_\varnothing$ is different from any other possible basis summand coming from $U_I\otimes V_I$ with $I\neq\varnothing$, and our claim holds. □

The cohomology of the upper-triangular McCool group $P\Sigma_n^+$ may be analyzed in a similar manner. The integral cohomology of $P\Sigma_n^+$ was computed by Cohen, Pakianathan, Vershinin, and Wu [2008, Theorem 1.4]. Their results yield:

Theorem 3.3 [Cohen et al. 2008]. *The rational cohomology algebra $H^*(P\Sigma_n^+)$ is isomorphic to E^+/I^+, where E^+ is the exterior algebra over $\mathbb{Q}$ generated by degree one elements $a_{i,j}$ for $1\le i<j\le n$, and I^+ is the homogeneous ideal generated by the degree two elements*

$$a_{i,j}a_{i,k} - a_{i,j}a_{j,k} \quad \textit{for } i<j<k.$$

This result may be used to exhibit an explicit basis for $H^q(P\Sigma_n^+)$ for each q with $0\le q\le n-1$; compare [Cohen et al. 2008, Section 7]. The group $H^q(P\Sigma_n^+)$ has a basis consisting of those q-fold products $a_{i_1,j_1}a_{i_2,j_2}\cdots a_{i_q,j_q}$ of the one-dimensional generators that satisfy $1 \le i_1 < i_2 < \cdots < i_q \le n-1$ and $i_p < j_p \le n$ for each p. It follows that $\sum_{q\ge 0} \dim H^q(P\Sigma_n^+)\cdot t^q = \prod_{k=1}^{n-1}(1+kt)$. In particular, $H^i(P\Sigma_n^+)=0$ for $i \ge n$, and the cup length of $H^*(P\Sigma_n^+)$ is $n-1$.

We analyze the zero-divisor cup length of the ring $H^*(P\Sigma_n^+)$ using these results.

Theorem 3.4. *Let $P\Sigma_n^+$ be the upper-triangular McCool group. Then the zero-divisor cup length of the rational cohomology algebra of $P\Sigma_n^+$ satisfies*

$$\mathsf{zcl}(H^*(P\Sigma_n^+)) \ge 2n-3.$$

Proof. Consider the zero-divisors $\bar{a}_{i,j} = a_{i,j}\otimes 1 - 1\otimes a_{i,j}$ and $a_{n-1,n}\otimes a_{n-1,n}$. We check that the product

$$\bar{a}_{1,n-1}\bar{a}_{1,n}\bar{a}_{2,n-1}\bar{a}_{2,n}\cdots\bar{a}_{n-2,n-1}\bar{a}_{n-2,n}\cdot(a_{n-1,n}\otimes a_{n-1,n}) \tag{3-3}$$

is nonzero. Note that

$$\bar{a}_{i,n-1}\cdot\bar{a}_{i,n} = a_{i,n}\otimes a_{i,n-1} - a_{i,n-1}\otimes a_{i,n} + a_{i,n-1}a_{i,n}\otimes 1 + 1\otimes a_{i,n-1}a_{i,n}$$

for any $i \le n-2$. The product (3-3) contains summands of the form

$$\pm a_{1,i_1}a_{2,i_2}\cdots a_{n-2,i_{n-2}}a_{n-1,n}\otimes a_{1,j_1}a_{2,j_2}\cdots a_{n-2,j_{n-2}}a_{n-1,n}, \tag{3-4}$$

where i_p and j_p take different values from the set $\{n-1, n\}$ for each p. Such summands represent distinct basis elements in the tensor product. These are, in fact, the only nonzero summands in the expression (3-3). Any other monomial, say μ, in this expression will contain a factor of the form $a_{i,n-1}a_{i,n}\otimes 1$ or $1\otimes a_{i,n-1}a_{i,n}$ for some i with $1 \le i \le n-2$. The relations $a_{i,n-1}a_{i,n} = a_{i,n-1}a_{n-1,n}$ in $H^*(P\Sigma_n^+)$ and the fact that $a_{n-1,n}\otimes a_{n-1,n}$ is also a factor of μ may be used to show that μ is trivial in $H^*(P\Sigma_n^+)\otimes H^*(P\Sigma_n^+)$. Thus the product (3-3) is a nontrivial linear combination of the terms given by (3-4), and is nonzero. □

Remark 3.5. It follows from the results of the next section that equality holds in Theorem 3.4, that is, $\mathsf{zcl}(H^*(P\Sigma_n^+)) = 2n-3$.

