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THE EQUIVARIANT CHOW RINGS OF QUOT SCHEMES

TOM BRADEN, LINDA CHEN AND FRANK SOTTILE

We give a presentation for the (integral) torus-equivariant Chow ring of the quot scheme, a smooth compactification of the space of rational curves of degree d in the Grassmannian. For this presentation, we refine Evain's extension of the method of Goresky, Kottwitz, and MacPherson to express the torus-equivariant Chow ring in terms of the torus-fixed points and explicit relations coming from the geometry of families of torus-invariant curves. As part of this calculation, we give a complete description of the torus-invariant curves on the quot scheme and show that each family is a product of projective spaces.

1. Introduction

Let k be an algebraically closed field and let d, n, r be nonnegative integers with $r < n$. We study the quot scheme $\mathcal{Q}_d := \mathcal{Q}_d(r, n)$ parametrizing quotient sheaves on $\mathbb{P}^1$ of the trivial vector bundle $\mathcal{O}^n_{\mathbb{P}^1}$ which have rank r and degree d. When $r > 0$, this is a compactification of the space $\mathcal{M}_d$ of parametrized rational curves of degree d on the Grassmannian $G(r, n)$ of r-dimensional quotients of k^n. Indeed, a morphism from $\mathbb{P}^1$ to $G(r, n)$ of degree d is equivalent to a quotient bundle $\mathcal{O}^n_{\mathbb{P}^1} \to \mathcal{T}$ of rank r and degree d.

Strømme [1987] showed that $\mathcal{Q}_d(r, n)$ is a smooth, projective, rational variety of dimension $r(n-r)+nd$. He described the decomposition into Białynicki-Birula cells induced by an action of a one-dimensional torus on $\mathcal{Q}_d$, thereby determining its Betti numbers. He also gave a presentation of its integral Chow ring (his Theorem 5.3) in terms of generators and relations. However, the set of generators is far from minimal, and the relations are given by the annihilator of a certain class, and are therefore nonexplicit. He also gave a more elementary set of generators for its rational Chow ring.

Later, Bertram [1997] used the geometry of $\mathcal{Q}_d$ to determine the (small) quantum cohomology ring of the Grassmannian. He used a recursive description of the

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boundary $\mathcal{Q}_d \setminus \mathcal{M}_d$ to show that the 3-point genus zero Gromov–Witten invariants of the Grassmannian are equal to particular intersection numbers on $\mathcal{Q}_d$. By studying certain types of intersections on quot schemes, he obtained a quantum Schubert calculus for the Grassmannian. However, he did not need to compute the full cohomology ring of $\mathcal{Q}_d$.

Our main result is a presentation of the T-equivariant Chow ring $A^*_T(\mathcal{Q}_d)$ for an action of the torus

$$T = T_{k^n} \times T_{\mathbb{P}^1}$$

on $\mathcal{Q}_d$, where $T_{k^n} = (\mathbb{G}_m)^n$ acts diagonally on k^n and $T_{\mathbb{P}^1} = \mathbb{G}_m$ acts primitively on $\mathbb{P}^1$. Since the ordinary Chow ring $A^*(\mathcal{Q}_d)$ is the quotient of $A^*_T(\mathcal{Q}_d)$ by the image of $A^{>0}_T(p)$, this also determines $A^*(\mathcal{Q}_d)$. When $k = \mathbb{C}$, the cycle class map induces an isomorphism between Chow and cohomology rings, so our result also determines the T-equivariant and ordinary cohomology rings of $\mathcal{Q}_d$.

Our presentation gives $A^*_T(\mathcal{Q}_d)$ as an explicit subring of a direct sum of polynomial rings. This arises from an analysis of the localization map. When an algebraic torus T acts on a smooth variety X with a finite fixed point set X^T, the inclusion $i\colon X^T \hookrightarrow X$ induces the localization map of (integral) equivariant Chow rings

$$i^*\colon A^*_T(X) \to A^*_T(X^T) = \bigoplus_{p \in X^T} A^*_T(p).$$

Each summand $A^*_T(p)$ is canonically isomorphic to the symmetric algebra S of the character group of T. If X is projective, then i^* is injective, so computing the ring $A^*_T(X)$ reduces to computing the image of i^*.

When $k = \mathbb{C}$, one can consider the corresponding localization map for rational equivariant cohomology. In this case, a result of Chang and Skjelbred [1974] implies that the image of the localization map is cut out by the images of the T-equivariant cohomology of components of the *one-skeleton* of X: the set of points that are fixed by some codimension one subtorus of T. These components are closures of families of T-invariant curves. In particular, when X has finitely many one-dimensional T-orbits (whose closures are T-invariant curves), Goresky, Kottwitz, and MacPherson [1998] used this to describe the image of the localization map for equivariant cohomology. Each T-invariant curve gives a relation, and these GKM relations cut out the image. Brion [1997] showed that this remains true for rational equivariant Chow rings of varieties over any algebraically closed field.

Our action of T on $\mathcal{Q}_d$ has finitely many fixed points, but there are infinitely many T-invariant curves. The GKM relations remain valid, but are now insufficient to cut out the image of the localization map, even over $\mathbb{Q}$; there will be extra relations coming from connected components of the one-skeleton of $\mathcal{Q}_d$. Brion [1997] adapted the result of Chang and Skjelbred to Chow groups, showing that the relations given by these families are sufficient to cut out $A^*_T(X)_{\mathbb{Q}}$. Determining these relations explicitly is more difficult than for the GKM relations, however, and few cases have been worked out in detail.

One case for which the relations are known is when X is a Hilbert scheme of points on a toric surface, which has families of T-invariant curves. Following a suggestion of Brion, Evain [2007] used Edidin and Graham's [1998b] version of the Atiyah–Bott–Berline–Vergne localization formula for equivariant Chow groups to give relations in terms of ideal-membership. The relations come from elements of the T-equivariant Chow rings of the families of T-invariant curves.

We discuss this in Section 5, and give a more explicit formula for Evain's relations which holds when each component Y of $X^{T'}$ for $T' \subset T$ of codimension one has smooth T-invariant subvarieties Z whose classes $[Z]$ generate $A_T^*(Y)$. We derive necessary and sufficient linear relations over $\mathbb{Q}$ from Evain's ideal-membership relations. Lastly, we show that if the T-weights of the tangent space at each fixed point are not too dependent (see Theorem 5.5), then Evain's relations also determine the integral Chow ring.

All these additional hypotheses hold for $\mathcal{Q}_d$. In fact, the components Y we get are quite simple: all are products of projective spaces. As a result, we obtain explicit descriptions of the equivariant Chow ring of $\mathcal{Q}_d$, both rationally and integrally.

To describe the combinatorics of fixed points in $\mathcal{Q}_d^T$, we use the following notations. For an element $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we define $|\mathbf{a}| = \sum_i a_i$. We use addition and subtraction on $\mathbb{Z}^n$ considered as an abelian group, and denote the identity element by $\mathbf{0} = (0, \dots, 0)$. If $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$, we set $\mathbf{a}! = \prod_i a_i!$, where $0! = 1$. Finally, we use the partial order $\mathbf{a} \leq \mathbf{b}$ to mean $a_i \leq b_i$ for all i.

In Section 2, we give an explicit parametrization of the fixed point set $\mathcal{Q}_d^T$ by a set $\mathcal{F}$ of triples $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$, where

- $\boldsymbol{\delta} \in \{0, 1\}^n$ takes the value 1 exactly $n - r$ times, so that $|\boldsymbol{\delta}| = n - r$, and
- $\mathbf{a}, \mathbf{b}$ are elements of $(\mathbb{Z}_{\geq 0})^n$ which satisfy $|\mathbf{a}| + |\mathbf{b}| = d$ and for which $\delta_i = 0$ implies $a_i = 0$ and $b_i = 0$.

The fixed points are maximally degenerate quotient sheaves supported at 0 and ∞; the data $\mathbf{a}$ and $\mathbf{b}$ describe the structure of the stalks as modules over $\mathcal{O}_{\mathbb{P}^1}$ and as representations of T.

Recall that $A_T^*(p) = S$, the symmetric algebra of the character group of T. We have $S = \mathbb{Z}[\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}]$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$, and $\mathbf{f}$ are dual to the obvious basis coming from the decomposition $T = T_{k^n} \times T_{\mathbb{P}^1}$. In particular, $\mathbf{f}$ restricts to the identity character on $T_{\mathbb{P}^1}$ and to the trivial character of T_{k^n}. We write $S_{\mathbb{Q}}$ for $S \otimes_{\mathbb{Z}} \mathbb{Q}$ and sometimes $S_{\mathbb{Z}}$ for S, when we wish to emphasize our ring of scalars. We write $S^{\mathcal{F}}$ for the set of tuples of polynomials $(f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} \in S \mid (\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}) \in \mathcal{F})$. Then $S^{\mathcal{F}} = A_T^*(Q_d^T)$, under the identification of $\mathcal{F}$ with Q_d^T. We exhibit the image of the localization map as a subring of $S^{\mathcal{F}}$.

In Section 3 we describe a finite set of T-invariant curves which span the tangent space at each fixed point. In Section 4, we describe the families of T-invariant curves and their closures. Based on this description, our relations for the image of $i^*_{\mathcal{Q}_d}$ are

I For any pair $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}), (\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}') \in \mathcal{F}$ with $\boldsymbol{\delta} = \boldsymbol{\delta}'$, $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$ except in positions i and j, and $\delta_i = \delta'_j = 1$ and $\delta_j = \delta'_i = 0$, we have

$$f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} \equiv f_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')} \mod \mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f}.$$

(Note that $(a'_j - a_i) = -(b'_j - b_i)$, since $a_i + b_i = a'_j + b'_j$.)

II (a) For any pair $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}), (\boldsymbol{\delta}, \mathbf{a}', \mathbf{b}) \in \mathcal{F}$ with $\mathbf{a}, \mathbf{a}'$ agreeing except in positions i and j, we have

$$f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} \equiv f_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b})} \mod \mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f}.$$

(b) For any pair $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}), (\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}') \in \mathcal{F}$ with $\mathbf{b}, \mathbf{b}'$ agreeing except in positions i and j, we have

$$f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} \equiv f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b}')} \mod \mathbf{e}_j - \mathbf{e}_i + (b_i - b'_j)\mathbf{f}.$$

(c) If we have $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}), (\boldsymbol{\delta}, \mathbf{a}', \mathbf{b}), (\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}') \in \mathcal{F}$ satisfying both of the previous conditions (with the same i and j), and in addition $a'_j - a_i = b_i - b'_j$, then

$$Df_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} - Df_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b})} - Df_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b}')} + Df_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b}')} \equiv 0 \mod \mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f},$$

where D is the differentiation in the direction of $\mathbf{e}_j^\vee$, the dual basis vector to $\mathbf{e}_j$.

(c)′ Under the hypotheses of **II(c)**,

$$f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} - f_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b})} - f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b}')} + f_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b}')} \equiv 0 \mod (\mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f})^2.$$

III For every $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}) \in \mathcal{F}$ and every $\mathbf{0} \le \mathbf{b}' < \mathbf{b}$,

$$\sum_{\mathbf{b}' \le \mathbf{c} \le \mathbf{b}} \frac{(-1)^{|\mathbf{c}|}}{(\mathbf{b}-\mathbf{c})!(\mathbf{c}-\mathbf{b}')!} D^{|\mathbf{b}|-|\mathbf{b}'|-1} f_{(\boldsymbol{\delta},\mathbf{a}+\mathbf{c},\mathbf{b}-\mathbf{c})} \equiv 0 \mod \mathbf{f},$$

where D is the differentiation in the direction of $\mathbf{f}^\vee$, the dual basis vector to $\mathbf{f}$.

III′ For every $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}) \in \mathcal{F}$, $\mathbf{b} \neq \mathbf{0}$,

$$\sum_{0 \le \mathbf{c} \le \mathbf{b}} \frac{(-1)^{|\mathbf{c}|}}{\mathbf{c}!(\mathbf{b}-\mathbf{c})!} f_{(\boldsymbol{\delta},\mathbf{a}+\mathbf{c},\mathbf{b}-\mathbf{c})} \equiv 0 \mod \mathbf{f}^{|\mathbf{b}|}.$$

By this, we mean that the left-hand side, which *a priori* is an element of $S_{\mathbb{Q}}$, actually lies in $\mathbf{f}^{|\mathbf{b}|} S_{\mathbb{Z}}$.

Relations **I**, **II**(a), and **II**(b) are standard GKM relations, while the rest come from families of T-invariant curves. In particular, relations **II**(c)/**II**(c)$'$ (respectively **III**/**III**$'$) come from certain families whose closures are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (respectively arbitrary products of projective spaces).

Theorem 1.1. *The rational equivariant Chow ring $A_T^*(\mathcal{Q}_d)_{\mathbb{Q}}$ is isomorphic to the set of tuples $f = (f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}) \in S_{\mathbb{Q}}^{\mathcal{F}}$ subject to the relations* **I**, **II**(a)–(c), *and* **III**.

The integral equivariant Chow ring $A_T^*(\mathcal{Q}_d)$ is isomorphic to the set of tuples $f = (f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}) \in S_{\mathbb{Z}}^{\mathcal{F}}$ subject to the relations **I**, **II**(a)(b)(c)$'$, *and* **III**$'$.

We prove Theorem 1.1 in Section 5E. Since the equivariant Chow ring determines the ordinary Chow ring for smooth spaces, this gives in principle a complete description of the Chow ring of $\mathcal{Q}_d$. The resulting computation of Betti numbers is the same as Strømme's computation.

Strømme's generators of the rational cohomology ring of $\mathcal{Q}_d$ were Künneth components of the Chern classes of the tautological vector bundle $\mathcal{S}$ on $\mathbb{P}^1 \times \mathcal{Q}_d$ defined by the universal exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}^n_{\mathbb{P}^1 \times \mathcal{Q}_d} \to \mathcal{T} \to 0,$$

where the restriction of $\mathcal{T}$ to $\mathbb{P}^1 \times \{p\} \cong \mathbb{P}^1$ is the quotient sheaf of $\mathcal{O}^n_{\mathbb{P}^1}$ represented by $p \in \mathcal{Q}_d$. In Section 6, we describe the equivariant Chern classes of $\mathcal{S}$ in $A_T^*(\mathcal{Q}_d)$ and thus lifts of Strømme's generators to the equivariant Chow groups. In Section 7 we work this out explicitly for $\mathcal{Q}_2(0, 2)$, using Theorem 1.1 to describe the equivariant and ordinary Chow rings and giving explicit lifts of Strømme's generators as localized classes.

2. Torus-fixed points of $\mathcal{Q}_d$

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the standard basis of k^n. Write T_{k^n} for the group of diagonal matrices in this basis. Let $[x, y]$ be coordinates on $\mathbb{P}^1$ with x vanishing at 0 and y at ∞. For $T_{\mathbb{P}^1}$ acting on $\mathbb{P}^1$ with fixed points 0 and ∞, the torus $T := T_{k^n} \times T_{\mathbb{P}^1}$ acts on $\mathcal{Q}_d$ naturally as indicated by the given splitting.

The T-fixed points are indexed by triples $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ in the set $\mathcal{F}$ of Section 1. The fixed point corresponding to $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ is the sequence of sheaves on $\mathbb{P}^1$,

$$\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} \to \mathcal{O}^n_{\mathbb{P}^1} \to \mathcal{T}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})},$$

where $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ is the image of the map

$$\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(-a_i - b_i) \xrightarrow{\operatorname{diag}(\delta_i x^{a_i} y^{b_i})} \mathcal{O}^n_{\mathbb{P}^1}.$$

We identify this fixed point with the subsheaf $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ of $\mathcal{O}^n_{\mathbb{P}^1}$.

We introduce the following notation. For natural numbers a, b, let $\mathcal{S}_{a,b}$ be the subsheaf of $\mathcal{O}_{\mathbb{P}^1}$ which is the image of the map

$$\mathcal{O}_{\mathbb{P}^1}(-a-b) \xrightarrow{x^a y^b} \mathcal{O}_{\mathbb{P}^1}.$$

Under the identification of modules over $\mathcal{O}_{\mathbb{P}^1}$ with saturated graded modules of the homogeneous coordinate ring $k[x, y]$, $\mathcal{S}_{a,b}$ is the ideal of $k[x, y]$ generated by $x^a y^b$. The quotient $\mathcal{O}_{\mathbb{P}^1}/\mathcal{S}_{a,b}$ is the skyscraper sheaf

$$\mathcal{T}_{a,b} = \mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_0^a \oplus \mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_\infty^b$$

on $\mathbb{P}^1$ supported at 0 and at ∞. Here, $\mathfrak{m}_p$ is the sheaf of ideals cutting out the point $p \in \mathbb{P}^1$. Then

$$\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} = \bigoplus_{\delta_i=1} \mathcal{S}_{a_i,b_i} \cdot \mathbf{e}_i, \quad \text{and}$$
$$\mathcal{T}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} = \bigoplus_{\delta_j=1} \mathcal{T}_{a_j,b_j} \cdot \mathbf{e}_j \oplus \bigoplus_{\delta_j=0} \mathcal{O}_{\mathbb{P}^1} \cdot \mathbf{e}_j.$$

In the sum, $\delta_j = 0$ means those j in $\{1, \ldots, n\}$ with $\delta_j = 0$, and the same for $\delta_j = 1$.

The tangent space to $\mathcal{Q}_d$ at this fixed point is

$$\operatorname{Hom}(\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}, \mathcal{T}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}).$$

Let $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_n$ be the basis dual to $\mathbf{e}_1, \ldots, \mathbf{e}_n$. For each i, j, set

$$\mathbf{E}_{ij} := \hat{\mathbf{e}}_i \otimes \mathbf{e}_j \in \operatorname{Hom}(k^n, k^n).$$

These $\mathbf{E}_{ij}$ form a basis for $\operatorname{Hom}(k^n, k^n)$.

Theorem 2.1. *The tangent space $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathcal{Q}_d$ is canonically identified with*

$$\bigoplus_{\delta_i=1} \bigoplus_{\delta_j=0} \operatorname{Hom}(\mathcal{S}_{a_i,b_i}, \mathcal{O}_{\mathbb{P}^1}) \cdot \mathbf{E}_{ij} \;\oplus\; \bigoplus_{\delta_i=1} \bigoplus_{\delta_j=1} \operatorname{Hom}(\mathcal{S}_{a_i,b_i}, \mathcal{T}_{a_j,b_j}) \cdot \mathbf{E}_{ij}. \tag{2-1}$$

We now give T-bases (bases of T-eigenvectors) for these summands and determine the corresponding weights. Fix a basis for the character group of T as follows. Extend the action of T_{k^n} on $\mathcal{O}_{\mathbb{P}^1}^n$ to T by letting the factor $T_{\mathbb{P}^1}$ act trivially. Then we abuse notation and denote the character of T acting on the i-th basis vector $\mathbf{e}_i$ by the same symbol $\mathbf{e}_i$. Thus the dual basis element $\hat{\mathbf{e}}_i$ has T-weight $-\mathbf{e}_i$.

Similarly, extend the action of $T_{\mathbb{P}^1}$ on $\mathbb{P}^1$ to an action of T by letting T_{k^n} act trivially, and denote by $\mathbf{f}$ the character of T corresponding to the action on the dense orbit $\mathbb{P}^1 \setminus \{0, \infty\}$. More precisely, we can let $T_{\mathbb{P}^1} \cong \mathbb{C}^*$ act on the homogeneous coordinates $k[x, y]$ of $\mathbb{P}^1$ by $q \cdot x = qx$ and $q \cdot y = y$. Thus T acts on the rational function $z := x/y$ with weight $\mathbf{f}$, and on the monomial $z^a = x^a y^{-a}$ with weight $a\mathbf{f}$.

The first sum of (2-1) involves spaces of the form $\mathrm{Hom}(\mathcal{S}_{a,b}, \mathcal{O}_{\mathbb{P}^1}) = H^0(\mathcal{S}^*_{a,b})$. This space of sections has a monomial T-basis

$$\{z^c \mid -a \leq c \leq b\}.$$

For example, $H^0(\mathcal{S}^*_{1,2}) = k \cdot \{z^{-1}, z^0 = 1, z, z^2\}$. Thus if $\delta_i = 1$ and $\delta_j = 0$, then the piece $\mathrm{Hom}(\mathcal{S}_{a_i,b_i}, \mathcal{O}_{\mathbb{P}^1}) \cdot \mathbf{E}_{ij}$ of the tangent space has a monomial T-basis $z^c \cdot \mathbf{E}_{ij}$ for all $-a_i \leq c \leq b_i$. The basis element $z^c \cdot \mathbf{E}_{ij}$ has T-weight

$$\mathbf{e}_j - \mathbf{e}_i + c\mathbf{f}. \tag{2-2}$$

The second sum of (2-1) involves spaces of the form $\mathrm{Hom}(\mathcal{S}_{a,b}, \mathcal{T}_{\alpha,\beta})$. Since $\mathcal{T}_{\alpha,\beta}$ is a skyscraper sheaf supported at 0 and ∞, a map $\phi \in \mathrm{Hom}(\mathcal{S}_{a,b}, \mathcal{T}_{\alpha,\beta})$ is determined by its actions at 0 and at ∞. At 0, $z = x/y$ is a local parameter, so the map ϕ becomes

$$\phi\colon z^a\mathbb{C}[z] \to \mathbb{C}[z]/\langle z^\alpha\rangle,$$

and thus has the form $z^{-a} f(z)$ where $f(z)$ has degree less than α. At ∞, z^{-1} is a local parameter, and the map ϕ has the form $z^b g(z^{-1})$ where $g(z)$ is a polynomial of degree less than β. Thus $\mathrm{Hom}(\mathcal{S}_{a,b}, \mathcal{T}_{\alpha,\beta})$ has the monomial T-basis

$$k \cdot \{z^{(\alpha-c)-a} \mid 1 \leq c \leq \alpha\} \oplus k \cdot \{z^{b-(\beta-c)} \mid 1 \leq c \leq \beta\},$$

where elements in the first summand act by zero on the stalk at ∞, and elements of the second summand act by zero at 0.

Thus, if $\delta_i = \delta_j = 1$, then the summand $\mathrm{Hom}(\mathcal{S}_{a_i,b_i}, \mathcal{T}_{a_j,b_j}) \cdot \mathbf{E}_{ij}$ of (2-1) has a monomial T-basis

$$\{z^{(a_j-c)-a_i} \cdot \mathbf{E}_{ij} \mid 1 \leq c \leq a_j\} \cup \{z^{(b_j-c)-b_i} \cdot \mathbf{E}_{ij} \mid 1 \leq c \leq b_j\}$$

with corresponding T-weights

$$\mathbf{e}_j - \mathbf{e}_i + ((a_j - c) - a_i)\mathbf{f} \quad \text{and} \quad \mathbf{e}_j - \mathbf{e}_i + (b_i - (b_j - c))\mathbf{f}. \tag{2-3}$$

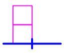
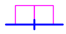
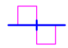
We note that this discussion gives a basis for $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathcal{Q}_d$ consisting of

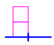
$$\begin{aligned}\sum_{\delta_j=0}\sum_{\delta_i=1}(a_i+b_i+1) + \sum_{\delta_i=1}\sum_{\delta_j=1}(a_j+b_j) &= (n-r)\cdot(d+r) + r\cdot d \\ &= r\cdot(n-r) + nd = \dim \mathcal{Q}_d\end{aligned}$$

elements, which shows that $\mathcal{Q}_d$ is smooth at the T-fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$, and hence everywhere, since $\mathcal{Q}_d$ is projective. In Section 3, we will describe T-invariant curves in $\mathcal{Q}_d$ containing $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ whose tangent directions at $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ coincide with this given T-basis of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathcal{Q}_d$.

Example 2.2. The quot scheme $\mathcal{Q} := \mathcal{Q}_2(0,2)$ of rank 0 and degree 2 quotients of $\mathcal{O}^2_{\mathbb{P}^1}$ has dimension $r(n-r) + dn = 2\cdot 0 + 2\cdot 2 = 4$. Note that the associated

Grassmannian is a point. Since $r = 0$, the index $\boldsymbol{\delta}$ is the same for each fixed point, $\delta_1 = \delta_2 = 1$, and so the fixed points are indexed by quadruples (a_1, a_2, b_1, b_2) of nonnegative integers whose sum is 2. Thus there are ten fixed points. We represent the fixed point (a_1, a_2, b_1, b_2) by two columns of boxes superimposed on a horizontal line where the i-th column has a_i boxes above the horizontal line and b_i boxes below it. For example,

 $\leftrightarrow (2, 0, 0, 0)$,  $\leftrightarrow (1, 1, 0, 0)$, and  $\leftrightarrow (1, 0, 0, 1)$.

The fixed point corresponding to  is the exact sequence of sheaves on $\mathbb{P}^1$,

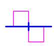
$$0 \longrightarrow \begin{matrix} \mathbb{O}(-2)\cdot\mathbf{e}_1 \\ \oplus \\ \mathbb{O}\cdot\mathbf{e}_2 \end{matrix} \begin{matrix} \xrightarrow{x^2} \\ \\ \xrightarrow{1} \end{matrix} \begin{matrix} \mathbb{O}\cdot\mathbf{e}_1 \\ \oplus \\ \mathbb{O}\cdot\mathbf{e}_2 \end{matrix} \begin{matrix} \longrightarrow \\ \\ \longrightarrow \end{matrix} \begin{matrix} \mathbb{O}/\mathfrak{m}_0^2\cdot\mathbf{e}_1 \\ \oplus \\ 0 \end{matrix} \longrightarrow 0,$$

where $\mathbb{O} = \mathbb{O}_{\mathbb{P}^1}$. The tangent space at this fixed point is the sum of the two 2-dimensional T-invariant spaces of homomorphisms having the indicated T-bases:

$$\operatorname{Hom}(\mathbb{O}, \mathbb{O}/\mathfrak{m}_0^2)\cdot\mathbf{E}_{21} = k\{1, z\}\cdot\mathbf{E}_{21},$$
$$\operatorname{Hom}(x^2\mathbb{O}, \mathbb{O}/\mathfrak{m}_0^2)\cdot\mathbf{E}_{11} = k\{z^{-2}, z^{-1}\}\cdot\mathbf{E}_{11}.$$

As before, $z := x/y$ is a local parameter at 0 and z^{-1} is a local parameter at ∞. These basis elements have four distinct T-weights

$$\mathbf{e}_1 - \mathbf{e}_2,\ \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{f},\ \text{and}\ -\mathbf{f},\ -2\mathbf{f}.$$

The fixed point corresponding to  is the exact sequence of sheaves on $\mathbb{P}^1$,

$$0 \longrightarrow \begin{matrix} \mathbb{O}(-1)\cdot\mathbf{e}_1 \\ \oplus \\ \mathbb{O}(-1)\cdot\mathbf{e}_2 \end{matrix} \begin{matrix} \xrightarrow{x} \\ \\ \xrightarrow{y} \end{matrix} \begin{matrix} \mathbb{O}\cdot\mathbf{e}_1 \\ \oplus \\ \mathbb{O}\cdot\mathbf{e}_2 \end{matrix} \begin{matrix} \longrightarrow \\ \\ \longrightarrow \end{matrix} \begin{matrix} \mathbb{O}/\mathfrak{m}_0\cdot\mathbf{e}_1 \\ \oplus \\ \mathbb{O}/\mathfrak{m}_\infty\cdot\mathbf{e}_2 \end{matrix} \longrightarrow 0.$$

The tangent space at this fixed point is the sum of four 1-dimensional T-invariant spaces of homomorphisms having bases and weights as indicated from the following table:

T-eigenspace	basis	T-weight
$\operatorname{Hom}(x\mathbb{O}, \mathbb{O}/\mathfrak{m}_0)\cdot\mathbf{E}_{11}$	$z^{-1}\cdot\mathbf{E}_{11}$	$-\mathbf{f}$
$\operatorname{Hom}(x\mathbb{O}, \mathbb{O}/\mathfrak{m}_\infty)\cdot\mathbf{E}_{12}$	$1\cdot\mathbf{E}_{12}$	$\mathbf{e}_2 - \mathbf{e}_1$
$\operatorname{Hom}(y\mathbb{O}, \mathbb{O}/\mathfrak{m}_0)\cdot\mathbf{E}_{21}$	$1\cdot\mathbf{E}_{21}$	$\mathbf{e}_1 - \mathbf{e}_2$
$\operatorname{Hom}(y\mathbb{O}, \mathbb{O}/\mathfrak{m}_\infty)\cdot\mathbf{E}_{22}$	$z\cdot\mathbf{E}_{22}$	$\mathbf{f}$

3. T-invariant curves

A flat family $\mathcal{S} \to \mathbb{P}^1$ equipped with a T-action whose fibre $\mathcal{S}(s,t)$ over a point $[s,t] \in \mathbb{P}^1$ is a free subsheaf of $\mathbb{O}^n_{\mathbb{P}^1}$ of rank $n-r$ and degree $-d$ gives a T-equivariant map $f_{\mathcal{S}} \colon \mathbb{P}^1 \to \mathcal{Q}_d$. When the family $\mathcal{S}$ is not trivial (T-equivariantly isomorphic to a product $\mathcal{S}_0 \times \mathbb{P}^1$ with T acting trivially on the base $\mathbb{P}^1$), then its image is a T-invariant curve in $\mathcal{Q}_d$.

Here, we describe a collection of T-invariant curves whose tangent directions at each T-fixed point $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ form a basis for $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathcal{Q}_d$. We exhibit each curve as a subsheaf $\mathcal{S}$ of $\mathbb{O}^n_{\mathbb{P}^1\times\mathbb{P}^1}$ of rank $n-r$ and degree $(-d,-1)$ with

$$\mathcal{S}(1,0) = \mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}, \quad \text{and} \quad \mathcal{S}(0,1) = \mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')},$$

where $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ is some other T-fixed point on $\mathcal{Q}_d$. Each subsheaf $\mathcal{S}$ is T-invariant, and so defines a T-invariant curve on $\mathcal{Q}_d$ connecting the two fixed points. Requiring that the second component of the degree is -1 guarantees that the family is not trivial.

We will show in the proof of Theorem 3.2 below that the differentials $df_{\mathcal{S}}$ at the fixed points $[0,1], [1,0] \in \mathbb{P}^1$ are nonzero for each subsheaf $\mathcal{S}$ that we consider. This implies that the maps $f_{\mathcal{S}}$ are closed immersions with image a smooth T-invariant curve. Further, we show that the tangent spaces to these curves form a basis of the tangent space at each T-fixed point which is compatible with the decomposition (2-1) (in fact, it coincides with the further decomposition of each summand of (2-1) into one-dimensional spaces spanned by monomials which was described following Theorem 2.1).

Each sheaf $\mathcal{S}$ has one of three types: I, II, or III. We describe them below and then argue that they have the desired properties. We write $\mathbb{O}$ for $\mathbb{O}_{\mathbb{P}^1\times\mathbb{P}^1}$. We use the correspondence between sheaves over $\mathbb{O}$ and saturated modules over the bihomogeneous coordinate ring $k[x,y][s,t]$ of $\mathbb{P}^1_{[x,y]} \times \mathbb{P}^1_{[s,t]}$. Then $\mathbb{O}^n$ is the rank n free module with basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$.

Type I: Let $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ be another fixed point where the data $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ and $(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b}')$ agree except in positions i and j, with $\delta_i = \delta'_j = 1$ and $\delta_j = \delta'_i = 0$. Note that $a_i + b_i = a'_j + b'_j$. Let $\mathcal{S}$ be the subsheaf of $\mathbb{O}^n$ which agrees with both $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$ except for its component in $\mathbb{O}\cdot\mathbf{e}_i + \mathbb{O}\cdot\mathbf{e}_j$, where it is the rank 1 and degree $-(a_i+b_i)$ subsheaf generated by the single element

$$sx^{a_i}y^{b_i}\cdot\mathbf{e}_i + tx^{a'_j}y^{b'_j}\cdot\mathbf{e}_j. \tag{3-1}$$

Type II: Let $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ be another T-fixed point where $\boldsymbol{\delta} = \boldsymbol{\delta}'$, $\mathbf{b} = \mathbf{b}'$, and the data $\mathbf{a}$ and $\mathbf{a}'$ agree except in positions i and j with $i \neq j$. We suppose that i and j have been chosen so that $a_i < a'_i$. Then $a_j > a'_j$ and $c := a'_i - a_i = a_j - a'_j > 0$. Set

$$\gamma := a_i + b_i + c - a_j - b_j.$$

If $\gamma \geq 0$, let $\mathcal{S}$ be the subsheaf of $\mathbb{O}^n$ which agrees with both $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$ except for its components in $\mathbb{O}\cdot\mathbf{e}_i + \mathbb{O}\cdot\mathbf{e}_j$, where it is the subsheaf generated by

$$x^{a_i+c}y^{b_i}\cdot\mathbf{e}_i,\quad x^{a_j}y^{b_j}\cdot\mathbf{e}_j,\quad \text{and}\quad sx^{a_i}y^{b_i}\cdot\mathbf{e}_i + tx^{a_j-c}y^{b_j+\gamma}\cdot\mathbf{e}_j. \tag{3-2}$$

When $\gamma < 0$, replace the third generator by $sx^{a_i}y^{b_i-\gamma}\cdot\mathbf{e}_i + tx^{a_j-c}y^{b_j}\cdot\mathbf{e}_j$.

The remaining sheaves of Type II are obtained by interchanging the roles of $\mathbf{a}$ and $\mathbf{b}$. That is, $\boldsymbol{\delta} = \boldsymbol{\delta}'$, $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b}$, $\mathbf{b}'$ agree except in positions $i \neq j$, and we further have that $c := b'_i - b_i = b_j - b'_j > 0$. Set

$$\gamma := a_i + b_i + c - a_j - b_j.$$

If $\gamma \geq 0$, let $\mathcal{S}$ be as before, except with the generators (3-2) replaced by

$$x^{a_i}y^{b_i+c}\cdot\mathbf{e}_i,\quad x^{a_j}y^{b_j}\cdot\mathbf{e}_j,\quad \text{and}\quad sx^{a_i}y^{b_i}\cdot\mathbf{e}_i + tx^{a_j+\gamma}y^{b_j-c}\cdot\mathbf{e}_j. \tag{3-3}$$

If $\gamma < 0$, then the third generator will be $sx^{a_i-\gamma}y^{b_i}\cdot\mathbf{e}_i + tx^{a_j}y^{b_j-c}\mathbf{e}_j$.

Type III: Let $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ be another T-fixed point where $\boldsymbol{\delta} = \boldsymbol{\delta}'$ and the data $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}', \mathbf{b}')$ agree except in position i. Thus $\delta_i \neq 0$ and $a_i + b_i = a'_i + b'_i$. Let $\mathcal{S}$ be the subsheaf of $\mathbb{O}^n$ which agrees with both $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$, except for its component in $\mathbb{O}\cdot\mathbf{e}_i$, where it is the rank 1 and degree $-(a_i + b_i)$ subsheaf generated by

$$sx^{a_i}y^{b_i}\cdot\mathbf{e}_i + tx^{a'_i}y^{b'_i}\cdot\mathbf{e}_i. \tag{3-4}$$

Theorem 3.1. *The subsheaves $\mathcal{S}$ of $\mathbb{O}^n_{\mathbb{P}^1\times\mathbb{P}^1}$ of types I, II, and III are T-invariant and free of rank $n-r$ and degree $(-d, -1)$. They satisfy*

$$\mathcal{S}(1,0) = \mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}, \quad \text{and} \quad \mathcal{S}(0,1) = \mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')},$$

and hence define T-invariant curves on $\mathfrak{Q}_d$.

Proof. The generators of $\mathcal{S}$ are T-invariant, except for those described by (3-1), (3-2), (3-3), and (3-4). But T acts transitively on those generators for $s\cdot t \neq 0$. Therefore, each sheaf $\mathcal{S}$ is T-invariant. In all cases, $\mathcal{S}$ has degree -1 with respect to $\mathbb{P}^1_{[s,t]}$.

The theorem is clear for the sheaves of types I and III, as they are constant on $\mathbb{P}^1_{[s,t]}$, except for the rank 1 components (3-1) and (3-4), each of which has degree $(-(a_i + b_i), -1)$. Specializing these generators at $[s,t] = [1,0]$ and $[0,1]$ shows that $\mathcal{S}(1,0) = \mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}(0,1) = \mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$.

We use a Gröbner basis argument for the sheaves of type II. The Hilbert function for a submodule M of $\mathbb{O}^2$ equals the Hilbert function for the module of leading terms of any Gröbner basis of M. As explained in [Eisenbud 1995, Chapter 15], a weight ω selecting these leading terms induces a $\mathbb{G}_m$-action on $\mathbb{O}^2$ whose restriction to the Gröbner basis of M generates a flat family over $\mathbb{A}^1$ whose special fibre is the module of leading terms.

For now, set $s = t = 1$. If $\mathbf{e}_i > \mathbf{e}_j$, then the generators (3-2) form a Gröbner basis for any position-over-monomial ordering, and the third generator has leading term $x^{a_i} y^{b_i} \cdot \mathbf{e}_i$. As $c > 0$, the module of leading terms is generated by $x^{a_i} y^{b_i} \cdot \mathbf{e}_i$ and $x^{a_j} y^{b_j} \cdot \mathbf{e}_j$, and so it has rank 2 and degree $-(a_i + b_i + a_j + b_j)$. The weight ω with $\omega(\mathbf{e}_i) = 0$ and $\omega(\mathbf{e}_j) = -1$ induces the leading terms and has corresponding $\mathbb{G}_m$-action $t \cdot (\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{e}_i, t\mathbf{e}_j)$, for $t \in \mathbb{G}_m$. This action on $\mathcal{S}(1, 1)$ is the flat family of modules over $\mathbb{A}^1$ generated by

$$x^{a_i+c} y^{b_i} \cdot \mathbf{e}_i, \quad x^{a_j} y^{b_j} \cdot \mathbf{e}_j, \quad \text{and} \quad x^{a_i} y^{b_i} \cdot \mathbf{e}_i + t x^{a_j-c} y^{b_j+\gamma} \cdot \mathbf{e}_j,$$

which is just the part of $\mathcal{S}$ in $\mathbb{O} \cdot \mathbf{e}_i + \mathbb{O} \cdot \mathbf{e}_j$ restricted to the affine subset U of $\mathbb{P}^1_{[s,t]}$ where $s \neq 0$. Thus $\mathcal{S}|_U$ is a flat family over U of free subsheaves of $\mathbb{O}^n_{\mathbb{P}^1}$ of rank $n - r$ and degree $-d$, and $\mathcal{S}(1, 0) = \mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$.

When $s = t = 1$ the generators (3-2) form a Gröbner basis when $\mathbf{e}_i < \mathbf{e}_j$, where the third generator has leading term $x^{a_j-c} y^{b_j+\gamma} \cdot \mathbf{e}_j$. The module of leading terms is generated by

$$x^{a_i+c} y^{b_i} \cdot \mathbf{e}_i, \quad x^{a_j} y^{b_j} \cdot \mathbf{e}_j, \quad \text{and} \quad x^{a_j-c} y^{b_j+\gamma} \cdot \mathbf{e}_j.$$

Since $\gamma \geq 0$ and $c > 0$, saturating the ideal of $k[x, y]$ generated by $x^{a_j} y^{b_j}$ and $x^{a_j-c} y^{b_j+\gamma}$ by the irrelevant maximal ideal generated by x and y gives the ideal generated by $x^{a_j-c} y^{b_j}$. Thus the module of leading terms is generated by

$$x^{a_i+c} y^{b_i} \cdot \mathbf{e}_i, \quad \text{and} \quad x^{a_j-c} y^{b_j} \cdot \mathbf{e}_j.$$

As before, restricting $\mathcal{S}$ to the affine set of points $[s, t]$ of $\mathbb{P}^1$ where $t \neq 0$ gives a flat family of subsheaves of $\mathbb{O}^n_{\mathbb{P}^1}$ of rank $n - r$ and degree $-d$ with special fibre $\mathcal{S}(0, 1) = \mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$. The same arguments suffice for the module generated by (3-3). □

Theorem 3.2. *All of the T-invariant curves induced by the sheaves $\mathcal{S}$ of types I, II, and III are smooth. For any T-fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ in $\mathcal{Q}_d$, the set of tangent directions to the curves which contain this point corresponds to the T-basis of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} \mathcal{Q}_d$ defined in Section 2, and this correspondence is T-equivariant, respecting the weights. More specifically, at the T-fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$,*

I. *The weight of Type I curve (3-1) is*

$$\mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f},$$

and such curves correspond to the first summand of (2-1).

II. *The weight of* Type II *curve* (3-2) *is*

$$\mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f},$$

the weight of Type II *curve* (3-3) *is*

$$\mathbf{e}_j - \mathbf{e}_i + (b_i - b'_j)\mathbf{f},$$

and such curves correspond to the second summand of (2-1) *when* $i \neq j$.

III. *The weight of* Type III *curve* (3-4) *is*

$$(a'_j - a_i)\mathbf{f},$$

and such curves correspond to the second summand of (2-1) *when* $i = j$.

Proof. In what follows, we work locally near $[1,0]$ by setting $s = 1$. The same arguments handle the other fixed point $[0, 1]$.

Note that the generator (3-1) of a Type I sheaf may be rewritten

$$x^{a_i} y^{b_i} \cdot \big(\mathbf{e}_i + t \cdot x^{a'_j - a_i} y^{b'_j - b_i} \mathbf{E}_{ij}(\mathbf{e}_i)\big).$$

This shows that the differential $df_{\mathcal{S}}$ at $[1,0]$ maps onto the span of the T-basis element $x^{a'_j - a_i} y^{b'_j - b_i} \mathbf{E}_{ij}$ of $\mathrm{Hom}(\mathcal{S}_{a_i,b_i} \cdot \mathbf{e}_i, \mathbb{O}_{\mathbb{P}^1} \cdot \mathbf{e}_j)$. Thus the Type I curves are smooth, and their tangent spaces at $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ span the component of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathfrak{Q}_d$ given by the first summand of (2-1). (Recall that in Type I, we have $\delta_i = \delta'_j = 1$ and $\delta_j = \delta'_i = 0$.)

A similar analysis shows that the tangent space at $t = 0$ of the Type III curve defined by (3-4) is spanned by $x^{a'_i - a_i} y^{b'_i - b_i} \mathbf{E}_{ii}$, and so the tangent spaces at $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ of Type III curves span the component of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathfrak{Q}_d$ given by the second summand of (2-1) when $i = j$.

For a curve of type II, note that the family of sheaves described by (3-2) is constant in a neighborhood of ∞. In a neighborhood of 0, it is given by

$$x^{a_i + c}\mathbf{e}_i, \quad x^{a_j}\mathbf{e}_j, \quad \text{and} \quad x^{a_i} \cdot \big(\mathbf{e}_i + t x^{(a_j - c) - a_i} \mathbf{E}_{ij}(\mathbf{e}_i)\big).$$

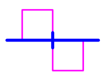
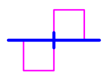
Thus $x^{(a_j - c) - a_i}\mathbf{E}_{ij}$ spans the tangent space at $t = 0$. A similar argument near ∞ for the sheaves described by (3-3) shows that the tangent spaces of Type II curves at $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ span the component of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathfrak{Q}_d$ given by the second summand of (2-1) when $i \neq j$. □

A *moment graph* of a T-variety is a graph whose vertices correspond to T-fixed points and whose edges correspond to T-invariant curves, embedded into $\mathbb{R} \otimes \mathrm{Hom}(T, \mathbb{Z})$ so that the edge corresponding to a T-invariant curve is parallel to the weight of the action of T on the curve. More specifically, if C is a T-invariant curve joining fixed points p and q, then the edge from p to q in the moment graph is a positive multiple of the T-weight of T_pC. When $k = \mathbb{C}$ and we fix a Kähler form, there is a moment map $\mu\colon \mathcal{Q}_d \to \mathfrak{t}^*$ and the image of the T-fixed points and T-invariant curves is a moment graph.

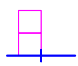
When there are finitely many T-invariant curves, the Goresky–Kottwitz–MacPherson method to compute equivariant cohomology is conveniently expressed in terms of a moment graph, with one relation for each edge. When there are infinitely many T-invariant curves, there are additional relations coming from families of T-invariant curves, so it is better to work with the moment *multigraph*, where each family of T-invariant curves (which will appear as a connected component of parallel edges in the moment graph) is considered to form a single multiedge with more than 2 vertices, given by the fixed points in the closure of the family. To have a structure which determines the equivariant cohomology or Chow groups, we should label each multiedge with the topological type of the corresponding family.

Guillemin and Zara [2001; 2002; 2003] have explored the combinatorial properties of moment graphs.

Example 3.3. Figure 1 represents a moment multigraph of $\mathcal{Q}_2(0, 2)$. Since the T-fixed points

 and 

have the same image in this and in any moment multigraph, we displace their images from their true positions for clarity. Similarly, some images of T-invariant curves are displaced or drawn as arcs.

The T-basis of the tangent space at  has weights

$$\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{f}, -\mathbf{f}, -2\mathbf{f}\}.$$

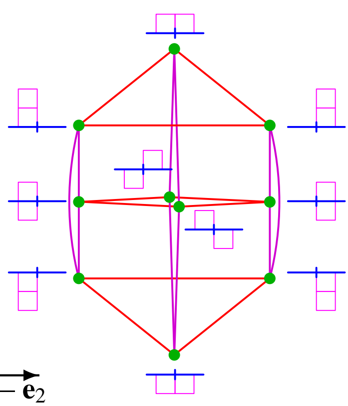
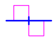


Figure 1. A moment multigraph of $\mathcal{Q}_2(0, 2)$.

These correspond to the following four T-invariant curves:

Generators of submodules of $\mathbb{O}^2$	$t=0$	$s=0$	Weight	Type
$x^2\mathbf{e}_1, x^2\mathbf{e}_2, t\mathbf{e}_1+s\mathbf{e}_2$	$x^2\mathbf{e}_1, 1\cdot\mathbf{e}_2$	$\mathbf{e}_1, x^2\mathbf{e}_2$	$1\cdot\mathbf{e}_1-\mathbf{e}_2$	II
$x^2\mathbf{e}_1, x^2\mathbf{e}_2, tx\mathbf{e}_1+sy\mathbf{e}_2$	$x^2\mathbf{e}_1, 1\cdot\mathbf{e}_2$	$x\mathbf{e}_1, x\mathbf{e}_2$	$\mathbf{e}_1-\mathbf{e}_2+\mathbf{f}$	II
$(sx^2+txy)\mathbf{e}_1, \mathbf{e}_2$	$x^2\mathbf{e}_1, 1\cdot\mathbf{e}_2$	$xy\mathbf{e}_1, 1\cdot\mathbf{e}_2$	$-\mathbf{f}$	III
$(sx^2+ty^2)\mathbf{e}_1, \mathbf{e}_2$	$x^2\mathbf{e}_1, 1\cdot\mathbf{e}_2$	$y^2\mathbf{e}_1, 1\cdot\mathbf{e}_2$	$-2\mathbf{f}$	III

The T-basis to the tangent space at  has weights

$$\{\pm\mathbf{f}, \pm(\mathbf{e}_2-\mathbf{e}_1)\}.$$

These correspond to the following four T-invariant curves:

Generators of submodules of $\mathbb{O}^2$	$t=0$	$s=0$	Weight	Type
$(sx+ty)\mathbf{e}_1, y\mathbf{e}_2$	$x\mathbf{e}_1, y\mathbf{e}_2$	$y\mathbf{e}_1, y\mathbf{e}_2$	$-\mathbf{f}$	III
$xy\mathbf{e}_1, y\mathbf{e}_2, sx\mathbf{e}_1+tx\mathbf{e}_2$	$x\mathbf{e}_1, y\mathbf{e}_2$	$xy\mathbf{e}_1, \mathbf{e}_2$	$\mathbf{e}_2-\mathbf{e}_1$	II
$x\mathbf{e}_1, xy\mathbf{e}_2, ty\mathbf{e}_1+sy\mathbf{e}_2$	$x\mathbf{e}_1, y\mathbf{e}_2$	$\mathbf{e}_1, xy\mathbf{e}_2$	$\mathbf{e}_1-\mathbf{e}_2$	II
$x\mathbf{e}_1, (sy+tx)\mathbf{e}_2$	$x\mathbf{e}_1, y\mathbf{e}_2$	$x\mathbf{e}_1, x\mathbf{e}_2$	$\mathbf{f}$	III

4. Families of T-invariant curves

Suppose that Z is a T-invariant curve on $\mathfrak{Q}_d$. Let T' be the identity component of the stabilizer in T of a general point of Z. It is a codimension one subtorus of T, so we can choose an isomorphism $T/T' \simeq \mathbb{G}_m$. The action of T on Z factors through the resulting quotient

$$T \longrightarrow T/T' \simeq \mathbb{G}_m.$$

This composition $\eta\colon T \to \mathbb{G}_m$ is a primitive weight. When T is smooth, it is a multiple of the weight of the action of T on the tangent space to Z at either fixed point.

Let Y be the component of the T'-fixed point locus $\mathfrak{Q}_d^{T'}$ which contains Z; since $\mathfrak{Q}_d$ is smooth, so is Y. Then $Y \setminus Y^T$ is foliated by one-dimensional orbits of T whose closures are T-invariant curves. We call Y the family of T-invariant curves on $\mathfrak{Q}_d$ which contains Z. If

$$p \in Y^T \subset \mathfrak{Q}_d^T$$

is a T-fixed point of Y, then T_pY is a T-invariant linear subspace of $T_p\mathfrak{Q}_d$ which is fixed pointwise by T'. In particular, all weights of the T-action on T_pY are multiples of η.

To classify families of T-invariant curves, we first determine which T-weights of $T_p\mathfrak{Q}_d$ are parallel, using Theorem 3.2. If the weight of a curve is not parallel to the weight of any other curve, the curve is isolated.

Theorem 4.1. *Two T-invariant curves $\mathcal{S}$, $\mathcal{S}'$ of types I, II, or III containing the fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ have parallel T-weights if and only if either*

(1) *both $\mathcal{S}$ and $\mathcal{S}'$ have Type III or*

(2) *both $\mathcal{S}$ and $\mathcal{S}'$ have Type II, and*
   - (a) *$\mathcal{S}$ connects $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ to $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b})}$,*
   - (b) *$\mathcal{S}'$ connects $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ to $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b}')}$, and*
   - (c) *$\mathbf{a}$ and $\mathbf{a}'$ agree except in positions i and j with $i \neq j$, $\mathbf{b}$ and $\mathbf{b}'$ agree except in positions i and j (same i, j), $a_i + b_i = a'_j + b'_j$.*

Proof. For (1), note that the weight of T on a Type III curve is parallel to $\mathbf{f}$.

If a curve does not have type III, then its weight has the form

$$\mathbf{e}_j - \mathbf{e}_i + c\,\mathbf{f},$$

where $\delta_i = 1$ and either $\delta_j = 0$ if it has type I or $\delta_j = 1$ if it has type II. Thus $\mathcal{S}$ and $\mathcal{S}'$ have the same weight and type, $\boldsymbol{\delta} = \boldsymbol{\delta}'$, and the indices i and j in their definitions coincide.

Weights of curves of types I and II correspond to (2-2) and (2-3), respectively. Inspecting (2-2) shows that no two curves of type I can have the same weight. Inspecting (2-3) reveals that either a given curve $\mathcal{S}$ of type II has a unique weight, or else there is exactly one other Type II curve $\mathcal{S}'$ with the same weight, and the two curves are as described in the statement of the theorem. □

We show that all Type III curves at a fixed point lie in a single family of T-invariant curves, and if two Type II curves have the same weight then they lie in a 2-dimensional family. Together with the isolated T-invariant curves, this shows that the tangent spaces at a given fixed point p to families of T-invariant curves are the subspaces of $T_p\mathcal{Q}_d$ which are stabilized by codimension 1 subtori of T. It follows that these families contain all T-invariant curves in $\mathcal{Q}_d$.

To see this, fix a weight and consider the family Y of all T-invariant curves meeting p that have that weight or a parallel weight. The common kernel of these parallel weights is a codimension 1 subtorus T' of T, which stabilises this family pointwise. Since $\mathcal{Q}_d$ is smooth, Y is smooth, and its tangent space at p is necessarily the T'-fixed subspace of the tangent space of $\mathcal{Q}_d$ at p. In particular the dimension of Y is the dimension of this T'-fixed subspace. Since the families described in Theorem 4.1 whose weights annihilate T' have dimension equal to the dimension of the T'-fixed subspace, there can be no other T'-stable curves.

Vertical families. The *vertical family* containing $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ is parametrized by the product of projective spaces

$$\prod_{\delta_i=1} \mathbb{P}H^0(\mathcal{O}(a_i+b_i)) \simeq \prod_{\delta_i=1} \mathbb{P}^{a_i+b_i}.$$

It contains exactly the fixed points $\mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$ where $\boldsymbol{\delta}=\boldsymbol{\delta}'$, and $\mathbf{a}'+\mathbf{b}'=\mathbf{a}+\mathbf{b}$, along with all Type III curves which connect them. This includes all Type III curves at each of these fixed points.

Consider the family $\mathcal{S}$ of submodules of $\mathbb{O}^n_{\mathbb{P}^1}$ generated by

$$\{\mathbf{e}_i s_i \mid \delta_i = 1,\, s_i \in H^0(\mathbb{O}(a_i+b_i))\}.$$

The base of this family is

$$\prod_{\delta_i=1} \mathbb{P}H^0(\mathbb{O}(a_i+b_i)),$$

all subsheaves have rank $n-r$ and degree $-d$, and the foliation by T-invariant curves is given by the T-action on the base.

Horizontal families. If there exist i, j, c, c' such that $1 \le c \le a_j$, $1 \le c' \le b_j$, and $a_i+b_i+c+c' = a_j+b_j$, then the point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ lies in a *horizontal family* parametrized by the product of two projective lines. Let

$$a_i' = a_i + c,\;\; a_j' = a_j - c,\;\; b_i' = b_i + c',\;\; b_j' = b_j - c'.$$

Let $([s,t],[\sigma,\tau])$ be the coordinates of $\mathbb{P}^1\times\mathbb{P}^1$, and let $\mathcal{S}$ be the submodule of $\mathbb{O}^n_{\mathbb{P}^1}$ which, except for its components in $\mathbb{O}\cdot\mathbf{e}_i+\mathbb{O}\cdot\mathbf{e}_j$, agrees with $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$. The component of $\mathcal{S}$ in $\mathbb{O}\cdot\mathbf{e}_i+\mathbb{O}\cdot\mathbf{e}_j$ is the subsheaf generated by

$$\mathbf{e}_j x^{a_j} y^{b_j},\;\; s\mathbf{e}_i x^{a_i'} y^{b_i} + t\mathbf{e}_j x^{a_j} y^{b_j'},\;\; \sigma\mathbf{e}_i x^{a_i} y^{b_i'} + \tau\mathbf{e}_j x^{a_j'} y^{b_j},\;\text{ and } \mathbf{e}_i x^{a_i'} y^{b_i'}.$$

Similar reasoning as for Theorem 3.1 shows that this defines a family of T-invariant curves over the base $\mathbb{P}^1\times\mathbb{P}^1$ with coordinates $([s,t],[\sigma,\tau])$. It contains four T-fixed points: setting $t=\tau=0$ gives the fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$, setting $t=\sigma=0$ gives the fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b})}$, setting $s=\tau=0$ gives the fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b}')}$, and setting $s=\sigma=0$ gives the fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b}')}$. This family also contains the four Type II curves connecting these four fixed points, given by setting exactly one of s, t, σ, or τ equal to zero. Furthermore, the data $(\boldsymbol{\delta},\mathbf{a},\mathbf{b})$ and $(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')$ satisfy Theorem 4.1(2)(c), and any two T-invariant curves $\mathcal{S}$ and $\mathcal{S}'$ as in Theorem 4.1(2) lie in a unique horizontal family.

We summarize the results of this section, in which we identified the fixed point loci of codimension 1 subtori of T.

Theorem 4.2. *The connected components of the fixed point loci of codimension* 1 *subtori of T which contain the fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ are*

(1) Type I *curves*, *which are isolated*;

(2) Type II *curves whose weight at $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ is unique*, *which are also isolated*;

(3) *horizontal families*, *which occur only when there are two curves of type* II *containing* $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ *with the same weight* (*these are isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$ *and contain these two* Type II *curves*);

(4) *one vertical family*, *which is isomorphic to a product of* $n-r$ *projective spaces and has dimension equal to* d (*all curves of type* III *which contain* $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ *lie in this family*).

5. Algebraic extension of GKM theory

We discuss equivariant localization and an extension of the Goresky–Kottwitz–MacPherson relations when there are finitely many fixed points but infinitely many T-invariant curves.

We work with equivariant Chow rings; similar results hold for equivariant cohomology. In fact, when $k=\mathbb{C}$, X is smooth and projective, and X^T is finite, the two theories coincide.

We first recall some properties of T-equivariant Chow rings as developed by Edidin and Graham [1998a] and Brion [1997]. Next, we outline Evain's [2007] development of ideas of Brion which extends the GKM relations to describe the T-equivariant Chow ring of a smooth variety with finitely many T-fixed points when there are infinitely many T-invariant curves. This description involves ideal-membership relations, one for each generator of the equivariant Chow ring of each family of T-invariant curves. When the generators are given by smooth subvarieties, these relations may be expressed in terms of tangent weights. This gives one form of our presentation for $A^*_T(\mathcal{Q}_d)$ in Theorem 1.1.

We next give a variant of these relations using differential operators, which gives the other form of our presentation for $A^*_T(\mathcal{Q}_d)$.

We then compute these relations for products of projective spaces, and finally deduce Theorem 1.1.

5A. ***Torus equivariant Chow rings.*** When a linear algebraic group G acts on a smooth scheme X, Edidin and Graham [1998a] defined the equivariant Chow ring $A^*_G(X)$, using Totaro's algebraic approximation to the classifying space of G. It satisfies functorial properties under equivariant maps analogous to those for ordinary Chow rings [Fulton 1998], including proper pushforwards and pullbacks by local complete intersection morphisms.

When the group is a torus T, Brion [1997] gave an alternative development of this theory which includes versions of the localization theorems that hold for equivariant cohomology. He gave the following presentation for the equivariant Chow ring, analogous to the usual presentation of Chow groups. The equivariant Chow ring $A^*_T(p)$ of a point p is the integral symmetric algebra S of the character group $\hat{T}$ of T. Equivariant pullback makes $A^*_T(X)$ into an S-module.

Proposition 5.1 [Brion 1997, Theorem 2.1]. *The S-module $A_T^*(X)$ is defined by generators* $[Y]$, *for each T-invariant subvariety Y, and by relations*

$$[div_Y(f)] - \chi[Y]$$

for each rational function f on Y which is a T-eigenvector of weight χ; here χ is considered as an element of S in degree 1.

The usual Chow ring may be recovered from the S-module $A_T^*(X)$, as the quotient by the ideal S^+ of S generated by the character group $\hat{T}$.

Proposition 5.2 [Brion 1997, Corollary 2.3.1].

$$A^*(X) = A_T^*(X) \otimes_S \mathbb{Z} = A_T^*(X)/S^+ A_T^*(X).$$

The analogous statement in equivariant cohomology requires stronger hypotheses.

When $k = \mathbb{C}$, the connection between Chow groups and cohomology is given by the cycle map

$$A_T^*(X) \to H_{T,c}^*(X, \mathbb{Z})$$

to compactly supported (Borel–Moore) equivariant cohomology. If X is projective and the fixed point set X^T is finite, then the cycle map is an isomorphism.

Some statements below hold only for the rational equivariant Chow ring

$$A_T^*(X)_{\mathbb{Q}} := A_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This is a module over the rational equivariant Chow ring $A_T^*(p)_{\mathbb{Q}}$ of a point p, which is the symmetric algebra $S_{\mathbb{Q}}$ of $\hat{T}_{\mathbb{Q}} := \hat{T} \otimes_{\mathbb{Z}} \mathbb{Q}$.

5B. *Localization.* We now assume that X^T is finite, and that X has a decomposition into T-invariant affine cells $C_1, \ldots, C_m$ which can be ordered so that for $i = 1, \ldots, m$, the union $C_1 \cup \cdots \cup C_i$ is Zariski open. We will call such varieties *filtrable*; this is close to the terminology of Brion [1997], but he did not require X^T to be finite, and his cells were allowed to be vector bundles over components of X^T. If X is smooth and projective and X^T is finite, then Białynicki-Birula [1973] showed that it is filtrable.

Let $i : X^T \to X$ be the inclusion of the subscheme of T-fixed points of X.

Proposition 5.3 [Brion 1997, Corollary 3.2.1]. *The S-module $A_T^*(X)$ is free. The map*

$$i^* : A_T^*(X) \to A_T^*(X^T)$$

is an injection.

Brion also established Chow ring versions of the results of Chang and Skjelbred and of Goresky, Kottwitz, and MacPherson concerning the image of the localization map.

Proposition 5.4 [Brion 1997, Sections 3.3 and 3.4]. (a) *The image of the localization map $i^*\colon A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$ is the intersection of the images of the localization maps*

$$i_{T'}^*\colon A_T^*(X^{T'})_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}},$$

where T' runs over all codimension one subtori of T.

(b) *When T acts with finitely many fixed points and has finitely many invariant curves, then the image of the localization map*

$$i^*\colon A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}} \simeq (S_{\mathbb{Q}})^{X^T}$$

is the set of all tuples $(f_p)_{p\in X^T} \in (S_{\mathbb{Q}})^{X^T}$ such that whenever p and q belong to the same irreducible T-invariant curve C, we have $f_p \equiv f_q$ modulo χ, where χ is the weight of the action of T on T_pC.

Statement (a) is analogous to a theorem of Chang and Skjelbred [1974] for equivariant cohomology. This result, together with the easy calculation of the equivariant Chow groups of $\mathbb{P}^1$, immediately gives (b), which is the Chow analog of the GKM relations for equivariant cohomology.

In general this result does not hold with $\mathbb{Z}$ coefficients. For instance, suppose that $\dim X = 2$, $x \in X^T$, and the weights of T on the tangent space T_xX are $a\chi$ and $a'\chi'$, where $\chi, \chi' \in \hat{T}$ are linearly independent primitive characters. Then condition (b) would say that if $(f_p) \in A_T^*(X^T)$ has $f_p = 0$ for $p \neq x$, then it is in the image of i^* if f_x is a multiple of $\operatorname{lcm}(a, a')\chi\chi'$. In fact, f_x must be a multiple of $aa'\chi\chi'$.

This is essentially the only obstruction to working with $\mathbb{Z}$ coefficients, at least if the fixed point set is finite. We say that the tangent weights of a T-variety X are *almost coprime* if whenever two T-weights of T_pX for $p \in X^T$ are divisible by the same integer $a > 1$, then they are parallel. With this added hypothesis, Brion's proof of Proposition 5.4 works over $\mathbb{Z}$.

Theorem 5.5. *Let X be a smooth filtrable T-variety whose tangent weights are almost coprime. Then Proposition 5.4 holds with rational Chow groups replaced by integral Chow groups.*

5C. *Evain's relations.* When T does not have finitely many invariant curves on X, then statement (b) of Proposition 5.4 fails, but by (a) we can still compute $A_T^*(X)$ if we know the images of $i_{T'}^*$ for all codimension one subtori T' of T. A finite set of such T' suffices, namely those which fix at least one T-invariant curve pointwise. Brion [1997] and Goldin and Holm [2001] have computed cases where the components of $X^{T'}$ are low-dimensional. Evain [2007] recently described relations in the general case:

Let $Y = X^{T'}$ for T' a codimension one subtorus of T. By [Iversen 1972], Y is smooth. For $p \in Y^T = X^T$, let

$$e_p^T(Y) = e^T(TY)|_p$$

be the localization of the equivariant Euler class of TY at p. Under the identification $A_T^*(p) = S$, this is the product of the T-weights on the tangent space T_pY.

Proposition 5.6 [Evain 2007, Corollary 27]. *A class $\alpha = (\alpha_p)_{p\in Y^T}$ in S^{Y^T} lies in $i_{T'}^*A_T^*(Y)$ if and only if*

$$\sum_{p\in Y^T} \frac{\alpha_p\beta_p}{e_p^T(Y)} \in S \tag{5-1}$$

for every $\beta \in i_{T'}^*A_T^*(Y)$.

Remark (On Evain's proof). The condition (5-1) is necessary, since if π is the projection of Y to a point, then the sum is simply $\pi_*(\alpha\cdot\beta)$, by the integration formula of Edidin and Graham [1998b]. Note that since π_* is S-linear, it is enough to take β in a generating set of the S-module $i^*A_T^*(Y)$.

By [Białynicki-Birula 1973], there are two T-invariant cell decompositions C_p^+ and C_p^- for $p \in Y^T$ of Y and an ordering of the fixed points Y^T such that the matrix with entries in S whose (p, q)-entry is

$$\pi_*([C_p^+]\cdot[C_q^-])$$

is unitriangular. Either set of classes $[C_p^+]$ or $[C_p^-]$ forms a basis for the S-module $A_T^*(Y)$, and expressing the elements α and β in these two bases proves sufficiency.

Combining Proposition 5.6 with Proposition 5.4 and Theorem 5.5 gives the criterion for membership in $i^*A_T^*(X)$:

Theorem 5.7. *A class $\alpha = (\alpha_p)_{p\in X^T} \in (S_{\mathbb{Q}})^{X^T}$ lies in the image $i^*A_T^*(X)_{\mathbb{Q}}$ of the localization map if and only if for all $Y = X^{T'}$ for T' a codimension one subtorus of T we have*

$$\sum_{p\in Y^T} \frac{\alpha_p\beta_p}{e_p^T(Y)} \in S_{\mathbb{Q}},$$

for all β in a set of $S_{\mathbb{Q}}$-module generators for $i_{T'}^*A_T^*(Y)_{\mathbb{Q}}$.

If the tangent weights of X are almost coprime, the same statement holds over $\mathbb{Z}$.

Remark. When X is smooth, the relations in Theorem 5.7 can also be taken for Y running over all irreducible components of the union of the fixed points and the T-invariant curves, since such Y are just the connected components of the T'-fixed loci $X^{T'}$ for some codimension one subtorus T' of T. We call this union of fixed points and T-invariant curves the *one-skeleton* of X.

To apply Theorem 5.7, we need to know explicit generators of $A_T^*(Y)$, or more precisely their localizations to Y^T. By Proposition 5.1, one class of generators are the equivariant fundamental cycles $[Z]$ of T-invariant subvarieties Z of the components Y. These are easy to compute when Z is smooth, since if $p \in Z^T$ we have

$$[Z]_p = e_p^T(N_Z Y),$$

the equivariant Euler class of the normal bundle to Z in Y, while if $p \in Y^T \setminus Z^T$, then $[Z]_p = 0$. It follows that

$$\frac{[Z]_p}{e_p^T(Y)} = \frac{1}{e_p^T(Z)}$$

if $p \in Z^T$.

To see this, note that $[Z]_p \in A_T^*(p)$ is the pullback of $[Z]$ along the regular embedding $i_{p,Y}\colon p \to Y$. We factor $i_{p,Y}$ as the composition

$$p \xrightarrow{i_{p,Z}} Z \xrightarrow{i_Z} Y.$$

The class $[Z] \in A_T^*(Y)$ is the pushforward along i_Z of the unit class

$$1 = [Z] \in A_T^*(Z),$$

and so we have

$$[Z]_p = i_{p,Z}^* i_Z^* i_{Z,*} 1 = i_{p,Z}^* e^T(N_Z Y) = e_p^T(N_Z Y),$$

by the self-intersection formula for Chow rings.

Thus if we can find for each Y a collection $\mathfrak{X}_Y$ of smooth T-invariant subvarieties of Y so that the classes $[Z]$ for $Z \in \mathfrak{X}_Y$ generate $A_T^*(Y)$ as an S-module, we get the more explicit version of Theorem 5.7:

Theorem 5.8. *A class $\alpha = (\alpha_p)_{p \in X^T} \in (S_{\mathbb{Q}})^{X^T}$ lies in $i^* A_T^*(X)_{\mathbb{Q}}$ if and only if*

$$\sum_{p \in Z^T} \frac{\alpha_p}{e_p^T(Z)} \in S_{\mathbb{Q}}, \tag{5-2}$$

for all $Z \in \mathfrak{X}_Y$ and all components Y of the one-skeleton of X. If the tangent weights of X are almost coprime, the same statement holds over $\mathbb{Z}$.

The necessity of (5-2) does not require the argument above, since if $(\alpha_p) = i_*\alpha$ for $\alpha \in A_T^*(X)$, then the sum is just $\pi_*(\alpha|_Z)$, where π is the projection of Z to a point.

Obvious candidates for the subvarieties $[Z]$ are the closures of the Białynicki-Birula cells, since their classes form an S-basis for $A_T^*(Y)$. Unfortunately, they are not in general smooth — this was the case for Evain. However, for the quot schemes we study, they are smooth, as the connected components Y are products of projective spaces. More generally we can ask that for each component Y of $X^{T'}$ there is a torus T_Y containing T which acts on Y with finitely many orbits, so that Y is a smooth toric variety. The closures of the cells will be T_Y-orbit closures, and therefore smooth.

The relations of Theorem 5.8 are the same as those found by Goldin and Holm [2001] for equivariant cohomology of Hamiltonian T-spaces in the case where the spaces $X^{T'}$ are at most four-dimensional (over $\mathbb{R}$).

5D. *Evain's relations as differential operators.* We rewrite this algebraic criterion in a different form. Suppose that Y is a component of $X^{T'}$ and $Z \subset Y$ is a smooth T-invariant subvariety. The action of T on Y factors through a character $\eta\colon T \to \mathbb{C}^*$, so the weights of T on T_pZ for $p \in Z^T$ are nonzero scalar multiples of η. Thus there exist numbers $d_p = d_p(Z)$ so that

$$e_p^T(Z) = d_p(Z) \cdot \eta^{\dim Z}.$$

The terms in (5-2) have a common denominator $\eta^{\dim Z}$, and so we may rewrite it as

$$\sum_{p \in Z^T} \frac{\alpha_p}{d_p(Z)} \in \eta^{\dim Z} S_{\mathbb{Q}}.$$

We can rewrite this condition using a linear differential operator. The ring $S_{\mathbb{Q}}$ is the symmetric algebra of $\hat{T}_{\mathbb{Q}}$, or dually the ring of polynomial functions on $\hat{T}^*_{\mathbb{Q}}$. Choose $\zeta \in \hat{T}^*_{\mathbb{Q}}$ for which $\zeta(\eta) \neq 0$. Then the operator $D = D_\zeta$ of differentiation in the direction of ζ acts on $S_{\mathbb{Q}}$. If $f \in S_{\mathbb{Q}}$ is divisible by η, then η^k divides f if and only if η^{k-1} divides Df, so the relation (5-2) is equivalent to

$$\sum_{p \in Z^T} d_p(Z)^{-1} D^j \alpha_p \equiv 0 \mod \eta,$$

for all $0 \leq j < \dim Z$.

We give a variant of Theorem 5.8 which uses the last relation, but only with the maximum order derivative $j = \dim Z - 1$. In exchange, we must apply it using more subvarieties Z.

Suppose that Y is a smooth component of the one-skeleton of X, let η be the associated character of T, and consider the two Białynicki-Birula cell decompositions

$$\{C_p^- \mid p \in Y^T\}, \quad \text{and} \quad \{C_p^+ \mid p \in Y^T\}$$

induced by the T-action. Each cell $C = C_p^+$, C_p^- is isomorphic to the T-vector space $T_pC \subset T_pY$. Suppose that within each cell $C = C_p^+$ we can find T-invariant affine subspaces $C_{p,1}, \ldots C_{p,\dim C}$ with $\dim C_{p,i} = i$ and which have smooth closures

$Z_{p,i} = \overline{C_{p,i}}$. As before, this will be true if each Y is a toric variety for a larger torus T_Y containing T, since we can take each $Z_{p,i}$ to be the closure of a T_Y-orbit.

Theorem 5.9. *With these assumptions, a class $\alpha = (\alpha_p)_{p \in X^T} \in (S_{\mathbb{Q}})^{X^T}$ lies in $i^* A_T^*(X)_{\mathbb{Q}}$ if and only if*

$$\sum_{q \in Z^T} d_q(Z)^{-1} D^{\dim Z - 1} \alpha_q \equiv 0 \mod \eta, \tag{5-3}$$

for all $Z = Z_{p,i}$ and for all components Y of the one-skeleton of X.

Proof. The necessity of the conditions (5-3) follows from the previous discussion.

To show they are sufficient, let U be an open union of the cells C_p^- and note that $Z_{p,i} \subset U$ if and only if $p \in U$. We use induction on the number of cells in U to show that the image of $i_U^* \colon A_T^*(U)_{\mathbb{Q}} \to A_T^*(U^T)_{\mathbb{Q}}$ is the set $(\alpha_x)|_{x \in U^T}$ satisfying (5-3) for all $Z_{p,i} \subset U$.

When U is a single cell, this is immediate, as i_U^* is an isomorphism and there are no $Z_{p,i}$'s contained in U. Otherwise, suppose $\alpha = (\alpha_x)|_{x \in U^T}$ satisfies (5-3) for all $Z_{p,i} \subset U$. Let $C_p^- \subset U$ be a closed cell, and put $U' = U \setminus C_p^-$. There is an exact sequence[Brion 1997, Proposition 3.2]

$$0 \to A_T^*(U)_{\mathbb{Q}} \xrightarrow{\rho} A_T^*(U')_{\mathbb{Q}} \times A_T^*(C_p^-)_{\mathbb{Q}} \to A_T^*(C_p^-)_{\mathbb{Q}}/(e^T(N)) \to 0, \tag{5-4}$$

where $e^T(N)$ is the equivariant Euler class of the normal bundle N to C_p^- in X. Under the isomorphism $A_T^*(C_p^-) \cong S$, this is just the product of all the T-weights of N. The components of ρ are the restriction maps, while the map

$$A_T^*(C_p^-)_{\mathbb{Q}} \to A_T^*(C_p^-)_{\mathbb{Q}}/(e^T(N))$$

is the natural quotient.

By the inductive hypothesis, $\alpha|_{(U')^T}$ lies in the image of $i_{U'}^*$. The map $A_T^*(U) \to A_T^*(U')$ is surjective and so we can write $\alpha = i_U^* \beta + \gamma$, with $\beta \in A_T^*(U)$ and $\gamma|_{(U')^T} = 0$. Since $i_U^* \beta$ satisfies the relations (5-3), so does γ. It will be enough to show that γ is in the image of i_U^*. But using the exact sequence (5-4), we see that this holds if and only if γ_p is a multiple of $c_d^T(N)$, which is a nonzero multiple of η^d, where $d = \operatorname{codim} C_p^- = \dim C_p^+$. But the relation (5-3) implies that $D^k \gamma_p \equiv 0 \pmod{\eta}$ for $0 \leq k < d$. The result follows. □

Example 5.10. Let $T = \mathbb{G}_m$ act on $X = \mathbb{P}^r$ by

$$t \cdot [x_0 : x_1 : \cdots : x_r] = [x_0 : t x_1 : \cdots : t^r x_r]$$

in homogeneous coordinates, where $t \in T$. For each $0 \leq j \leq r$, let $p_j \in X^T$ denote the T-fixed point corresponding to the j-th standard basis vector $\mathbf{e}_j$. The tangent space $T_{p_j}X$ is $\mathbb{C}^n/\mathbb{C}\cdot\mathbf{e}_j$, with the action of T given by $t\cdot\overline{\mathbf{e}}_k = t^{k-j}\overline{\mathbf{e}}_k$. Thus

$$e_{p_j}(X) = (-1)^j j!(r-j)!\eta^r,$$

where η is the identity character. More generally, if $0 \le l \le n \le r$, let

$$Z_{l,m} = \mathbb{P}\,\mathrm{Span}\{\mathbf{e}_l, \mathbf{e}_{l+1}, \ldots, \mathbf{e}_m\}.$$

The same calculation gives

$$e_{p_j}(Z_{l,m}) = (-1)^{j-l}(j-l)!(m-j)!\eta^{m-l}.$$

We can apply Theorem 5.8 using the smooth subvarieties $Z_{0,l}$, for $1 \le l \le r$. Then $i^*A_T^*(X) \subset A_T^*(X^T)$ is the set of tuples $\alpha = (\alpha_0, \ldots, \alpha_r)$ where

$$\sum_{0\le j\le l} \frac{(-1)^j\alpha_j}{j!(l-j)!} \in \eta^l S, \tag{5-5}$$

for all $1 \le l \le r$.

On the other hand, we can apply Theorem 5.9 using all the subvarieties $Z_{l,m}$. If D is differentiation on S in the direction of $\eta^\vee$, then α lies in the image of i^* if and only if

$$\sum_{l\le j\le m} \frac{(-1)^{j-l}D^{m-l-1}\alpha_j}{(j-l)!(m-j)!} \in \eta S, \tag{5-6}$$

for all $0 \le l < m \le r$. We could take one more derivative and ask that the resulting sums vanish, but this would not generalize to actions of higher-dimensional tori.

When $X = \mathbb{P}^1$ we get exactly the GKM relation for a primitive action.

Example 5.11. These same arguments apply to products of projective spaces. Let $\mathbf{r} = (r_1, \ldots, r_n)$, and let $X = \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$, where the action of $t \in T = \mathbb{G}_m$ on a point $([x_0^1 : x_1^1 : \cdots : x_{r_1}^1], \ldots, [x_0^n : x_1^n : \cdots : x_{r_n}^n])$ is given by multiplying x_j^i by t^j. The fixed points have the form $p_{\mathbf{j}} = (p_{j_1}, \ldots, p_{j_n})$, where p_{j_i} is the j_i-th fixed point in $\mathbb{P}^{r_i}$, in the notation of Example 5.10, and $\mathbf{j} = (j_1, \ldots, j_n)$ satisfies $0 \le \mathbf{j} \le \mathbf{r}$, meaning that $0 \le j_i \le r_i$ for all $1 \le i \le n$.

For $\mathbf{l}, \mathbf{m} \in \mathbb{Z}^n$ with $0 \le l_i \le m_i \le r_i$ for all i, set $Z_{\mathbf{l},\mathbf{m}} = Z_{l_1,m_1} \times \cdots \times Z_{l_n,m_n}$. For each $\mathbf{l} \le \mathbf{j} \le \mathbf{m}$, the tangent space to $Z_{\mathbf{l},\mathbf{m}}$ at $p_{\mathbf{j}}$ is $\bigoplus_{i=1}^n T_{p_{j_i}} Z_{l_i,m_i}$. Using the computation from Example 5.10, we see that

$$e^T_{p_{\mathbf{j}}}(Z_{\mathbf{l},\mathbf{m}}) = (-1)^{|\mathbf{j}|}(\mathbf{j}-\mathbf{l})!\,(\mathbf{m}-\mathbf{j})!\,\eta^{|\mathbf{m}|-|\mathbf{l}|}.$$

Recall that for an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$, we put $\mathbf{a}! = a_1! \cdots a_n!$.

As in Example 5.10, we can either apply Theorem 5.8 using the subvarieties $Z_{\mathbf{0},\mathbf{l}}$, or Theorem 5.9 using all the $Z_{\mathbf{l},\mathbf{m}}$. The resulting conditions for a tuple $\alpha =$

$(\alpha_{\mathbf{j}})_{\mathbf{0}\le\mathbf{j}\le\mathbf{r}}$ to be in the image of the localization map are just (5-5) and (5-6), where the variables now represent elements of $\mathbb{Z}^n$ rather than scalars.

5E. *Proof of Theorem 1.1.*

Proof. We combine these localization results with the geometry of the quot scheme from Sections 2, 3, and 4 to produce a proof of Theorem 1.1.

The only nonprimitive tangent weights are those with $i = j$ in (2-3), which are multiples of $\mathbf{f}$. These correspond to Type III curves. Thus the tangent weights are almost coprime, so Theorem 5.8 gives a correct description of the integral equivariant Chow ring.

The relations **I**, **II**(a) and **II**(b) are the GKM relations for the T-invariant curves of types I and II, as described in Section 3 using the identification of the T-weights of these curves with the tangent weights given by Theorem 3.2.

The relations **II**(c)/**II**(c)$'$ come from the horizontal families of T-invariant curves of type II. As described in Section 4, these are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, where the action on each factor is by the same primitive character. As in Example 5.11, we can apply the relation in Theorem 5.8 with $Y = \mathbb{P}^1 \times \mathbb{P}^1$ to get the relations **II**(c)/**II**(c)$'$. We get one new relation using $Z = Y$; smaller T-invariant subvarieties contained in Y are either T-invariant curves, whose relations are already covered by **II**(a) and **II**(b), or points, which give no relation.

Finally, the relations **III** and **III**$'$ come from the vertical families. As described in Section 4, the vertical family containing $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ is isomorphic to

$$\prod_{\delta_i=1} \mathbb{P}H^0(\mathbb{O}(a_i+b_i)) \simeq \prod_{\delta_i=1} \mathbb{P}^{a_i+b_i},$$

and the fixed points in the family are those $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b}')}$ with $\mathbf{a}'+\mathbf{b}' = \mathbf{a}+\mathbf{b}$. The codimension one subtorus $T_{k^n} \subset T$ acts trivially on the family, and the remaining action of $T_{\mathbb{P}^1}$ is the one described in Example 5.11, using the monomial basis of $H^0(\mathbb{O}(a_i+b_i))$. Example 5.11 then gives exactly the relations **III** and **III**$'$. This proves Theorem 1.1. □

6. Equivariant Chern classes on $\mathcal{Q}_d$

Recall that $\mathbb{P}^1 \times \mathcal{Q}_d$ has a universal exact sequence of sheaves

$$0 \to \mathcal{S} \to \mathbb{O}^n_{\mathbb{P}^1\times\mathcal{Q}_d} \to \mathcal{T} \to 0$$

with $\mathcal{S}$ the tautological vector bundle of rank $n-r$.

Since both $\mathcal{Q}_d$ and $\mathbb{P}^1$ have cell decompositions, we have a Künneth decomposition of Chow rings,

$$A^*(\mathbb{P}^1 \times \mathcal{Q}_d) \simeq A^*(\mathbb{P}^1) \otimes_{\mathbb{Z}} A^*(\mathcal{Q}_d).$$

Let $\pi : \mathbb{P}^1 \times \mathcal{Q}_d \to \mathcal{Q}_d$ be the projection. For each $1 \le i \le n-r$ we may decompose the Chern class $c_i(\mathcal{S})$ as

$$c_i(\mathcal{S}) = \pi^* t_i + h\pi^* u_{i-1}, \tag{6-1}$$

where $t_i, u_i \in A^i(\mathcal{Q}_d)$ and h is the class of a point in $A^1(\mathbb{P}^1)$ pulled back to $\mathbb{P}^1 \times \mathcal{Q}_d$. Note that $u_0 = -d$. Strømme [1987] proved that $A^*(\mathcal{Q}_d)_{\mathbb{Q}}$ is generated by the classes

$$\{t_1, \dots, t_k, u_1, \dots, u_{k-1}\}.$$

For each $1 \le i \le n-r$, the equivariant Chern class $c_i^T(\mathcal{S})$ localizes at a fixed point p to the i-th elementary symmetric polynomial e_i in the T-weights of the fibre of $\mathcal{S}$ at p. The fixed points of $\mathbb{P}^1 \times \mathcal{Q}_d$ correspond to $\{0, \infty\} \times \mathcal{F}$. For $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}) \in \mathcal{F}$, the bundle $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ on $\mathbb{P}^1$ is a sum of line bundles $\mathcal{S}_{a_j,b_j}\mathbf{e}_j$ for $\delta_j = 1$. Since $\mathcal{S}_{a,b}$ has weight $a\mathbf{f}$ at 0 and $-b\mathbf{f}$ at ∞, the localizations of $c_i^T(\mathcal{S})$ are

$$c^T_{i,0,(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} = e_i(\{\mathbf{e}_j + a_j\mathbf{f} \mid \delta_j = 1\}) \quad \text{at } (0, (\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})),$$
$$c^T_{i,\infty,(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} = e_i(\{\mathbf{e}_j - b_j\mathbf{f} \mid \delta_j = 1\}) \quad \text{at } (\infty, (\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})).$$

We also have a Künneth decomposition in equivariant Chow cohomology:

$$A_T^*(\mathbb{P}^1 \times \mathcal{Q}_d) \xleftarrow{\sim} A_T^*(\mathbb{P}^1) \otimes_S A_T^*(\mathcal{Q}_d).$$

To see this, just imitate the argument for ordinary Chow cohomology; the equivariant Chow cohomology of a variety with an algebraic cell decomposition will be a free S-module, with a module basis given by the closures of the cells.