4. Topological complexity

In this section, we recall several necessary properties of topological complexity and prove the main results of the paper.

Let X be a path-connected topological space. We are interested in the case where X is an Eilenberg–Mac Lane space of type $K(G, 1)$ for $G = P\Sigma_n$ or $G = P\Sigma_n^+$, so assume that X has the homotopy type of a finite CW-complex. Let PX denote the space of all continuous paths $\gamma\colon [0, 1] \to X$, equipped with the compact-open topology. The map $\pi : PX \to X \times X$, $\gamma \mapsto (\gamma(0), \gamma(1))$, which sends a path to its endpoints, is a fibration, with fiber ΩX, the based loop space of X.

Recall from Section 1 that the motion planning problem asks for a (continuous) section of this fibration, a map $s\colon X \times X \to PX$ satisfying $\pi \circ s = \mathrm{id}_{X\times X}$. As shown by Farber [2003, Theorem 1], in most cases such a section cannot exist.

Proposition 4.1 [Farber 2003]. *The path space fibration $\pi : PX \to X \times X$ admits a section if and only if X is contractible.*

Definition 4.2. The *topological complexity* $\mathsf{TC}(X)$ of X is the smallest positive integer k for which $X \times X = U_1 \cup \cdots \cup U_k$, where U_j is open and there exists a continuous section $s_j\colon U_i \to PX$ satisfying $\pi \circ s_j = \mathrm{id}_{U_i}$ for each j with $1 \le j \le k$. In other words, the topological complexity of X is the Schwarz genus (or sectional category) of the path space fibration $\pi\colon PX \to X \times X$.

The topological complexity of X is a homotopy-type invariant; see [Farber 2003, Theorem 3]. If G is a discrete group, define $\mathsf{TC}(G)$, the topological complexity of G, to be that of an Eilenberg–Mac Lane space of type $K(G, 1)$. Farber [2006, Section 31] poses the problem of determining the topological complexity of G in terms of other invariants of G such as $\mathrm{cd}(G)$, the cohomological dimension. In this section, we solve this problem for the basis-conjugating automorphism groups $P\Sigma_n$ and $P\Sigma_n^+$.

We will require several properties of topological complexity. We briefly record these and refer to the survey [Farber 2006] for further details.

First, if X is a finite-dimensional cell complex, then $\mathsf{TC}(X) \le 2\dim(X) + 1$; see [Farber 2006, Section 3]. Consequently, if G is a group of finite geometric dimension, then

$$\mathsf{TC}(G) \le 2\,\mathrm{geom\,dim}(G) + 1. \tag{4-1}$$

Second, as noted in Section 1, a lower bound for the topological complexity of a group G is provided by the zero-divisor cup length of the cohomology ring $H^*(G) = H^*(G; \mathbb{Q})$:

$$\mathsf{TC}(G) \ge 1 + \mathsf{zcl}(H^*(G)); \tag{4-2}$$

see [Farber 2006, Section 15]. Finally, if X and Y are path-connected paracompact locally contractible topological spaces (in particular, CW-complexes), then

$$\mathsf{TC}(X \times Y) \le \mathsf{TC}(X) + \mathsf{TC}(Y) - 1;$$

see [Farber 2006, Section 12]. Consequently, if G_1 and G_2 are groups (of finite geometric dimension), then

$$\mathsf{TC}(G_1 \times G_2) \le \mathsf{TC}(G_1) + \mathsf{TC}(G_2) - 1. \tag{4-3}$$

With these facts at hand, we now prove our main theorems.

Theorem 4.3. *The topological complexity of the basis-conjugating automorphism group $P\Sigma_n$ is* $\mathsf{TC}(P\Sigma_n) = 2n-1$.

Proof. By Theorem 3.2, the zero-divisor cup length of $H^*(P\Sigma_n)$ is given by $\mathsf{zcl}(H^*(P\Sigma_n)) = 2n-2$. So the lower bound (4-2) yields $\mathsf{TC}(P\Sigma_n) \geq 2n-1$. For the reverse inequality, recall from Proposition 2.5 that

$$\text{geom dim}(P\Sigma_n) = \text{cd}(P\Sigma_n) = n-1.$$

Consequently, the upper bound (4-1) yields $\mathsf{TC}(P\Sigma_n) \leq 2n-1$. □

Theorem 4.4. *The topological complexity of the upper triangular McCool group $P\Sigma_n^+$ is* $\mathsf{TC}(P\Sigma_n^+) = 2n-2$.