An equivariant Künneth decomposition of $c_i^T(\mathcal{S})$ analogous to (6-1) requires the choice of a lift of the class of a point to $A_T^1(\mathbb{P}^1)$. Localizing a class $x \in A_T^1(\mathbb{P}^1)$ gives an ordered pair $(x_0, x_\infty) \in \mathbb{Z}\mathbf{f} \oplus \mathbb{Z}\mathbf{f}$. Lifts of classes from $A^1(\mathbb{P}^1)$ are only well-defined modulo the span of $(\mathbf{f}, \mathbf{f})$. Three possible choices for lifting the class of a point are

$$(i)\ (-\mathbf{f}, 0), \quad (ii)\ (0, \mathbf{f}), \quad \text{and} \quad (iii)\ \tfrac{1}{2}(-\mathbf{f}, \mathbf{f}).$$

The symmetric lift (iii) requires rational coefficients. We will use this lift to express our formulas.

Given a lift $h \in A_T^1(\mathbb{P}^1)$ of the class of a point, the formula (6-1) defines equivariant lifts of the classes t_i, u_{i-1}. Strømme's result together with Proposition 5.2 implies that these classes generate $A_T^*(X)_{\mathbb{Q}}$ as an $S_{\mathbb{Q}}$-algebra.

Proposition 6.1. *The symmetric choice (iii) of lift $h \in A_T^1(\mathbb{P}^1)$ of the class of a point gives the following formula for the Künneth components t_i, u_{i-1} of the equivariant Chern class $c_i^T(\mathcal{S})$ of the bundle $\mathcal{S}$, expressed in terms of their localizations* *at the fixed point of* $\mathcal{Q}_d$ *indexed by* $p := (\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$:

$$t_{i,p} = \tfrac{1}{2}\big(c^T_{i,0,p} + c^T_{i,\infty,p}\big), \quad \textit{and} \quad u_{i-1,p} = \frac{1}{\mathbf{f}}\big(c^T_{i,\infty,p} - c^T_{i,0,p}\big).$$

7. Equivariant Chow ring of the quot scheme $\mathcal{Q}_2(0, 2)$

We use Theorem 1.1 to describe the equivariant Chow ring of $\mathcal{Q}_2 := \mathcal{Q}_2(0, 2)$. We first give a basis for $A^*_T(\mathcal{Q}_2)$ as a module over $S = \mathbb{Z}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}]$.

The equivariant Chow ring is the collection of tuples $(f_p \mid p \in \mathcal{Q}_2^T) \in S^{\mathcal{Q}_2^T}$ which satisfy the relations of Theorem 1.1. If p and q are connected by an edge in the moment multigraph (see Figure 1) with weight χ, then $f_p - f_q$ lies in the ideal generated by χ. These are the standard GKM relations.

There are two multiedges with four vertices in the multigraph, namely the vertical and horizontal lines of symmetry. They should be seen as flattened quadrangles, since they are images of subvarieties isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Each gives rise to an additional relation as follows. Suppose that the quadrangle has four vertices a, b, c, d:

$$\xrightarrow{\chi} \qquad \begin{tikzcd}[row sep=tiny] & c \arrow[dr, dash] & \\ a \arrow[ur, dash] \arrow[dr, dash] & & d. \\ & b \arrow[ur, dash] & \end{tikzcd}$$

(Here, the edges are parallel with direction χ.) Then the tuples (f_p) must satisfy

$$f_a - f_b - f_c + f_d \in \chi^2 S.$$

These are relations of types **II(c)**$'$ (horizontal) and **III**$'$ (vertical) of Theorem 1.1.

The remaining multiedges with more than two vertices are the left and right vertical edges, both with three vertices. They should be seen as flattened triangles, since they come from subvarieties isomorphic to $\mathbb{P}^2$. The additional relations they induce are described as follows. Let the three fixed points on the multiedge be a, b, c, with b between a and c. Then

$$\tfrac{1}{2} f_a - f_b + \tfrac{1}{2} f_c \in \mathbf{f}^2 S.$$

This is relation of type **III**$'$ of Theorem 1.1.

When $k = \mathbb{C}$, we can construct an S-module basis for equivariant cohomology using equivariant Morse theory; as is well-known, a generic projection of the moment map to a line will give a Morse function which is perfect for equivariant cohomology. This results in an inductive algorithm to produce a basis, which is nicely expounded in [Tymoczko 2005] (see also [Guillemin and Zara 2001]). Pick a direction vector $\mathbf{v}$ (corresponding to an element of the Lie algebra $\mathfrak{t}_{\mathbb{R}}$) which does not annihilate the direction vector of any edge. Orient each edge to have positive pairing with $\mathbf{v}$; this is the Hasse diagram of a partial order on the fixed points induced by $\mathbf{v}$. Then, using the relations described above, we can inductively

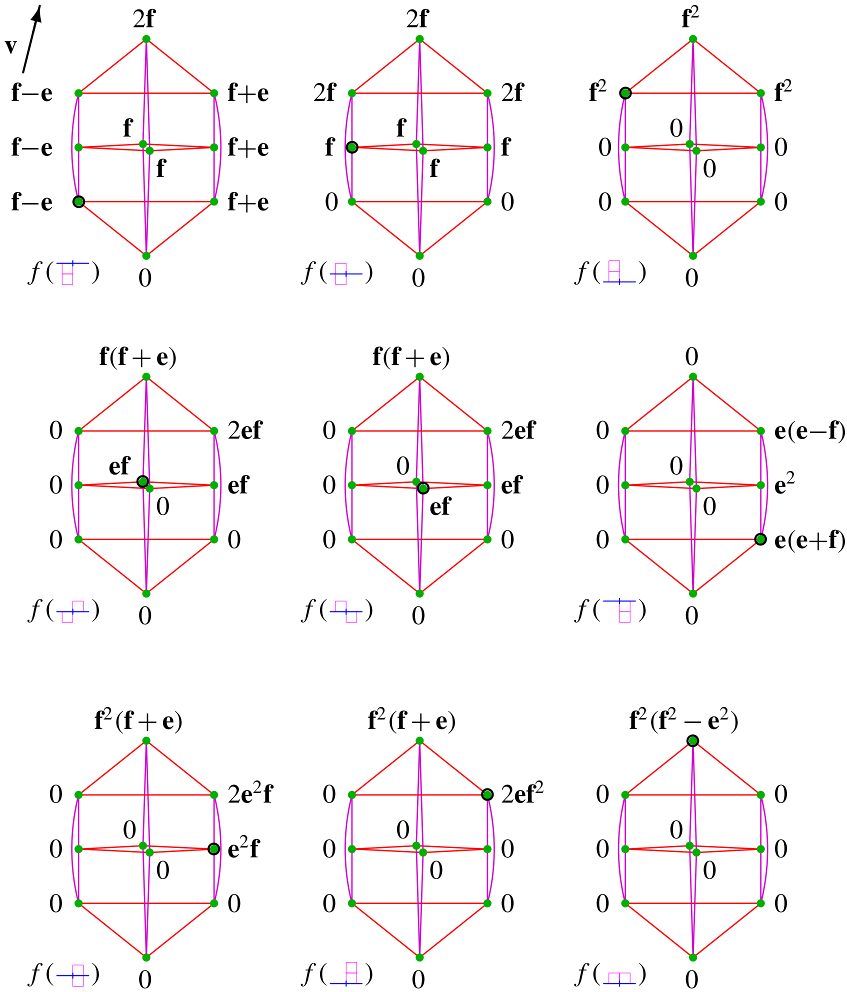


Figure 2. An S-module basis for $A_T^*(\mathcal{Q}_2(0,2))$.

construct a triangular basis with respect to this ordering. That is, if $f = f(p)$ corresponds to the fixed point p, it vanishes at q ($f_q = 0$) unless $p < q$, and f_p is the product of weights of edges pointing down from p. While this algorithm was motivated by Morse theory, it makes sense over any field k, if $\mathbf{v}$ is a linear function on the character group of T which does not annihilate any edge of the moment graph.

Set $\mathbf{e} := \mathbf{e}_1 - \mathbf{e}_2$ and pick the vector $\mathbf{v} = \mathbf{f} + \epsilon\mathbf{e}$, where $\epsilon > 0$ is small. One basis element is the identity $f(\;)$, which localizes to 1 at each fixed point. We display each of the remaining nine in Figure 2 as a localization diagram, writing its localizations on a copy of a moment multigraph.

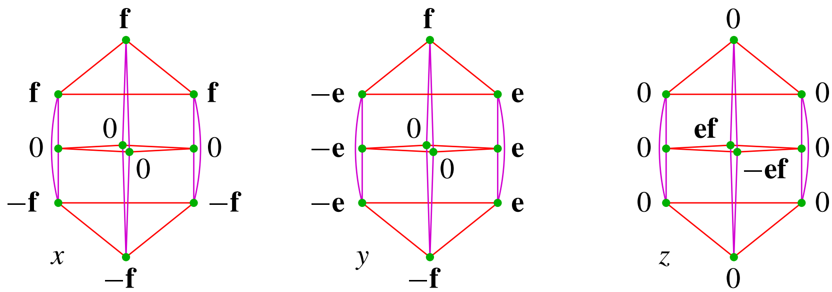


Figure 3. Generators for the $S_{\mathbb{Q}}$-algebra $A_T^*(\mathcal{Q}_2(0,2))_{\mathbb{Q}}$.

Set $x := f(\;) - \mathbf{f}$, $y := f(\;) - \mathbf{f}$, and $z := f(\;) - f(\;)$. Figure 3 shows their localization diagrams. We show that they generate $A_T^*(\mathcal{Q}_2)_{\mathbb{Q}}$ as an $S_{\mathbb{Q}}$-algebra by showing that each basis element $f(p)$ of degree greater than 2 lies in $S_{\mathbb{Q}}[x,y,z]$. Since

$$
\begin{aligned}
f(\;) &= \tfrac{1}{2}x(x+\mathbf{f}),\\
f(\;) &= \tfrac{1}{2}(y+\mathbf{e})(y-x), \text{ and}\\
f(\;) + f(\;) &= (y+\mathbf{e})(x+\mathbf{f}),
\end{aligned}
\tag{7-1}
$$

the four degree 2 basis elements lie in $S_{\mathbb{Q}}[x,y,z]$. The remaining three basis elements also lie in $S_{\mathbb{Q}}[x,y,z]$,

$$
yf(\;) = f(\;), \quad xf(\;) = f(\;), \quad \text{and} \quad (y-\mathbf{e})f(\;) = f(\;).
$$

Inspecting the localization diagrams of x, y, and z shows that the following 5 expressions vanish in $A_T^*(\mathcal{Q}_2)$:

$$
xz,\ \ yz,\ \ x(x^2-\mathbf{f}^2),\ \ (y^2-\mathbf{e}^2)(y-x),\ \text{and}\ z^2-(y^2-\mathbf{e}^2)(x^2-\mathbf{f}^2).
$$

In the lexicographic term order where $z > y > x > e > f$, these five polynomials form a Gröbner basis for the ideal $\mathcal{I}$ of $\mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, x, y, z]$ they generate with leading terms xz, yz, x^3, y^3, and z^2. There are ten standard monomials

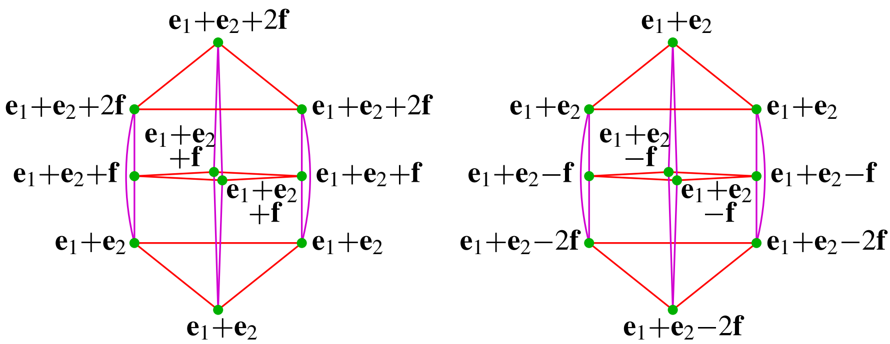
$$
1,\ x,\ y,\ x^2,\ xy,\ y^2,\ z,\ x^2y,\ xy^2,\ \text{and}\ x^2y^2.
$$

Since $A_T^*(\mathcal{Q}_2)_{\mathbb{Q}}$ is free over $S_{\mathbb{Q}} = \mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}]$ of rank 10, we conclude that

$$
A_T^*(\mathcal{Q}_2)_{\mathbb{Q}} \simeq \mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, x, y, z]/\mathcal{I}.
$$

Using Proposition 5.2, we obtain the presentation of the rational Chow ring

$$
A^*(\mathcal{Q}_2)_{\mathbb{Q}} \simeq \mathbb{Q}[x,y,z]/\langle xz,\ yz,\ x^3,\ xy^2-y^3,\ z^2-x^2y^2\rangle.
$$



Localization of $c_1^T(\mathcal{S})$ at $0 \in \mathbb{P}^1$ Localization of $c_1^T(\mathcal{S})$ at $\infty \in \mathbb{P}^1$

Figure 4. First Chern class of $\mathcal{S}$.

Looking at (7-1), we see that the integral Chow ring is not generated by x, y, and z, so its presentation will be considerably more complicated.

We now consider Strømme's generators. Figure 4 shows the localization diagram of the first Chern class of $\mathcal{S}$. By the formula of Proposition 6.1, we have

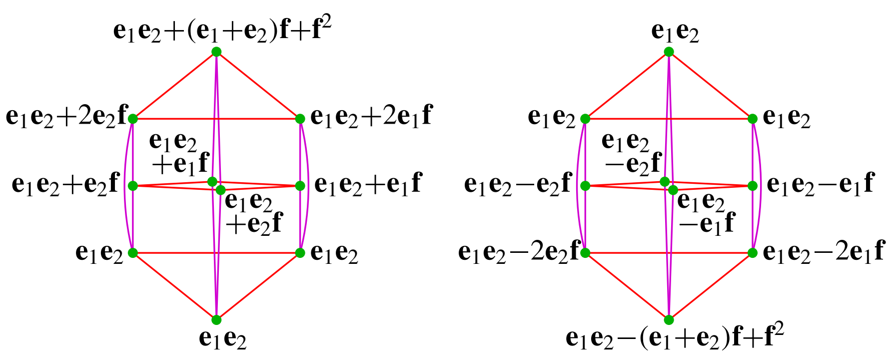
$$c_1^T(\mathcal{S}) = \mathbf{e}_1 + \mathbf{e}_2 + x - 2h,$$

so that

$$t_1 = \mathbf{e}_1 + \mathbf{e}_2 + x, \quad \text{and} \quad u_0 = -2.$$

Figure 5 shows the localization diagram of the second Chern class of $\mathcal{S}$. By the formula of Proposition 6.1, we have

$$c_2^T(\mathcal{S}) = \mathbf{e}_1\mathbf{e}_2 + \tfrac{1}{2}x(y + \mathbf{e}_1 + \mathbf{e}_2) + \tfrac{1}{2}z - h(y + \mathbf{e}_1 + \mathbf{e}_2),$$



Localization of $c_2^T(\mathcal{S})$ at $0 \in \mathbb{P}^1$ Localization of $c_2^T(\mathcal{S})$ at $\infty \in \mathbb{P}^1$

Figure 5. Second Chern class of $\mathcal{S}$.

so that

$$t_2 = \mathbf{e}_1\mathbf{e}_2 + \tfrac{1}{2}x(y+\mathbf{e}_1+\mathbf{e}_2) + \tfrac{1}{2}z \quad \text{and} \quad u_1 = -(y+\mathbf{e}_1+\mathbf{e}_2).$$

The corresponding classes in $A^*(\mathcal{Q}_2)$ are

$$t_1 = x, \quad u_1 = -y, \quad \text{and} \quad t_2 = \tfrac{1}{2}(z+xy).$$

Remark. This shows that the claim in [Strømme 1987, Theorem 5.3] that the classes t_i, u_{i-1} generate the integral Chow ring is false.

References

[Bertram 1997] A. Bertram, "Quantum Schubert calculus", *Adv. Math.* **128**:2 (1997), 289–305. MR 98j:14067 Zbl 0945.14031

[Białynicki-Birula 1973] A. Białynicki-Birula, "Some theorems on actions of algebraic groups", *Ann. of Math.* (2) **98** (1973), 480–497. MR 51 #3186 Zbl 0275.14007

[Brion 1997] M. Brion, "Equivariant Chow groups for torus actions", *Transform. Groups* **2**:3 (1997), 225–267. MR 99c:14005 Zbl 0916.14003

[Chang and Skjelbred 1974] T. Chang and T. Skjelbred, "The topological Schur lemma and related results", *Ann. of Math.* (2) **100** (1974), 307–321. MR 51 #11552 Zbl 0288.57021

[Edidin and Graham 1998a] D. Edidin and W. Graham, "Equivariant intersection theory", *Invent. Math.* **131**:3 (1998), 595–634. MR 99j:14003a Zbl 0940.14003

[Edidin and Graham 1998b] D. Edidin and W. Graham, "Localization in equivariant intersection theory and the Bott residue formula", *Amer. J. Math.* **120**:3 (1998), 619–636. MR 99g:14005 Zbl 0980.14004

[Eisenbud 1995] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, New York, 1995. MR 97a:13001 Zbl 0819.13001

[Evain 2007] L. Evain, "The Chow ring of punctual Hilbert schemes on toric surfaces", *Transform. Groups* **12**:2 (2007), 227–249. MR 2008f:14008 Zbl 1128.14004

[Fulton 1998] W. Fulton, *Intersection theory*, 2nd ed., Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics **2**, Springer, Berlin, 1998. MR 99d:14003 Zbl 0885.14002

[Goldin and Holm 2001] R. Goldin and T. S. Holm, "The equivariant cohomology of Hamiltonian G-spaces from residual S^1 actions", *Math. Res. Lett.* **8**:1-2 (2001), 67–77. MR 2002f:53140 Zbl 0987.57011

[Goresky et al. 1998] M. Goresky, R. Kottwitz, and R. MacPherson, "Equivariant cohomology, Koszul duality, and the localization theorem", *Invent. Math.* **131**:1 (1998), 25–83. MR 99c:55009 Zbl 0897.22009

[Guillemin and Zara 2001] V. Guillemin and C. Zara, "1-skeleta, Betti numbers, and equivariant cohomology", *Duke Math. J.* **107**:2 (2001), 283–349. MR 2002j:53110 Zbl 1020.57013

[Guillemin and Zara 2002] V. Guillemin and C. Zara, "Combinatorial formulas for products of Thom classes", pp. 363–405 in *Geometry, mechanics, and dynamics*, edited by P. Newton et al., Springer, New York, 2002. MR 2003i:53121 Zbl 1009.57034

[Guillemin and Zara 2003] V. Guillemin and C. Zara, "The existence of generating families for the cohomology ring of a graph", *Adv. Math.* **174**:1 (2003), 115–153. MR 2004f:53099 Zbl 1020.57014

[Iversen 1972] B. Iversen, "A fixed point formula for action of tori on algebraic varieties", *Invent. Math.* **16** (1972), 229–236. MR 45 #8656 Zbl 0246.14010

[Strømme 1987] S. A. Strømme, "On parametrized rational curves in Grassmann varieties", pp. 251–272 in *Space curves* (Rocca di Papa, 1985), edited by F. Ghione et al., Lecture Notes in Math. **1266**, Springer, Berlin, 1987. MR 88i:14020 Zbl 0625.14027

[Tymoczko 2005] J. S. Tymoczko, "An introduction to equivariant cohomology and homology, following Goresky, Kottwitz, and MacPherson", pp. 169–188 in *Snowbird lectures in algebraic geometry* (Snowbird, UT, 2004), edited by R. Vakil, Contemp. Math. **388**, Amer. Math. Soc., Providence, RI, 2005. MR 2006m:55019

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TOPOLOGICAL COMPLEXITY OF BASIS-CONJUGATING AUTOMORPHISM GROUPS

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We compute the topological complexity of Eilenberg–Mac Lane spaces associated to the group of automorphisms of a finitely generated free group that act by conjugation on a given basis, and to certain subgroups.

1. Introduction

Given a mechanical system, a motion planning algorithm is a function that assigns to any pair of states of the system, an initial state and a desired state, a continuous motion of the system starting at the initial state and ending at the desired state. Interest in such algorithms arises in robotics; see Latombe [1991] as a general reference. In a sequence of recent papers [2003; 2004; 2006], Farber develops a topological approach to the problem of motion planning, introducing a numerical invariant that gives a measure of the "navigational complexity" of the system.

Let X be a path-connected topological space, the space of all possible configurations of a mechanical system. In topological terms, the motion planning problem consists of finding an algorithm that takes pairs of configurations, that is, points $(x_0, x_1) \in X \times X$, and produces a continuous path $\gamma\colon [0, 1] \to X$ from the initial configuration $x_0 = \gamma(0)$ to the terminal configuration $x_1 = \gamma(1)$. Let PX be the space of all continuous paths in X, equipped with the compact-open topology. The map $\pi : PX \to X \times X$, $\gamma \mapsto (\gamma(0), \gamma(1))$, which sends a path to its endpoints, is a fibration. The motion planning problem then asks for a section of this fibration, a map $s\colon X \times X \to PX$ satisfying $\pi \circ s = \mathrm{id}_{X\times X}$. It would be desirable for a motion planning algorithm to depend continuously on the input. However, one can show that there exists a globally continuous motion planning algorithm $s\colon X \times X \to PX$ if and only if X is contractible; see [Farber 2003, Theorem 1]. One is thus led to study the discontinuities of such algorithms.

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For the space X, define the *topological complexity* $\mathsf{TC}(X)$ to be the Schwarz genus, or sectional category, of the path-space fibration:

$$\mathsf{TC}(X) := \operatorname{secat}(\pi \colon PX \to X \times X).$$

In other words, $\mathsf{TC}(X)$ is the smallest number k for which there exists an open cover $X \times X = U_1 \cup \cdots \cup U_k$ such that the map π admits a continuous section $s_j \colon U_j \to PX$ over each U_j satisfying $\pi \circ s_j = \mathrm{id}_{U_j}$. One can show that $\mathsf{TC}(X)$ is an invariant of the homotopy type of X; see [Farber 2003, Theorem 3].

Let X be an aspherical space, that is, a space whose higher homotopy groups vanish: $\pi_i(X) = 0$ for $i \geq 2$. Farber [2006, Section 31] poses the problem of computing the topological complexity of such a space in terms of algebraic properties of the fundamental group $G = \pi_1(X)$. In other words, given a discrete group G, define the topological complexity of G to be $\mathsf{TC}(G) := \mathsf{TC}(K(G, 1))$, the topological complexity of an Eilenberg–Mac Lane space of type $K(G, 1)$, and express $\mathsf{TC}(G)$ in terms of invariants such as the cohomological or geometric dimension of G if possible.

A number of results in the literature may be interpreted in the context of this problem. For a right-angled Artin group G, the topological complexity of an associated $K(G, 1)$-complex was computed in [Cohen and Pruidze 2008]. For the Artin pure braid group $G = P_n$, the configuration space $F(\mathbb{C}, n)$ of n ordered points in $\mathbb{C}$ is an associated Eilenberg–Mac Lane space. Similarly, the configuration space $F(\mathbb{C}_m, n)$ of n ordered points in $\mathbb{C}_m = \mathbb{C} \setminus \{m \text{ points}\}$ is an Eilenberg–Mac Lane space for the group $P_{n,m} = \ker(P_n \to P_m)$, the kernel of the homomorphism that forgets the last $n - m$ strands of a pure braid. In [Farber and Yuzvinsky 2004] and [Farber et al. 2007], Farber, Grant, and Yuzvinsky determine the topological complexity of these configuration spaces. All these results may be expressed in terms of the cohomological dimension, $\operatorname{cd}(G)$, of the underlying group G. For instance, one has $\mathsf{TC}(P_n) = \mathsf{TC}(F(\mathbb{C}, n)) = 2n - 2 = 2\operatorname{cd}(P_n)$.

The pure braid group P_n and the group $P_{n,m}$ may be realized as subgroups of $\operatorname{Aut}(F_n)$, the automorphism group of the finitely generated free group $F_n = \langle x_1, \ldots, x_n \rangle$. The purpose of this note is to determine the topological complexity of several other subgroups of $\operatorname{Aut}(F_n)$.

Let $G = P\Sigma_n$ be the "group of loops", the group of motions of a collection of $n \geq 2$ unknotted, unlinked circles in 3-space, where each (oriented) circle returns to its original position. This group may be realized as the basis-conjugating automorphism group, or pure symmetric automorphism group, of F_n, consisting of all automorphisms that, for the fixed basis $\{x_1, \ldots, x_n\}$ for F_n, send each generator to a conjugate of itself. A presentation for $P\Sigma_n$ was found by McCool [1986]. In particular, this group is generated by automorphisms $\alpha_{i,j} \in \operatorname{Aut}(F_n)$ for $1 \leq i \neq j \leq n$, defined by $\alpha_{i,j}(x_i) = x_j x_i x_j^{-1}$ and $\alpha_{i,j}(x_k) = x_k$ for $k \neq i$. Also of interest is the "upper triangular McCool group", the subgroup $P\Sigma_n^+$ of $P\Sigma_n$ generated by $\alpha_{i,j}$ for $i < j$. The main results of this note may be summarized as follows.

Theorem. *The topological complexity of the basis-conjugating automorphism group is*

$$\mathsf{TC}(P\Sigma_n) = 2n - 1.$$

The topological complexity of the upper triangular McCool group is

$$\mathsf{TC}(P\Sigma_n^+) = 2n - 2.$$

Let X be an Eilenberg–Mac Lane complex of type $K(G, 1)$ for either $G = P\Sigma_n$ or $G = P\Sigma_n^+$. Since the topological complexity $\mathsf{TC}(X) = \mathsf{TC}(G)$ of X is the Schwarz genus of the path-space fibration, it admits several useful bounds. For instance, one has

$$\mathsf{TC}(X) = \operatorname{secat}(\pi : PX \to X \times X) \leq \operatorname{cat}(X \times X) \leq 2\operatorname{cat}(X) - 1 \leq 2\dim(X) + 1,$$

where $\operatorname{cat}(X)$ denotes the Lusternik–Schnirelmann category of X; see Schwarz [1961; 1962] and James [1978] as classical references. One also has a cohomological lower bound

$$\mathsf{TC}(X) \geq 1 + \mathsf{cl}(\ker(\pi^* : H^*(X \times X; \mathbb{Q}) \to H^*(PX; \mathbb{Q}))),$$

where $\mathsf{cl}(A)$ denotes the cup length of a graded ring A, the largest integer q for which there are homogeneous elements $a_1, \ldots, a_q$ of positive degree in A such that $a_1 \cdots a_q \neq 0$. Using the Künneth formula, the fact that $PX \simeq X$, and the equality $H^*(X; \mathbb{Q}) = H^*(G; \mathbb{Q})$, the kernel of $\pi^* : H^*(X \times X; \mathbb{Q}) \to H^*(PX; \mathbb{Q})$ may be identified with the kernel $Z = Z(H^*(G; \mathbb{Q}))$ of the cup-product map

$$H^*(G; \mathbb{Q}) \otimes H^*(G; \mathbb{Q}) \xrightarrow{\cup} H^*(G; \mathbb{Q});$$

see [Farber 2003, Theorem 7]. We call the cup length of the ideal Z of zero-divisors the *zero-divisor cup length* of $H^*(G; \mathbb{Q})$ and denote it by $\mathsf{zcl}(H^*(G; \mathbb{Q})) = \mathsf{cl}(Z)$. In this notation, the cohomological lower bound reads

$$\mathsf{TC}(G) \geq 1 + \mathsf{zcl}(H^*(G; \mathbb{Q})).$$

This note is organized as follows. After a discussion of basis-conjugating automorphism groups in Section 2, including the determination of their geometric dimensions, we use the (known) structure of the cohomology rings of these groups to compute the zero-divisor cup lengths of these rings in Section 3. These results are used in Section 4 to find the topological complexity of these groups. We conclude with some remarks concerning formality in Section 5.

2. Basis-conjugating automorphism groups

Let N be a compact set contained in the interior of a manifold M. Generalizing the familiar interpretation of a braid as the motion of $N = \{n$ distinct points$\}$ in $M = \mathbb{R}^2$, Dahm [1962] defines a motion of N in M as a path h_t in $\mathcal{H}_c(M)$, the space of homeomorphisms of M with compact support, satisfying $h_0 = \mathrm{id}_M$ and $h_1(N) = N$. With an appropriate notion of equivalence, the set of equivalence classes of motions of N in M is a group, and, furthermore, there is a homomorphism from this group to the automorphism group of the fundamental group $\pi_1(M \setminus N)$.

Goldsmith [1981] gives an exposition of Dahm's (unpublished) work, with particular attention paid to the case where $N = \mathcal{L}_n$ is a collection of n unknotted, unlinked circles in $M = \mathbb{R}^3$. Let $\mathcal{G}_n$ denote the corresponding motion group. Goldsmith shows that $\mathcal{G}_n$ is generated by three types of motions — flipping a single circle, interchanging two (adjacent) circles, and pulling one circle through another — and that the Dahm homomorphism $\phi\colon \mathcal{G}_n \to \mathrm{Aut}(\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n))$ is an embedding.

Choose a basepoint $e \in \mathbb{R}^3$ that is disjoint from $\mathcal{L}_n = C_1 \cup \cdots \cup C_n$, and for each i, let x_i be (the homotopy class of) a loop based at e linking C_i once. This identifies $\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n, e) = F_n$ with the free group generated by $x_1, \ldots, x_n$. With this identification, the generators of the motion group $\mathcal{G}_n \hookrightarrow \mathrm{Aut}(F_n)$ correspond to automorphisms ρ_i (flip C_i), τ_i (switch C_i and C_{i+1}), and $\alpha_{i,j}$ (pull C_i through C_j) defined by

$$\rho_i(x_k) = \begin{cases} x_k^{-1} & \text{if } k = i, \\ x_k & \text{if } k \neq i, \end{cases} \qquad \tau_i(x_k) = \begin{cases} x_{k+1} & \text{if } k = i, \\ x_{k-1} & \text{if } k = i+1, \\ x_k & \text{if } k \neq i, i+1, \end{cases}$$

and

$$\alpha_{i,j}(x_k) = \begin{cases} x_j x_k x_j^{-1} & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases} \tag{2-1}$$

Let $\varphi : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/[F_n, F_n]) \cong \mathrm{GL}(n, \mathbb{Z})$ denote the epimorphism induced by the abelianization homomorphism $F_n \to F_n/[F_n, F_n] \cong \mathbb{Z}^n$. There is a corresponding short exact sequence

$$1 \longrightarrow \mathrm{IA}_n \longrightarrow \mathrm{Aut}(F_n) \xrightarrow{\varphi} \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1\ ,$$

where $\mathrm{IA}_n = \ker \varphi$ is the well-known group of automorphisms of F_n that induce the identity on $H_1(F_n; \mathbb{Z})$. Brownstein and Lee [1993] considered the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \ker(\varphi \circ \phi) \arrow[r] \arrow[d] & \mathcal{G}_n \arrow[r, "\varphi\circ\phi"] \arrow[d, "\phi"] & \mathbb{Z}/2 \wr \Sigma_n \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{IA}_n \arrow[r] & \mathrm{Aut}(F_n) \arrow[r, "\varphi"] & \mathrm{GL}(n, \mathbb{Z}) \arrow[r] & 1,
\end{tikzcd}$$

where the vertical maps are embeddings. They showed that the image of $\mathcal{G}_n$ under $\varphi \circ \phi$ is the wreath product $\mathbb{Z}/2 \wr \Sigma_n$, the reflection group of type D_n. The kernel of $\varphi \circ \phi$ corresponds to the group $\mathcal{C}_n$ of "pure motions" of $\mathcal{L}_n$, motions that bring each oriented circle back to its original position. The isomorphic image of $\ker(\varphi \circ \phi)$ in $\mathrm{Aut}(F_n)$, that is, the intersection $\mathrm{IA}_n \cap \phi(\mathcal{G}_n)$, is the basis-conjugating automorphism group of the free group.

Definition 2.1. The *basis-conjugating automorphism group* of the free group F_n is the subgroup of $\mathrm{Aut}(F_n)$ generated by the elements $\alpha_{i,j}$ from (2-1) with $1 \le i, j \le n$, and $i \ne j$. Following [Jensen et al. 2006], we denote this group by $P\Sigma_n$.

McCool [1986] showed that $P\Sigma_n$ admits a presentation with the aforementioned generators and defining relations

$$\left\{ \begin{array}{ll} [\alpha_{i,j}, \alpha_{k,l}] & \text{for } i, j, k, l \text{ distinct,} \\ [\alpha_{i,j}, \alpha_{k,j}] & \text{for } i, j, k \text{ distinct,} \\ [\alpha_{i,j}, \alpha_{i,k}\alpha_{j,k}] & \text{for } i, j, k \text{ distinct,} \end{array} \right\} \tag{2-2}$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ denotes the commutator.

An "upper triangular" version of the basis-conjugating automorphism group has been studied in a number of recent works; see [Bardakov and Mikhailov 2008; Cohen et al. 2007; Cohen et al. 2008].

Definition 2.2. The *upper triangular McCool group* $P\Sigma_n^+$ is the subgroup of $P\Sigma_n$ generated by the elements $\alpha_{i,j}$ with $i < j$, subject to the relevant relations (2-2).

The upper triangular McCool group $P\Sigma_n^+$ shares a number of features with the Artin pure braid group P_n. For instance, both groups may be realized as iterated semidirect products of free groups:

$$P_n = F_{n-1} \rtimes_{\eta_{n-1}} \cdots \rtimes_{\eta_2} \rtimes F_1 \quad \text{and} \quad P\Sigma_n^+ = F_{n-1} \rtimes_{\mu_{n-1}} \cdots \rtimes_{\mu_2} \rtimes F_1.$$

For the pure braid group, the action of the free group F_k on F_m with $1 \le k < m \le n-1$ is given by the restriction of the Artin representation $\eta_m \colon P_m \to \mathrm{Aut}(F_m)$; see for instance [Birman 1974]. For the upper triangular McCool group, the action of $F_k = \langle \alpha_{n-k,j} \mid n-k+1 \le j \le n \rangle$ on $F_m = \langle \alpha_{n-m,j} \mid n-m+1 \le j \le n \rangle$, that is, the homomorphism $\mu_m \colon \rtimes_{j=1}^{m-1} F_j \to \mathrm{Aut}(F_m)$, was determined in [Cohen et al. 2008] (with different notation). Using the relations (2-2), one can check that

$$\mu_m(\alpha_{j,p})(\alpha_{i,q}) = \alpha_{j,p}^{-1}\alpha_{i,q}\alpha_{j,p} = \begin{cases} \alpha_{i,p}\alpha_{i,q}\alpha_{i,p}^{-1} & \text{if } q = j, \\ \alpha_{i,q} & \text{otherwise,} \end{cases}$$

where $i = n-m,\ \ j = n-k,\ \ 1 \le i < j < p \le n$, and $i+1 \le q \le n$.

Consideration of centers provides another similarity between these groups. For a group G, let $Z(G)$ denote the center of G, and let $\overline{G} = G/Z(G)$. It is well known that the center of the pure braid group is infinite cyclic and that $P_n \cong \overline{P}_n \times Z(P_n) = \overline{P}_n \times \mathbb{Z}$. The analogous result holds for the upper triangular McCool group.

Proposition 2.3. *The center of the upper triangular McCool group $P\Sigma_n^+$ is infinite cyclic, the quotient $\overline{P\Sigma}_n^+ = F_{n-1} \rtimes_{\mu_{n-1}} \cdots \rtimes_{\mu_3} F_2$ is an iterated semidirect product of free groups, and $P\Sigma_n^+ \cong \overline{P\Sigma}_n^+ \times Z(P\Sigma_n^+) = \overline{P\Sigma}_n^+ \times \mathbb{Z}$.*

Proof. Consider the element $c = \alpha_{1,n}\alpha_{2,n} \cdots \alpha_{n-1,n}$ of the group $P\Sigma_n^+$. Using (2-2), it is readily checked that c commutes with all the generators of $P\Sigma_n^+$, and so $c \in Z(P\Sigma_n^+)$. Also it is clear that $c \in \mathrm{Aut}(F_n)$ has infinite order. Consequently, the infinite cyclic subgroup $C = \langle c\rangle$ is contained in the center $Z(P\Sigma_n^+)$.

Since $\alpha_{n-1,n} = (\alpha_{1,n}\alpha_{2,n} \cdots \alpha_{n-2,n})^{-1} \cdot c$, the group $P\Sigma_n^+$ admits a presentation with generators c and $\alpha_{i,j}$ for $1 \le i < j \le n$ and $(i, j) \ne (n-1, n)$, relations $[c, \alpha_{i,j}]$ for all $i < j$, and the relations (2-2) (not involving $\alpha_{n-1,n}$). Thus, $P\Sigma_n^+ \cong C \times (P\Sigma_n^+/C)$. Since the free group F_1 in the iterated semidirect product decomposition $P\Sigma_n^+ = \rtimes_{j=1}^{n-1} F_j$ is generated by $\alpha_{n-1,n}$, it is clear from the above discussion that $P\Sigma_n^+/C = F_{n-1} \rtimes_{\mu_{n-1}} \cdots \rtimes_{\mu_3} F_2$. An easy inductive argument reveals that the center of this quotient is trivial. It follows that $C = Z(P\Sigma_n^+)$, which completes the proof. $\square$

Despite the aforementioned similarities, the groups P_n and $P\Sigma_n^+$ are not isomorphic; see Bardakov and Mikhailov [2008].

Definition 2.4. Let G be a group. The *cohomological dimension* $\mathrm{cd}(G)$ of G is the smallest integer n such that $H^q(G; M) = 0$ for any G-module M and all $q > n$. The *geometric dimension* $\mathrm{geom\,dim}(G)$ of the group G is the smallest dimension of an Eilenberg–Mac Lane complex of type $K(G, 1)$.

Proposition 2.5. *Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then*

$$\mathrm{geom\,dim}(P\Sigma_n) = \mathrm{cd}(P\Sigma_n) = n - 1.$$

Proof. Collins [1989] showed that, for each n, the cohomological dimension of $P\Sigma_n$ is as asserted: $\mathrm{cd}(P\Sigma_n) = n - 1$. A classical result of Eilenberg and Ganea [1957] states that, for groups of cohomological dimension at least 3, the geometric dimension is equal to the cohomological dimension. Thus, the assertion holds for $P\Sigma_n$ with $n > 3$.

Since $P\Sigma_2 = F_2$ is the free group generated by $\alpha_{2,1}$ and $\alpha_{1,2}$, the case $n = 2$ is immediate.

It remains to consider the case $n = 3$. The group $P\Sigma_3$ is generated by six elements $\alpha_{i,j}$ with $1 \le i \ne j \le 3$. Let $\beta_1 = \alpha_{2,1}\alpha_{3,1}$, $\beta_2 = \alpha_{1,2}\alpha_{3,2}$, and $\beta_3 = \alpha_{1,3}\alpha_{2,3}$, and observe that these elements generate the inner automorphism group $\mathrm{Inn}(F_3)$ of F_3, which is isomorphic to F_3. As noted in [Brownstein and Lee 1993], the group $P\Sigma_3 = \mathrm{Inn}(F_3) \rtimes F$ is a semidirect product, where $F = \langle \alpha_{1,2}, \alpha_{2,1}, \alpha_{3,1}\rangle$ is also a free group on 3 generators. Thus, $P\Sigma_3 \cong F_3 \rtimes F_3$ is a semidirect product of two finitely generated free groups.

For an arbitrary iterated semidirect product of finitely generated free groups G, Cohen and Suciu [1998, Section 1.3] give an explicit construction of a $K(G, 1)$-complex X_G. If $G = \rtimes_{i=1}^{\ell} F_{d_i}$, the complex X_G is ℓ-dimensional. In particular, for the group $G = P\Sigma_3$, this construction yields a 2-dimensional $K(G, 1)$-complex. We therefore have $\operatorname{geom}\dim(P\Sigma_3) = \operatorname{cd}(P\Sigma_3) = 2$. □

A similar result holds for the upper triangular McCool groups.

Proposition 2.6. *Suppose $P\Sigma_n^+$ is the upper triangular McCool group, and let $\overline{P\Sigma}_n^+ = P\Sigma_n^+/Z(P\Sigma_n^+)$. Then*

$$\operatorname{geom}\dim(P\Sigma_n^+) = \operatorname{cd}(P\Sigma_n^+) = n-1 \textit{ and } \operatorname{geom}\dim(\overline{P\Sigma}_n^+) = \operatorname{cd}(\overline{P\Sigma}_n^+) = n-2.$$

Proof. Since $\overline{P\Sigma}_n^+ = F_{n-1} \rtimes_{\mu_{n-1}} \cdots \rtimes_{\mu_3} \rtimes F_2$ and $P\Sigma_n^+ = \overline{P\Sigma}_n^+ \times \mathbb{Z}$ are iterated semidirect products of finitely generated free groups, this follows immediately from the results of [Cohen and Suciu 1998]. □

3. Structure of the cohomology ring

As noted in Section 1, the zero-divisor cup length of the cohomology ring of a group provides a lower bound for the topological complexity. In this section, we determine this lower bound for the groups $P\Sigma_n$ and $P\Sigma_n^+$.

Let $A = \bigoplus_{k=0}^{\ell} A^k$ be a graded algebra over a field $\Bbbk$, and recall that the cup length $\mathsf{cl}(A)$ is the largest integer q for which there are homogeneous elements $a_1, \dots, a_q$ of positive degree in A such that $a_1 \cdots a_q \neq 0$. The tensor product $A \otimes A$ has a natural graded algebra structure, with multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1|\cdot|u_2|} u_1 u_2 \otimes v_1 v_2.$$

Let $\mu : A \otimes A \to A$ denote the multiplication homomorphism, and let $Z = \ker(\mu)$ be the ideal of zero-divisors. The zero-divisor cup length of A, denoted by $\mathsf{zcl}(A)$, is the cup length of this ideal: $\mathsf{zcl}(A) = \mathsf{cl}(Z)$. Observe that if $a \in A$, then the element $\bar{a} = a \otimes 1 - 1 \otimes a \in Z$ is a zero-divisor.

In [1993], Brownstein and Lee determined the low-dimensional cohomology $H^{\leq 2}(P\Sigma_n; \mathbb{Z})$ of the basis-conjugating automorphism group, and conjectured the general ring structure in terms of generators and relations. This conjecture was recently proved by Jensen, McCammond, and Meier [2006, Theorem 6.7]. For our purposes, it suffices to work with coefficients in the field $\Bbbk = \mathbb{Q}$ of rational numbers. So we suppress coefficients and denote the rational cohomology of a group G by $H^*(G) = H^*(G; \mathbb{Q})$ throughout this section and the next.

Theorem 3.1 [Jensen et al. 2006]. *The rational cohomology algebra $H^*(P\Sigma_n)$ is isomorphic to E/I, where E is the exterior algebra over $\mathbb{Q}$ generated by degree* *one elements $a_{i,j}$ for $1 \le i \ne j \le n$, and I is the homogeneous ideal generated by the degree two elements*

$$\begin{aligned} &a_{i,j}a_{j,i} && \textit{for } i,\ j \textit{ distinct, and} \\ &a_{k,j}a_{j,i} - a_{k,j}a_{k,i} - a_{i,j}a_{k,i} && \textit{for } i,\ j,k \textit{ distinct.} \end{aligned}$$

This result may be used to exhibit an explicit basis for $H^q(P\Sigma_n)$ for each q with $0 \le q \le n-1$; see [Jensen et al. 2006, Section 6]. Call an element of the form $a_{i,j}a_{j,k}\cdots a_{s,t}a_{t,i}$ a cyclic product. Then $H^q(P\Sigma_n)$ has a basis consisting of those q-fold products $a_{i_1,j_1}a_{i_2,j_2}\cdots a_{i_q,j_q}$ of the one-dimensional generators that do not contain any cyclic products and have distinct first indices $i_1,\dots,i_q$. It follows that the Poincaré polynomial of $P\Sigma_n$ is $\sum_{q\ge 0}\dim H^q(P\Sigma_n)\cdot t^q = (1+nt)^{n-1}$. In particular, $H^i(P\Sigma_n) = 0$ for $i \ge n$, and the cup length of $H^*(P\Sigma_n)$ is $n-1$.

We use these results to find the zero-divisor cup length of the ring $H^*(P\Sigma_n)$.

Theorem 3.2. *Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then the zero-divisor cup length of the rational cohomology algebra of $P\Sigma_n$ is*

$$\mathsf{zcl}(H^*(P\Sigma_n)) = 2n-2.$$

Proof. In general, the zero-divisor cup length of an algebra A cannot exceed the cup length of the tensor product $A \otimes A$, which is twice the cup length of A itself: $\mathsf{zcl}(A) \le \mathsf{cl}(A\otimes A) = 2\,\mathsf{cl}(A)$. Since $\mathsf{cl}(H^*(P\Sigma_n)) = n-1$ by Theorem 3.1, it follows that $\mathsf{zcl}(H^*(P\Sigma_n)) \le 2n-2$.

For the reverse inequality, we work in the aforementioned basis for $H^*(P\Sigma_n)$ and the corresponding induced basis for the tensor product $H^*(P\Sigma_n)\otimes H^*(P\Sigma_n)$. Observe that any monomial in the generators of $H^*(P\Sigma_n)$ that contains a cyclic product must vanish, and that any finite expression in $H^*(P\Sigma_n)$ can be reduced to an expression in the basis elements after finitely many applications of the relation

$$a_{k,j}a_{k,i} = a_{k,j}a_{j,i} + a_{i,j}a_{k,i} \tag{3-1}$$

by eliminating, step-by-step, repetition in the first index.

For each $i < n$, consider the elements $\mathbf{x}_i = a_{i,i+1}$ and $\mathbf{y}_i = a_{i+1,i}$ in $H^*(P\Sigma_n)$ and the corresponding zero divisors $\bar{\mathbf{x}}_i = \mathbf{x}_i \otimes 1 - 1 \otimes \mathbf{x}_i$ and $\bar{\mathbf{y}}_i = \mathbf{y}_i \otimes 1 - 1 \otimes \mathbf{y}_i$ in the tensor product $H^*(P\Sigma_n)\otimes H^*(P\Sigma_n)$. We claim that the product

$$M = \prod_{i=1}^{n-1}\bar{\mathbf{x}}_i \cdot \prod_{i=1}^{n-1}\bar{\mathbf{y}}_i = \bar{\mathbf{x}}_1\bar{\mathbf{x}}_2\cdots\bar{\mathbf{x}}_{n-1}\bar{\mathbf{y}}_1\bar{\mathbf{y}}_2\cdots\bar{\mathbf{y}}_{n-1}$$

of these $2n-2$ zero divisors is different from zero. To prove this, we use the relation (3-1) to express M in terms of the specified basis of the tensor product, and identify at least one monomial left unchanged by the reduction process.

If I is a subset of $[n-1]=\{1,2,\dots,n-1\}$, let $|I|$ denote the cardinality of I, and let $U_I = z_1\cdots z_{n-1}$ and $V_I = \hat{z}_1\cdots\hat{z}_{n-1}$, where

$$z_i = \begin{cases} \mathbf{y}_i, & \text{if } i \notin I, \\ \mathbf{x}_i, & \text{if } i \in I \end{cases} \quad \text{and} \quad \hat{z}_i = \begin{cases} \mathbf{y}_i, & \text{if } i \in I, \\ \mathbf{x}_i, & \text{if } i \notin I. \end{cases}$$

Then, using the fact that $\bar{\mathbf{x}}_i\bar{\mathbf{y}}_i = \mathbf{y}_i \otimes \mathbf{x}_i - \mathbf{x}_i \otimes \mathbf{y}_i$, we have

$$M = \sum_{I\subseteq[n-1]} (-1)^{|I|} U_I \otimes V_I. \tag{3-2}$$

When $I=\varnothing$ is the empty set, the summand $U_\varnothing \otimes V_\varnothing$ in (3-2) is

$$U_\varnothing \otimes V_\varnothing = \mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_{n-1} \otimes \mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_{n-1} = a_{2,1}a_{3,2}\cdots a_{n,n-1} \otimes a_{1,2}a_{2,3}\cdots a_{n-1,n}.$$

This monomial is already a basis element of $H^{n-1}(P\Sigma_n)\otimes H^{n-1}(P\Sigma_n)$.

We claim that the expression of any other summand $(-1)^{|I|}U_I \otimes V_I$ of (3-2) in terms of our basis for $H^*(P\Sigma_n)\otimes H^*(P\Sigma_n)$ will avoid the specified basis element $U_\varnothing \otimes V_\varnothing$. Clearly, if the monomial U_I is already a basis element of $H^*(P\Sigma_n)$, there is nothing to prove. Otherwise, U_I contains a factor $a_{k,j}a_{k,i}$ for at least one k with $1<k<n$, and these are the only generators in the product U_I involving index k. Applying the relation (3-1) to the product $a_{k,j}a_{k,i}$, we obtain (up to sign)

$$U_I = (a_{k,j}a_{j,i} + a_{i,j}a_{k,i}) \cdot (\text{other factors}) = a_{k,j}P + a_{k,i}Q,$$

where P and Q are monomials in the generators $a_{r,s}$ of $H^*(P\Sigma_n)$ with $r \neq k$ and $s \neq k$. Further application of reductive relation (3-1) to P and Q will result in no further appearance of k in the indices. Hence writing $U_I = a_{k,j}P + a_{k,i}Q$ in the specified basis for $H^*(P\Sigma_n)$ will yield a linear combination of basis elements, each with exactly one factor involving index k. On the other hand, our fixed monomial $U_\varnothing = a_{n,n-1}\cdots a_{k+1,k}a_{k,k-1}\cdots a_{2,1}$ contains two factors involving index k. Therefore the basis monomial $U_\varnothing \otimes V_\varnothing$ is different from any other possible basis summand coming from $U_I \otimes V_I$ with $I \neq \varnothing$, and our claim holds. □

The cohomology of the upper-triangular McCool group $P\Sigma_n^+$ may be analyzed in a similar manner. The integral cohomology of $P\Sigma_n^+$ was computed by Cohen, Pakianathan, Vershinin, and Wu [2008, Theorem 1.4]. Their results yield:

Theorem 3.3 [Cohen et al. 2008]. *The rational cohomology algebra $H^*(P\Sigma_n^+)$ is isomorphic to E^+/I^+, where E^+ is the exterior algebra over $\mathbb{Q}$ generated by degree one elements $a_{i,j}$ for $1 \le i < j \le n$, and I^+ is the homogeneous ideal generated by the degree two elements*

$$a_{i,j}a_{i,k} - a_{i,j}a_{j,k} \quad \textit{for } i<j<k.$$

This result may be used to exhibit an explicit basis for $H^q(P\Sigma_n^+)$ for each q with $0 \le q \le n-1$; compare [Cohen et al. 2008, Section 7]. The group $H^q(P\Sigma_n^+)$ has a basis consisting of those q-fold products $a_{i_1,j_1}a_{i_2,j_2}\cdots a_{i_q,j_q}$ of the one-dimensional generators that satisfy $1 \le i_1 < i_2 < \cdots < i_q \le n-1$ and $i_p < j_p \le n$ for each p. It follows that $\sum_{q\ge 0} \dim H^q(P\Sigma_n^+)\cdot t^q = \prod_{k=1}^{n-1}(1+kt)$. In particular, $H^i(P\Sigma_n^+)=0$ for $i \ge n$, and the cup length of $H^*(P\Sigma_n^+)$ is $n-1$.

We analyze the zero-divisor cup length of the ring $H^*(P\Sigma_n^+)$ using these results.

Theorem 3.4. *Let $P\Sigma_n^+$ be the upper-triangular McCool group. Then the zero-divisor cup length of the rational cohomology algebra of $P\Sigma_n^+$ satisfies*

$$\mathsf{zcl}(H^*(P\Sigma_n^+)) \ge 2n-3.$$

Proof. Consider the zero-divisors $\bar{a}_{i,j} = a_{i,j}\otimes 1 - 1\otimes a_{i,j}$ and $a_{n-1,n}\otimes a_{n-1,n}$. We check that the product

$$\bar{a}_{1,n-1}\bar{a}_{1,n}\bar{a}_{2,n-1}\bar{a}_{2,n}\cdots\bar{a}_{n-2,n-1}\bar{a}_{n-2,n}\cdot(a_{n-1,n}\otimes a_{n-1,n}) \tag{3-3}$$

is nonzero. Note that

$$\bar{a}_{i,n-1}\cdot\bar{a}_{i,n} = a_{i,n}\otimes a_{i,n-1} - a_{i,n-1}\otimes a_{i,n} + a_{i,n-1}a_{i,n}\otimes 1 + 1\otimes a_{i,n-1}a_{i,n}$$

for any $i \le n-2$. The product (3-3) contains summands of the form

$$\pm a_{1,i_1}a_{2,i_2}\cdots a_{n-2,i_{n-2}}a_{n-1,n}\otimes a_{1,j_1}a_{2,j_2}\cdots a_{n-2,j_{n-2}}a_{n-1,n}, \tag{3-4}$$

where i_p and j_p take different values from the set $\{n-1, n\}$ for each p. Such summands represent distinct basis elements in the tensor product. These are, in fact, the only nonzero summands in the expression (3-3). Any other monomial, say μ, in this expression will contain a factor of the form $a_{i,n-1}a_{i,n}\otimes 1$ or $1\otimes a_{i,n-1}a_{i,n}$ for some i with $1 \le i \le n-2$. The relations $a_{i,n-1}a_{i,n} = a_{i,n-1}a_{n-1,n}$ in $H^*(P\Sigma_n^+)$ and the fact that $a_{n-1,n}\otimes a_{n-1,n}$ is also a factor of μ may be used to show that μ is trivial in $H^*(P\Sigma_n^+)\otimes H^*(P\Sigma_n^+)$. Thus the product (3-3) is a nontrivial linear combination of the terms given by (3-4), and is nonzero. □

Remark 3.5. It follows from the results of the next section that equality holds in Theorem 3.4, that is, $\mathsf{zcl}(H^*(P\Sigma_n^+)) = 2n-3$.

4. Topological complexity

In this section, we recall several necessary properties of topological complexity and prove the main results of the paper.

Let X be a path-connected topological space. We are interested in the case where X is an Eilenberg–Mac Lane space of type $K(G,1)$ for $G = P\Sigma_n$ or $G = P\Sigma_n^+$, so assume that X has the homotopy type of a finite CW-complex. Let PX denote the space of all continuous paths $\gamma\colon [0,1]\to X$, equipped with the compact-open topology. The map $\pi : PX \to X\times X$, $\gamma \mapsto (\gamma(0),\gamma(1))$, which sends a path to its endpoints, is a fibration, with fiber ΩX, the based loop space of X.

Recall from Section 1 that the motion planning problem asks for a (continuous) section of this fibration, a map $s\colon X \times X \to PX$ satisfying $\pi \circ s = \mathrm{id}_{X\times X}$. As shown by Farber [2003, Theorem 1], in most cases such a section cannot exist.

Proposition 4.1 [Farber 2003]. *The path space fibration $\pi : PX \to X \times X$ admits a section if and only if X is contractible.*

Definition 4.2. The *topological complexity* $\mathsf{TC}(X)$ of X is the smallest positive integer k for which $X \times X = U_1 \cup \cdots \cup U_k$, where U_j is open and there exists a continuous section $s_j\colon U_i \to PX$ satisfying $\pi \circ s_j = \mathrm{id}_{U_i}$ for each j with $1 \le j \le k$. In other words, the topological complexity of X is the Schwarz genus (or sectional category) of the path space fibration $\pi\colon PX \to X \times X$.

The topological complexity of X is a homotopy-type invariant; see [Farber 2003, Theorem 3]. If G is a discrete group, define $\mathsf{TC}(G)$, the topological complexity of G, to be that of an Eilenberg–Mac Lane space of type $K(G, 1)$. Farber [2006, Section 31] poses the problem of determining the topological complexity of G in terms of other invariants of G such as $\mathrm{cd}(G)$, the cohomological dimension. In this section, we solve this problem for the basis-conjugating automorphism groups $P\Sigma_n$ and $P\Sigma_n^+$.

We will require several properties of topological complexity. We briefly record these and refer to the survey [Farber 2006] for further details.

First, if X is a finite-dimensional cell complex, then $\mathsf{TC}(X) \le 2\dim(X) + 1$; see [Farber 2006, Section 3]. Consequently, if G is a group of finite geometric dimension, then

$$\mathsf{TC}(G) \le 2\,\mathrm{geom\,dim}(G) + 1. \tag{4-1}$$

Second, as noted in Section 1, a lower bound for the topological complexity of a group G is provided by the zero-divisor cup length of the cohomology ring $H^*(G) = H^*(G; \mathbb{Q})$:

$$\mathsf{TC}(G) \ge 1 + \mathsf{zcl}(H^*(G)); \tag{4-2}$$

see [Farber 2006, Section 15]. Finally, if X and Y are path-connected paracompact locally contractible topological spaces (in particular, CW-complexes), then

$$\mathsf{TC}(X \times Y) \le \mathsf{TC}(X) + \mathsf{TC}(Y) - 1;$$

see [Farber 2006, Section 12]. Consequently, if G_1 and G_2 are groups (of finite geometric dimension), then

$$\mathsf{TC}(G_1 \times G_2) \le \mathsf{TC}(G_1) + \mathsf{TC}(G_2) - 1. \tag{4-3}$$

With these facts at hand, we now prove our main theorems.

Theorem 4.3. *The topological complexity of the basis-conjugating automorphism group $P\Sigma_n$ is* $\mathsf{TC}(P\Sigma_n) = 2n - 1$.

Proof. By Theorem 3.2, the zero-divisor cup length of $H^*(P\Sigma_n)$ is given by $\mathsf{zcl}(H^*(P\Sigma_n)) = 2n - 2$. So the lower bound (4-2) yields $\mathsf{TC}(P\Sigma_n) \geq 2n - 1$. For the reverse inequality, recall from Proposition 2.5 that

$$\text{geom dim}(P\Sigma_n) = \text{cd}(P\Sigma_n) = n - 1.$$

Consequently, the upper bound (4-1) yields $\mathsf{TC}(P\Sigma_n) \leq 2n - 1$. □

Theorem 4.4. *The topological complexity of the upper triangular McCool group* $P\Sigma_n^+$ *is* $\mathsf{TC}(P\Sigma_n^+) = 2n - 2$.

Proof. By Theorem 3.4, the zero-divisor cup length of $H^*(P\Sigma_n^+)$ is no less than $2n - 3$. So the lower bound (4-2) yields $\mathsf{TC}(P\Sigma_n^+) \geq 2n - 2$.

For the reverse inequality, recall from Proposition 2.3 that $P\Sigma_n^+ \cong \overline{P\Sigma}_n^+ \times \mathbb{Z}$. Since the circle S^1 is a $K(\mathbb{Z}, 1)$-space, and $\mathsf{TC}(\mathbb{Z}) = \mathsf{TC}(S^1) = 2$ (see, for instance, [Farber 2003, Section 5]), the product inequality (4-3) yields

$$\mathsf{TC}(P\Sigma_n^+) \leq \mathsf{TC}(\overline{P\Sigma}_n^+) + \mathsf{TC}(\mathbb{Z}) - 1 = \mathsf{TC}(\overline{P\Sigma}_n^+) + 1.$$

By Proposition 2.6, we have geom $\dim(\overline{P\Sigma}_n^+) = \text{cd}(\overline{P\Sigma}_n^+) = n-2$. Consequently, the upper bound (4-1) yields $\mathsf{TC}(\overline{P\Sigma}_n^+) \leq 2n - 3$. Thus $\mathsf{TC}(P\Sigma_n^+) \leq 2n - 2$. □

Corollary 4.5. *The zero-divisor cup length of the rational cohomology algebra of* $P\Sigma_n^+$ *is* $\mathsf{zcl}(H^*(P\Sigma_n^+)) = 2n - 3$.

5. Formality

If X is an Eilenberg–Mac Lane space of type $K(G, 1)$, where either $G = P\Sigma_n$ or $G = P\Sigma_n^+$, the results of the previous section imply that the topological complexity of X is given by the cohomological lower bound, that is,

$$\mathsf{TC}(X) = 1 + \mathsf{zcl}(H^*(X; \mathbb{Q})).$$

This equality holds for a number of spaces of interest in topology, including certain configuration spaces, complements of certain complex hyperplane arrangements, and Eilenberg–Mac Lane spaces corresponding to right-angled Artin groups; see [Cohen and Pruidze 2008; Farber et al. 2007; Farber and Yuzvinsky 2004; Yuzvinsky 2007]. Since all of these spaces are formal in the sense of Sullivan [1977], it is natural to speculate that such an equality holds for an arbitrary formal space X. Conjecturally, $\mathsf{TC}(X) = 1 + \mathsf{zcl}(H^*(X; R))$ for appropriate coefficients R. This conjecture is explicitly made by Yuzvinsky [2007] for the complement of an arbitrary hyperplane arrangement. Related problems are studied in [Fernández Suárez et al. 2006] and [Lechuga and Murillo 2007]. In this section, we show that the upper triangular McCool group $P\Sigma_n^+$ provides evidence in favor of such a conjecture.

Theorem 5.1. *Let X be an Eilenberg–Mac Lane space of type $K(G, 1)$, where $G = P\Sigma_n^+$ is the upper triangular McCool group. Then X is a formal space.*

To prove this theorem, we will need some definitions and facts concerning formality and related notions.

Let X be a space with the homotopy type of a connected, finite-type CW-complex. Loosely speaking, X is *formal* if the rational homotopy type of X is determined by the rational cohomology ring $H^*(X; \mathbb{Q})$. Examples of formal spaces include spheres, simply-connected Eilenberg–Mac Lane spaces, and those mentioned above.

Let G be a finitely presented group. Following Quillen [1969], call G *1-formal* if the Malcev Lie algebra of G is quadratic; see [Papadima and Suciu 2004] for details. As shown by Sullivan [1977] and Morgan [1978], the fundamental group $G = \pi_1(X)$ of a formal space X is a 1-formal group. There are, however, nonformal spaces with 1-formal fundamental groups; see [Kohno 1983; Morgan 1978].

Papadima and Suciu [2006, Proposition 2.1] provide a sufficient condition for the formality of a CW-complex. Recall that a connected, graded algebra A over a field $\Bbbk$ is said to be a *Koszul algebra* if $\mathrm{Tor}^A_{p,q}(\Bbbk, \Bbbk) = 0$ for all $p \neq q$, where p is the homological degree of the Tor groups and q is the internal degree coming from the grading of A. A necessary condition is that A be a quadratic algebra, the quotient of a free algebra on generators in degree 1 by an ideal generated in degree 2.

Proposition 5.2 [Papadima and Suciu 2006]. *Let X be a connected, finite-type CW-complex. If $H^*(X; \mathbb{Q})$ is a Koszul algebra and $G = \pi_1(X)$ is a 1-formal group, then X is a formal space.*

Berceanu and Papadima [2007, Remark 5.5] have recently shown that the upper triangular McCool group $P\Sigma_n^+$ is 1-formal. Thus, to prove Theorem 5.1, it suffices to show that the rational cohomology algebra $H^*(P\Sigma_n^+; \mathbb{Q})$ is Koszul. For this, we will use [Jambu and Papadima 1998, Proposition 6.3].

Let $A = \bigoplus_{k \geq 0} A^k$ be a connected, graded $\Bbbk$-algebra, and denote the augmentation ideal of A by $A^+ = \bigoplus_{k \geq 1} A^k$. Call a subalgebra B of A normal if $AB^+ = B^+A$. If $B \subset A$ is normal, there is a canonical projection $\pi : A \to F$, where $F = A/AB^+$.

Proposition 5.3 [Jambu and Papadima 1998]. *Let $B \subset A$ be a normal subalgebra such that A is free as a right B-module, and assume that the $\Bbbk$-algebras A, B and $F = A/AB^+$ are quadratic. If B and F are Koszul algebras, then A is a Koszul algebra.*

We apply this result to the rational cohomology algebra $H^*(P\Sigma_n^+; \mathbb{Q})$.

Proposition 5.4. *The rational cohomology algebra $H^*(P\Sigma_n^+;\mathbb{Q})$ of the upper triangular McCool group is a Koszul algebra.*

Proof. Write $A_n = H^*(P\Sigma_n^+;\mathbb{Q})$.

The proof consists of an inductive application of Proposition 5.3. As $P\Sigma_2^+ \cong \mathbb{Z}$, the base case A_2 is trivial.

Inductively assume that A_{n-1} is Koszul. For $k<n$, observe that A_k is isomorphic to the subalgebra $\tilde{A}_k$ of A_n generated by the elements $a_{i,j}$ with $n-k<i<j\le n$. Thus, we may assume that the subalgebra $\tilde{A}_{n-1}$ of A_n is Koszul. Since the algebras under consideration are graded commutative, $\tilde{A}_{n-1}$ is a normal subalgebra of A_n. Furthermore, A_n is free as a right $\tilde{A}_{n-1}$-module. Namely,

$$A_n = 1\cdot\tilde{A}_{n-1} \oplus a_{1,2}\cdot\tilde{A}_{n-1} \oplus \cdots \oplus a_{1,n}\cdot\tilde{A}_{n-1}.$$

This follows from the fact that in any monomial of the algebra A_n, the factor $a_{1,i}$ with minimal i always survives, since $a_{1,i}a_{1,j} = a_{1,i}a_{i,j}$ in A_n for any $1<i<j$; see Theorem 3.3.

Analyzing again the relations in A_n, we observe that the algebra $A_n/A_n\tilde{A}_{n-1}^+$ is a graded algebra generated by the elements $a_{1,i}$ for $2\le i\le n$, where all the terms in degree 2 and higher die. Consequently, the algebra $A_n/A_n\tilde{A}_{n-1}^+$ is quadratic and, moreover, Koszul. Thus, all the algebras under consideration are quadratic, and the conditions of Proposition 5.3 are satisfied. The result follows immediately. □

Since the upper triangular McCool group $P\Sigma_n^+$ is 1-formal (see [Berceanu and Papadima 2007]) and $H^*(P\Sigma_n^+;\mathbb{Q})$ is Koszul, Proposition 5.2 implies that an Eilenberg–Mac Lane space of type $K(P\Sigma_n^+,1)$ is formal, proving Theorem 5.1. Such a space X provides an example of a non-simply-connected formal space with $\mathsf{TC}(X) = 1+\mathsf{zcl}(H^*(X;\mathbb{Q}))$.

Remark 5.5. Berceanu and Papadima [2007, Theorem 5.4] also showed that the basis-conjugating automorphism group $P\Sigma_n$ is 1-formal. Using the realizations $P\Sigma_2 \cong F_2$ and $P\Sigma_3 \cong F_3 \rtimes F_3$ noted in the proof of Proposition 2.5, one can show that $H^*(P\Sigma_n;\mathbb{Q})$ is Koszul and hence a $K(P\Sigma_n,1)$-space is formal for $n\le 3$. We do not know if the cohomology algebra $H^*(P\Sigma_n;\mathbb{Q})$ is Koszul for $n>3$.

References

[Bardakov and Mikhailov 2008] V. G. Bardakov and R. Mikhailov, "On certain questions of the free group automorphisms theory", *Comm. Algebra* **36** (2008), 1489–1499. MR 2406602 Zbl 05293085

[Berceanu and Papadima 2007] B. Berceanu and S. Papadima, "Universal representations of braid and braid-permutation groups", preprint, 2007. arXiv 0708.0634v1

[Birman 1974] J. S. Birman, *Braids, links, and mapping class groups*, Annals of Mathematics Studies **82**, Princeton University Press, 1974. MR 51 #11477 Zbl 0305.57013

[Brownstein and Lee 1993] A. Brownstein and R. Lee, "Cohomology of the group of motions of n strings in 3-space", pp. 51–61 in *Mapping class groups and moduli spaces of Riemann surfaces*

(Göttingen and Seattle, 1991), edited by C.-F. Bödigheimer and R. M. Hain, Contemp. Math. **150**, Amer. Math. Soc., Providence, RI, 1993. MR 94k:20073 Zbl 0804.20033

[Cohen and Pruidze 2008] D. C. Cohen and G. Pruidze, "Motion planning in tori", *Bull. Lond. Math. Soc.* **40**:2 (2008), 249–262. MR 2414784 Zbl 05309674

[Cohen and Suciu 1998] D. C. Cohen and A. I. Suciu, "Homology of iterated semidirect products of free groups", *J. Pure Appl. Algebra* **126**:1-3 (1998), 87–120. MR 99e:20064 Zbl 0908.20033

[Cohen et al. 2007] D. C. Cohen, F. R. Cohen, and S. Prassidis, "Centralizers of Lie algebras associated to descending central series of certain poly-free groups", *J. Lie Theory* **17**:2 (2007), 379–397. MR 2008e:20049 Zbl 1135.20025

[Cohen et al. 2008] F. R. Cohen, J. Pakianathan, V. Vershinin, and J. Wu, "Basis-conjugating automorphisms of a free group and associated Lie algebras", pp. 147–168 in *Groups, homotopy and configuration spaces* (Tokyo 2005), Geom. Topol. Monogr. **13**, Geom. Topol., Coventry, 2008. Zbl 05261770

[Collins 1989] D. J. Collins, "Cohomological dimension and symmetric automorphisms of a free group", *Comment. Math. Helv.* **64**:1 (1989), 44–61. MR 90e:20035 Zbl 0669.20027

[Dahm 1962] D. Dahm, *A generalization of braid theory*, thesis, Princeton University, 1962.

[Eilenberg and Ganea 1957] S. Eilenberg and T. Ganea, "On the Lusternik–Schnirelmann category of abstract groups", *Ann. of Math.* (2) **65** (1957), 517–518. MR 19,52d Zbl 0079.25401

[Farber 2003] M. Farber, "Topological complexity of motion planning", *Discrete Comput. Geom.* **29**:2 (2003), 211–221. MR 2004c:68132 Zbl 1038.68130

[Farber 2004] M. Farber, "Instabilities of robot motion", *Topology Appl.* **140**:2-3 (2004), 245–266. MR 2005g:68166 Zbl 1106.68107

[Farber 2006] M. Farber, "Topology of robot motion planning", pp. 185–230 in *Morse theoretic methods in nonlinear analysis and in symplectic topology*, edited by P. Biran et al., NATO Sci. Ser. II Math. Phys. Chem. **217**, Springer, Dordrecht, 2006. MR 2008d:68141 Zbl 1089.68131

[Farber and Yuzvinsky 2004] M. Farber and S. Yuzvinsky, "Topological robotics: subspace arrangements and collision free motion planning", pp. 145–156 in *Geometry, topology, and mathematical physics*, edited by V. M. Buchstaber and I. M. Krichever, Amer. Math. Soc. Transl. Ser. 2 **212**, Amer. Math. Soc., Providence, RI, 2004. MR 2005i:55019 Zbl 1088.68171

[Farber et al. 2007] M. Farber, M. Grant, and S. Yuzvinsky, "Topological complexity of collision free motion planning algorithms in the presence of multiple moving obstacles", pp. 75–83 in *Topology and robotics*, edited by M. Farber et al., Contemp. Math. **438**, Amer. Math. Soc., Providence, RI, 2007. MR 2008i:55015 Zbl 05250982

[Fernández Suárez et al. 2006] L. Fernández Suárez, P. Ghienne, T. Kahl, and L. Vandembroucq, "Joins of DGA modules and sectional category", *Algebr. Geom. Topol.* **6** (2006), 119–144. MR 2006k:55011 Zbl 1097.55006

[Goldsmith 1981] D. L. Goldsmith, "The theory of motion groups", *Michigan Math. J.* **28**:1 (1981), 3–17. MR 82h:57007 Zbl 0462.57007

[Jambu and Papadima 1998] M. Jambu and S. Papadima, "A generalization of fiber-type arrangements and a new deformation method", *Topology* **37**:6 (1998), 1135–1164. MR 99g:52019 Zbl 0988.52031

[James 1978] I. M. James, "On category, in the sense of Lusternik–Schnirelmann", *Topology* **17**:4 (1978), 331–348. MR 80i:55001 Zbl 0408.55008

[Jensen et al. 2006] C. Jensen, J. McCammond, and J. Meier, "The integral cohomology of the group of loops", *Geom. Topol.* **10** (2006), 759–784. MR 2007c:20121 Zbl 05052489

[Kohno 1983] T. Kohno, "On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces", *Nagoya Math. J.* **92** (1983), 21–37. MR 85d:14032 Zbl 0503.57001

[Latombe 1991] J.-C. Latombe, *Robot Motion Planning*, Kluwer, Dordrecht, 1991.

[Lechuga and Murillo 2007] L. Lechuga and A. Murillo, "Topological complexity of formal spaces", pp. 105–114 in *Topology and robotics*, Contemp. Math. **438**, Amer. Math. Soc., Providence, RI, 2007. MR 2359032 Zbl 05250984

[McCool 1986] J. McCool, "On basis-conjugating automorphisms of free groups", *Canad. J. Math.* **38**:6 (1986), 1525–1529. MR 87m:20093 Zbl 0613.20024

[Morgan 1978] J. W. Morgan, "The algebraic topology of smooth algebraic varieties", *Inst. Hautes Études Sci. Publ. Math.* 48 (1978), 137–204. MR 80e:55020 Zbl 0401.14003

[Papadima and Suciu 2004] S. Papadima and A. I. Suciu, "Chen Lie algebras", *Int. Math. Res. Not.* 21 (2004), 1057–1086. MR 2004m:17043 Zbl 1076.17007

[Papadima and Suciu 2006] S. Papadima and A. I. Suciu, "Algebraic invariants for right-angled Artin groups", *Math. Ann.* **334**:3 (2006), 533–555. MR 2006k:20078 Zbl 05013674

[Quillen 1969] D. Quillen, "Rational homotopy theory", *Ann. of Math.* (2) **90** (1969), 205–295. MR 41 #2678 Zbl 0191.53702

[Schwarz 1961] A. S. Švarc, "The genus of a fibered space", *Trudy Moskov. Mat. Obšč.* **10** (1961), 217–272. In Russian; translated, with [Schwarz 1962], in *Eleven papers on topology and algebra*, pp. 49–140, Amer. Math. Soc. Transl. Ser. 2, **55**, Amer. Math. Soc., Providence, RI, 1966. MR 27 #4233 Zbl 0178.26202

[Schwarz 1962] A. S. Švarc, "The genus of a fibre space", *Trudy Moskov Mat. Obšč.* **11** (1962), 99–126. MR 27 #1963

[Sullivan 1977] D. Sullivan, "Infinitesimal computations in topology", *Inst. Hautes Études Sci. Publ. Math.* 47 (1977), 269–331 (1978). MR 58 #31119 Zbl 0374.57002

[Yuzvinsky 2007] S. Yuzvinsky, "Topological complexity of generic hyperplane complements", pp. 115–119 in *Topology and robotics* (Zurich, 2006), edited by M. Farber et al., Contemp. Math. **438**, Amer. Math. Soc., Providence, RI, 2007. MR 2008i:52025 Zbl 05250985

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RIGIDITY OF REPRESENTATIONS IN SO(4, 1) FOR DEHN FILLINGS ON 2-BRIDGE KNOTS

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We prove that, for a hyperbolic two-bridge knot, infinitely many Dehn fillings are rigid in $\mathrm{SO}_0(4,1)$. Here rigidity means that any discrete and faithful representation in $\mathrm{SO}_0(4,1)$ is conjugate to the holonomy representation in $\mathrm{SO}_0(3,1)$. We also show local rigidity for almost all Dehn fillings.

1. Introduction

In this paper we consider compact, orientable three-manifolds M, whose boundary is a torus $\partial M \cong T^2$ and whose interior admits a complete hyperbolic metric of finite volume. We will specialize to the case where M is the exterior of a hyperbolic two-bridge knot and in particular its fundamental group is generated by two peripheral elements.

We tacitly assume that a basis for $H_1(\partial M; \mathbf{Z}) \cong \mathbf{Z}^2$ has been fixed, and for $(p, q) \in \mathbf{Z}^2$ coprime, we denote by $M_{p/q}$ the manifold obtained by Dehn filling with meridian curve (p, q). According to Thurston's hyperbolic Dehn filling theorem, for all but finitely many $p/q \in \mathbf{Q} \cup \{\infty\}$, the Dehn filled manifold $M_{p/q}$ is hyperbolic. In particular, its holonomy representation is the only discrete and faithful representation of $\pi_1(M_{p/q})$ in $\mathrm{SO}_0(3,1)$ up to conjugation. Here $\mathrm{SO}_0(3,1)$ denotes the identity component of $\mathrm{SO}(3,1)$ and is isomorphic to $\mathrm{Isom}^+(\mathbf{H}^3)$, the orientation preserving isometry group of hyperbolic three space.

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In this paper we address the question of whether $M_{p/q}$ has other discrete and faithful representations in $\mathrm{SO}_0(4,1) \cong \mathrm{Isom}^+(\mathbf{H}^4)$.

For the unfilled manifold M, M. Kapovich [1994] proved global rigidity. That is to say, every discrete and faithful representation of $\pi_1(M)$ in $\mathrm{SO}_0(4,1)$ is conjugated to a representation in $\mathrm{SO}_0(3,1)$, and therefore to the holonomy of the hyperbolic metric. Moreover Kapovich [1994] also proved infinitesimal rigidity for infinitely many Dehn fillings $M_{p/q}$, which are therefore locally rigid: there is no continuous nontrivial deformation in $\mathrm{SO}_0(4,1)$ of the holonomy of $M_{p/q}$. That result was then generalized by K. Scannell [2002] to Dehn fillings on a larger class of manifolds.

Here we prove global rigidity for infinitely many Dehn fillings on M, and local rigidity for almost all of them.

Theorem 1.1. *Let M be the exterior of a hyperbolic two-bridge knot. Then, for infinitely many $p/q \in \mathbf{Q} \cup \{\infty\}$, the Dehn filled manifold $M_{p/q}$ has no discrete and faithful representation in* $\mathrm{SO}_0(4,1)$ *other than its holonomy in* $\mathrm{SO}_0(3,1)$.

The following definition can be found in [Scannell 2002]. Here we consider the action of the holonomy representation in the Lie algebra $\mathfrak{so}(4,1)$ via the adjoint representation.

Definition 1.2. Let M be a compact three manifold with boundary consisting of tori and whose interior is hyperbolic. Its *parabolic cohomology* is defined to be the kernel:

$$\mathrm{PH}^1(M, \mathfrak{so}(4,1)) = \ker\big(H^1(M, \mathfrak{so}(4,1)) \to H^1(\partial M, \mathfrak{so}(4,1))\big).$$

Scannell [2002] proves that, for two-bridge knot exteriors,

$$\mathrm{PH}^1(M, \mathfrak{so}(4,1)) = 0.$$

The following improves a theorem of [Scannell 2002]:

Theorem 1.3. *Let M be a compact three manifold with boundary a torus whose interior is hyperbolic. If* $\mathrm{PH}^1(M, \mathfrak{so}(4,1)) = 0$, *then almost all Dehn filled manifolds $M_{p/q}$ are locally rigid in* $\mathrm{SO}(4,1)$.

Theorem 1.1 follows from Theorems 1.4 and 1.5 below. Theorem 1.3 is a consequence of the local analysis of the variety of representations made in the proof of Theorem 1.5.

Convergence of representations will be understood in the variety of representations of M, in particular algebraic convergence. For a discrete group Γ, the set of all representations of Γ in $\mathrm{SO}_0(n,1)$ is denoted by $R(\Gamma, \mathrm{SO}_0(n,1))$. It is well known that it is a real algebraic variety; see [Johnson and Millson 1987; Morgan 1986]. We are interested in representations up to conjugacy, but the space of conjugacy classes does not seem to have a structure easy to work with. For results about this set of conjugacy classes we refer to [Johnson and Millson 1987; Kapovich 1994; Morgan 1986; Scannell 2000].

When $\Gamma = \pi_1(M)$, it is customary to write

$$R(M, \mathrm{SO}_0(n, 1)) = R\big(\pi_1(M), \mathrm{SO}_0(n, 1)\big).$$

Theorem 1.4. *Let M be a hyperbolic two-bridge knot exterior. Let $\rho_0 : \pi_1(M) \to$ $\mathrm{Isom}(\mathbf{H}^4)$ be the holonomy of the (Fuchsian) complete hyperbolic structure of M. Let $\{\rho_n\}_{n\in\mathbf{N}}$ be a sequence of representations*

$$\rho_n : \pi_1(M) \to \mathrm{Isom}(\mathbf{H}^4).$$

Suppose that, for each n, the representation ρ_n factorizes through a discrete and faithful representation of $\pi_1(M_{p_n/q_n})$ with $p_n^2 + q_n^2 \to \infty$. Then, up to conjugation, ρ_n converges to ρ_0.

Theorem 1.5. *Let M be hyperbolic with one cusp with $\mathrm{PH}^1(M, \mathfrak{so}(4, 1)) = 0$. Let $\{\rho_n\}$ be a sequence of representations of $\pi_1(M)$ in $\mathrm{SO}_0(4, 1)$ that converges to the holonomy of M.*

If each ρ_n comes from a discrete and faithful representation of $\pi_1(M_{p_n/q_n})$, not conjugated to a representation in $\mathrm{SO}_0(3, 1)$, then $p_n/q_n \to l$ for some $l \in \mathbf{R} \cup \{\infty\}$ depending only on M.

Here is the idea of the proof of Theorem 1.4. Take a sequence $\{\rho_n\}_{n\in\mathbf{N}}$, so that each ρ_n is a discrete and faithful representation of M_{p_n/q_n} with $p_n^2 + q_n^2 \to \infty$. By theorems of [Bestvina 1988; Morgan and Shalen 1984; Paulin 1997], the space of discrete and faithful representations of M in SO(4, 1) is compact. Even if ρ_n is not a faithful representation of M, the proof can be adapted to say that a subsequence of ρ_n converges to ρ, a representation of M. Moreover, the so-called Chuckrow–Wielenberg's theorem can also be adapted to say that ρ is discrete and faithful. Since M is generated by two peripheral elements, by a theorem of M. Kapovich [1994] ρ must be the holonomy representation of M in SO(3, 1).

Theorem 1.5 involves an analysis of $R(M, \mathrm{SO}(4, 1))$ in a neighborhood of ρ_0, following closely the results obtained by Scannell [2002]. The tangent space to

$$R\big(\pi_1(M), \mathrm{SO}_0(4, 1)\big)$$

is the space of cocycles valued on the Lie algebra $\mathfrak{so}(4, 1)$, and the space tangent to the orbits by conjugation is the space of coboundaries. Thus, to get relevant information, we study the cohomology group $H^1(M, \mathfrak{so}(4, 1))$. We need to understand how a sequence of representations in SO(4, 1) can approach ρ_0, and we shall study which elements in $H^1(M, \mathfrak{so}(4, 1))$ are tangent vectors of deformations of

ρ_0 in $R\big(\pi_1(M), \mathrm{SO}_0(4, 1)\big)$, what in Section 4A is called the *differentiable tangent cone*.

Since ρ_0 is contained in SO(3, 1), the Lie algebra splits as $\pi_1(M)$-module as

$$\mathfrak{so}(4, 1) = \mathfrak{so}(3, 1) \oplus \mathbf{R}^{3,1},$$

where $\mathbf{R}^{3,1}$ denotes the Minkowski space equipped with SO(3, 1) linear action. That splitting induces a direct sum of cohomology groups

$$H^1(M, \mathfrak{so}(4, 1)) = H^1(M, \mathfrak{so}(3, 1)) \oplus H^1(M, \mathbf{R}^{3,1}).$$

The subspace $H^1(M, \mathfrak{so}(3, 1))$ has dimension 2 and it is the tangent space to the variety of representations in SO(3, 1) up to conjugation, described by Thurston's proof of the hyperbolic Dehn filling theorem; see [Kapovich 2001]. By a theorem of Scannell [2002], $H^1(M, \mathbf{R}^{3,1})$ has dimension one. In Proposition 4.2, we show that the tangent cone is contained in the union

$$H^1(M, \mathfrak{so}(4, 1)) = H^1(M, \mathfrak{so}(3, 1)) \cup H^1(M, \mathbf{R}^{3,1}).$$

With the help of the curve selection lemma, this implies that if we have a sequence of non-Fuchsian representations approaching ρ_0, then the sequence must be contained in a semialgebraic set tangent to $H^1(M, \mathbf{R}^{3,1})$.

In the proof we need to understand how elements in $H^1(\partial M, \mathbf{R}^{3,1})$ are realized by deformations in the boundary ∂M, viewed as isometries of $\mathbf{R}^3 = \partial\mathbf{H}^4 \setminus \{\infty\}$. Those are realized by deforming a lattice in $\mathbf{R}^2 \subset \mathbf{R}^3$ as a group of screw motions whose axis is contained in $\mathbf{R}^2$. This kind of deformation imposes some restriction on the Dehn filling coefficients, that we use to prove the theorem.