Proof. By Theorem 3.4, the zero-divisor cup length of $H^*(P\Sigma_n^+)$ is no less than $2n-3$. So the lower bound (4-2) yields $\mathsf{TC}(P\Sigma_n^+) \geq 2n-2$.

For the reverse inequality, recall from Proposition 2.3 that $P\Sigma_n^+ \cong \overline{P\Sigma}_n^+ \times \mathbb{Z}$. Since the circle S^1 is a $K(\mathbb{Z}, 1)$-space, and $\mathsf{TC}(\mathbb{Z}) = \mathsf{TC}(S^1) = 2$ (see, for instance, [Farber 2003, Section 5]), the product inequality (4-3) yields

$$\mathsf{TC}(P\Sigma_n^+) \leq \mathsf{TC}(\overline{P\Sigma}_n^+) + \mathsf{TC}(\mathbb{Z}) - 1 = \mathsf{TC}(\overline{P\Sigma}_n^+) + 1.$$

By Proposition 2.6, we have $\text{geom dim}(\overline{P\Sigma}_n^+) = \text{cd}(\overline{P\Sigma}_n^+) = n-2$. Consequently, the upper bound (4-1) yields $\mathsf{TC}(\overline{P\Sigma}_n^+) \leq 2n-3$. Thus $\mathsf{TC}(P\Sigma_n^+) \leq 2n-2$. □

Corollary 4.5. *The zero-divisor cup length of the rational cohomology algebra of $P\Sigma_n^+$ is* $\mathsf{zcl}(H^*(P\Sigma_n^+)) = 2n-3$.

5. Formality

If X is an Eilenberg–Mac Lane space of type $K(G, 1)$, where either $G = P\Sigma_n$ or $G = P\Sigma_n^+$, the results of the previous section imply that the topological complexity of X is given by the cohomological lower bound, that is,

$$\mathsf{TC}(X) = 1 + \mathsf{zcl}(H^*(X; \mathbb{Q})).$$

This equality holds for a number of spaces of interest in topology, including certain configuration spaces, complements of certain complex hyperplane arrangements, and Eilenberg–Mac Lane spaces corresponding to right-angled Artin groups; see [Cohen and Pruidze 2008; Farber et al. 2007; Farber and Yuzvinsky 2004; Yuzvinsky 2007]. Since all of these spaces are formal in the sense of Sullivan [1977], it is natural to speculate that such an equality holds for an arbitrary formal space X. Conjecturally, $\mathsf{TC}(X) = 1 + \mathsf{zcl}(H^*(X; R))$ for appropriate coefficients R. This conjecture is explicitly made by Yuzvinsky [2007] for the complement of an arbitrary hyperplane arrangement. Related problems are studied in [Fernández Suárez et al. 2006] and [Lechuga and Murillo 2007]. In this section, we show that the upper triangular McCool group $P\Sigma_n^+$ provides evidence in favor of such a conjecture.

Theorem 5.1. *Let X be an Eilenberg–Mac Lane space of type $K(G, 1)$, where $G = P\Sigma_n^+$ is the upper triangular McCool group. Then X is a formal space.*

To prove this theorem, we will need some definitions and facts concerning formality and related notions.

Let X be a space with the homotopy type of a connected, finite-type CW-complex. Loosely speaking, X is *formal* if the rational homotopy type of X is determined by the rational cohomology ring $H^*(X; \mathbb{Q})$. Examples of formal spaces include spheres, simply-connected Eilenberg–Mac Lane spaces, and those mentioned above.

Let G be a finitely presented group. Following Quillen [1969], call G *1-formal* if the Malcev Lie algebra of G is quadratic; see [Papadima and Suciu 2004] for details. As shown by Sullivan [1977] and Morgan [1978], the fundamental group $G = \pi_1(X)$ of a formal space X is a 1-formal group. There are, however, nonformal spaces with 1-formal fundamental groups; see [Kohno 1983; Morgan 1978].

Papadima and Suciu [2006, Proposition 2.1] provide a sufficient condition for the formality of a CW-complex. Recall that a connected, graded algebra A over a field $\Bbbk$ is said to be a *Koszul algebra* if $\mathrm{Tor}^A_{p,q}(\Bbbk, \Bbbk) = 0$ for all $p \neq q$, where p is the homological degree of the Tor groups and q is the internal degree coming from the grading of A. A necessary condition is that A be a quadratic algebra, the quotient of a free algebra on generators in degree 1 by an ideal generated in degree 2.