Our study of the variety of representations relies on previous work of Scannell [2002], where he shows that this is a singular point of the variety of representations.

The paper is organized in four sections. In Section 2 we prove Theorem 1.4. Section 3 is devoted to the preliminaries about infinitesimal deformations, and Section 4 to the analysis of a neighborhood of the variety of representations. The results of both sections are used in Section 5, where Theorem 1.5 is proved.

2. Convergence of representations

In this section we prove Theorem 1.4. Even if most of the arguments and techniques we use can be found in the literature, we give a proof for completeness, stressing the changes required in our situation. We mention in particular the recent work of M. Kapovich [2007] on convergence of groups.

Proof of Theorem 1.4. The proof goes through two main steps:

I The sequence $\{\rho_n\}$ is bounded in $R\big(\pi_1(M), \mathrm{Isom}(\mathbf{H}^4)\big)$;

II each accumulation point of $\{\rho_n\}$ is discrete and faithful, whence conjugate to ρ_0 by [Kapovich 1994; Scannell 2002].

Each step is explained in a different subsection.

2A. ***Ultralimits and asymptotic cones.*** We concentrate here on Step I of the proof of Theorem 1.4.

We start by recalling the following result of Morgan and Shalen [1988].

Theorem 2.1. *If M is a complete hyperbolic 3-manifold, then there is no action of $\pi_1(M)$ on an $\mathbf{R}$-tree by isometries having*

- *no global fixed points, and*
- *small arc-stabilizers (that is, arc-stabilizers do not contain rank-two free subgroups).*

We will show that, if $\{\rho_n\}$ was unbounded, then it would induce an action of $\pi_1(M)$ on an $\mathbf{R}$-tree as the one forbidden by Theorem 2.1, deducing therefore that $\{\rho_n\}$ must be bounded. The way to do that uses standard techniques of asymptotic cones (see for example [Kapovich 2001] for details on asymptotic cones).

Let ω be a nonprincipal ultrafilter, that we think of as a family of subsets of natural numbers such that:

(1) Given any subset $S \subset \mathbf{N}$, either $S \in \omega$ or $\mathbf{N} \setminus S \in \omega$;

(2) if $S \in \omega$, and $S' \supset S$, then $S' \in \omega$;

(3) if $S \subset \mathbf{N}$ is a finite subset, then $S \notin \omega$;

(4) if $S, S' \in \omega$, then $S \cap S' \in \omega$.

We say that a sequence $\{x_n\}$ in a topological space *ω-converges* to a point x, and we write $\omega \lim x_n = x$, if for each open neighborhood U of x, the set $\{i \in \mathbf{N} : x_i \in U\}$ belongs to ω. It is an easy exercise that any sequence in a compact space has a unique ω-limit.

Let $\{\gamma_i\}$ be a finite set of generators of $\pi_1(M)$. Let $* = \{*_n\}$ be a sequence of points such that $*_n \in \mathbf{H}^4$ realizes the minimum

$$\min_{p \in \mathbf{H}^4} \max_i d(\rho_n(\gamma_i)p, p)$$

and let

$$\lambda_n = \max_i d(\rho_n(\gamma_i)*_n, *_n).$$

Let (X, d) be the asymptotic cone of $\mathbf{H}^4$ made using the ultrafilter ω, the sequence of rescaling parameters $\{\lambda_n\}$ and the base-points sequence $\{*_n\}$. Namely,

$$X = \left\{ \{x_n\} \subset \mathbf{H}^4 : \omega \lim \frac{d(x_n, *_n)}{\lambda_n} < \infty \right\},$$

where we identify two sequences $\{x_n\}$ and $\{y_n\}$ whenever

$$\omega \lim \frac{d(x_n, y_n)}{\lambda_n} = 0,$$

and we set:

$$d(\{x_n\}, \{y_n\}) = \omega \lim \frac{d(x_n, y_n)}{\lambda_n}.$$

The following lemma is a standard fact about asymptotic cones of hyperbolic spaces; see for example [Kapovich 2001].

Lemma 2.2. *If $\{\rho_n\}$ is unbounded, that is, if $\lambda_n \to \infty$, then (X, d) is an $\mathbf{R}$-tree.*

Lemma 2.3. *The ω-limit ρ_ω of $\{\rho_n\}$ is an isometric action of $\pi_1(M)$ on X, that is, a representation of $\pi_1(M)$ on* $\mathrm{Isom}(X)$.

Proof. The action ρ_ω on X is tautologically defined by

$$\rho_\omega(\gamma)(\{x_n\}) = \{\rho_n(\gamma)(x_n)\}.$$

We have to check that such a definition is well-posed; namely, that for all $\{x_n\} \in X$ and all $\gamma \in \pi_1(M)$ we have $\{\rho_n(\gamma)(x_n)\} \in X$. In other words, we need to check that

$$\omega \lim \frac{d(x_n, *_n)}{\lambda_n} < \infty \quad \Longrightarrow \quad \omega \lim \frac{d(\rho_n(\gamma)x_n, *_n)}{\lambda_n} < \infty.$$

Let $\gamma = \gamma_{i_1} \cdots \gamma_{i_k}$ be a decomposition of γ in terms of the fixed generators of $\pi_1(M)$. Then

$$\begin{aligned}
d(\rho_n(\gamma)x_n, *_n) &\le d(\rho_n(\gamma)x_n, \rho_n(\gamma)*_n) + d(\rho_n(\gamma)*_n, *_n)\\
&= d(x_n, *_n) + d(\rho_n(\gamma)*_n, *_n)\\
&\le d(x_n, *_n) + \sum_{j=1}^{k} d\big(\rho_n(\gamma_{i_1} \cdots \gamma_{i_j})*_n, \rho_n(\gamma_{i_1} \cdots \gamma_{i_{j-1}})*_n\big)\\
&\le d(x_n, *_n) + k \max_i d(\rho_n(\gamma_i)*_n, *_n) = d(x_n, *_n) + k\lambda_n.
\end{aligned}$$

Therefore ρ_ω is well defined. The fact that it is an action by isometries is obvious. □

Lemma 2.4. *The representation*

$$\rho_\omega : \pi_1(M) \to \mathrm{Isom}(X)$$

has no global fixed point.

Proof. Let $\{x_n\}$ be any point of X. By construction of $*_n$ and λ_n, the index i_n that realizes $\max_i d(\rho_n(\gamma_i)(x_n), x_n)$ satisfies

$$d(\rho_n(\gamma_{i_n})(x_n), x_n) \ge \lambda_n.$$

Thus

$$\frac{d(\rho_n(\gamma_{i_n})(x_n), x_n)}{\lambda_n} \geq 1$$

and therefore $d(\rho_\omega(\gamma_\omega)\{x_n\}, \{x_n\}) \geq 1$, where $\gamma_\omega = \omega \lim \gamma_{i_n} \in \{\gamma_i\}$. It follows that $\{x_n\} \in X$ is not globally fixed. $\square$

Notice that the previous two lemmas, as well as the following one, do not use the hypothesis that $\{\rho_n\}$ is unbounded. Indeed, such hypothesis is only needed to show that X is an **R**-tree.

Lemma 2.5. *The representation ρ_ω has small arc-stabilizers.*

Proof. Let $I \subset X$ be an arc, and let $\Gamma < \pi_1(M)$ be its ρ_ω-stabilizer:

$$\rho_\omega(\Gamma)(I) = I.$$

Let $\{x_n\}$ and $\{y_n\}$ be the end-points of I, and let $\gamma \in \Gamma$. Up to replacing γ by γ^2 we have $\rho_\omega(\gamma)\{x_n\} = \{x_n\}$ and $\rho_\omega(\gamma)\{y_n\} = \{y_n\}$ as elements of X. That is to say,

$$\omega \lim \frac{d(\rho_n(\gamma)(x_n), x_n)}{\lambda_n} = 0$$

and the same holds true for y_n. Since the ω-limit of $d(x_n, y_n)/\lambda_n$ is the length of I, there exists a subsequence of indices $\{n_k\}$ such that

$$\frac{d(\rho_{n_k}(\gamma)(x_{n_k}), x_{n_k}) + d(\rho_{n_k}(\gamma)(y_{n_k}), y_{n_k})}{d(x_{n_k}, y_{n_k})} \to 0.$$

Given any other $\psi \in \Gamma$, the same limit holds true up to subsequence. Now we use [Bestvina 1988, Proposition 4.5] (a Margulis-type argument, together with some hyperbolic trigonometry) to deduce that the group generated by $\rho_{n_k}(\gamma)$ and $\rho_{n_k}(\psi)$ is abelian for large enough k.

It follows that the commutator $[\gamma, \psi]$ belongs to the kernel of ρ_{n_k} for large enough k.

Since ρ_n factorizes through a faithful representation of $\pi_1(M_{p_n/q_n})$, the following lemma shows that $[\gamma, \psi]$ is in fact trivial in Γ. This implies that Γ is abelian, and therefore cannot contain rank-two free subgroups. $\square$

Lemma 2.6. *Let*

$$P_n : \pi_1(M) \to \pi_1(M_{p_n/q_n})$$

denote the natural surjection induced by Dehn filling. If $p_n^2 + q_n^2 \to \infty$, then, for all $m \in \mathbf{N}$,

$$\bigcap_{n>m} \ker(P_n) = 1.$$

Proof. For large enough n, the Dehn filling on M with parameters (p_n, q_n) is hyperbolic by Thurston's hyperbolic Dehn filling theorem. Let

$$\gamma \in \bigcap \ker(P_n).$$

Then, the holonomy of γ is trivial in each M_{p_n/q_n}. Again, by Thurston's theorem, the holonomy of γ in M_{p_n/q_n} converges to the holonomy of γ in the complete hyperbolic structure of M, which is therefore trivial. This is possible only if $\gamma = 1$. $\square$

So, ρ_ω is an isometric action on X with no global fixed points and small arc stabilizers. By Theorem 2.1 such an action cannot exist. Therefore X cannot be an $\mathbf{R}$-tree, whence we get that λ_n must be bounded. Then, up to conjugation by isometries of $\mathbf{H}^4$, we can suppose that $*_n$ is constant, and the sequence $\{\rho_n\}$ is in that case bounded. This ends the proof of Step I.

2B. ***Accumulation point of Dehn fillings.*** We deal now with Step II of the proof of Theorem 1.4.

Let ρ be an accumulation point of $\{\rho_n\}$. Let V be an open neighborhood of the identity in $\mathrm{Isom}(\mathbf{H}^4)$ such that any discrete group finitely generated by elements in V is virtually nilpotent. Such a V exists by the Margulis lemma. Let U be an open neighborhood of the identity such that $\overline{U} \subset V$.

Lemma 2.7. *Let $\Gamma < \pi_1(M)$ be the subgroup generated by the elements γ such that $\rho(\gamma) \in U$. Then Γ is abelian.*

Proof. Let $\Gamma_0 < \Gamma$ be a group finitely generated by elements whose ρ-image is in U. For n large enough, $\rho_n(\Gamma_0)$ is a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^4)$, finitely generated by elements of V. Then, $\rho_n(\Gamma_0)$ is virtually nilpotent.

Since $\rho(\Gamma_0)$ is virtually nilpotent, it is also elementary, because the limit set of a nontrivial normal subgroup is the same as the limit set of the whole group. In particular, $\rho_n(\Gamma_0)$ is elementary. Moreover, $\rho_n(\Gamma_0)$ is torsion-free because ρ_n is faithful as a representation of $\pi_1(M_{p_n/q_n})$.

An elementary and torsion-free group of isometries in $\mathbf{H}^4$ must be either

- a subgroup of the stabilizer of a geodesic, $\mathbf{R} \rtimes O(3)$, or
- a parabolic subgroup fixing a point at $\partial\mathbf{H}^4$, that is, a subgroup of $\mathrm{Isom}(\mathbf{R}^3) \cong \mathbf{R}^3 \rtimes O(3)$.

In particular, all such groups are nilpotent of order two; see [Wolf 1984]. Thus, for n large enough,

$$[[\rho_n(\Gamma_0), \rho_n(\Gamma_0)], \rho_n(\Gamma_0)] = 1.$$

It follows that any $\gamma \in \Gamma_0$ of the form $\gamma = [[\gamma_1, \gamma_2], \gamma_3]$ belongs eventually to $\ker \rho_n$, and by Lemma 2.6, this forces γ to be trivial. That is to say, Γ_0 itself is virtually nilpotent. Since Γ_0 is a subgroup of the fundamental group of a hyperbolic manifold, this implies that Γ_0 is abelian. Since this holds for any Γ_0, we get that Γ itself is abelian. $\square$

Corollary 2.8. *The representation ρ is faithful.*

Proof. If $\ker(\rho)$ were not trivial, then it would be abelian by Lemma 2.7, but $\pi_1(M)$ has no abelian, nontrivial, normal subgroups. $\square$

The very same argument shows

Corollary 2.9. *The representation ρ is discrete.*

Proof. Let H_0 be the connected component of the identity of the topological closure

$$\overline{\rho(\pi_1(M))}.$$

As $\overline{\rho(\pi_1(M))}$ is a Lie group, H_0 is normal in $\overline{\rho(\pi_1(M))}$. It follows that

$$\Gamma_0 := H_0 \cap \rho(\pi_1(M))$$

is normal in $\rho(\pi_1(M))$, and since ρ is faithful, $\Gamma := \rho^{-1}(\Gamma_0)$ is normal in $\pi_1(M)$.

On the other hand, Γ_0 is generated by elements in U, and then, by Lemma 2.7, Γ is abelian. Therefore, $\Gamma = \{1\}$ and $H_0 = \{1\}$, that is to say, ρ is discrete. $\square$

This concludes Step II and so the proof of Theorem 1.4. $\square$

3. Infinitesimal isometries and deformations

This section contains the background material and tools that we need in the proof of Theorem 1.5.

Let $\mathbf{R}^{n,1}$ denote the Minkowski space, that is, $\mathbf{R}^{n+1}$ equipped with the usual Lorentz product, that has matrix

$$J = \begin{pmatrix} -1 & 0 & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

We shall use the hyperboloid model for hyperbolic space

$$\mathbf{H}^n = \left\{ x = \begin{pmatrix} x^0 \\ \vdots \\ x^n \end{pmatrix} \in \mathbf{R}^{n,1} \mid x^t J x = -1, x^0 > 0 \right\}$$

so that the orientation preserving isometry group of $\mathbf{H}^n$ is identified with the identity component of the group of linear transformations of $\mathbf{R}^{n+1}$ that preserve J as a bilinear form:

$$\mathrm{Isom}^+(\mathbf{H}^n) = \mathrm{SO}_0(n, 1).$$

We are interested in the cases $n = 3$ and $n = 4$. We shall consider the inclusion $\mathrm{SO}_0(3, 1) \subset \mathrm{SO}_0(4, 1)$ induced by the inclusion $\mathbf{R}^{3,1} \subset \mathbf{R}^{4,1}$ consisting in adding a fifth coordinate x^4. Namely,

$$\begin{aligned} \mathrm{SO}(3,1) &\to \mathrm{SO}(4,1), \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3-1}$$

3A. *Infinitesimal isometries.* The Lie algebra of $\mathrm{SO}(n, 1)$ is

$$\mathfrak{so}(n, 1) = \{a \in M_{n+1}(\mathbf{R}) \mid a^t J = -Ja\}.$$

Elements in $\mathfrak{so}(n, 1)$ are viewed as *infinitesimal isometries*: a matrix $a \in \mathfrak{so}(n, 1)$ is the tangent vector to the path $\exp(ta) \in \mathrm{SO}_0(n, 1)$ at $t = 0$. The action of the isometry group on itself by conjugation induces the adjoint action of $\mathrm{SO}_0(n, 1)$ on the Lie algebra $\mathfrak{so}(n, 1)$. Since we have an inclusion $\mathrm{SO}_0(3, 1) \subset \mathrm{SO}_0(4, 1)$, $\mathfrak{so}(4, 1)$ is also a $\mathrm{SO}_0(3, 1)$-module.

Lemma 3.1. *We have an isomorphism of* $\mathrm{SO}_0(3, 1)$*-modules*

$$\mathfrak{so}(4, 1) = \mathfrak{so}(3, 1) \oplus \mathbf{R}^{3,1}$$

where $\mathrm{SO}(3, 1)$ *acts on* $\mathfrak{so}(3, 1)$ *by the adjoint action and on* $\mathbf{R}^{3,1}$ *by the usual linear action.*

Proof. Explicit construction. Given a matrix $a \in \mathfrak{so}(3, 1)$ and a (column) vector $v \in \mathbf{R}^{3,1}$, we consider the matrix

$$\begin{pmatrix} a & v \\ -v^t J & 0 \end{pmatrix} \in \mathfrak{so}(4, 1), \tag{3-2}$$

where v^t is the transpose vector. It is easy to check that this gives the isomorphism of the lemma, compatible with inclusion (3-1). □

The Lie algebra of infinitesimal deformations can be identified with the Lie algebra of Killing vector fields.

Proposition 3.2. *The subspace* $\mathbf{R}^{3,1} \subset \mathfrak{so}(4, 1)$ *corresponds to the Killing vector fields orthogonal to* $\mathbf{H}^3 \subset \mathbf{H}^4$.

Proof. Given $a \in \mathfrak{so}(4, 1)$, the corresponding field V evaluated at $x \in \mathbf{H}^4$ is

$$V_x = \frac{d}{dt}\Big|_{t=0} \exp(ta)x.$$

Since we are working in a linear model, for $x \in \mathbf{H}^4 \subset \mathbf{R}^{4,1}$,

$$V_x = ax \in \mathbf{R}^{4,1}.$$

In this model, $\mathbf{H}^3 = \mathbf{H}^4 \cap \{x^4 = 0\}$. Thus the Killing vector fields perpendicular to $\mathbf{H}^3$ correspond to matrices in $\mathfrak{so}(4,1)$ whose entries are zero, except for the last column or the last row, which is the image of the embedding of $\mathbf{R}^{3,1}$ in $\mathfrak{so}(4,1)$. $\square$

The splitting of Lemma 3.1 can be also understood by using the action on the *de Sitter space*

$$\mathbf{S}^{3,1} = \{v \in \mathbf{R}^{4,1} : v^t J v = 1\},$$

which is naturally identified to the space of oriented hyperplanes in $\mathbf{H}^4$; see [Epstein and Penner 1988]. Since SO(4, 1) acts transitively on $\mathbf{S}^{3,1}$ with stabilizers SO(3, 1), we have that

$$\mathrm{SO}(4,1)/\mathrm{SO}(3,1) = \mathbf{S}^{3,1}.$$

Moreover the fibration

$$\mathrm{SO}(3,1) \to \mathrm{SO}(4,1) \to \mathbf{S}^{3,1},$$

whose projection maps $A \in \mathrm{SO}(4,1)$ to $A \cdot p$ for some fixed $p \in \mathbf{S}^{3,1}$, induces an exact sequence of $\mathfrak{so}(3,1)$-modules:

$$0 \to \mathfrak{so}(3,1) \to \mathfrak{so}(4,1) \to T_p\mathbf{S}^{3,1} \to 0,$$

which splits, by using either the Killing form or Lemma 3.1.

Remark 3.3. We have a canonical identification

$$T_p\mathbf{S}^{3,1} \cong \mathbf{R}^{3,1}$$

where $p \in \mathbf{S}^{3,1}$ is the hyperplane stabilized by $\mathfrak{so}(3,1) \subset \mathfrak{so}(4,1)$.

In order to be compatible with the previous computations, we shall assume that we choose the point

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbf{S}^{3,1}.$$

3B. ***The Zariski tangent space to the variety of representations.*** For a representation $\rho:\Gamma\to \mathrm{SO}_0(n,1)$, the Zariski tangent space of the variety of representations at ρ is naturally identified to the space of cocycles [Weil 1964]. Namely, the space of cocycles is defined as

$$Z^1(\Gamma,\mathfrak{so}(n,1)_\rho)=\big\{d:\Gamma\to\mathfrak{so}(n,1)\;\big|\;d(\gamma_1\gamma_2)=d(\gamma_1)+\mathrm{Ad}_{\rho(\gamma_1)}(d(\gamma_2))\big\}.$$

Weil's correspondence maps the cocycle $d\in Z^1(\Gamma,\mathfrak{so}(n,1)_\rho)$ to the infinitesimal deformation

$$\rho_t(\gamma)=(1+t\,d(\gamma))\rho(\gamma),$$

for all $\gamma\in\Gamma$, which is a representation modulo t^2, hence a Zariski tangent vector; see [Lubotzky and Magid 1985].

The space of coboundaries is

$$B^1(\Gamma,\mathfrak{so}(n,1)_\rho)=\\ \{d_a:\Gamma\to\mathfrak{so}(n,1)\mid d_a(\gamma)=a-\mathrm{Ad}_{\rho(\gamma)}(a),\ \text{for some } a\in\mathfrak{so}(n,1)\},$$

and it is identified to the Zariski tangent space of orbits by conjugation.

The quotient is the cohomology group

$$H^1(\Gamma,\mathfrak{so}(n,1)_\rho)=Z^1(\Gamma,\mathfrak{so}(n,1)_\rho)/B^1(\Gamma,\mathfrak{so}(n,1)_\rho).$$

Under some circumstances, $H^1(\Gamma,\mathfrak{so}(n,1)_\rho)$ can be viewed as the tangent space to the space of conjugacy classes of representations. However, for technical reasons, since $\mathbf{R}$ is not algebraically closed, it is easier to work with the variety of representations.

Now we focus on the case $\Gamma=\pi_1(M)$, where M is a cusped manifold and ρ_0 is the holonomy representation of its complete structure. We omit the representation ρ_0 when writing the Lie algebra as $\pi_1(M)$-modules via the adjoint action of ρ_0. We also write $H^1(M,V)$ for $H^1(\pi_1(M),V)$. Recall that the parabolic cohomology is defined as

$$\mathrm{PH}^1(M,\mathfrak{so}(4,1))=\ker\big(H^1(M,\mathfrak{so}(4,1))\to H^1(\partial M,\mathfrak{so}(4,1))\big).$$

Next lemma is due to Scannell [2002].

Lemma 3.4. *Let $\rho_0:\pi_1(M)\to\mathrm{SO}_0(3,1)\subset\mathrm{SO}_0(4,1)$ be the holonomy representation of a hyperbolic two-bridge knot. Then*

$$\mathrm{PH}^1(M,\mathfrak{so}(4,1))=0.$$

To prove that

$$\ker\big(H^1(M,\mathfrak{so}(4,1))\to H^1(\partial M,\mathfrak{so}(4,1))\big)$$

vanishes, the idea is that an element in the kernel corresponds to a deformation that keeps the generators parabolic. By a geometric argument due to Kapovich and Scannell, such a representation must preserve a hyperbolic space of dimension three.

Lemma 3.5. *Suppose that* $\mathrm{PH}^1(M, \mathfrak{so}(4,1))$ *vanishes. Then the image of the inclusion*

$$0 \to H^1(M, \mathfrak{so}(4,1)) \to H^1(\partial M, \mathfrak{so}(4,1))$$

has half dimension.

The proof consists in applying the long exact sequence of the pair and Poincaré duality.

Recall from Lemma 3.1 that, as ρ_0-module by the adjoint action, we have a decomposition

$$\mathfrak{so}(4,1) = \mathfrak{so}(3,1) \oplus \mathbf{R}^{3,1},$$

where $\mathbf{R}^{3,1}$ is the four-dimensional real vector space equipped with the linear action of $\mathrm{SO}(3,1) \subset GL(\mathbf{R}, 4)$. In particular,

$$H^1(M, \mathfrak{so}(4,1)) = H^1(M, \mathfrak{so}(3,1)) \oplus H^1(M, \mathbf{R}^{3,1}).$$

The dimensions of those spaces for the torus are well known.

Lemma 3.6 [Kapovich 1994; Scannell 2002].

$$\dim\big(H^1(\partial M, \mathfrak{so}(3,1))\big) = 4, \quad \textit{and} \quad \dim(H^1(\partial M, \mathbf{R}^{3,1})) = 2.$$

From Lemmas 3.6 and 3.5, we get:

Corollary 3.7. *If* $\mathrm{PH}^1(M, \mathfrak{so}(4,1)) = 0$, *then*

$$\dim\big(H^1(M, \mathfrak{so}(3,1))\big) = 2, \quad \textit{and} \quad \dim(H^1(M, \mathbf{R}^{3,1})) = 1.$$

The following lemma can be easily proved using the formalism of the previous section. Recall that the projection $\mathrm{SO}(4,1) \to \mathbf{S}^{3,1}$ maps $A \in \mathrm{SO}(4,1)$ to $A \cdot p \in \mathbf{S}^{3,1}$, for some fixed $p \in \mathbf{S}^{3,1}$.

Lemma 3.8. *Let* $\rho_t : \Gamma \to \mathrm{SO}(4,1)$ *be a smooth path of representations of a group* Γ *in* $\mathrm{SO}(4,1)$, *and let* $\bar\rho_t$ *be its projection to* $\mathbf{S}^{3,1}$. *Then, the first nontrivial derivative of* $\bar\rho_t$ *defines a cocycle in* $Z^1(\Gamma, \mathbf{R}^{3,1})$.

If the cocycle is nontrivial in $H^1(\Gamma, \mathbf{R}^{3,1})$, *then for small* t *the representation* ρ_t *does not fix any hyperplane in* $\mathbf{H}^4$.

Notice that this lemma uses the natural interpretation of points in $\mathbf{S}^{3,1}$ as hyperplanes in $\mathbf{H}^4$.

3C. *Trace functions.* For $\gamma \in \Gamma$, let $\operatorname{tr}_\gamma : R(\Gamma, \mathrm{SO}_0(4,1)) \to \mathbf{R}$ denote the trace function. Since tr_γ is constant on orbits by conjugation, $d\operatorname{tr}_\gamma : Z^1(\Gamma, \mathfrak{so}(4,1)) \to \mathbf{R}$ vanishes on $B^1(\Gamma, \mathfrak{so}(4,1))$, and it induces a linear map, $d\operatorname{tr}_\gamma : H^1(\Gamma, \mathfrak{so}(4,1)) \to \mathbf{R}$. (See also [Bart and Scannell 2007] for properties of the trace function.)

Lemma 3.9. *For a representation* $\rho : \Gamma \to \mathrm{SO}_0(3,1) \subset \mathrm{SO}_0(4,1)$, *it holds*

$$H^1(\Gamma, \mathbf{R}^{3,1}) \subseteq \ker d\operatorname{tr}_\gamma,$$

for all $\gamma \in \Gamma$.

Proof. Using the embedding $\mathrm{SO}(3,1) \subset \mathrm{SO}(4,1)$ of (3-1), the corresponding embedding of $\mathbf{R}^{3,1}$ in $\mathfrak{so}(4,1)$ maps the vector $v \in \mathbf{R}^{3,1}$ to the matrix

$$\begin{pmatrix} 0 & v \\ Jv^t & 0 \end{pmatrix}.$$

Then a path of representations ρ_t tangent to a vector in $H^1(\Gamma, \mathbf{R}^{3,1})$ can be written, up to fist order and up to conjugation, as

$$\rho_t(\gamma) = \left(\mathrm{Id} + t\begin{pmatrix} 0 & v \\ Jv^t & 0 \end{pmatrix}\right)\begin{pmatrix} \rho(\gamma) & 0 \\ 0 & 1 \end{pmatrix} + o(t^2) = \begin{pmatrix} \rho(\gamma) & tv \\ -tJv^t\rho(\gamma) & 1 \end{pmatrix} + o(t^2),$$

for all $\gamma \in \Gamma$. Hence $\operatorname{tr}(\rho_t(\gamma)) = \operatorname{tr}(\rho_0(\gamma)) + o(t^2)$ and $\frac{d}{dt}\operatorname{tr}(\rho_t(\gamma))|_{t=0} = 0$. $\square$

Recall that $\mathrm{Isom}^+(\mathbf{H}^3) \cong PSL_2(\mathbf{C})$. This isomorphism comes from identifying the conformal sphere $S^2 = \partial_\infty \mathbf{H}^3$ with the projective line $\mathbf{P}^1\mathbf{C}$. The relation between traces in $\mathrm{SO}(3,1)$ and $SL_2(\mathbf{C})$ is:

Remark 3.10. Let $A \in \mathrm{Isom}^+(\mathbf{H}^3)$. Then,

$$\operatorname{tr}_{\mathrm{SO}(3,1)}(A) = |\operatorname{tr}_{SL_2(\mathbf{C})}(A)|^2 \quad \text{and} \quad \operatorname{tr}_{\mathrm{SO}(4,1)}(A) = |\operatorname{tr}_{SL_2(\mathbf{C})}(A)|^2 + 1.$$

Lemma 3.11. *Let* $\phi : \pi_1(M) \to PSL_2(\mathbf{C})$ *be the holonomy representation of a hyperbolic manifold with one cusp. Then*

(a) $H^1(M, \mathfrak{sl}_2(\mathbf{C})_{\mathrm{Ad}\,\phi}) \cong T_\phi X(M, PSL_2(\mathbf{C})) \cong \mathbf{C}$;

(b) *for every* $\gamma \in \pi_1(\partial M)$, $\gamma \neq 1$, $d\operatorname{tr}_\gamma : H^1(M, \mathfrak{sl}_2(\mathbf{C})_{\mathrm{Ad}\,\phi}) \to \mathbf{C}$ *is nonzero*;

(c) *given* $\gamma_1, \gamma_2 \in \pi_1(\partial M)$ *so that*

$$\phi(\gamma_i) = \pm\begin{pmatrix} 1 & x(\gamma_i) \\ 0 & 1 \end{pmatrix},$$

then

$$\frac{d\operatorname{tr}_{\gamma_1}}{d\operatorname{tr}_{\gamma_2}} = \left(\frac{x(\gamma_1)}{x(\gamma_2)}\right)^2.$$

Proof. Assertions (a) and (b) are the contents of hyperbolic Dehn filling theorem; see [Kapovich 2001]. To prove (c), we follow [Boileau and Porti 2001, Appendix B] and, of course, Thurston's notes [1980]. We write a deformation as

$$\phi_t(\gamma_i) = \pm \begin{pmatrix} e^{u_i(t)} & x(\gamma_i)(t) \\ 0 & e^{-u_i(t)} \end{pmatrix}.$$

The parameter t is not a parameter of the deformation space because the trace function

$$\mathrm{tr}_{\gamma_i} = e^{u_i(t)} + e^{-u_i(t)} = 2\cosh(u_i(t))$$

defines a local parameter of $R(M, SL_2(\mathbf{C}))//SL_2(\mathbf{C})$. However one can take t and $u_i(t)$ as analytic functions, by working in a double branched covering; see [Boileau and Porti 2001]. Following Thurston's notes, the commutativity relation becomes $x(\gamma_1)\sinh u_2 = x(\gamma_2)\sinh u_1$. Thus

$$\lim_{t\to 0} \frac{\dot{u}_1}{\dot{u}_2} = \lim_{t\to 0} \frac{u_1}{u_2} = \frac{x(\gamma_1)(0)}{x(\gamma_2)(0)},$$

and (c) follows from straightforward computation. □

4. The variety of representations around ρ_0

In this section we study the geometry of $R(M, \mathrm{SO}(4, 1))$ in a neighborhood of the holonomy representation ρ_0 for the complete structure of M, namely the differentiable tangent cone in Section 4A, and the partial slice in Section 4B. Both tools are going to be used in the proof of Theorems 1.3 and 1.5.

4A. ***The differentiable tangent cone.*** There are several notions of tangent cone. For our purpose, we consider

Definition 4.1. The *differentiable tangent cone* of the variety of representations at ρ_0 is defined to be the subset of vectors in the Zariski tangent space $Z^1(M, \mathfrak{so}(4, 1))$ that are the tangent vectors to a curve of representations, parametrized by $[0, \varepsilon)$ and which is differentiable to the right at the origin.

As the cocycles project to cohomology, we look at the image of the differentiable tangent cone in

$$H^1(M, \mathfrak{so}(4, 1)) \cong H^1(M, \mathfrak{so}(3, 1)) \oplus H^1(M, \mathbf{R}^{3,1}).$$

The aim of this section is to prove

Proposition 4.2. *If* $\mathrm{PH}^1(M, \mathfrak{so}(4, 1)) = 0$, *then the image of the differentiable tangent cone in cohomology is contained in* $H^1(M, \mathfrak{so}(3, 1)) \cup H^1(M, \mathbf{R}^{3,1})$.

Before to prove it, we need a remark and a couple of lemmas. The remark can be proved by straightforward computation.

Remark 4.3. Let $A \in \mathrm{SO}_0(4, 1)$.

– If A is parabolic and induces a translation in $\partial\mathbf{H}^4 \setminus \{\infty\} \cong \mathbf{R}^3$, then

$$\operatorname{trace}(A) = 5.$$

– If A is parabolic and induces a screw motion of angle α in $\partial\mathbf{H}^4 \setminus \{\infty\}$, then

$$\operatorname{trace}(A) = 3 + 2\cos\alpha.$$

– If A is elliptic with rotational angles α and β, then

$$\operatorname{trace}(A) = 1 + 2\cos\alpha + 2\cos\beta.$$

– If A is loxodromic with translation length λ and with angle α, then

$$\operatorname{trace}(A) = 1 + 2\cosh\lambda + 2\cos\alpha.$$

In particular, if $A \in \mathrm{SO}_0(4, 1)$ satisfies $\operatorname{trace}(A) > 5$, then A is loxodromic.

The following lemma can be found in [Scannell 2002, Lemma 4.2].

Lemma 4.4. *A representation of $\mathbf{Z} \times \mathbf{Z}$ in $\mathrm{SO}_0(4, 1)$ containing loxodromic elements and obtained by perturbing a parabolic one (that consists only of translations in $\mathbf{R}^3$) must be conjugated to $\mathrm{SO}_0(3, 1)$.*

The next lemma is the analogue of Proposition 4.2 for $\pi_1(\partial M) \cong \mathbf{Z} \oplus \mathbf{Z}$, instead of M.

Lemma 4.5. *Let $c : [0, 1] \to R\big(\pi(\partial M), \mathrm{SO}(4, 1)\big)$ be a path differentiable to the right at 0, and such that $c(0)$ is the restriction of the holonomy of the complete structure on M. Assume that the $\mathfrak{so}(3, 1)$-component of the projection of $\dot c(0)$ in $H^1(\partial M, \mathfrak{so}(3, 1)) \subset H^1(\partial M, \mathfrak{so}(4, 1))$ is not zero.*

Then, there is $\varepsilon > 0$ such that for all $0 < t < \varepsilon$, $c(t)$ is conjugate to $\mathrm{SO}(3, 1)$. In particular, the $\mathbf{R}^{3,1}$-component of $\dot c(0)$ vanishes.

Proof. Let us identify $\dot c(0)$ with its projection in cohomology. Let u be the $\mathfrak{so}(3, 1)$-component of $\dot c(0)$, that is, $\dot c(0) - u \in H^1(\partial M, \mathbf{R}^{3,1})$. By Lemma 3.9, for all $\gamma \in \pi_1(\partial M)$,

$$d\,\mathrm{tr}_\gamma(\dot c(0)) = d\,\mathrm{tr}_\gamma(u).$$

As a representation in $PSL_2(\mathbf{C})$, $c(0)$ is conjugated to

$$\gamma \mapsto \pm\begin{pmatrix} 1 & x(\gamma) \\ 0 & 1 \end{pmatrix},$$

for all $\gamma \in \pi_1(\partial M)$. The map $x : \pi_1(\partial M) \to \mathbf{C}$ is a morphism and it defines a lattice of $\mathbf{C}$. From the formula

$$\mathrm{tr}_{\mathrm{SO}(4,1)} = \mathrm{tr}_{\mathrm{SO}(3,1)} + 1 = |\,\mathrm{tr}_{SL(2,\mathbf{C})}\,|^2 + 1,$$

and taking a lift such that $\mathrm{tr}_{SL(2,\mathbf{C}),\gamma}(c(0)) = 2$, we get

$$d\,\mathrm{tr}_{\mathrm{SO}(4,1),\gamma}(u) = 4\,\mathrm{Re}(d\,\mathrm{tr}_{SL(2,\mathbf{C}),\gamma}(u)).$$

Since $x : \pi_1(\partial M) \to \mathbf{C}$ is a lattice, the set of arguments

$$\left\{ \frac{x(\gamma)}{|x(\gamma)|} \in S^1 : \gamma \in \pi_1(\partial M) \setminus \{1\} \right\}$$

is dense in the unit circle S^1. By Lemma 3.11 (c), and since $d\,\mathrm{tr}_{SL(2,\mathbf{C}),\gamma}(u) \neq 0$, one can find γ such that $\mathrm{Re}(d\,\mathrm{tr}_{SL(2,\mathbf{C}),\gamma}(u)) > 0$.

Therefore, for small enough t the representation $c(t)$ contains loxodromic elements because of Remark 4.3, and is conjugated to SO(3, 1) by Lemma 4.4. □

Proof of Proposition 4.2. Since $\mathrm{PH}^1(M, \mathfrak{so}(4,1))$ vanishes, the restriction from $\pi_1(M)$ to $\pi_1(\partial M)$ induces an inclusion in cohomology

$$0 \to H^1(M, \mathfrak{so}(4,1)) \to H^1(\partial M, \mathfrak{so}(4,1)),$$

and it is sufficient to show the corresponding statement for ∂M instead of M. Namely, we have to check that "mixed" elements

$$(u, v) \in H^1(\partial M, \mathfrak{so}(3,1)) \oplus H^1(\partial M, \mathbf{R}^{3,1})$$

with $u \neq 0$ and $v \neq 0$ are not contained in the image of the differentiable tangent cone, and this is true by Lemma 4.5. □

Remark 4.6. Further work would yield that the inverse image of the subspace $H^1(M, \mathbf{R}^{3,1}) \subset H^1(M, \mathfrak{so}(4,1))$ in $Z^1(M, \mathfrak{so}(4,1))$ is integrable, using the computations of Section 5C and an argument analogue to [Boileau et al. 2005, Theorem 9.4].

4B. *The partial slice.*

Instead of working with the space of conjugacy classes of representations in $R(M, \mathrm{SO}(4,1))$, we shall construct a partial slice to the orbit by conjugation. Since SO(4, 1) is a real group, the space of conjugacy classes is not an algebraic variety, though the partial slice is.

For $\rho \in R(M, \mathrm{SO}(4,1))$, the orbit of ρ by conjugation is denoted by $O(\rho)$.

Proposition 4.7. *There exists an algebraic subvariety $S \subset R(M, \mathrm{SO}(4,1))$ and an open neighborhood $U \subset R(M, \mathrm{SO}(4,1))$ of ρ_0, with the following properties*:

(1) $S \cap O(\rho_0) = \{\rho_0\}$.

(2) *If $\rho \in U$ satisfies that $\rho|_{\partial M}$ fixes a point in $\partial \mathbf{H}^4$, then $O(\rho) \cap S \cap U \neq \varnothing$ and consists of a single point.*

(3) *The map $TS \to H^1(M, \mathfrak{so}(4,1))$ is injective.*

(4) *If* $\rho \in U$ *satisfies that* $O(\rho) \cap R(M, \mathrm{SO}(3,1)) \neq \varnothing$, *then*

$$O(\rho) \cap S \cap U \subset R(M, \mathrm{SO}(3,1)).$$

Proof. Let γ_1, γ_2 be a pair of peripheral elements that generate a nonelementary subgroup of $\pi_1(M)$. Define S as the subset of representations ρ in $R(M, \mathrm{SO}(4,1))$ that satisfy the conditions (a)–(d) below:

(a) $\rho(\gamma_1)$ fixes the same point of $\partial\mathbf{H}^4$ as $\rho_0(\gamma_1)$.

(b) $\rho(\gamma_2)$ fixes the same point of $\partial\mathbf{H}^4$ as $\rho_0(\gamma_2)$.

Let $\mu_1 \in \pi_1(M)$ be an element such that γ_1 and μ_1 generate a peripheral subgroup $\pi_1(\partial M)$. We fix an identification of $\partial\mathbf{H}^4 \setminus \mathrm{Fix}(\rho_0(\gamma_1))$ with $\mathbf{R}^3$, so that $\rho \in S$ restricted to $\mathbf{R}^3$ acts by affine transformations. Let $\vec{0} \in \mathbf{R}^3$ denote the origin, $\rho_0(\gamma_1)(\vec{0}) \neq \vec{0}$. The remaining conditions defining S are:

(c) $\rho(\gamma_1)(\vec{0}) = \rho_0(\gamma_1)(\vec{0})$.

(d) The ordered pairs of vectors

$$\big(\rho(\gamma_1)(\vec{0}), \rho(\mu_1)(\vec{0})\big), \quad \text{and} \quad \big(\rho_0(\gamma_1)(\vec{0}), \rho_0(\mu_1)(\vec{0})\big)$$

span the same oriented plane of $\mathbf{R}^3$.

Given a representation such that its restriction to ∂M fixes a point in $\partial\mathbf{H}^4$, conditions (a) to (d) can be achieved by conjugation. Notice also that this determines the representation up to conjugacy. Namely, conditions (a) and (b) fix a representation of the conjugacy class, up to isometries that preserve a pair of points in $\partial\mathbf{H}^4 = \mathbf{R}^3 \cup \{\infty\}$. Hence we may assume that the fixed points are $\{\infty\}$ for $\rho(\gamma_1)$ in (a) and $\vec{0}$ for $\rho(\gamma_2)$ in (b). Thus the group of elements that fix those points is the product of the orthogonal group with the group of homotethies in $\mathbf{R}^3$, but this indeterminacy is eliminated by (c) and (d). Hence assertions (1) and (2) of the proposition follow.

The restrictions (a)–(d) can be written as $F^{-1}(0)$ for some map $F : U \to \mathbf{R}^{10}$ transverse to the orbit $O(\rho_0)$; hence $O(\rho_0)$ and S are transverse.

$$B^1(M, \mathfrak{so}(4,1)) \cap T_{\rho_0} S = 0,$$

since $B^1(M, \mathfrak{so}(4,1))$ is the tangent space to $O(\rho_0)$, and assertion (3) follows.

Finally, assertion (4) holds because uniqueness of assertion (2), and the fact that properties (a)–(d) may be achieved by conjugation in $\mathrm{SO}(3,1)$. □

Notice that representations such that its restriction to ∂M is contained in $\mathrm{SO}(4)$ are excluded by this set S, this is why we call it *partial* slice.

5. Non-Fuchsian representations

This last section is devoted to the proof of Theorem 1.5. In Section 5A we deform a parabolic group of translations in the plane as a group of screw motions in Euclidean space. Viewing $\mathbf{R}^3$ as $\partial\mathbf{H}^4 \setminus \{\infty\}$, those give infinitesimal deformations in $\mathfrak{so}(4,1)$ that take values in $\mathbf{R}^{3,1}$. We claim in Lemma 5.4 that those are all possible deformations of $\mathbf{Z}\oplus\mathbf{Z}$ that take values in $\mathbf{R}^{3,1}$. Section 5B is devoted to the proof of Theorem 1.5, assuming Lemma 5.4 which is proved in Section 5C.

5A. ***Deformations with peripheral screw motions.*** In this subsection we construct explicit examples of deformations of a parabolic representation of $\mathbf{Z}\oplus\mathbf{Z}$ that give cocycles valued in $\mathbf{R}^{3,1}$. The aim will be to show later that those are all the possible infinitesimal non-Fuchsian deformations.

Example 5.1 (*Translation along a line as limits of rotations in the plane).* Consider the translation that maps $x \in \mathbf{R}$ to $x+a$. We extend it to a translation of the plane with vector $(a,0)^t \in \mathbf{R}^2$. For $a \in \mathbf{R}$, consider the family of rotations of $\mathbf{R}^2$ parametrized by $0<t<\varepsilon$, centered at the point $\left(0, \frac{1}{t}\right)^t$ and of angle $\alpha = at$. They can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos(at) & -\sin(at) \\ \sin(at) & \cos(at) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \sin(at)/t \\ \big(1-\cos(at)\big)/t \end{pmatrix},$$

for all $(x,y)^t \in \mathbf{R}^2$. Obviously, when $t \to 0$, this converges to the translation of vector $(a,0)^t$.

We now want to compute the derivative of this expression with respect to t. Consider a family of representations of $\mathbf{Z}$ that map 1 to the previous example. The corresponding cocycle maps 1 to the infinitesimal rotation

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

plus a vertical infinitesimal translation. The corresponding Killing vector field is perpendicular to the horizontal coordinate axis $\{(x,0) \mid x \in \mathbf{R}\}$.

Example 5.2 (*Translation along a plane as limits of rotations in the space).* The same picture as above can be adapted for a group of translations of the plane by decomposing it as the orthogonal sum of two lines:

$$\mathbf{R}^2 = \langle\omega\rangle \oplus \langle\mathbf{i}\,\omega\rangle,$$

where $\omega \in \mathbf{R}^2$ satisfies $|\omega| = 1$, $\mathbf{i}\,\omega$ denotes the result of rotating ω by $\pi/2$, and identifying $\mathbf{R}^2 \cong \mathbf{C}$. Then in the direction ω we do not make any deformation, and in the direction of $\mathbf{i}\,\omega$ we do the construction of Example 5.1.

Assume we have a representation $\phi : \Gamma \to \mathbf{R}^2$ into the group of translations. Let rot : $\mathfrak{Isom}(\mathbf{R}^3) \to \mathfrak{so}(3)$ denote the projection induced by taking the linear part of an isometry. There is a natural identification $\mathfrak{so}(3) \cong \mathbf{R}^3$. An elementary computation then shows:

Lemma 5.3. *If d is the cocycle of deformation in Example 5.2, then*

$$\operatorname{rot}(d(\gamma)) = (\phi(\gamma) \cdot \mathbf{i}\,\omega)\,\omega,$$

for all $\gamma \in \Gamma$.

Here the dot denotes the Euclidean scalar product, so $\phi(\gamma) \cdot \mathbf{i}\,\omega$ denotes the orthogonal projection of $\phi(\gamma)$ in the direction perpendicular to ω.

We are interested in the restriction of the holonomy ρ_0 of the complete structure, that we view as a representation into the group of translations $\rho_0 : \pi_1(\partial M) \to \mathbf{R}^2$.

Lemma 5.4. (a) *Any cocycle $d \in Z^1(\partial M, \mathbf{R}^{3,1})$ takes values in $\mathbf{R}^{3,1} \cap \mathfrak{Isom}(\mathbf{R}^3)$.*

(b) *Every cohomology class is represented by a cocycle d, so there exists a unit vector $\omega \in \mathbf{R}^2$, and a parameter $\lambda \in \mathbf{R}$ such that*

$$\operatorname{rot}(d(\gamma)) = (\phi(\gamma) \cdot \mathbf{i}\,\omega)\lambda\,\omega,$$

for all $\gamma \in \pi_1(\partial M)$. Moreover the cohomology class is trivial if and only if $\operatorname{rot} \circ d$ *is trivial.*

The proof is postponed to Section 5C.

5B. *Sequences of non-Fuchsian representations.* For all $n \in \mathbf{N}$, let $\rho_n : \pi_1(M) \to \mathrm{SO}(4, 1)$ be a representation induced by a discrete and faithful representation of M_{p_n/q_n} not conjugated to $\mathrm{SO}(3, 1)$.

We claim that the restriction of ρ_n to $\pi_1(\partial M)$ fixes a point in $\partial\mathbf{H}^4$, because its restriction to $\pi_1(\partial M)$ is a discrete and faithful representation of $\mathbf{Z}$, and therefore it cannot fix an interior point of $\mathbf{H}^4$. Thus we can apply Proposition 4.7, and assume, possibly up to conjugation, that ρ_n belongs to the partial slice S.

From now on, all statements are up to subsequence. This can be done because the limit l will depend only on M.

Lemma 5.5. *There exists a cocycle $d \in T_{\rho_0}S$ and a sequence of positive real numbers $\varepsilon_n \to 0$ satisfying*

$$\rho_n(\gamma) = (1 + \varepsilon_n d(\gamma) + o(\varepsilon_n))\rho_0(\gamma),$$

for all $\gamma \in \pi_1 M$.

Here $o(\varepsilon_n)$ denotes a term such that $o(\varepsilon_n)/\varepsilon_n \to 0$.

Proof. This is a compactness argument. Embed $R(M, \mathrm{SO}(4, 1))$ in some $\mathbf{R}^N$ as an algebraic subvariety, let ε_n be the distance between ρ_n and ρ_0 and take a converging subsequence of unitary vectors $\frac{1}{\varepsilon_n}(\rho_n - \rho_0)$. The limit must be a vector Zariski tangent to S, and therefore it is a cocycle d. □

Lemma 5.6. *The cocycle d projects to a nontrivial element in $H^1(M, \mathfrak{so}(4, 1))$ contained in $H^1(M, \mathbf{R}^{3,1})$.*

Proof. By Proposition 4.7 (3) the cocycle d is nontrivial in cohomology. Assume that it is not contained in $H^1(M, \mathbf{R}^{3,1})$, we look for a contradiction by applying the curve selection lemma for semialgebraic sets.

Let $S \subset R(M, \mathrm{SO}(4, 1))$ denote the slice of Proposition 4.7. Working with an embedding of $S \subset R(M, \mathrm{SO}(4, 1))$ in the Euclidean space $\mathbf{R}^N$ and putting ρ_0 as the origin, let $\alpha > 0$ denote the angle between d and the linear subspace $Z^1(M, \mathbf{R}^{3,1})$. Consider the semialgebraic cone C consisting of those vectors of $\mathbf{R}^N$ whose angle with $Z^1(M, \mathbf{R}^{3,1})$ is greater than or equal to $\alpha/2$. By the curve selection lemma applied to $C \cap S \setminus R(M, \mathrm{SO}(3, 1))$, there exists a semialgebraic curve $c : [0, 1] \to S$ such that

(1) $c(0) = \rho_0$, and

(2) $c((0, 1]) \subset C \cap S \setminus R(M, \mathrm{SO}(3, 1))$.

The first nontrivial derivative $c^{(n)}(0)$ gives an element of the differential tangent cone whose projection to $H^1(M, \mathfrak{so}(3, 1))$ is nontrivial, by the choice of α. Thus, by Proposition 4.2 applied to $c(t^{1/n})$, the cohomology class of d must be contained in $H^1(M, \mathfrak{so}(3, 1))$.

Now, we argue with the inclusion of ∂M in M and the projection of SO(4, 1) to the de Sitter space $\mathbf{S}^{3,1} = \mathrm{SO}(4, 1)/\mathrm{SO}(3, 1)$.

On one hand, by Lemma 4.5 the restriction of c to ∂M gives a path $c_\partial : [0, 1] \to R(\partial M, \mathrm{SO}(4, 1))$ that must be contained in SO(3, 1). In particular, the projection $\bar{c}_\partial$ of such path to $\mathbf{S}^{3,1}$ is the trivial path.

On the other hand, since c is not contained in $R(M, \mathrm{SO}(3, 1))$, its projection $\bar{c}$ to $\mathbf{S}^{3,1}$ must have some nontrivial derivative. This defines, by Lemma 3.8, a nontrivial cocycle $b \in Z^1(M, \mathbf{R}^{3,1})$, which is nontrivial in $H^1(M, \mathbf{R}^{3,1})$, because in the proof of Proposition 4.7, it is shown that $T_{\rho_0} S \cap B^1(M, \mathfrak{so}(4, 1)) = 0$, that is, the tangent space to the partial slice S and the coboundary space are transverse. By Lemma 3.4, b would give a nontrivial element when restricted to the boundary, contradicting the fact that $\bar{c}_\partial$ is the constant path. □

Proof of Theorem 1.3. By contradiction, assume that there exists an infinite sequence p_n/q_n so that M_{p_n/q_n} is not locally rigid. Thus the holonomy of M_{p_n/q_n} can be perturbed to $\tilde{\rho}_n : \pi_1(M_{p_n/q_n}) \to \mathrm{SO}(4, 1)$, not contained in SO(3, 1). Notice that since the holonomy of M_{p_n/q_n} maps the core of the filling torus to a loxodromic element, we can assume that the restriction $\tilde\rho_n|_{\partial M}$ is not elliptic. Thus $\tilde\rho_n$ belongs to the partial slice S and the arguments of Lemmas 5.5 and 5.6 apply. In other words, there is a cocycle d which is tangent to $\tilde\rho_n$, and that cocycle is contained in $H^1(M, \mathbf{R}^{3,1})$.

However, since the $\tilde\rho_n$ are obtained by perturbation of the fuchsian holonomies of the manifolds M_{p_n/q_n}, and since the sequence of such holonomies defines a cocycle in $H^1(M, \mathfrak{so}(3,1))$, we can choose the perturbations small enough so that the cocycle d is not contained in $H^1(M, \mathbf{R}^{3,1})$. Hence we get a contradiction. $\square$

Proof of Theorem 1.5. Let $\rho_n : \pi_1(M) \to \mathrm{SO}(4,1)$ be a sequence of representations induced by a discrete and faithful representation of $\pi_1(M_{p_n/q_n})$, and let d be as in Lemma 5.5. Since d is tangent to S, we may assume that its restriction to $\pi_1(\partial M)$ takes values in the infinitesimal isometries of the Euclidean space $\mathbf{R}^3 = \partial\mathbf{H}^4 \setminus \{p_0\}$ by Lemma 5.4 (a). Thus, the composition with the projection $\mathfrak{Isom}(\mathbf{R}^3) \to \mathfrak{so}(3)$ gives a cocycle valued in infinitesimal rotations $\delta : \pi_1(\partial M) \to \mathfrak{so}(3)$.

Since d is not trivial in $H^1(\partial M, \mathbf{R}^{3,1})$, by Lemma 5.4 (b), the image of δ is nontrivial and it has an invariant direction $\omega \in \mathbf{R}^2 \subset \mathbf{R}^3$ with $|\omega| = 1$.

The restriction of ρ_n to ∂M consists of screw motions of $\mathbf{R}^3$, and for each element $\gamma \in \pi_1(\partial M)$, the translation length of this screw motion is the product

$$\mathrm{trans}(\rho_n(\gamma)) = \big(\rho_n(\gamma)(0) - (0)\big) \cdot \omega_n,$$

where $\omega_n \in \mathbf{R}^3$ is a unitary vector in the direction of the axis of the screw motion. Since the projection of d is δ, it follows that $\omega_n \to \omega$. Thus, given a system of peripheral generators $\langle \gamma_1, \gamma_2 \rangle$, and the restriction

$$p_n \ \mathrm{trans}(\rho_n(\gamma_1)) + q_n \ \mathrm{trans}(\rho_n(\gamma_2)) = 0,$$

we deduce the limit

$$\lim_{n\to\infty} \frac{p_n}{q_n} = -\frac{\big(\rho_0(\gamma_2)(0) - (0)\big) \cdot \omega}{\big(\rho_0(\gamma_1)(0) - (0)\big) \cdot \omega}$$

which is a well-defined element $l \in \mathbf{R} \cup \infty$, depending on the cusped manifold M only. $\square$

Proof of Theorem 1.1. Let C be the set of coefficients (p, q) so that $\pi_1(M_{p/q})$ has a discrete and faithful representation in $\mathrm{SO}(4,1)$, other than the holonomy of the complete hyperbolic structure of $M_{p/q}$. By Theorem 1.4, any sequence of pairwise distinct such representations must converge to the holonomy of the complete hyperbolic structure of M. By Theorem 1.5 it now follows that the set C is asymptotic to the line $p/q = l$, where l is a number — possibly ∞ — depending only on M. Thus C cannot be cofinite in the set of all filling coefficients. $\square$

5C. *Cohomology of $\mathbf{Z}\oplus\mathbf{Z}$ with coefficients in* $\mathbf{R}^{3,1}$. The aim of this subsection is to prove Lemma 5.4.

Before the proof we fix some notation. The restriction of the holonomy representation ρ_0 of $\pi_1(M)$ to $\pi_1(\partial M)\cong\mathbf{Z}\oplus\mathbf{Z}$ is a parabolic representation. Identifying the fixed point p_0 of $\rho_0|_{\pi(\partial M)}$ with ∞, so that $\partial\mathbf{H}^3=\mathbf{R}^2\cup\{p_0\}$, the restriction is a representation by translations, that defines a lattice in the plane $\mathbf{R}^2$.

We choose

$$p_0=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

to be the point of the light cone invariant by the holonomy of ∂M. With that choice, for $\gamma\in\pi_1(\partial M)$ if the translation vector of γ is

$$\operatorname{trans}(\rho_0(\gamma))=(x,y)\in\mathbf{R}^2,$$

then the holonomy (as an element of SO(3, 1)) is

$$\rho_0(\gamma)=\exp\begin{pmatrix}0&0&x&y\\0&0&x&y\\x&-x&0&0\\y&-y&0&0\end{pmatrix}=\begin{pmatrix}1+\frac{1}{2}(x^2+y^2)&-\frac{1}{2}(x^2+y^2)&x&y\\\frac{1}{2}(x^2+y^2)&1-\frac{1}{2}(x^2+y^2)&x&y\\x&-x&1&0\\y&-y&0&1\end{pmatrix}.$$

Elements in $\mathbf{R}^{3,1}$ may be written as

$$v=\begin{pmatrix}z+\lambda\\z-\lambda\\-\beta\\\alpha\end{pmatrix},\tag{5-1}$$

with $\lambda,z,\beta,\alpha\in\mathbf{R}$. Using the inclusion $\mathbf{R}^{3,1}\subset\mathfrak{so}(4,1)$ of (3-2), the parameter λ corresponds to the length of an infinitesimal displacement of p_0 in $\partial\mathbf{H}^4$ in the direction perpendicular to $\partial\mathbf{H}^3$.

In particular $\lambda=0$ defines the subspace of infinitesimal isometries that fix p_0, and therefore they restrict to infinitesimal similarities of $\mathbf{R}^3=\partial\mathbf{H}^4\setminus\{p_0\}$. Since our deformations vanish on the direction tangent to $\mathbf{R}^2\times\{0\}$ they must be infinitesimal isometries of Euclidean space. In fact we have:

$$\mathbf{R}^{3,1}\cap\mathfrak{Isom}(\mathbf{R}^3)=\{v\in\mathbf{R}^{3,1}\mid\lambda=0\}.$$

The parameter z describes the length of an infinitesimal translation in the direction perpendicular to $\mathbf{R}^2$. Finally, β and α correspond to an infinitesimal rotation of vector $(\alpha, \beta, 0) \in \mathbf{R}^2 \times \{0\}$. Using the coordinates in (5-1), the projection

$$\mathrm{rot} : \mathfrak{Isom}(\mathbf{R}^3) \to \mathfrak{so}(3) \cong \mathbf{R}^3$$

restricts to

$$\begin{aligned} \mathrm{rot} : \{\lambda = 0\} \subset \mathbf{R}^{3,1} &\to \mathbf{R}^2 \times \{0\}, \\ v &\mapsto (\alpha, \beta, 0). \end{aligned}$$

Here rot denotes the tangent map of the epimorphism $\mathrm{Isom}(\mathbf{R}^3) \to O(3)$.

Proof of Lemma 5.4. Fix a system of generators g_1, g_2 for $\pi_1(\partial M)$, so that $\rho_0(g_1)$ is a translation of vector $(x_1, y_1) \in \mathbf{R}^2$, and $\rho_0(g_2)$, of vector $(x_2, y_2) \in \mathbf{R}^2$.

For a cocycle $d \in Z^1(\partial M; \mathbf{R}^{3,1})$, define $\lambda_1, \lambda_2, \alpha_1, \alpha_2, \beta_1, \beta_2, z_1$ and $z_2 \in \mathbf{R}$ so that

$$d(g_i) = \begin{pmatrix} z_i + \lambda_i \\ z_i - \lambda_i \\ -\beta_i \\ \alpha_i \end{pmatrix},$$

for $i = 1, 2$. By Fox calculus (see, for example, [Lubotzky and Magid 1985]) the parameters α_i, β_i, z_i and λ_i are subject to the relation $(g_1-1)d(g_2)=(g_2-1)d(g_1)$, which is equivalent to

$$\lambda_1 = \lambda_2 = 0, \quad \text{and} \quad -\beta_2 x_1 + \alpha_2 y_1 = -\beta_1 x_2 + \alpha_1 y_2,$$

that give a five-dimensional real space on the parameters subject to these relations. In particular since $\lambda_1 = \lambda_2 = 0$, statement (a) is proved.

Notice that d is a coboundary if and only if there exist parameters $A, B, L \in \mathbf{R}$ such that

$$d(g_i) = \begin{pmatrix} L(x_i^2 + y_i^2) - B x_i + A y_i \\ L(x_i^2 + y_i^2) - B x_i + A y_i \\ 2 L x_i \\ 2 L y_i \end{pmatrix},$$

for $i = 1, 2$.

It follows immediately that if $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, then d is a coboundary.

Now the remaining of the proof is an elementary but tricky computation. The equality

$$-\beta_2 x_1 + \alpha_2 y_1 = -\beta_1 x_2 + \alpha_1 y_2$$

may be seen as an equality of imaginary parts:

$$\mathrm{Im}\big((x_1 - \mathbf{i}\, y_1)(\alpha_2 + \mathbf{i}\, \beta_2)\big) = \mathrm{Im}\big((x_2 - \mathbf{i}\, y_2)(\alpha_1 + \mathbf{i}\, \beta_1)\big).$$

Next we claim that, by adding a cocycle, we can remove imaginary parts. Namely the expression

$$(x_1 - \mathbf{i}\, y_1)(\alpha_2 + \mathbf{i}\, \beta_2) - (x_2 - \mathbf{i}\, y_2)(\alpha_1 + \mathbf{i}\, \beta_1) \tag{5-2}$$

may have nontrivial real part, but we can assume that it vanishes, because adding the cocycle such that $\alpha_j = L y_j$ and $\beta_j = -L x_j$ for some $L \in \mathbf{R}$ and $j = 1, 2$, it means changing the expression (5-2) by adding $2L(x_1 y_2 - y_1 x_2) \neq 0$.

Since (5-2) vanishes, there exist $\lambda \in \mathbf{R}$ and $\omega \in \mathbf{C}$ with $|\omega| = 1$ such that

$$\frac{\alpha_2 + \mathbf{i}\,\beta_2}{x_2 - \mathbf{i}\,y_2} = \frac{\alpha_1 + \mathbf{i}\,\beta_1}{x_1 - \mathbf{i}\,y_1} = \frac{\lambda}{2}\,\mathbf{i}\,\omega^2.$$

Adding the coboundary such that $\alpha_j = \frac{\lambda}{2} y_j$ and $\beta_j = -\frac{\lambda}{2} x_j$, for $j = 1, 2$, we deduce:

$$\alpha_1 + \mathbf{i}\,\beta_1 = (x_1 - \mathbf{i}\,y_1)\frac{\lambda}{2}\,\mathbf{i}\,\omega^2 + \frac{\lambda}{2}(y_1 - \mathbf{i}\,x_1),$$
$$\alpha_2 + \mathbf{i}\,\beta_2 = (x_2 - \mathbf{i}\,y_2)\frac{\lambda}{2}\,\mathbf{i}\,\omega^2 + \frac{\lambda}{2}(y_2 - \mathbf{i}\,x_2).$$

Hence, expressing the scalar product $\cdot$ in terms of real parts, we have:

$$(\alpha_1, \beta_1) = \alpha_1 + \mathbf{i}\,\beta_1 = \mathrm{Re}((x_1 - \mathbf{i}\,y_1)\,\mathbf{i}\,\omega)\lambda\,\omega = ((x_1, y_1)\cdot \mathbf{i}\,\omega)\,\lambda\,\omega,$$
$$(\alpha_2, \beta_2) = \alpha_2 + \mathbf{i}\,\beta_2 = \mathrm{Re}((x_2 - \mathbf{i}\,y_2)\,\mathbf{i}\,\omega)\lambda\,\omega = ((x_2, y_2)\cdot \mathbf{i}\,\omega)\,\lambda\,\omega.$$

And we conclude the proof of the lemma by linearity. □

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References

[Bart and Scannell 2007] A. Bart and K. P. Scannell, "A note on stamping", *Geom. Dedicata* **126** (2007), 283–291. MR 2008e:57014 Zbl 1130.57022

[Bestvina 1988] M. Bestvina, "Degenerations of the hyperbolic space", *Duke Math. J.* **56**:1 (1988), 143–161. MR 89m:57011 Zbl 0652.57009

[Boileau and Porti 2001] M. Boileau and J. Porti, "Geometrization of 3-orbifolds of cyclic type", *Astérisque* 272 (2001), 208. MR 2002f:57034 Zbl 0971.57004

[Boileau et al. 2005] M. Boileau, B. Leeb, and J. Porti, "Geometrization of 3-dimensional orbifolds", *Ann. of Math.* (2) **162**:1 (2005), 195–290. MR 2007f:57028 Zbl 1087.57009

[Epstein and Penner 1988] D. B. A. Epstein and R. C. Penner, "Euclidean decompositions of non-compact hyperbolic manifolds", *J. Differential Geom.* **27**:1 (1988), 67–80. MR 89a:57020 Zbl 0611.53036

[Johnson and Millson 1987] D. Johnson and J. J. Millson, "Deformation spaces associated to compact hyperbolic manifolds", pp. 48–106 in *Discrete groups in geometry and analysis* (New Haven, CT, 1984), edited by R. Howe, Progr. Math. **67**, Birkhäuser, Boston, 1987. MR 88j:22010 Zbl 0664.53023

[Kapovich 1994] M. Kapovich, "Deformations of representations of discrete subgroups of SO(3, 1)", *Math. Ann.* **299**:2 (1994), 341–354. MR 95d:57010 Zbl 0828.57009

[Kapovich 2001] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics **183**, Birkhäuser, Boston, 2001. MR 2002m:57018 Zbl 0958.57001

[Kapovich 2007] M. Kapovich, "On sequences of finitely generated discrete groups", preprint, 2007. arXiv 0708.2671

[Lubotzky and Magid 1985] A. Lubotzky and A. R. Magid, "Varieties of representations of finitely generated groups", *Mem. Amer. Math. Soc.* **58**:336 (1985), xi+117. MR 87c:20021 Zbl 0598.14042

[Morgan 1986] J. W. Morgan, "Group actions on trees and the compactification of the space of classes of $\mathrm{SO}(n, 1)$-representations", *Topology* **25**:1 (1986), 1–33. MR 87h:20062 Zbl 0595.57030

[Morgan and Shalen 1984] J. W. Morgan and P. B. Shalen, "Valuations, trees, and degenerations of hyperbolic structures. I", *Ann. of Math.* (2) **120**:3 (1984), 401–476. MR 86f:57011 Zbl 0583.57005

[Morgan and Shalen 1988] J. W. Morgan and P. B. Shalen, "Degenerations of hyperbolic structures. III. Actions of 3-manifold groups on trees and Thurston's compactness theorem", *Ann. of Math.* (2) **127**:3 (1988), 457–519. MR 89e:57010b Zbl 0661.57004

[Paulin 1997] F. Paulin, "Sur les automorphismes extérieurs des groupes hyperboliques", *Ann. Sci. École Norm. Sup.* (4) **30**:2 (1997), 147–167. MR 98c:20070 Zbl 0877.20014

[Scannell 2000] K. P. Scannell, "Infinitesimal deformations of some SO(3, 1) lattices", *Pacific J. Math.* **194**:2 (2000), 455–464. MR 2001c:57018 Zbl 1019.57007

[Scannell 2002] K. P. Scannell, "Local rigidity of hyperbolic 3-manifolds after Dehn surgery", *Duke Math. J.* **114**:1 (2002), 1–14. MR 2003e:57031 Zbl 1025.57019

[Thurston 1980] W. P. Thurston, "The Geometry and Topology of Three-Manifolds", lecture notes, 1980, Available at http://www.msri.org/publications/books/gt3m/.

[Weil 1964] A. Weil, "Remarks on the cohomology of groups", *Ann. of Math.* (2) **80** (1964), 149–157. MR 30 #199 Zbl 0192.12802

[Wolf 1984] J. A. Wolf, *Spaces of constant curvature*, 5th ed., Publish or Perish, Houston, TX, 1984. MR 88k:53002 Zbl 0556.53033

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UNIVALENCE OF EQUIVARIANT RIEMANN DOMAINS OVER THE COMPLEXIFICATIONS OF RANK-ONE RIEMANNIAN SYMMETRIC SPACES

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Let G/K be a noncompact, rank-one, Riemannian symmetric space, and let $G^{\mathbb{C}}$ be the universal complexification of G. We prove that a holomorphically separable, G-equivariant Riemann domain over $G^{\mathbb{C}}/K^{\mathbb{C}}$ is necessarily univalent, provided that G is not a covering of $\mathrm{SL}(2, \mathbb{R})$. As a consequence, one obtains a univalence result for holomorphically separable, $G \times K$-equivariant Riemann domains over $G^{\mathbb{C}}$. Here $G \times K$ acts on $G^{\mathbb{C}}$ by left and right translations. The proof of such results involves a detailed study of the G-invariant complex geometry of the quotient $G^{\mathbb{C}}/K^{\mathbb{C}}$, including a complete classification of all its Stein G-invariant subdomains.

1. Introduction

Let Y be a domain in a Stein manifold X. By a classical result of H. Rossi [1963], the envelope of holomorphy of Y exists and can be realized as a Riemann domain $\hat{p} : \hat{Y} \to X$. In general it is a difficult problem to explicitly determine $\hat{Y}$ and to establish whether $\hat{p}$ is injective, that is, whether the envelope of holomorphy $\hat{Y}$ is *univalent*. However, in the presence of a large group of symmetries, some results are known. For instance, let the vector group $G = (\mathbb{R}^n, +)$ act on its universal complexification $G^{\mathbb{C}} = (\mathbb{C}^n, +)$ by left multiplication. Bochner's tube theorem characterizes the envelope of holomorphy of a G-invariant domain Y in $G^{\mathbb{C}}$ as the smallest, convex, G-invariant domain in $G^{\mathbb{C}}$ containing Y. In particular it shows that such envelope is univalent. An analogous statement holds true for G a compact torus, that is, for envelopes of holomorphy of Reinhardt domains in $(\mathbb{C}^*)^n$.

Let G be a connected Lie group, and let Y be a complex G-manifold, that is, a complex manifold endowed with a real-analytic action of G by holomorphic transformations. A *G-equivariant Riemann domain over* $G^{\mathbb{C}}$ is by definition a G-equivariant local biholomorphism $p : Y \to G^{\mathbb{C}}$. A motivation for determining conditions under which p is injective in this more general context comes from the theory of globalization of local actions. Namely, given a reduced complex

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space endowed with a holomorphic G-action, one can consider the induced local $G^{\mathbb{C}}$-action. It turns out that the univalence of G-equivariant Riemann domains over $G^{\mathbb{C}}$ is a necessary condition for extending such a local action to a global one; see [Palais 1957; Heinzner and Iannuzzi 1997; Casadio Tarabusi et al. 2000].

For a certain class of groups, including for example the product of a compact and a simply connected nilpotent Lie group, univalence results were obtained for arbitrary holomorphically separable, G-equivariant Riemann domains over $G^{\mathbb{C}}$ by Cœuré and Loeb [1986]. Note that since $G^{\mathbb{C}}$ is Stein (see [Heinzner 1993]), holomorphic separability of Y is a necessary condition for p to be injective.

When G is a noncompact, real semisimple Lie group, univalence of holomorphically, separable G-equivariant Riemann domains over $G^{\mathbb{C}}$ does not hold in general. For $G = \mathrm{SL}(2, \mathbb{R})$, a Stein counterexample was pointed out to us by K. Oeljeklaus; see Section 8. The image of this Riemann domain in $G^{\mathbb{C}}$ is also invariant under right K-translations, and its construction is based on the existence of nontrivial coverings of the K-orbits in $G^{\mathbb{C}}$. Here K is a maximal compact subgroup in G. Observe also that $\mathrm{SL}(2, \mathbb{C})/\mathrm{SL}(2, \mathbb{R})$ is simply connected. Thus this example gives a negative answer to the question of whether the simple-connectivity of the quotient $G^{\mathbb{C}}/G$ is a sufficient condition for univalence of G-equivariant Riemann domains over $G^{\mathbb{C}}$; see [Cœuré and Loeb 1986].

Let G be a connected, noncompact, real simple Lie group, and let K be a maximal compact subgroup of G. The group G is not necessarily embedded in $G^{\mathbb{C}}$, but it is assumed to have finite center. Consider the action of the product group $G \times K$ on $G^{\mathbb{C}}$ by left and right translations. One of the results of this paper is the following theorem, Theorem 8.1.

Theorem. *Let G/K be a noncompact, rank-one, Riemannian symmetric space. A holomorphically separable, $G \times K$-equivariant Riemann domain $p : Y \to G^{\mathbb{C}}$ is univalent, provided that G is not a covering of* $\mathrm{SL}(2, \mathbb{R})$.

Note that since Y embeds equivariantly into its envelope of holomorphy (see Section 2), there is no loss of generality in assuming that Y is Stein. Then a result of P. Heinzner [1991] implies that the categorical quotient $Y /\!/ K$ is also Stein. By performing categorical K-reduction on both Y and $G^{\mathbb{C}}$, one can associate to $p : Y \to G^{\mathbb{C}}$ a Stein, G-equivariant Riemann domain $q : Y /\!/ K \to G^{\mathbb{C}}/K^{\mathbb{C}}$. A suitable characterization of the univalence of q (see Proposition 3.1) implies that p is univalent if q is univalent. Then the above theorem is a consequence of the following one, which is the main result of the paper; see Theorem 7.6 and Remark 7.8.

Theorem. *Let G/K be a noncompact, rank-one, Riemannian symmetric space. A holomorphically separable, G-equivariant Riemann domain $q : \Sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is univalent, provided that G is not a covering of* $\mathrm{SL}(2, \mathbb{R})$.

The proof of this theorem is carried out as follows. First we show that, with few exceptions, the map q is injective on every G-orbit. For principal G-orbits this is done by determining their topology. The result is then extended to the remaining G-orbits by a general argument in Section 5. As a consequence there exists a G-invariant domain in Σ on which q is injective.

Next we show that such domain can be enlarged to the whole Σ. This is done by successively lifting to Σ local slices for principal G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Since such slices are one-dimensional and q is injective on G-orbits, each lifting determines a G-invariant domain in Σ on which q is injective. The main difficulty is in ensuring monodromy around singular G-orbits. For this we combine a detailed description of the G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ with the complex-geometric properties of certain non-Stein, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

By the above univalence result, all Stein, G-equivariant Riemann domains over $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be regarded as Stein, invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$. We carry out their classification in Theorem 6.1. These results also provide examples of Kobayashi hyperbolic domains whose envelopes of holomorphy are not Kobayashi hyperbolic; see Example 7.9.

For G/K of arbitrary rank, recent investigations due to several authors have indicated an interplay between the complex geometry of distinguished Stein, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and the harmonic analysis on the G-symmetric spaces contained in $G^{\mathbb{C}}/K^{\mathbb{C}}$; see [Krötz and Stanton 2005; Fels et al. 2006] and references therein. A better understanding of envelopes of holomorphy of G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ might give new insights in this context. We hope the present paper to be a first step for further investigations in higher rank.

The paper is organized as follows. In Section 2, we recall some basic notions and results from geometric invariant theory. In Section 3, from a Stein $G\times K$-equivariant Riemann domain $p : Y \to G^{\mathbb{C}}$ we obtain a Stein, G-equivariant, Riemann domain $q : Y /\!/ K \to G^{\mathbb{C}}/K^{\mathbb{C}}$. We also show that p is univalent if q is univalent. In Section 4, we give a detailed description of the G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ when G/K is a noncompact, rank-one, Riemannian symmetric space. We also describe an explicit model for the space $G^{\mathbb{C}}/K^{\mathbb{C}}$ in the cases $G = \mathrm{SO}_0(n, 1)$ and $G = \mathrm{SU}(n, 1)$. In Section 5, we show that, with few exceptions, a G-equivariant Riemann domain $q : Y \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is univalent on every G-orbit. In Section 6, we carry out a complete classification of Stein, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$. When $G = \mathrm{SU}(n, 1)$ some of these domains appear to be new. In Section 7, we prove the univalence result for holomorphically separable, G-equivariant Riemann domains over $G^{\mathbb{C}}/K^{\mathbb{C}}$. In Section 8, we obtain the result for holomorphically separable, $G\times K$-equivariant Riemann domains over $G^{\mathbb{C}}$. We also discuss some examples. In the appendix, Section 9, we compute the Levi form of all nonclosed hypersurface G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. The results of this computation are used in Sections 6 and 7.

2. Preliminaries

Let G be a connected, real Lie group. A complex Lie group $G^{\mathbb{C}}$ together with a Lie group homomorphism $\iota : G \to G^{\mathbb{C}}$ is called a *universal complexification* of G if, for every Lie group homomorphism ψ from G to a complex Lie group H, there exists a holomorphic homomorphism $\psi^{\mathbb{C}} : G^{\mathbb{C}} \to H$ such that $\psi = \psi^{\mathbb{C}} \circ \iota$. A universal complexification of G always exists and is unique up to biholomorphisms; see [Hochschild 1965].

Assume that G is a connected, real semisimple Lie group. Denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$ its complexification. Then the universal complexification of G is a complex semisimple Lie group $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. If G is a real form of a simply connected complex semisimple Lie group $G^{\mathbb{C}}$, then its universal complexification is $G^{\mathbb{C}}$. Furthermore, if Γ is a central subgroup of G, then the universal complexification of the quotient group G/Γ is given by $G^{\mathbb{C}}/\Gamma$. Note that every automorphism of G uniquely extends to a holomorphic automorphism of its universal complexification $G^{\mathbb{C}}$.

Let K be a compact Lie group and X a Stein K-space, that is, a reduced Stein space with a real-analytic action of K by holomorphic transformations. The *categorical quotient* $X /\!/ K$ of X is defined by the equivalence relation in which $x \sim y$ if and only if $f(x) = f(y)$ for every K-invariant holomorphic function f on X. We recall some basic properties of the categorical quotient; see [Heinzner 1991].

Theorem 2.1. *Let K be a compact Lie group and X a Stein K-space. Then*

(i) *the categorical quotient $X /\!/ K$ equipped with the algebra $\mathbb{O}(X)^K$ of holomorphic K-invariant functions on X is a Stein space, and the projection $\pi : X \to X /\!/ K$ is holomorphic; and*

(ii) *for every K-invariant holomorphic map ψ from X to a complex space Y, there exists a unique holomorphic map $\hat{\psi} : X /\!/ K \to Y$ making the diagram*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\psi"'] & X /\!/ K \arrow[dl, "\hat{\psi}"] \\ Y & \end{tikzcd}$$

commute.

If the K-action on X is the restriction of a $K^{\mathbb{C}}$-action, then the algebras of K-invariant and of $K^{\mathbb{C}}$-invariant holomorphic functions on X coincide. In particular, they induce the same equivalence relation on X and $X /\!/ K \cong X /\!/ K^{\mathbb{C}}$. In this case, if all $K^{\mathbb{C}}$-orbits are closed, then $X /\!/ K^{\mathbb{C}}$ coincides with the usual orbit space $X/K^{\mathbb{C}}$; see [Snow 1982, Theorem 3.8]. A K-action on a Stein space can always be extended to a $K^{\mathbb{C}}$-action, as shown by the following theorem.

Theorem 2.2 [Heinzner 1991]. *Let K be a compact Lie group and X a Stein K-space. Then there exist a Stein $K^{\mathbb{C}}$-space X^* and a K-equivariant holomorphic embedding $\iota : X \hookrightarrow X^*$ with the following properties*:

(i) *the map ι is open, and $K^{\mathbb{C}} \cdot \iota(X) = X^*$*;

(ii) *for every K-equivariant holomorphic map φ from X into a complex $K^{\mathbb{C}}$-space Z, there exists a unique $K^{\mathbb{C}}$-equivariant holomorphic map $\varphi^* : X^* \to Z$ making the diagram*

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[d, "\varphi"'] & X^* \arrow[dl, "\varphi^*"] \\ Z & \end{tikzcd}$$

commute;

(iii) *the inclusion $X \hookrightarrow X^*$ induces an isomorphism between the categorical quotients $X /\!/ K$ and $X^* /\!/ K^{\mathbb{C}}$.*

Observe that, since $K^{\mathbb{C}} \cdot \iota(X) = X^*$, if X is nonsingular, then X^* is also nonsingular.

Let X be a complex manifold, and let G be a Lie group. A Riemann domain over X is a complex manifold Y together with a locally biholomorphic map $p : Y \to X$. If both X and Y are G-manifolds and the map p is G-equivariant, then we refer to $p : Y \to X$ as a *G-equivariant Riemann domain*. If X is Stein and Y is holomorphically separable, then Y embeds as an open domain in its envelope of holomorphy $\widehat{Y}$, and the map p extends to a local biholomorphism $\hat{p} : \widehat{Y} \to X$; see [Rossi 1963]. Moreover the G-action on Y extends to $\widehat{Y}$, and the map $\hat{p}$ is G-equivariant, that is, $\hat{p} : \widehat{Y} \to X$ is a Stein, G-equivariant Riemann domain.

A Riemann domain $p : Y \to X$ is called *univalent* if the map p is injective. Assume X is Stein and Y is holomorphically separable. If $\hat{p}$ is univalent, then p is also univalent. Aiming at univalence results for holomorphically separable Riemann domains over $G^{\mathbb{C}}$, it is therefore not restrictive to start with Riemann domains that are Stein.

3. From Riemann domains over $G^{\mathbb{C}}$ to Riemann domains over $G^{\mathbb{C}}/K^{\mathbb{C}}$

Let G be a connected, noncompact, real semisimple Lie group, let $K \subset G$ be a maximal compact subgroup, and let $G^{\mathbb{C}}$ be the universal complexification of G. Let $G \times K$ act on $G^{\mathbb{C}}$ by left and right translations, that is,

$$(g, k) \cdot z := gzk^{-1} \quad \text{for } (g, k) \in G \times K \text{ and } z \in G^{\mathbb{C}}.$$

In this section, to every Stein, $G \times K$-equivariant Riemann domain $p : Y \to G^{\mathbb{C}}$ we associate a Stein, G-equivariant Riemann domain $q : \Sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}$. We also show that the univalence of q implies that of p.

Let X be a Stein $K^{\mathbb{C}}$-manifold and let $p : Y \to X$ be a Stein, K-equivariant Riemann domain. By Theorem 2.2, there exist a Stein $K^{\mathbb{C}}$-manifold Y^*, a K-equivariant holomorphic open embedding $\iota : Y \hookrightarrow Y^*$, and a $K^{\mathbb{C}}$-equivariant holomorphic map $p^* : Y^* \to X$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, hook, "\iota"] \arrow[d, "p"'] & Y^* \arrow[dl, "p^*"] \\ X & \end{tikzcd}$$

commutes. Since p is locally biholomorphic, p^* is $K^{\mathbb{C}}$-equivariant, and $Y^* = K^{\mathbb{C}} \cdot Y$, one has that p^* is locally biholomorphic as well, that is, it defines a Stein $K^{\mathbb{C}}$-equivariant Riemann domain. By Theorem 2.2, the spaces $Y^* /\!/ K^{\mathbb{C}}$ and $Y /\!/ K$ are biholomorphic. Therefore Theorem 2.1 implies there exists a holomorphic map $q : Y /\!/ K \to X /\!/ K^{\mathbb{C}}$ making the diagram

$$\begin{tikzcd} Y^* \arrow[r] \arrow[d, "p^*"'] & Y^* /\!/ K^{\mathbb{C}} \cong Y /\!/ K \arrow[d, "q"] \\ X \arrow[r] & X /\!/ K^{\mathbb{C}} \end{tikzcd}$$

commute. Here the horizontal arrows denote the categorical quotient maps.

Assume that all $K^{\mathbb{C}}$-orbits in X are closed and all $K^{\mathbb{C}}$-isotropies are connected. We claim that all $K^{\mathbb{C}}$-orbits in Y^* are closed as well. Suppose by contradiction that there exists a nonclosed orbit $K^{\mathbb{C}} \cdot y$ in Y^*. Let $K^{\mathbb{C}} \cdot z$ be a lower dimensional orbit in its closure; see [Snow 1982, Proposition 2.3]. Since p^* is locally biholomorphic and $K^{\mathbb{C}}$-equivariant, the orbit $K^{\mathbb{C}} \cdot p^*(z)$ lies in the closure of $K^{\mathbb{C}} \cdot p^*(y)$ and has lower dimension. In particular such orbits are distinct. It follows that the orbit $K^{\mathbb{C}} \cdot p^*(y)$ is not closed, contradicting the assumption.

By the above claim, the categorical quotients $X /\!/ K^{\mathbb{C}}$ and $Y^* /\!/ K^{\mathbb{C}}$ coincide with the orbit spaces $X/K^{\mathbb{C}}$ and $Y^*/K^{\mathbb{C}}$, respectively. If we also assume that the $K^{\mathbb{C}}$-orbits have connected isotropy subgroups, such orbit spaces are nonsingular and the map $q : Y /\!/ K \to X/K^{\mathbb{C}}$ defines a Stein Riemann domain. We refer to it as the *Riemann domain induced by* $p : Y \to X$. Next we prove a general univalence result for Stein, K-equivariant Riemann domains.