Proposition 5.2 [Papadima and Suciu 2006]. *Let X be a connected, finite-type CW-complex. If $H^*(X; \mathbb{Q})$ is a Koszul algebra and $G = \pi_1(X)$ is a 1-formal group, then X is a formal space.*

Berceanu and Papadima [2007, Remark 5.5] have recently shown that the upper triangular McCool group $P\Sigma_n^+$ is 1-formal. Thus, to prove Theorem 5.1, it suffices to show that the rational cohomology algebra $H^*(P\Sigma_n^+; \mathbb{Q})$ is Koszul. For this, we will use [Jambu and Papadima 1998, Proposition 6.3].

Let $A = \bigoplus_{k \geq 0} A^k$ be a connected, graded $\Bbbk$-algebra, and denote the augmentation ideal of A by $A^+ = \bigoplus_{k \geq 1} A^k$. Call a subalgebra B of A normal if $AB^+ = B^+A$. If $B \subset A$ is normal, there is a canonical projection $\pi : A \to F$, where $F = A/AB^+$.

Proposition 5.3 [Jambu and Papadima 1998]. *Let $B \subset A$ be a normal subalgebra such that A is free as a right B-module, and assume that the $\Bbbk$-algebras A, B and $F = A/AB^+$ are quadratic. If B and F are Koszul algebras, then A is a Koszul algebra.*

We apply this result to the rational cohomology algebra $H^*(P\Sigma_n^+; \mathbb{Q})$.

Proposition 5.4. *The rational cohomology algebra $H^*(P\Sigma_n^+;\mathbb{Q})$ of the upper triangular McCool group is a Koszul algebra.*

Proof. Write $A_n = H^*(P\Sigma_n^+;\mathbb{Q})$.

The proof consists of an inductive application of Proposition 5.3. As $P\Sigma_2^+ \cong \mathbb{Z}$, the base case A_2 is trivial.

Inductively assume that A_{n-1} is Koszul. For $k<n$, observe that A_k is isomorphic to the subalgebra $\tilde{A}_k$ of A_n generated by the elements $a_{i,j}$ with $n-k<i<j\le n$. Thus, we may assume that the subalgebra $\tilde{A}_{n-1}$ of A_n is Koszul. Since the algebras under consideration are graded commutative, $\tilde{A}_{n-1}$ is a normal subalgebra of A_n. Furthermore, A_n is free as a right $\tilde{A}_{n-1}$-module. Namely,

$$A_n = 1\cdot\tilde{A}_{n-1}\oplus a_{1,2}\cdot\tilde{A}_{n-1}\oplus\cdots\oplus a_{1,n}\cdot\tilde{A}_{n-1}.$$

This follows from the fact that in any monomial of the algebra A_n, the factor $a_{1,i}$ with minimal i always survives, since $a_{1,i}a_{1,j}=a_{1,i}a_{i,j}$ in A_n for any $1<i<j$; see Theorem 3.3.

Analyzing again the relations in A_n, we observe that the algebra $A_n/A_n\tilde{A}_{n-1}^+$ is a graded algebra generated by the elements $a_{1,i}$ for $2\le i\le n$, where all the terms in degree 2 and higher die. Consequently, the algebra $A_n/A_n\tilde{A}_{n-1}^+$ is quadratic and, moreover, Koszul. Thus, all the algebras under consideration are quadratic, and the conditions of Proposition 5.3 are satisfied. The result follows immediately. □

Since the upper triangular McCool group $P\Sigma_n^+$ is 1-formal (see [Berceanu and Papadima 2007]) and $H^*(P\Sigma_n^+;\mathbb{Q})$ is Koszul, Proposition 5.2 implies that an Eilenberg–Mac Lane space of type $K(P\Sigma_n^+,1)$ is formal, proving Theorem 5.1. Such a space X provides an example of a non-simply-connected formal space with $\mathsf{TC}(X)=1+\mathsf{zcl}(H^*(X;\mathbb{Q}))$.

Remark 5.5. Berceanu and Papadima [2007, Theorem 5.4] also showed that the basis-conjugating automorphism group $P\Sigma_n$ is 1-formal. Using the realizations $P\Sigma_2\cong F_2$ and $P\Sigma_3\cong F_3\rtimes F_3$ noted in the proof of Proposition 2.5, one can show that $H^*(P\Sigma_n;\mathbb{Q})$ is Koszul and hence a $K(P\Sigma_n,1)$-space is formal for $n\le 3$. We do not know if the cohomology algebra $H^*(P\Sigma_n;\mathbb{Q})$ is Koszul for $n>3$.

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