Proposition 3.1. *Let X be a Stein $K^{\mathbb{C}}$-manifold, all of whose $K^{\mathbb{C}}$-orbits are closed and have connected isotropy subgroups. Let $p : Y \to X$ be a Stein, K-equivariant Riemann domain, and let $p^* : Y^* \to X$ be its extension to the $K^{\mathbb{C}}$-globalization Y^* of Y.*

(i) *The induced Stein, Riemann domain $q : Y /\!/ K \to X/K^{\mathbb{C}}$ is univalent if and only if $p^* : Y^* \to X$ is univalent.*

(ii) *If* $q : Y /\!/ K \to X/K^{\mathbb{C}}$ *is univalent, then* $p : Y \to X$ *is univalent.*

Proof. (i) If p^* is injective, then it maps distinct $K^{\mathbb{C}}$-orbits in Y^* onto distinct $K^{\mathbb{C}}$-orbits in X. As we already noticed, since all $K^{\mathbb{C}}$-orbits in X are closed, the categorical quotients $X/\!/K^{\mathbb{C}}$ and $Y^*/\!/K^{\mathbb{C}}$ coincide with the orbit spaces $X/K^{\mathbb{C}}$ and $Y^*/K^{\mathbb{C}}$, respectively. It follows that the induced map $Y^*/K^{\mathbb{C}} \to X/K^{\mathbb{C}}$ is injective. Moreover, by Theorem 2.2, the space $Y /\!/ K$ can be identified with $Y^* /\!/ K^{\mathbb{C}}$. As a result the induced Riemann domain $q : Y /\!/ K \to X/K^{\mathbb{C}}$ is univalent.

Conversely, assume that $q : Y /\!/ K \to X/K^{\mathbb{C}}$ is univalent, that is, that the map $Y^*/K^{\mathbb{C}} \to X/K^{\mathbb{C}}$ is injective. By assumption, the $K^{\mathbb{C}}$-isotropy subgroups in X are connected; thus p^* is injective on every $K^{\mathbb{C}}$-orbit in Y^*. It follows that $p^* : Y^* \to X$ is globally injective. This proves (i); statement (ii) is a direct consequence. □

Remark 3.2. In general, under the assumptions of the above proposition, the univalence of $p : Y \to X$ does not imply that of $q : Y /\!/ K \to X/K^{\mathbb{C}}$. For instance, let $\mathbb{C}^*$ act on $\mathbb{C} \times \mathbb{C}^*$ and on $X := \mathbb{C}^* \times \mathbb{C}^*$ by multiplication on the second factor. Define $p^* : \mathbb{C} \times \mathbb{C}^* \to X$ by $(z, w) \mapsto (e^z, w)$ and consider

$$Y := \{(z, w) \in \mathbb{C} \times \mathbb{C}^* : \operatorname{Im} z < |w| < 2\pi + \operatorname{Im} z\}.$$

Then Y is a Stein S^1-invariant subdomain of $Y^* = \mathbb{C} \times \mathbb{C}^*$ and the map $p := p^*|_Y$ is injective. Nevertheless the induced map $q : Y /\!/ S^1 \cong \mathbb{C} \to X/\mathbb{C}^* \cong \mathbb{C}^*$, given by $z \mapsto e^z$, is not injective.

Consider now the case when X is the group $G^{\mathbb{C}}$ endowed with the $G \times K$-action by left and right translations. Let $p : Y \to G^{\mathbb{C}}$ be a Stein, $G \times K$-equivariant Riemann domain. Note that the actions of G and K commute on $G^{\mathbb{C}}$. Thus they also commute on Y, because p is equivariant and locally injective. Since the K-action on $G^{\mathbb{C}}$ is the restriction of a $K^{\mathbb{C}}$-action all of whose orbits are closed, the spaces $G^{\mathbb{C}} /\!/ K$ and $G^{\mathbb{C}}/K^{\mathbb{C}}$ are biholomorphic.

By the universality property of the categorical quotient (see Theorem 2.1), the G-actions on Y and on $G^{\mathbb{C}}$ induce G-actions on $Y /\!/ K$ and on $G^{\mathbb{C}}/K^{\mathbb{C}}$, respectively. Moreover the induced Stein, Riemann domain $q : Y /\!/ K \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is G-equivariant. By applying Proposition 3.1 to this situation, we obtain the following.

Corollary 3.3. *Let* $p : Y \to G^{\mathbb{C}}$ *be a Stein,* $G \times K$*-equivariant Riemann domain over* $G^{\mathbb{C}}$*, and let* $q : Y /\!/ K \to G^{\mathbb{C}}/K^{\mathbb{C}}$ *be the induced Stein,* G*-equivariant Riemann domain over* $G^{\mathbb{C}}/K^{\mathbb{C}}$*. If* q *is univalent, then* p *is univalent.*

4. G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$

Let G be a connected, noncompact, real simple Lie group, let $K \subset G$ be a maximal compact subgroup, and let $G^{\mathbb{C}}$ be the universal complexification of G. Assume that G is embedded in $G^{\mathbb{C}}$. The quotient G/K is a Riemannian symmetric space of the noncompact type. In this section we obtain a complete description of the G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ in the case when G/K has rank one.

We recall some basic facts holding for G/K of arbitrary rank. Denote by σ the antiholomorphic involution of $G^{\mathbb{C}}$ relative to G and by $\tau : G^{\mathbb{C}} \to G^{\mathbb{C}}$ the holomorphic extension of the Cartan involution θ of G with respect to K. Note that the fixed point set of τ in $G^{\mathbb{C}}$ contains the complexification $K^{\mathbb{C}}$ of K. The commuting composition $\sigma \circ \tau = \tau \circ \sigma$ is a Cartan involution of $G^{\mathbb{C}}$. Denote by U the corresponding compact real form. The U-orbit of the base point $eK^{\mathbb{C}}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ is diffeomorphic to the compact dual symmetric space U/K, and is embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ transversally to G/K.

Remark 4.1. (i) For every triple $(G, K, G^{\mathbb{C}})$ as above, the manifold $G^{\mathbb{C}}/K^{\mathbb{C}}$ is simply connected. To see this, denote by $\widetilde{G}^{\mathbb{C}}$ and $\widetilde{U} \subset \widetilde{G}^{\mathbb{C}}$ the universal coverings of $G^{\mathbb{C}}$ and U, respectively. Let $\widehat{G}$ be the real form of $\widetilde{G}^{\mathbb{C}}$ relative to the lifting of σ to $\widetilde{G}^{\mathbb{C}}$. The group $\widehat{G}$ is connected (see [Steinberg 1968]) and is a finite covering of G. Hence $G = \widehat{G}/\Gamma$, where Γ is a finite central subgroup of $\widehat{G}$. Similarly $K = \widehat{K}/\Gamma$, where $\widehat{K}$ is a maximal compact subgroup of $\widehat{G}$. One has $G^{\mathbb{C}} \cong \widetilde{G}^{\mathbb{C}}/\Gamma$ (see Section 2) and consequently $U = \widetilde{U}/\Gamma$. As a consequence there are isomorphisms

$$U/K \cong \widetilde{U}/\Gamma/\widehat{K}/\Gamma \cong \widetilde{U}/\widehat{K}.$$

Since $\widehat{K}$ is connected, the quotient $\widetilde{U}/\widehat{K}$ is simply connected. Moreover U/K is a topological retract of $G^{\mathbb{C}}/K^{\mathbb{C}}$. Hence the claim follows.

(ii) From different triples $(G, K, G^{\mathbb{C}})$ as above associated with the same Riemannian symmetric space, one obtains the same complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$. Indeed the map $\widetilde{G}^{\mathbb{C}}/\widehat{K}^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$, given by $g\widehat{K}^{\mathbb{C}} \mapsto g\Gamma K^{\mathbb{C}}$, defines a biholomorphism. Moreover the center of G acts trivially on $G^{\mathbb{C}}/K^{\mathbb{C}}$. As a consequence, different triples $(G, K, G^{\mathbb{C}})$ yield the same G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ and G-equivariantly diffeomorphic orbits.

Closed G-orbits of maximal dimension form an open dense subset of $G^{\mathbb{C}}/K^{\mathbb{C}}$ and come in a finite number of orbit types. We refer to them as *principal G-orbits*. They have real codimension equal to the rank of G/K. *Singular orbits* are closed G-orbits that are not principal.

The G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ is closely related to the $G \times K^{\mathbb{C}}$-orbit structure of $G^{\mathbb{C}}$. Then, slices for the closed G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be obtained by applying Matsuki's results [1997, Section 4] on double coset decompositions of groups arising from two involutions.

Let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to K, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Following Matsuki, we denote by $\mathcal{A} := \exp i\mathfrak{a}K^{\mathbb{C}}$ the image of the compact torus $\exp i\mathfrak{a}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$. The set $\mathcal{A}$ is a slice for all closed G-orbits intersecting the compact dual symmetric space U/K in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

It is called the *fundamental Cartan subset*. The remaining slices for closed G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ are of the form $\mathscr{C} := \exp i\mathfrak{c} \cdot z$, where $\mathfrak{c}$ is an abelian semisimple subalgebra of $\mathfrak{g}$ of the same dimension as $\mathfrak{a}$ and $z \in \mathscr{A}$ is a *base point* sitting on a singular closed G-orbit. Such sets $\mathscr{C}$ are called *standard Cartan subsets*.

By [Geatti 2006], every standard Cartan subset $\mathscr{C}$ admits a base point z with the following properties:

- there exists a subgroup $G' \subseteq G$ (possibly G' is equal to G) such that the isotropy subgroup of z in G' coincides with the isotropy subgroup G_z of z in G;
- the quotient G'/G_z is a pseudo-Riemannian symmetric space of the same rank as G/K;
- the slice representation of G_z at z is equivalent to the isotropy representation of G'/G_z.

More precisely, let $\mathfrak{g}' = \mathfrak{g}_z \oplus \mathfrak{q}'$ be the decomposition of the Lie algebra of G' corresponding to the symmetric space G'/G_z (when $G' = G$, $\mathfrak{g} = \mathfrak{g}_z \oplus \mathfrak{q}$). Denote by $T(G \cdot z)_z$ the tangent space of the orbit $G \cdot z$ at z and by N_z a complementary subspace of $T(G \cdot z)_z$ in $T(G^{\mathbb{C}}/K^{\mathbb{C}})_z$. Then $N_z \cong \mathfrak{q}'$ and the slice representation at z is equivalent to the Adjoint representation of G_z on $\mathfrak{q}'$. Moreover, both $\mathfrak{a}$ and $\mathfrak{c}$ are maximal abelian subalgebras in $\mathfrak{q}'$.

Consider the twisted bundle $G \times_{G_z} \mathfrak{q}'$ defined as the orbit space of $G \times \mathfrak{q}'$ under the action of G_z given by $h \cdot (g, X) := (gh^{-1}, \mathrm{Ad}_h X)$. The group G acts on $G \times_{G_z} \mathfrak{q}'$ by $\hat{g} \cdot [g, X] := [\hat{g}g, X]$. By Luna's slice theorem [1975, Proposition 1.2], there exists an open Ad_{G_z}-invariant neighborhood V of 0 in $\mathfrak{q}'$ such that the map

$$G \times_{G_z} V \to G^{\mathbb{C}}/K^{\mathbb{C}}, \qquad [g, X] \mapsto g \exp iX \cdot z \tag{4-1}$$

is a G-equivariant diffeomorphism onto an open G-invariant neighborhood of z in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Nonclosed G-orbits in $G \times_{G_z} V$ correspond to nonclosed Ad_{G_z}-orbits in V. The standard Cartan subset $\mathscr{C}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ is the image of the set $\{e\} \times \mathfrak{c}$ via the above map.

Let us now assume that G/K has rank one. Then the G-orbit space of $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be completely determined. Let $\Delta_{\mathfrak{a}}$ be the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$, and let

$$\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{a}) \bigoplus_{\alpha \in \Delta_{\mathfrak{a}}} \mathfrak{g}^{\alpha}, \quad \text{with } Z_{\mathfrak{g}}(\mathfrak{a}) = Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a},$$

be the corresponding root decomposition. Here $Z_{\mathfrak{g}}(\mathfrak{a})$ and $Z_{\mathfrak{k}}(\mathfrak{a})$ denote the centralizers of $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Let Γ be the lattice in $\mathfrak{a}$ given by the kernel of the map $\mathfrak{a} \to U/K$ defined by $X \mapsto \exp(iX)K$. Since the symmetric space U/K

is simply connected (see Remark 4.1), the lattice Γ is given by

$$\Gamma = \bigoplus_{\alpha \in \Delta_{\mathfrak{a}}} \mathbb{Z} i\pi h_\alpha,$$

where $h_\alpha \in \mathfrak{a}$ is uniquely determined by $\alpha(h_\alpha) = 2$; see [Helgason 1978, Theorem 8.5, page 322]. Denote by $W_K(\mathfrak{a})$ the Weyl group of $\mathfrak{a}$, and let the semidirect product $W_K(\mathfrak{a}) \ltimes \Gamma$ act on $\mathfrak{a}$ by $(k, \gamma) \cdot A := \mathrm{Ad}_k\, A + \gamma$. Denote by $\mathfrak{a}_0$ a fundamental domain for this action, and define $\mathscr{A}_0 := \exp i\mathfrak{a}_0 K^{\mathbb{C}}$. Then every closed G-orbit through the fundamental Cartan subset $\mathscr{A}$ intersects $\mathscr{A}_0$ in a single point; see [Matsuki 1997, Theorem 3].

Let $z \in \mathscr{A}_0$ be a base point for a standard Cartan subset $\mathscr{C}$. By [Geatti 2006] and by the local linearization (4-1), the G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ in a neighborhood of z is modeled on the orbit structure of the tangent space of a rank-one, pseudo-Riemannian symmetric space under the isotropy representation. It can be described as follows.

Remark 4.2. Let G/H a rank-one, pseudo-Riemannian symmetric space. Assume that the group H is connected. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the corresponding Lie algebra decomposition and $\mathfrak{q} \cap \mathfrak{k} \oplus \mathfrak{q} \cap \mathfrak{p}$ the Cartan decomposition of $\mathfrak{q}$. The isotropy representation of G/H is equivalent to the Adjoint representation of H on $\mathfrak{q}$. Denote by B both the Killing form of $\mathfrak{g}$ and its restriction to $\mathfrak{q} \setminus \{0\}$. The signature of B on $\mathfrak{q}$ is given by (s^+, s^-), with

$$s^+ := \dim(\mathfrak{q} \cap \mathfrak{p}) \quad \text{and} \quad s^- := \dim(\mathfrak{q} \cap \mathfrak{k}).$$

For $r \in \mathbb{R}$, denote by B_r the level hypersurface $\{B = r\}$ in $\mathfrak{q} \setminus \{0\}$. In diagonalized form one has $B_r = \{x_1^2 + \cdots + x_{s^+}^2 - y_1^2 - \cdots - y_{s^-}^2 = r\}$. Since G/K has rank one, every Ad_H-orbit in $\mathfrak{q} \setminus \{0\}$ is a hypersurface. Thus, by the connectedness of H and the Ad_H-invariance of B, such an orbit coincides with a connected component of some B_r. We distinguish four cases.

(a) Assume $s^+ = s^- = 1$. For every $r \neq 0$, the level set B_r consists of two connected components. They intersect either $\mathfrak{a} = \mathfrak{q} \cap \mathfrak{p}$ or $\mathfrak{c} = \mathfrak{q} \cap \mathfrak{k}$ in opposite points, depending on whether $r > 0$ or $r < 0$. The nilcone B_0 consists of four nonclosed Ad_H-orbits.

(b) Assume $s^+ > 1$ and $s^- = 1$. For $r > 0$, the level set B_r consists of a single component intersecting $\mathfrak{q} \cap \mathfrak{p}$ in a sphere. Thus, for every nonzero vector $A \in \mathfrak{q} \cap \mathfrak{p}$ and every $t > 0$, the points tA and $-tA$ belong to the same Ad_H-orbit. If $r < 0$ the level set B_r consists of two connected components, which intersect $\mathfrak{c} = \mathfrak{q} \cap \mathfrak{k}$ in opposite points. The nilcone B_0 consists of two nonclosed Ad_H-orbits.

(c) Assume $s^+ = 1$ and $s^- > 1$. If $r > 0$, the level set B_r consists of two connected components, which intersect $\mathfrak{a} = \mathfrak{q} \cap \mathfrak{p}$ in opposite points. If $r < 0$, the level set B_r intersects $\mathfrak{q} \cap \mathfrak{k}$ in a sphere. Thus for every nonzero vector $C \in \mathfrak{q} \cap \mathfrak{k}$ and every $s > 0$, the points sC and $-sC$ belong to the same Ad_H-orbit. The nilcone B_0 consists of two nonclosed Ad_H-orbits.

(d) Assume $s^+ > 1$ and $s^- > 1$. For every $r \neq 0$ the level set B_r consists of a single connected component. It intersects either $\mathfrak{q} \cap \mathfrak{p}$ or $\mathfrak{q} \cap \mathfrak{k}$ in a sphere, depending on whether $r > 0$ or $r < 0$. Thus for every nonzero vector $A \in \mathfrak{q} \cap \mathfrak{p}$ and every $t > 0$, the points tA and $-tA$ belong to the same Ad_H-orbit. A similar statement holds true for points sC and $-sC$, with C a nonzero vector in $\mathfrak{q} \cap \mathfrak{k}$ and $s > 0$. The nilcone B_0 consists of a unique nonclosed Ad_H-orbit.

In order to give further details, we recall the classification of rank-one, Riemannian symmetric spaces of the noncompact type. For each space M, we list its real dimension, its standard presentation G/K, and the dimensions of the restricted roots spaces of $\mathfrak{g}$; see [Wolf 1984, page 294] and [Helgason 1978, page 532].

M	$\dim M$	G/K		$\dim \mathfrak{g}^{\alpha}$	$\dim \mathfrak{g}^{2\alpha}$
$H^n(\mathbb{R})$	n	$\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$,	$n \geq 2$	$n-1$	0
$H^n(\mathbb{C})$	$2n$	$\mathrm{SU}(n,1)/\mathrm{U}(n)$,	$n \geq 2$	$2(n-1)$	1
$H^n(\mathbb{H})$	$4n$	$\mathrm{Sp}(n,1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$,	$n \geq 2$	$4(n-1)$	3
$H^2(\mathbb{C}ay)$	16	$F_4^*/\mathrm{Spin}(9)$		8	7

Table 4.0

Remark. The two-dimensional symmetric space $\mathrm{SO}_0(2,1)/\mathrm{SO}(2)$ can alternately be identified with $\mathrm{SU}(1,1)/\mathrm{U}(1)$ or $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, and the symmetric space $\mathrm{SO}_0(3,1)/\mathrm{SO}(3)$ can be identified with $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$.

4.1. ***The reduced case.*** Assume that the restricted root system of $\mathfrak{g}$ is *reduced*, that is, it consists of two roots $\{\pm\alpha\}$. This is the case of the spaces $H^n(\mathbb{R})$ in Table 4.0. A fundamental domain for the action of $W_K(\mathfrak{a}) \ltimes \Gamma$ on $\mathfrak{a}$ is given by $\mathfrak{a}_0 = \{A \in \mathfrak{a} : 0 \leq \alpha(A) \leq \pi\}$, and there are three singular orbits intersecting $\mathcal{A}_0 := \exp i\mathfrak{a}_0 K^{\mathbb{C}}$. Their base points are given by $z_j = g_j K^{\mathbb{C}}$ for $j = 1, 2, 3$. Here $g_j = \exp iA_j$ and the elements $A_j \in \mathfrak{a}_0$ satisfy the conditions

$$\text{(4-2)} \qquad \alpha(A_1) = 0, \qquad \alpha(A_2) = \pi/2, \qquad \alpha(A_3) = \pi,$$

respectively. The G-orbits through z_1 and z_3 are diffeomorphic to the symmetric space G/K and are embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as totally real submanifolds of maximal dimension. Moreover, the G-orbit through z_2 is a rank-one, pseudo-Riemannian symmetric space G/H with involution $\tau_{z_2} = \mathrm{Ad}_{g_2} \circ \tau \circ \mathrm{Ad}_{g_2^{-1}}$. The space G/H is embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as a closed, totally real submanifold of maximal dimension; see [Geatti 2002, Lemma 2.11 and Remark 2.13]. A standard Cartan subset starting at z_2 is given by $\mathscr{C} = \exp i\mathfrak{c} \cdot z_2$, where $\mathfrak{c} = \mathbb{R}(X + \theta(X))$ and X is a nonzero vector in $\mathfrak{g}^{\alpha}$. In the next lemma we determine the G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ in a neighborhood of z_2. Fix a generator C of $\mathfrak{c}$.

Lemma 4.3. *Assume that the restricted root system of $\mathfrak{g}$ is reduced. Let $z_2 \in \mathscr{A}_0$ be the base point of the Cartan subset $\mathscr{C}$.*

(i) *If* $\dim G/K > 2$, *then the orbit $G \cdot z_2$ is simply connected. In particular*, *the isotropy subgroup H of z_2 in G is connected.*

(ii) *For every* $s > 0$, *the points* $\exp(isC) \cdot z_2$ *and* $\exp(-isC) \cdot z_2$ *lie on the same G-orbit in $G^{\mathbb{C}}/K^{\mathbb{C}}$ if and only if* $\dim \mathfrak{g}^{\alpha} > 1$.

(iii) *If* $\dim \mathfrak{g}^{\alpha} > 1$, *there are two nonclosed G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing $G \cdot z_2$ in their closure. If* $\dim \mathfrak{g}^{\alpha} = 1$, *such orbits are four.*

Proof. (i) Using the hyperquadric model (see Example 4.4), one can verify that the orbit of z_2 is diffeomorphic to $\mathrm{SO}_0(n,1)/\mathrm{SO}_0(n-1,1)$. In particular, it is topologically equivalent to a sphere of dimension $n-1$ and is simply connected for $n > 2$. In that case, the isotropy subgroup H is connected, since G is connected by assumption. When $n = 2$, the orbit G/H is not simply connected. The isotropy subgroup of z_2 is either connected (when $G = \mathrm{SO}_0(2,1)$) or its quotient by the ineffectivity subgroup is connected (when G is a nontrivial covering of $\mathrm{SO}_0(2,1)$).

As a result, (ii) and (iii) follow from Remark 4.2, provided that $\dim(\mathfrak{q} \cap \mathfrak{p}) = 1$ and $\dim(\mathfrak{q} \cap \mathfrak{k}) = \dim \mathfrak{g}^{\alpha}$. To show this, define $\mathfrak{g}[\alpha] := \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}$. Then $\mathfrak{g}[\alpha]$ is a θ-stable subspace of $\mathfrak{g}$ of dimension equal to $2\dim \mathfrak{g}^{\alpha}$. Let $\mathfrak{g}[\alpha] = \mathfrak{g}[\alpha]_{\mathfrak{k}} \oplus \mathfrak{g}[\alpha]_{\mathfrak{p}}$ be its Cartan decomposition. The components $\mathfrak{g}[\alpha]_{\mathfrak{k}}$ and $\mathfrak{g}[\alpha]_{\mathfrak{p}}$ are generated by vectors of the form

$$X + \theta(X) \quad \text{and} \quad X - \theta(X),$$

respectively, where X ranges through the elements of a basis of $\mathfrak{g}^{\alpha}$. In particular, $\dim \mathfrak{g}[\alpha]_{\mathfrak{k}} = \dim \mathfrak{g}[\alpha]_{\mathfrak{p}} = \dim \mathfrak{g}^{\alpha}$. Consider the decomposition $\mathfrak{g} = Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \mathfrak{g}[\alpha]$, and note that $\tau_{z_2} = \mathrm{Ad}_{g_2} \circ \tau \circ \mathrm{Ad}_{g_2^{-1}} = \mathrm{Ad}_{g_2^2} \circ \theta$. Since $\mathrm{Ad}_{\exp iA_2} = \exp(\mathrm{ad}(iA_2))$, one has $\tau_{z_2} = \mathrm{Id}$ on $Z_{\mathfrak{k}}(\mathfrak{a})$ and $\tau_{z_2} = -\mathrm{Id}$ on $\mathfrak{a}$. Since $\alpha(A_2) = \pi/2$, one has $\tau_{z_2} = -\theta$ on $\mathfrak{g}[\alpha]$. It follows that $\mathfrak{q} := \mathrm{Fix}(-\tau_{z_2}, \mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{g}[\alpha]_{\mathfrak{k}}$. In particular, $\dim(\mathfrak{q} \cap \mathfrak{p}) = \dim \mathfrak{a} = 1$ and $\dim(\mathfrak{q} \cap \mathfrak{k}) = \dim \mathfrak{g}[\alpha]_{\mathfrak{k}} = \dim \mathfrak{g}^{\alpha}$, as wished. □

From the above discussion and Table 4.0, it follows that in the reduced case the G-orbit space of $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be described by the following diagrams. For

$G/K = \mathrm{SO}_0(2,1)/\mathrm{SO}(2)$, the diagram is

$$
\begin{array}{ccccc}
 & & | & & \\
 & & \ell_2(I_2) & & \\
 & w_1 \bullet & | & \bullet\, w_2 & \\
\bullet & \text{———} & \bullet & \text{———} & \bullet \\
z_1 & \ell_1(I_1) & z_2 & \ell_3(I_3) & z_3 \\
 & w_4 \bullet & | & \bullet\, w_3 & \\
 & & \ell_4(I_4) & & \\
 & & | & &
\end{array}
\tag{4-3}
$$

For $G/K = \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ with $n > 2$, the diagram is

$$
\begin{array}{ccccc}
 & & | & & \\
 & & \ell_2(I_2) & & \\
 & w_1 \bullet & | & \bullet\, w_2 & \\
\bullet & \text{———} & \bullet & \text{———} & \bullet \\
z_1 & \ell_1(I_1) & z_2 & \ell_3(I_3) & z_3
\end{array}
\tag{4-4}
$$

Set $I_1 = I_3 = (0,1)$. For $j = 1, 3$, the maps $\ell_j : I_j \to G^{\mathbb{C}}/K^{\mathbb{C}}$, defined by

$$\ell_1(t) := \exp(-itA_2)\cdot z_2 \quad \text{and} \quad \ell_3(t) := \exp(itA_2)\cdot z_2, \tag{4-5}$$

parametrize the principal G-orbits through $\mathscr{A}_0$. One has

$$\mathscr{A}_0 \;=\; z_1 \cup \ell_1(I_1) \cup z_2 \cup \ell_3(I_3) \cup z_3.$$

Set $I_2 = I_4 = (0,\infty)$. For $j = 2, 4$, the maps $\ell_j : I_j \to G^{\mathbb{C}}/K^{\mathbb{C}}$, defined by

$$\ell_2(s) := \exp(isC)\cdot z_2 \quad \text{and} \quad \ell_4(s) := \exp(-isC)\cdot z_2, \tag{4-6}$$

parametrize the principal closed G-orbits through the standard Cartan subset $\mathscr{C}$ and $\mathscr{C} = \ell_2(I_2) \cup z_2 \cup \ell_4(I_4)$. The points w_1, w_2, w_3, and w_4 represent the nonclosed G-orbits containing the singular orbit $G \cdot z_2$ in their closure.

Example 4.4. *The complex hyperquadric.* Let $G = \mathrm{SO}_0(n,1)$, with $n \geq 2$, and let $G^{\mathbb{C}} = \mathrm{SO}(n,1,\mathbb{C})$ be its universal complexification. By definition $G^{\mathbb{C}}$ is the subgroup of $\mathrm{SL}(n+1,\mathbb{C})$ leaving invariant the quadratic form of signature $(n,1)$. The space $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be identified with the $G^{\mathbb{C}}$-orbit through $(0,\ldots,1)$ that coincides with the n-dimensional complex hyperquadric

$$M^{\mathbb{C}} = \{(\xi_1,\ldots,\xi_{n+1}) \in \mathbb{C}^{n+1} : \xi_1^2 + \cdots + \xi_n^2 - \xi_{n+1}^2 = -1\}.$$

Fix the elements

$$A_2 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \pi/2 \\ 0 & \dots & \pi/2 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -2 & 0 \\ 0 & \dots & 2 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

in $\mathfrak{g}$ as generators of $\mathfrak{a}$ and $\mathfrak{c}$, respectively. Then points on the singular orbits in $M^{\mathbb{C}}$ satisfying conditions (4-2) are given by

$$z_1 = (0, \dots, 0, 1), \qquad z_2 = (0, \dots, 0, i, 0), \qquad z_3 = (0, \dots, 0, -1).$$

The G-orbit of z_2 is diffeomorphic to the pseudo-Riemannian symmetric space $G/H \cong \mathrm{SO}_0(n, 1)/\mathrm{SO}_0(n-1, 1)$. For $t \in (0, 1)$, the slices ℓ_1 and ℓ_3 are given by

$$\ell_1(t) = (0, \dots, 0, i \sin(\pi(1-t)/2), \cos(\pi(1-t)/2)),$$
$$\ell_3(t) = (0, \dots, 0, i \sin(\pi(1+t)/2), \cos(\pi(1+t)/2)).$$

For $s > 0$, the slices ℓ_2 and ℓ_4 are given by

$$\ell_2(s) = (0, \dots, 0, \quad \sinh 2s, i \cosh 2s, 0),$$
$$\ell_4(s) = (0, \dots, 0, -\sinh 2s, i \cosh 2s, 0).$$

The slice representation at z_2 is equivalent to the linear action of $\mathrm{SO}_0(n-1, 1)$ on $\mathbb{R}^n$. When $n = 2$, we can choose representatives of the four nonclosed hypersurface G-orbits containing $G \cdot z_2$ in their closure to be

$$w_1 = (-1, i, -1), \quad w_2 = (1, i, -1), \quad w_3 = (1, i, 1), \quad w_4 = (-1, i, 1).$$

When $n > 2$, the slice representation identifies ℓ_2 and ℓ_4 and representatives of the two nonclosed hypersurface G-orbits containing $G \cdot z_2$ in their closure are for example

$$w_1 = (-1, 0, \dots, 0, i, -1) \quad \text{and} \quad w_2 = (1, 0, \dots, 0, i, -1).$$

4.2. ***The nonreduced case.*** Assume that the restricted root system of $\mathfrak{g}$ is nonreduced, that is, it consists of four roots $\{\pm\alpha, \ \pm 2\alpha\}$. This is the case of $H^n(\mathbb{C})$, $H^n(\mathbb{H})$ and $H^2(\mathbb{C}ay)$ in Table 4.0. Then $\mathfrak{a}_0 = \{A \in \mathfrak{a} : 0 \leq \alpha(A) \leq \pi/2\}$ is a fundamental domain for the action of $W_K(\mathfrak{a}) \ltimes \Gamma$ in $\mathfrak{a}$, and there are three singular orbits intersecting $\mathscr{A}_0$. Their base points are given by $z_j = g_j K^{\mathbb{C}}$ for $j = 1, 2, 3$. Here $g_j = \exp i A_j$ and the elements $A_j \in \mathfrak{a}_0$ satisfy the conditions

$$\alpha(A_1) = 0, \qquad \alpha(A_2) = \pi/4, \qquad \alpha(A_3) = \pi/2, \tag{4-7}$$

respectively. The G-orbit through z_1 is diffeomorphic to the symmetric space G/K, and the one through z_3 is diffeomorphic to a rank-one, pseudo-Riemannian symmetric space G/H. Both orbits are embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as totally real submanifolds of maximal dimension; see [Geatti 2002, Lemma 2.11 and Remark 2.13]. The orbit of z_2 is a homogeneous space G/H', with $H' := G_{z_2}$ and $\dim G/H' > \dim G/K$; see [Geatti 2002, Lemma 2.14 and Remark 2.15]. Set $G' := Z_G(g_2^4)$, where $Z_G(g_2^4)$ denotes the centralizer of g_2^4 in G. Then H' is contained in G' and G'/H' is a rank-one, pseudo-Riemannian symmetric space with involution $\tau_{z_2} = \mathrm{Ad}_{g_2} \circ \tau \circ \mathrm{Ad}_{g_2^{-1}}$. Moreover, the slice representation at z_2 is equivalent to the isotropy representation of G'/H'; see [Geatti 2006]. The standard Cartan subset starting at z_2 is given by $\mathscr{C}' = \exp i\,\mathfrak{c}' \cdot z_2$, where $\mathfrak{c}' = \mathbb{R}(X + \theta(X))$ and X is a nonzero vector in $\mathfrak{g}^{2\alpha}$. If $Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \mathfrak{g}^{\pm\alpha} \oplus \mathfrak{g}^{\pm 2\alpha}$ is the restricted root decomposition of $\mathfrak{g}$, then the Lie algebra of G' is given by

$$\mathfrak{g}' = Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \mathfrak{g}^{\pm 2\alpha}. \tag{4-8}$$

Moreover, if $\mathfrak{h}' \oplus \mathfrak{q}'$ is the τ_{z_2}-decomposition of $\mathfrak{g}'$, then $\mathfrak{c}'$ is a maximal abelian subalgebra in $\mathfrak{q}'$. Fix a generator C' of $\mathfrak{c}'$.

Lemma 4.5. *Assume that the restricted root system of $\mathfrak{g}$ is nonreduced. Let $z_2 \in \mathscr{A}_0$ be the base point of the Cartan subset $\mathscr{C}'$.*

(i) *The isotropy subgroup H' of z_2 in G is connected.*

(ii) *For every $t > 0$, the points $\exp(itC') \cdot z_2$ and $\exp(-itC') \cdot z_2$ sit on the same G-orbit if and only if $\dim \mathfrak{g}^{2\alpha} > 1$.*

(iii) *If $\dim \mathfrak{g}^{2\alpha} > 1$, there are two nonclosed G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing $G \cdot z_2$ in their closure. If $\dim \mathfrak{g}^{2\alpha} = 1$, such orbits are four.*

Proof. (i) The group H' is connected if and only if $H' \cap K$ is connected. Note that $G' = Z_G(g_2^4)$ is θ-stable, since so is G and $\theta(g_2^4) = g_2^{-4}$. Therefore $H' \cap K$ is the common fixed point subgroup of the two involutions τ_{z_2} and θ of G'. As a result, $H' \cap K = Z_K(g_2^2)$. Now regard z_2 as a point on the compact dual symmetric space U/K endowed with the K-action by left translations. Denote by K_{z_2} the isotropy subgroup of z_2 in K. On the one hand, $K_{z_2} = Z_K(g_2^2)$. On the other hand, since the isotropy subalgebra $\mathfrak{k}_{z_2}$ is given by $\mathfrak{k} \cap \mathrm{Ad}_{z_2}(\mathfrak{k})$, one sees that $\mathfrak{k}_{z_2}$ has minimal dimension and coincides with $Z_{\mathfrak{k}}(\mathfrak{a})$ if and only if $\alpha(A_2) \neq m\pi$ for $m \in \mathbb{Z}$. By (4-7), it follows that K_{z_2} is principal and consequently is equal to $Z_K(\mathfrak{a})$. Finally $Z_K(\mathfrak{a})$ is connected for all rank-one, Riemannian symmetric spaces of dimension greater than two; see [Knapp 1996] or Lemma 5.1 for a direct proof. In conclusion, $H' \cap K = Z_K(g_2^2) = K_{z_2} = Z_K(\mathfrak{a})$, which implies (i).

Parts (ii) and (iii) follow by applying Remark 4.2 to the symmetric space G'/H', provided that $\dim \mathfrak{q}' \cap \mathfrak{p} = 1$ and $\dim \mathfrak{q}' \cap \mathfrak{k} = \dim \mathfrak{g}^{2\alpha}$. In order to show this, define $\mathfrak{g}[2\alpha] := \mathfrak{g}^{2\alpha} \oplus \mathfrak{g}^{-2\alpha}$. Then $\mathfrak{g}[2\alpha]$ is θ-stable subspace of $\mathfrak{g}$ of dimension equal to $2\dim \mathfrak{g}^{2\alpha}$. Let $\mathfrak{g}[2\alpha] = \mathfrak{g}[2\alpha]_{\mathfrak{k}} \oplus \mathfrak{g}[2\alpha]_{\mathfrak{p}}$ be its Cartan decomposition. The components $\mathfrak{g}[2\alpha]_{\mathfrak{k}}$ and $\mathfrak{g}[2\alpha]_{\mathfrak{p}}$ are generated by vectors of the form $X+\theta(X)$ and $X-\theta(X)$, respectively, where X ranges through the elements of a basis of $\mathfrak{g}^{2\alpha}$. In particular $\dim \mathfrak{g}[2\alpha]_{\mathfrak{k}} = \dim \mathfrak{g}[2\alpha]_{\mathfrak{p}} = \dim \mathfrak{g}^{2\alpha}$. One sees that

$$\tau_{z_2} = \mathrm{Id} \quad \text{on } Z_{\mathfrak{k}}(\mathfrak{a}), \qquad \tau_{z_2} = -\mathrm{Id} \quad \text{on } \mathfrak{a}, \qquad \tau_{z_2} = -\theta \quad \text{on } \mathfrak{g}[2\alpha].$$

Consequently $\mathfrak{q}' := \mathrm{Fix}(-\tau_{z_2}, \mathfrak{g}') = \mathfrak{a} \oplus \mathfrak{g}[2\alpha]_{\mathfrak{k}}$, and $\dim(\mathfrak{q}' \cap \mathfrak{p}) = \dim \mathfrak{a} = 1$. Similarly, $\dim(\mathfrak{q}' \cap \mathfrak{k}) = \dim \mathfrak{g}[2\alpha]_{\mathfrak{k}} = \dim \mathfrak{g}^{2\alpha}$, as wished. $\square$

By [Geatti 2002, Lemma 2.11 and Remark 2.13], the G-orbit of z_3 is a rank-one, pseudo-Riemannian symmetric space G/H with involution $\tau_{z_3} = \mathrm{Ad}_{g_3} \circ \tau \circ \mathrm{Ad}_{g_3^{-1}}$. The space G/H is embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as a closed, totally real submanifold of maximal dimension. The standard Cartan subset that starts at z_3 is given by $\mathscr{C} = \exp i\mathfrak{c} \cdot z_3$, where $\mathfrak{c} = \mathbb{R}(X+\theta(X))$ and X is a nonzero vector in $\mathfrak{g}^{\alpha}$. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the τ_{z_3}-decomposition of $\mathfrak{g}$, then $\mathfrak{c}$ is a maximal abelian subalgebra in $\mathfrak{q}$. Fix a generator C of $\mathfrak{c}$.

Lemma 4.6. *Assume that the restricted root system of $\mathfrak{g}$ is nonreduced. Let $z_3 \in \mathscr{A}_0$ be the base point of the Cartan subset $\mathscr{C}$.*

(i) *The orbit $G \cdot z_3$ is simply connected. In particular the isotropy subgroup H of z_3 in G is connected.*

(ii) *For every $t > 0$, the points $\exp(itC) \cdot z_3$ and $\exp(-itC) \cdot z_3$ sit on the same G-orbit in $G^{\mathbb{C}}/K^{\mathbb{C}}$.*

(iii) *There is precisely one nonclosed G-orbit in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing $G \cdot z_3$ in its closure.*

Proof. (i) Since by assumption G is connected, we prove that H is connected by showing that the orbit $G \cdot z_3$ is simply connected. To do this, Remark 4.1 says it suffices to choose G as in the standard presentation in Table 4.0. Let $G = \mathrm{SU}(n, 1)$. By direct computations (see Example 4.7) one finds $G \cdot z_3 \cong \mathrm{SU}(n, 1)/\mathrm{U}(n-1, 1)$. This quotient is topologically equivalent to the complex projective space $\mathbb{CP}^{n-1}$. In particular, it is simply connected.

Consider then $G = \mathrm{Sp}(n, 1)$ or $G = F_4^*$. In both cases the group G is simply connected. Since H is the fixed point subgroup of an involution of G, it is connected [Steinberg 1968]. It follows that the quotient is simply connected.

Parts (ii) and (iii) follow from Remark 4.2, provided that $\dim(\mathfrak{q} \cap \mathfrak{p}) = 1 + \mathfrak{g}^{2\alpha}$ and $\dim(\mathfrak{q} \cap \mathfrak{k}) = \dim \mathfrak{g}^{\alpha}$. In order to show this, define $\mathfrak{g}[\alpha] := \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}$ and $\mathfrak{g}[2\alpha] := \mathfrak{g}^{2\alpha} \oplus \mathfrak{g}^{-2\alpha}$. Then both $\mathfrak{g}[\alpha]$ and $\mathfrak{g}[2\alpha]$ are θ-stable subspaces of $\mathfrak{g}$ of dimension equal to $\dim \mathfrak{g}^{\alpha}$ and $2\dim \mathfrak{g}^{2\alpha}$, respectively. Let $\mathfrak{g}[\alpha]_{\mathfrak{k}}$, $\mathfrak{g}[\alpha]_{\mathfrak{p}}$, $\mathfrak{g}[2\alpha]_{\mathfrak{k}}$, and $\mathfrak{g}[2\alpha]_{\mathfrak{p}}$ be the components of the respective Cartan decompositions. The same arguments as in the proof of Lemmas 4.3 and 4.5 show that

$$\dim \mathfrak{g}[\alpha]_{\mathfrak{k}} = \dim \mathfrak{g}[\alpha]_{\mathfrak{p}} = \dim \mathfrak{g}^{\alpha} \quad \text{and} \quad \dim \mathfrak{g}[2\alpha]_{\mathfrak{k}} = \dim \mathfrak{g}[2\alpha]_{\mathfrak{p}} = \dim \mathfrak{g}^{2\alpha}.$$

Moreover, one sees that

$$\tau_{z_3} = \mathrm{Id} \text{ on } Z_{\mathfrak{k}}(\mathfrak{a}), \quad \tau_{z_3} = -\mathrm{Id} \text{ on } \mathfrak{a}, \quad \tau_{z_3} = -\theta \text{ on } \mathfrak{g}[\alpha], \quad \tau_{z_3} = \theta \text{ on } \mathfrak{g}[2\alpha].$$

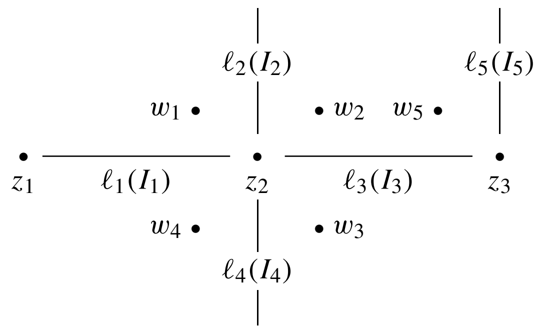
Since

$$\mathfrak{g} = Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \mathfrak{g}[\alpha] \oplus \mathfrak{g}[2\alpha],$$

it follows that $\mathfrak{q} := \mathrm{Fix}(-\tau_{z_3}, \mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{g}[\alpha]_{\mathfrak{k}} \oplus \mathfrak{g}[2\alpha]_{\mathfrak{p}}$. In particular, $\dim(\mathfrak{q} \cap \mathfrak{p}) = 1 + \dim \mathfrak{g}^{2\alpha}$ and $\dim(\mathfrak{q} \cap \mathfrak{k}) = \dim \mathfrak{g}^{\alpha}$, as claimed. □

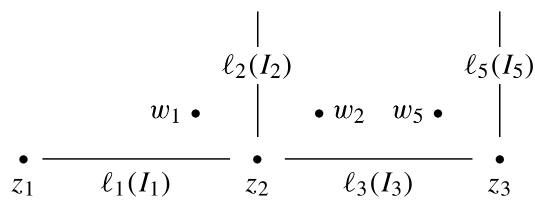
As a consequence of the above lemmas and Table 4.0, in the nonreduced case the G-orbit space of $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be represented by the following diagrams. For $G/K = \mathrm{SU}(n, 1)/\mathrm{U}(n)$ with $n \geq 2$, the diagram is

(4-9)



If $G/K = \mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ for $n \geq 2$, or if $G/K = F_4^*/\mathrm{Spin}(9)$, then the diagram is

(4-10)



Set $I_1 = I_3 = (0, 1)$. For $j = 1, 3$, define $\ell_j : I_j \to G^{\mathbb{C}}/K^{\mathbb{C}}$ by

$$\ell_1(t) = \exp(-itA_2) \cdot z_2 \quad \text{and} \quad \ell_3(t) = \exp(itA_2) \cdot z_2. \tag{4-11}$$

The slices ℓ_1 and ℓ_3 parametrize the principal G-orbits through $\mathscr{A}_0$ and

$$\mathscr{A}_0 = z_1 \cup \ell_1(I_1) \cup z_2 \cup \ell_3(I_3) \cup z_3.$$

Set $I_2 = I_4 = (0, \infty)$. For $j = 2, 4$, define $\ell_j : I_j \to G^{\mathbb{C}}/K^{\mathbb{C}}$ by

$$\ell_2(s) = \exp(sC') \cdot z_2 \quad \text{and} \quad \ell_4(s) = \exp(-sC') \cdot z_2. \tag{4-12}$$

The slices ℓ_2 and ℓ_4 parametrize the principal G-orbits through the Cartan subset $\mathscr{C}'$ with base point z_2 and $\mathscr{C}' = \ell_2(I_2) \cup z_2 \cup \ell_4(I_4)$. Finally, set $I_5 = (0, \infty)$, and define $\ell_5 : I_5 \to G^{\mathbb{C}}/K^{\mathbb{C}}$ by

$$\ell_5(s) = \exp(sC) \cdot z_3. \tag{4-13}$$

The slice ℓ_5 parametrizes the principal G-orbits through the standard Cartan subset $\mathscr{C}$ with base point z_3. The points $w_1, \ldots, w_4$ represent the nonclosed orbits containing $G \cdot z_2$ in their closure. The point w_5 represents the nonclosed orbit containing $G \cdot z_3$ in its closure.

Example 4.7. *A model in the nonreduced case.* Let $G = \mathrm{SU}(n, 1)$, with $n \geq 2$, be the subgroup of $\mathrm{SL}(n+1, \mathbb{C})$ leaving invariant the hermitian form $\langle z, w\rangle_{n,1} = z_1\overline{w}_1 + \ldots + z_n\overline{w}_n - z_{n+1}\overline{w}_{n+1}$ in $\mathbb{C}^{n+1}$. Denote by σ the conjugation of $G^{\mathbb{C}} = \mathrm{SL}(n+1, \mathbb{C})$ relative to G, namely $\sigma(g) = I_{n,1}{}^t\overline{g}^{-1} I_{n,1}$. Denote by $\overline{\mathbb{P}}^n$ the complex projective space endowed with the opposite complex structure, that is, the one for which the map $\mathbb{P}^n \to \overline{\mathbb{P}}^n$, $[z] \mapsto [\bar{z}]$ is holomorphic. The group $G^{\mathbb{C}}$ acts holomorphically on $\mathbb{P}^n \times \overline{\mathbb{P}}^n$ by $g \cdot ([z], [w]) := ([g \cdot z], [\sigma(g) \cdot w])$.

Under this action, $\mathbb{P}^n \times \overline{\mathbb{P}}^n$ consists of two orbits: a closed one given by

$$\{([z], [w]) \in \mathbb{P}^n \times \overline{\mathbb{P}}^n : \langle z, w\rangle_{n,1} = 0\}$$

and an open one given by its complement. The quotient $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be identified with the open orbit

$$M^{\mathbb{C}} := G^{\mathbb{C}} \cdot ([0 : \cdots : 0 : 1], [0 : \cdots : 0 : 1]) = \mathbb{P}^n \times \overline{\mathbb{P}}^n \setminus \{\langle z, w\rangle_{n,1} = 0\}.$$

Fix the elements

$$A_2 = \begin{pmatrix} 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \pi/4 \\ 0 & \ldots & \pi/4 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & i & 0 \\ 0 & \ldots & 0 & -i \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & -1 & 0 \\ 0 & \ldots & 1 & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 \end{pmatrix}$$

in $\mathfrak{g}$ as generators of $\mathfrak{a}$, $\mathfrak{c}'$ and $\mathfrak{c}$, respectively. Then points on the singular orbits in $M^{\mathbb{C}}$ satisfying conditions (4-7) are given by

$$\begin{aligned} z_1 &= ([0 : \ldots : 0 : 1], [0 : \ldots : 0 : 1]), \\ z_2 &= ([0 : \ldots : 0 : i : 1], [0 : \ldots : 0 : -i : 1]), \\ z_3 &= ([0 : \ldots : 0 : 1 : 0], [0 : \ldots : 0 : 1 : 0]). \end{aligned}$$

The G-orbit of z_2 is diffeomorphic to the homogeneous space G/H', where $H' \cong \mathrm{U}(n-1) \times \mathrm{SO}(1,1)$. The group G' is isomorphic to $\mathrm{U}(n-1) \times \mathrm{SU}(1,1)$, and the quotient G'/H' is diffeomorphic to the two-dimensional rank-one, pseudo-Riemannian symmetric space $\mathrm{SU}(1,1)/\mathrm{SO}(1,1)$. The G-orbit of z_3 is diffeomorphic to the pseudo-Riemannian symmetric space $\mathrm{SU}(n,1)/\mathrm{SU}(n-1,1)$. The slices ℓ_1 and ℓ_3 are given by

$$\ell_1(t) = ([0:\ldots:i \sin\tfrac{\pi}{4}(1-t):\cos\tfrac{\pi}{4}(1-t)], [0:\ldots:-i \sin\tfrac{\pi}{4}(1-t):\cos\tfrac{\pi}{4}(1-t)]),$$
$$\ell_3(t) = ([0:\ldots:i \sin\tfrac{\pi}{4}(1+t):\cos\tfrac{\pi}{4}(1+t)], [0:\ldots:-i \sin\tfrac{\pi}{4}(1+t):\cos\tfrac{\pi}{4}(1+t)]),$$

where $t \in (0,1)$. The slices ℓ_2, ℓ_4 and ℓ_5 are given by

$$\begin{aligned}
\ell_2(s) &= ([0:\ldots:ie^{-s}:e^s], [0:\ldots:-ie^s:e^{-s}]),\\
\ell_4(s) &= ([0:\ldots:ie^{s}:e^{-s}], [0:\ldots:-ie^{-s}:e^{s}]),\\
\ell_5(s) &= ([0:\ldots:\sinh s:i\cosh s:0], [0:\ldots:\sinh s:-i\cosh s:0]),
\end{aligned}$$

with $s > 0$. The slice representation at z_2 is equivalent to the standard action of $\mathrm{SO}(1,1)$ on $\mathbb{R}^2$. So there are four nonclosed G-orbits containing $G \cdot z_2$ in their closure. We can choose representatives of such orbits to be

$$\begin{aligned}
w_1 &= ([0:\ldots:0:1], [0:\ldots:-i:1]), & w_2 &= ([0:\ldots:i:1], [0:\ldots:1:0]),\\
w_3 &= ([0:\ldots:1:0], [0:\ldots:-i:1]), & w_4 &= ([0:\ldots:i:1], [0:\ldots:0:1]).
\end{aligned}$$

A representative for the unique nonclosed orbit containing $G \cdot z_3$ in its closure is given by $w_5 = ([0:\ldots:1:-i:1], [0:\ldots:1:i:1])$.

Remark 4.8. When $G = \mathrm{SU}(1,1)$, the restricted root system of $\mathfrak{g}$ is reduced. The quotient $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be identified with $\mathbb{P}^1 \times \overline{\mathbb{P}}^1 \setminus \{\langle z, w\rangle_{1,1} = 0\}$, and the G-orbit space can be described as above, except for the fact that the slice ℓ_5 and the point w_5 must be omitted. Moreover the G-orbit through z_3 is diffeomorphic to the symmetric space G/K. Note that $\mathrm{SU}(1,1)^{\mathbb{C}}/\mathrm{U}(1)^{\mathbb{C}}$ is biholomorphic to $\mathrm{SO}_0(2,1)^{\mathbb{C}}/\mathrm{SO}(2)^{\mathbb{C}}$. Thus it can also be identified with the two-dimensional hyperquadric described in Example 4.4.

5. Univalence on G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$

Let G be a connected, noncompact, real simple Lie group, let $K \subset G$ be a maximal compact subgroup, and let $G^{\mathbb{C}}$ be the universal complexification of G. Assume that G is embedded in $G^{\mathbb{C}}$. Consider a G-equivariant Riemann domain

$$q : \Sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}.$$

The main goal of this section is to prove that q is injective on G-orbits if G/K is a rank-one, Riemannian symmetric space of dimension greater than three. We first prove the result for principal G-orbits, and later we extend it to all G-orbits by a general argument. In most cases, the injectivity of q on principal G-orbits follows from their simple connectedness. The cases $\dim G/K = 2, 3$ are discussed separately.

Recall that by Remark 4.1(ii), different triples $(G, K, G^{\mathbb{C}})$ associated with the same Riemannian symmetric space G/K yield G-equivariantly diffeomorphic orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Let $\mathscr{A}_0$, $\mathscr{C}'$ and $\mathscr{C}$ be the standard Cartan subsets in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Let H be the isotropy subgroup of the base point of $\mathscr{C}$, and let H' be the isotropy subgroup of the base point of $\mathscr{C}'$; see Lemmas 4.3, 4.5 and 4.6. By [Geatti 2002, Propositions 3.4 and 3.15], the principal orbits intersecting $\mathscr{A}_0$, $\mathscr{C}$ and $\mathscr{C}'$ have isotropy type $Z_K(\mathfrak{a})$, $Z_H(\mathfrak{c})$ and $Z_{H'}(\mathfrak{c}')$, respectively.

Lemma 5.1. *Principal G-orbits of isotropy type $Z_K(\mathfrak{a})$ are simply connected if and only if* $\dim G/K > 2$.

Proof. An orbit $G/Z_K(\mathfrak{a})$ is topologically equivalent to $K/Z_K(\mathfrak{a})$. Consider the isotropy representation of K on $\mathfrak{p}$. The nonzero K-orbits in $\mathfrak{p}$ are diffeomorphic to $K/Z_K(\mathfrak{a})$. Since G/K has rank one, they are also diffeomorphic to spheres of dimension $\dim(G/K) - 1$. Hence the statement follows. □

Remark 5.2. When $G = \mathrm{SO}_0(2, 1)$, the isotropy subgroup $Z_K(\mathfrak{a})$ is trivial. Therefore principal orbits of type $G/Z_K(\mathfrak{a})$ are diffeomorphic to $\mathrm{SO}_0(2, 1)$ and topologically equivalent to $\mathrm{SO}(2)$. In particular, they are not simply connected.

Lemma 5.3. *Principal G-orbits of isotropy type $Z_H(\mathfrak{c})$ are simply connected, except when G is one of the groups* $\mathrm{SO}_0(2, 1)$, $\mathrm{SO}_0(3, 1)$ *or* $\mathrm{SU}(2, 1)$.

Proof. An orbit $G/Z_H(\mathfrak{c})$ is topologically equivalent to $K/Z_{K\cap H}(\mathfrak{c})$. We prove the lemma by discussing each case separately. Let $G = \mathrm{SO}_0(n, 1)$. Using the hyperquadric model given in Example 4.4, one checks that

$$H \cong \mathrm{SO}_0(n-1, 1), \quad Z_H(\mathfrak{c}) \cong \mathrm{SO}_0(n-2, 1), \quad K/Z_{H\cap K}(\mathfrak{c}) \cong \mathrm{SO}(n)/\mathrm{SO}(n-2).$$

In particular, $K/Z_{K\cap H}(\mathfrak{c})$ is diffeomorphic to a Stiefel manifold, which is simply connected for $n > 3$.

Consider next the case $G = \mathrm{SU}(n, 1)$, with $n \geq 3$. Direct computations on the model in Example 4.7 show that

$$H \cong \mathrm{U}(n-1, 1),$$
$$Z_{K\cap H}(\mathfrak{c}) \cong \mathrm{U}(n-2) \times \mathrm{U}(1),$$
$$K/Z_{K\cap H}(\mathfrak{c}) \cong \mathrm{U}(n)/(\mathrm{U}(n-2) \times \mathrm{U}(1)).$$

Since, for $n \geq 3$, the embedding $\mathrm{U}(n-2) \to \mathrm{U}(n)$ induces an epimorphism of fundamental groups, so does the embedding $\mathrm{U}(n-2) \times \mathrm{U}(1) \to \mathrm{U}(n)$. As a consequence, $K/Z_{K\cap H}(\mathfrak{c})$ is simply connected.

Finally, consider $G = \mathrm{Sp}(n, 1)$ or $G = F_4^*$. Note that in both cases K is simply connected. Therefore $K/Z_{K\cap H}(\mathfrak{c})$ is simply connected provided that $Z_{K\cap H}(\mathfrak{c})$ is connected. In order to show that this, consider the compact, rank-one, symmetric space $K/K\cap H$ and the corresponding isotropy representation of $K\cap H$ on $\mathfrak{k}\cap\mathfrak{q}$. The nonzero $K\cap H$-orbits in $\mathfrak{k}\cap\mathfrak{q}$ are of type $K\cap H/Z_{K\cap H}(\mathfrak{c})$ and are diffeomorphic to spheres of dimension $\dim(\mathfrak{k}\cap\mathfrak{q})-1$. Since $\dim(\mathfrak{k}\cap\mathfrak{q}) = \dim\mathfrak{g}^\alpha > 2$ (see Table 4.0), they are simply connected. By Lemma 4.3 or Lemma 4.6, the group H and likewise its maximal compact subgroup $K\cap H$ are connected. Then the exact homotopy sequence of the quotient $K\cap H/Z_{K\cap H}(\mathfrak{c})$, implies that the group $Z_{K\cap H}(\mathfrak{c})$ is connected, as wished. □

Remark 5.4. When $G = \mathrm{SO}_0(2, 1)$, direct computations using the model described in Example 4.4 show that the isotropy subgroup $Z_H(\mathfrak{c})$ is trivial. Therefore principal orbits of type $G/Z_H(\mathfrak{c})$ are diffeomorphic to $\mathrm{SO}_0(2, 1)$ and topologically equivalent to $\mathrm{SO}(2)$. In particular, they are not simply connected.

Similarly, when $G = \mathrm{SO}_0(3, 1)$ the isotropy subgroup $Z_H(\mathfrak{c})$ is isomorphic to $\mathrm{SO}_0(1, 1)$, which is connected. Therefore principal orbits of type

$$G/Z_H(\mathfrak{c}) \cong \mathrm{SO}_0(3, 1)/\mathrm{SO}_0(1, 1)$$

are topologically equivalent to $\mathrm{SO}(3)$ and are not simply connected.

When $G = \mathrm{SU}(2, 1)$, direct computations using the model described in Example 4.7 show that the isotropy subgroup $Z_{K\cap H}(\mathfrak{c})$ is isomorphic to $S(\mathrm{U}(1)\times\mathrm{U}(1))$, which is connected. Principal orbits of type $G/Z_H(\mathfrak{c})$ are topologically equivalent to $K/Z_{K\cap H}(\mathfrak{c}) \cong \mathrm{U}(2)/\mathrm{U}(1) \cong \mathrm{SO}(3)$. Hence they are not simply connected.

Note that in all the above cases, despite the fact that the orbits are not simply connected, the corresponding isotropy subgroups are connected.

Lemma 5.5. *All principal G-orbits of type $Z_{H'}(\mathfrak{c}')$ are simply connected.*

Proof. An orbit of type $G/Z_{H'}(\mathfrak{c}')$ is topologically equivalent to $K/Z_{H'\cap K}(\mathfrak{c}')$. We prove the latter quotient is simply connected by discussing each case separately.

Consider first $G = \mathrm{SU}(n, 1)$. Direct computations using the model constructed in Example 4.7 show that $Z_{H'\cap K}(\mathfrak{c}') \cong \mathrm{U}(n-1)$. Hence the quotient $K/Z_{H'\cap K}(\mathfrak{c}') \cong \mathrm{U}(n)/\mathrm{U}(n-1)$ is diffeomorphic to the sphere S^{2n-1}. In particular, it is simply connected for all $n \geq 2$.

Next let $G = \mathrm{Sp}(n, 1)$ or $G = F_4^*$. Both G and K are simply connected. So the quotient $K/Z_{H'\cap K}(\mathfrak{c}')$ is simply connected provided that $Z_{H'\cap K}(\mathfrak{c}')$ is connected. In order to show this, denote by K' the maximal compact subgroup of G'; see Section 4.2. Since H' is contained in G', the groups $H'\cap K$ and $H'\cap K'$ coincide and are both connected by Lemma 4.5. Consider the compact, rank-one, symmetric space $K'/(K'\cap H') \subset G'/H'$. The nonzero orbits of the isotropy representation of $K'\cap H'$ on $\mathfrak{k}'\cap\mathfrak{q}'$ are of type $K'\cap H'/Z_{K'\cap H'}(\mathfrak{c}')$ and are diffeomorphic to spheres of dimension equal to $\dim \mathfrak{g}^{2\alpha} - 1$. Since $\dim \mathfrak{g}^{2\alpha} > 2$ (see Table 4.0), they are simply connected. Since $H' \cap K'$ is connected, it follows from the exact homotopy sequence of the quotient $K' \cap H' / Z_{K' \cap H'}(\mathfrak{c}')$ that the groups $Z_{K' \cap H'}(\mathfrak{c}')$ and $Z_{K \cap H'}(\mathfrak{c}')$ are also connected. It follows that the quotients $K/Z_{H' \cap K}(\mathfrak{c}')$ and $G/Z_{H' \cap K}(\mathfrak{c}')$ are simply connected, as desired. □

Lemma 5.6. *Let $q : \Sigma \to Z$ be a G-equivariant Riemann domain. Assume that every z in Z admits an arbitrary small neighborhood V and a sequence $\{z_n\}$ converging to z with the property that both the isotropy subgroups G_{z_n} and the intersections $G \cdot z_n \cap V$ are connected. Then q is injective on every G-orbit of Σ.*

Proof. Assume by contradiction that the map q is not injective on the G-orbit through some ζ in Σ. Then there exists $h \in G$ with $h \cdot \zeta \neq \zeta$ such that $q(h \cdot \zeta) = q(\zeta)$. Since q is locally injective, one can choose an open neighborhood V of $z := q(\zeta)$ in Z as in the assumption, and open neighborhoods W_ζ and $W_{h\cdot\zeta}$ of ζ and $h \cdot \zeta$ in Σ, such that $W_\zeta \cap W_{h\cdot\zeta} = \varnothing$ and the restrictions $q|W_\zeta : W_\zeta \to V$ and $q|W_{h\cdot\zeta} : W_{h\cdot\zeta} \to V$ are bijective. Then there exists a sequence $\{z_n\}$ in Z, converging to z, with the property that both the isotropy subgroups G_{z_n} and the intersections $G \cdot z_n \cap V$ are connected.

Consider the sequence $\{\zeta_n := (q|W_\zeta)^{-1}(z_n)\}$ in W_ζ. Since $\{\zeta_n\}$ converges to ζ for n large enough, the points $h \cdot \zeta_n$ lie in $W_{h\cdot\zeta}$. Therefore their images $q(h \cdot \zeta_n) = h \cdot q(\zeta_n) = h \cdot z_n$ lie in V. Since both G_{z_n} and $G \cdot z_n \cap V$ are connected, the set $\Omega_n := \{g \in G : g \cdot z_n \in V\}$ is connected. Note that both e and h belong to Ω_n. Hence there exists a continuous path $\gamma : [0, 1] \to \Omega_n$ with $\gamma(0) = e$ and $\gamma(1) = h$. By the G-equivariance of q, both paths $t \mapsto (q|W_\zeta)^{-1}(\gamma(t) \cdot z_n)$ and $t \mapsto \gamma(t) \cdot \zeta_n$ in Σ are liftings of $t \mapsto \gamma(t) \cdot z_n$, with initial point ζ_n. On the other hand, $(q|W_\zeta)^{-1}(\gamma(1) \cdot z_n) \in W_\zeta$ while $\gamma(1) \cdot \zeta_n \in W_{h\cdot\zeta}$, giving a contradiction. □

As a consequence of these lemmas, we obtain the main result of this section.

Proposition 5.7. *Let G be a connected, noncompact, real simple Lie group such that the Riemannian symmetric space G/K has rank one. Assume that G is embedded in its universal complexification $G^{\mathbb{C}}$ and is different from the groups* $\mathrm{SL}(2, \mathbb{R})$ *and* $\mathrm{Spin}(3, 1)$. *Let $q : \Sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}$ be a G-equivariant Riemann domain. Then q is injective on every G-orbit.*

Proof. We begin by proving the following claim.

Claim. The isotropy subgroups of all principal G-orbits are connected.

Proof of the claim. Since G is connected, the isotropy subgroups of simply connected orbits are necessarily connected. Hence by Lemmas 5.1–5.5 we only need to discuss the isotropy types $Z_K(\mathfrak{a})$ when G has Lie algebra $\mathfrak{so}_0(2, 1)$ and the isotropy types $Z_H(\mathfrak{c})$ when G has Lie algebra $\mathfrak{so}_0(2, 1)$, $\mathfrak{so}_0(3, 1)$ and $\mathfrak{su}(2, 1)$.

Let $\mathfrak{g} = \mathfrak{so}(2, 1)$. When $G = \mathrm{SO}_0(2, 1)$ the isotropy subgroups of all principal G-orbits are connected, by Remarks 5.2 and 5.4. Observe that $\mathrm{SO}_0(2, 1)$ is centerless and that $\mathrm{SL}(2, \mathbb{R})$ is a double covering of $\mathrm{SO}_0(2, 1)$. Since the universal complexification of $\mathrm{SL}(2, \mathbb{R})$ is $\mathrm{SL}(2, \mathbb{C})$, which is simply connected, no covering of $\mathrm{SO}_0(2, 1)$ other than $\mathrm{SL}(2, \mathbb{R})$ admits an embedding into its universal complexification. Hence the claim follows for every group $G \neq \mathrm{SL}(2, \mathbb{R})$ that has Lie algebra $\mathfrak{so}(2, 1)$ and embeds in its universal complexification.

Let $\mathfrak{g} = \mathfrak{so}(3, 1)$. When $G = \mathrm{SO}_0(3, 1)$ the isotropy subgroup $Z_H(\mathfrak{c})$ is connected, by Remark 5.4. Note that $\mathrm{SO}_0(3, 1)$ is centerless and $\mathrm{Spin}(3, 1)$ is the only nontrivial covering of $\mathrm{SO}_0(3, 1)$ that embeds in its universal complexification. Hence the claim follows for every group $G \neq \mathrm{Spin}(3, 1)$ that has Lie algebra $\mathfrak{so}(3, 1)$ and embeds in its universal complexification.

Finally, let $\mathfrak{g} = \mathfrak{su}(2, 1)$. When $G = \mathrm{SU}(2, 1)$, the isotropy subgroup $Z_H(\mathfrak{c})$ is connected, by Remark 5.4. Thus the same holds true for every connected real Lie group covered by $\mathrm{SU}(2, 1)$. Since no covering group of $\mathrm{SU}(2, 1)$ admits an embedding in its universal complexification, the claim holds true for every G that has Lie algebra $\mathfrak{su}(2, 1)$ and embeds in its universal complexification. This concludes the proof of the claim.

In order to complete the proof of the proposition, recall that the union of principal G-orbits forms an open dense subset of $G^{\mathbb{C}}/K^{\mathbb{C}}$. Hence, by the above claim every point in $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be approximated by points with connected isotropy subgroups. Due to this fact and the description of the slice representation at closed G-orbits (see Remark 4.2 and the diagrams in Section 4), all assumptions of Lemma 5.6 are met and the statement follows. □

Remark 5.8. When $G = \mathrm{SL}(2, \mathbb{R})$, the isotropy subgroups of all principal G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ consist of the central elements $\{\pm I_2\}$. As we shall see in Example 7.7, in this case there exist Stein, G-equivariant Riemann domains that are not injective on G-orbits. Similarly, one can construct G-equivariant Riemann domains that are not injective on G-orbits in the case $G = \mathrm{Spin}(3, 1)$. However, by Theorem 7.6 such Riemann domains cannot be Stein.

6. G-invariant Stein domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$

Let G/K be a noncompact, rank-one, Riemannian symmetric space. In this section we exhibit a complete classification of Stein G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$. The main ingredient is the computation of the Levi form of hypersurface G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$, which is carried out in [Geatti 2002] and in the appendix, Section 9. Most of the Stein domains in our list are known. However, for $G = \mathrm{SU}(n, 1)$ we present some examples which appear to be new. By working out an explicit model of $G^{\mathbb{C}}/K^{\mathbb{C}}$, we show that they are all biholomorphic to $\mathbb{B}^n \times \mathbb{C}^n$.

The classification result is stated for the standard presentations of G/K given in Table 4.0. This is no loss of generality, since by Remark 4.1 the G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ as well as the CR-structure and topology of G-orbits do not depend on the presentation of the symmetric space G/K.

Retain the notation used in diagrams (4-3), (4-4), (4-9), and (4-10). Consider the G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ defined, for $0 \le a < 1$ and $0 \le b < \infty$, by

$$\begin{aligned} D_1(a) &= G\cdot(z_1 \cup \ell_1((a,1))), & D_2(a) &= G\cdot(z_3 \cup \ell_3((a,1))), \\ S_1(b) &= G\cdot \ell_2((b,\infty)), & S_2(b) &= G\cdot \ell_4((b,\infty)), \end{aligned} \tag{6-1}$$

Theorem 6.1. *Let G/K be a noncompact, rank-one, Riemannian symmetric space. All Stein G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ are given by the following table.*

Domain	$G =$	$SO_0(2,1)$	$SO_0(n,1)$, $n \ge 3$	$SU(n,1)$, $n \ge 2$	$Sp(n,1)$, $n \ge 2$, F_4^*
$D_1(a)$,	$0 \le a < 1$	*Stein*	*Stein*	*Stein*	*Stein*
$D_2(a)$,	$0 \le a < 1$	*Stein*	*Stein*	*no*	*no*
$S_1(b)$,	$0 \le b < \infty$	*Stein*	*no*	*no*	*no*
$S_2(b)$,	$0 \le b < \infty$	*Stein*	*no*	*no*	*no*
$D_1(0) \cup G\cdot w_1 \cup S_1(0)$		*Stein*	*no*	*Stein*	*no*
$D_1(0) \cup G\cdot w_4 \cup S_2(0)$		*Stein*	*no*	*Stein*	*no*
$D_2(0) \cup G\cdot w_2 \cup S_1(0)$		*Stein*	*no*	*no*	*no*
$D_2(0) \cup G\cdot w_3 \cup S_2(0)$		*Stein*	*no*	*no*	*no*

Table 6.0

Remark. The domains $D_1(0)$ and $D_2(0)$ are known as Akhiezer–Gindikin domains. They were introduced in [Akhiezer and Gindikin 1990] for G/K of arbitrary rank. In the two-dimensional case, the domains $S_1(0)$ and $S_2(0)$ are related to the causal structure of the symmetric space $G/H = SO_0(2,1)/SO(1,1)$. Domains of this type were studied in [Neeb 1999].

Proof. We first show that all the domains listed in the above table are Stein. The Akhiezer–Gindikin domain $D_1(0)$ is Stein by [Burns et al. 2003]. For $0 < a < 1$, the domains $D_1(a)$ are G-invariant subdomains of $D_1(0)$ containing the minimal orbit $G\cdot z_1 \cong G/K$. Their Steinness follows for example from the nonlinear convexity theorem in [Gindikin and Krötz 2002].

When $G = SO_0(n,1)$, with $n \ge 2$, the domain $D_2(0)$ and its subdomains $D_2(a)$ for $0 < a < 1$ are Stein since they are biholomorphic to $D_1(0)$ and $D_1(a)$, respectively. One such biholomorphism is given for example by the map

$$G^{\mathbb{C}}/K^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}, \quad gK^{\mathbb{C}} \mapsto g_3 g K^{\mathbb{C}},$$

where $g_3 = \exp iA_3$, with $\alpha(A_3) = \pi/2$; see (4-2). Note that $g_3 \in SO(n,1) \setminus \{SO_0(n,1)\}$; therefore $g_3 G = G g_3$. As a result, the above map exchanges the singular orbits $G \cdot z_1$ and $G \cdot z_3$ and maps $G \cdot \ell_1(a)$ onto $G \cdot \ell_3(a)$, for $0 < a < 1$.

When $G = SO_0(2,1)$, the domains $S_1(0)$ and $S_2(0)$ and their subdomains $S_1(b)$ and $S_2(b)$ for $0 < b < \infty$ were shown to be Stein in [Neeb 1999].

The last four domains in the list contain in their interior one of the nonclosed orbits $G \cdot w_i$ for some $i = 1, \dots, 4$. Their boundary consists of two nonclosed G-orbits and the singular orbit in their closure. All of them are Stein if $G = SO_0(2,1) \cong SU(1,1)/\{\pm I_2\}$. Only $D_1(0) \cup G \cdot w_1 \cup S_1(0)$ and $D_1(0) \cup G \cdot w_4 \cup S_2(0)$ are Stein when $G = SU(n,1)$ with $n > 1$. These facts are proved in Example 6.3 by constructing explicit models of such domains.

To complete the classification, it remains to show that no G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ are Stein other than the ones listed in Table 6.0. When $G = SO_0(2,1) \cong SU(1,1)/\{\pm \mathrm{Id}\}$ and $G = SU(n,1)$ with $n \geq 2$, this is proved in Example 6.3.

In all other cases, namely $SO_0(n,1)$ with $n > 1$, $Sp(n,1)$, and F_4^*, this follows from the description of the G-orbit space of $G^{\mathbb{C}}/K^{\mathbb{C}}$ given in diagrams (4-4), (4-9), (4-10) and from the computation of the Levi form of the hypersurface G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Indeed, by [Geatti 2002, Propositions 5.6 and 5.21], all principal orbits have indefinite Levi form, except for the ones intersecting the slice ℓ_1 (the domain $D_1(a)$ is Stein) and, only when the restricted root system of $\mathfrak{g}$ is reduced, the slice ℓ_3 (the domain $D_2(a)$ is Stein for $G = SO_0(n,1)$). Moreover, by Remarks 9.10 and 9.18, the Levi form of the nonclosed hypersurface orbits $G \cdot w_2$ and $G \cdot w_5$ is indefinite. Since the boundary of a Stein domain cannot have indefinite Levi form, the theorem follows. □

Let us illustrate the result of Theorem 6.1 on the model of $G^{\mathbb{C}}/K^{\mathbb{C}}$ described in Example 4.4. The Stein, G-invariant domains are studied by means of an appropriate G-invariant function on $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Example 6.2. Let $G = SO_0(n,1)$. By Example 4.4, the quotient $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be identified with $M^{\mathbb{C}} := \{\xi \in \mathbb{C}^{n+1} : \xi_1^2 + \dots + \xi_n^2 - \xi_{n+1}^2 = -1\}$. Assume $n > 2$. Consider the G-invariant function $f : M^{\mathbb{C}} \to \mathbb{R}$ defined by

$$f(\xi_1, \dots, \xi_{n+1}) := |\xi_1|^2 + \dots + |\xi_n|^2 - |\xi_{n+1}|^2 - 1.$$

For every $0 < a < 1$, the G-invariant domains $D_1(a)$ and $D_2(a)$ coincide with the two connected components of the set $\{\xi \in M^{\mathbb{C}} : f(\xi) < r\}$ for some $-2 < r < 0$. Every such domain is bounded by a single G-orbit on which the Levi form of f is positive definite. Hence it is Stein.

The G-invariant domains $D_1(0)$ and $D_2(0)$ coincide with the two connected components of the set $\{\xi \in M^{\mathbb{C}} : f(\xi) < 0\}$. They are bounded by the nonsmooth hypersurfaces $\partial D_1(0) = G \cdot (z_2 \cup w_1)$ and $\partial D_2(0) = G \cdot (z_2 \cup w_2)$, respectively.

At all smooth points of $\partial D_1(0)$ and $\partial D_2(0)$, the Levi form of f has $n-2$ positive eigenvalues and one zero eigenvalue. This is consistent with the fact that $D_1(0)$ and $D_2(0)$ are Stein. The Levi form of f is indefinite on all remaining hypersurface G-orbits. Thus there are no other Stein G-invariant domains in $M^{\mathbb{C}}$. $\square$

Next we determine all Stein, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ in the case $G = \mathrm{SU}(n, 1)$ by using the model of $G^{\mathbb{C}}/K^{\mathbb{C}}$ described in Example 4.7 and Remark 4.8. This settles the missing cases in the proof of Theorem 6.1.

Example 6.3. Let $G = \mathrm{SU}(n, 1)$ with $n \geq 1$. By Example 4.7, the quotient $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be identified with $M^{\mathbb{C}} := \mathbb{P}^n \times \overline{\mathbb{P}}^n \setminus \{\langle z, w\rangle_{n,1} = 0\}$. Consider the G-invariant function $f : M^{\mathbb{C}} \to \mathbb{R}$ defined by

$$f([z], [w]) = -\frac{\langle z, z\rangle_{n,1}\langle w, w\rangle_{n,1}}{|\langle z, w\rangle_{n,1}|^2}. \tag{6-2}$$

Consider first the case $G = \mathrm{SU}(1, 1)$.

By computing the Levi form of f on the G-orbits in the level set $\{f = r\}$ with $r < 0$, one shows that the domains $D_1(a)$ and $D_2(a)$ are Stein for all $0 < a < 1$. Similarly one shows that $S_1(b)$ and $S_2(b)$ are Stein for every $b > 0$. One can also verify that the Levi form of f on all nonclosed hypersurface orbits $G\cdot w_1, \ldots, G\cdot w_4$ is identically zero. This is consistent with the fact that the domains $D_1(0)$, $D_2(0)$, $S_1(0)$ and $S_2(0)$ are Stein. We claim that the domains

$$W_{1,1} := D_1(0) \cup G \cdot w_1 \cup S_1(0), \qquad W_{1,2} := D_1(0) \cup G \cdot w_4 \cup S_2(0),$$
$$W_{2,1} := D_2(0) \cup G \cdot w_2 \cup S_1(0), \qquad W_{2,2} := D_1(0) \cup G \cdot w_3 \cup S_2(0)$$

are Stein as well. By evaluating the hermitian forms $\langle z, z\rangle_{n,1}$ and $\langle w, w\rangle_{n,1}$ on the slices described in Example 4.7 and Remark 4.8, one sees that such domains can be characterized as follows:

$$W_{1,1} = \{\langle z, w\rangle_{1,1} \neq 0 \text{ and } \langle z, z\rangle_{1,1} < 0\},$$
$$W_{1,2} = \{\langle z, w\rangle_{1,1} \neq 0 \text{ and } \langle w, w\rangle_{1,1} < 0\},$$
$$W_{2,1} = \{\langle z, w\rangle_{1,1} \neq 0 \text{ and } \langle w, w\rangle_{1,1} > 0\},$$
$$W_{2,2} = \{\langle z, w\rangle_{1,1} \neq 0 \text{ and } \langle z, z\rangle_{1,1} > 0\}.$$

As a consequence, the maps defined by

$$\Delta \times \mathbb{C} \to W_{1,1}, \quad (u, v) \mapsto ([u : 1], [\bar{v} : 1 + \bar{u}\bar{v}]),$$
$$\mathbb{C} \times \Delta \to W_{1,2}, \quad (u, v) \mapsto ([u : 1 + uv], [\bar{v} : 1]),$$
$$\Delta \times \mathbb{C} \to W_{2,1}, \quad (u, v) \mapsto ([1 + uv : u], [1 : \bar{v}]),$$
$$\mathbb{C} \times \Delta \to W_{1,2}, \quad (u, v) \mapsto [1 : u], [1 + \bar{u}\bar{v} : \bar{v}])$$

are biholomorphisms. Here Δ denotes the unit disk in $\mathbb{C}$. In particular the domains $W_{1,1}, \ldots, W_{2,2}$ are Stein, as claimed.

Other G-domains in $M^{\mathbb{C}}$ that are possibly Stein can only be obtained as arbitrary unions of domains $W_{k,l}$ for $k, l = 1, 2$. We claim that such unions are not Stein. For instance, let us show that $W_{1,1} \cup W_{2,1}$ is not Stein. Consider the Stein local chart

$$\phi : \mathbb{C}^2 \to \mathbb{P}^1 \times \overline{\mathbb{P}}^1, \quad (u, v) \mapsto ([u : 1], [1 : \bar{v}]).$$

Since the preimage

$$\phi^{-1}(W_{1,1} \cup W_{2,1}) = \{(u, v) \in \mathbb{C}^2 : u \neq v \text{ and either } |u| < 1 \text{ or } |v| < 1\}$$

is not Stein, the domain $W_{1,1} \cup W_{2,1}$ is not Stein either. An analogous argument applies to the remaining cases.

Now consider the case $G = \mathrm{SU}(n, 1)$ with $n \geq 1$.

Using the G-invariant function f, one can prove that the domains $D_1(a)$ are Stein for $a > 0$. One can also verify that $D_1(0)$ coincides with a connected component of the set $\{z \in M^{\mathbb{C}} \mid f(z) < 0\}$ and that on the smooth part of its boundary $\partial D_1(0) = G \cdot (w_1 \cup z_2 \cup w_4)$, the Levi form of f has nonnegative eigenvalues. This is consistent with the fact that $D_1(0)$ is Stein.

Moreover, the Levi form of f is indefinite on the principal G-orbits through the slices ℓ_2, ℓ_3, ℓ_4, and ℓ_5 and on the nonclosed hypersurface orbit $G \cdot w_5$. On the other hand, the Levi form of f is definite on the nonclosed hypersurface orbits $G \cdot w_2$ and $G \cdot w_3$. As a result, the only other G-invariant domains in $M^{\mathbb{C}}$ that are possibly Stein are

$$W_{1,1} := D_1(0) \cup G \cdot w_1 \cup S_1(0), \quad W_{1,2} := D_1(0) \cup G \cdot w_4 \cup S_2(0), \quad W_{1,1} \cup W_{1,2}.$$

First we show that $W_{1,1}$ and $W_{1,2}$ are indeed Stein. By evaluating $\langle z, z\rangle_{n,1}$ and $\langle w, w\rangle_{n,1}$ on the slices described in Example 4.7, one sees that such domains can be characterized as follows:

$$W_{1,1} = \{([z], [w]) \in \mathbb{P}^n \times \overline{\mathbb{P}}^n : \langle z, w\rangle_{n,1} \neq 0 \text{ and } \langle z, z\rangle_{n,1} < 0\},$$
$$W_{1,2} = \{([z], [w]) \in \mathbb{P}^n \times \overline{\mathbb{P}}^n : \langle z, w\rangle_{n,1} \neq 0 \text{ and } \langle w, w\rangle_{n,1} < 0\}.$$

As a consequence the maps

$$\mathbb{B}^n \times \mathbb{C}^n \to W_{1,1}, \quad (u, v) \mapsto ([u : 1], [v : 1 + \bar{u}_1\bar{v}_1 + \cdots + \bar{u}_n\bar{v}_n]),$$
$$\mathbb{C}^n \times \mathbb{B}^n \to W_{1,2}, \quad (u, v) \mapsto ([u : 1 + u_1v_1 + \cdots + u_nv_n], [\bar{v} : 1])$$

are biholomorphisms. Here $\mathbb{B}^n$ denotes the unit ball in $\mathbb{C}^n$. In particular $W_{1,1}$ and $W_{1,2}$ are Stein, as claimed.

Next we show the domain $\Omega := W_{1,1} \cup W_{1,2}$ with $\partial\Omega = G \cdot (w_2 \cup z_2 \cup w_3)$ is not Stein. Assume by contradiction that Ω is Stein. Let $\mathfrak{c}'$ be the abelian subalgebra generating the Cartan subset $\mathscr{C}'$ (see Example 4.7). Let $T = \exp \mathfrak{c}'$ be the corresponding compact torus in G. Consider the T-action on Ω and the induced local holomorphic $T^{\mathbb{C}}$-action. By the globalization theorem in [Heinzner 1991, Section 6.6], the domain Ω embeds in its $T^{\mathbb{C}}$-globalization Ω^* as a T-invariant, orbit-convex subset. By definition, this means that the intersection of Ω with an $\exp i\mathfrak{c}$-orbit in Ω^* is connected.

Every $T^{\mathbb{C}}$-orbit through the slice ℓ_1 is contained in Ω. Indeed in $M^{\mathbb{C}}$ one can verify that

$$\begin{aligned}\exp(isC') \cdot \ell_1(t) = \big([0 : \ldots : e^s i \sin \tfrac{\pi}{4}(1-t) : e^{-s} \cos \tfrac{\pi}{4}(1-t)],\\ [0 : \ldots : -e^{-s} i \sin \tfrac{\pi}{4}(1-t) : e^s \cos \tfrac{\pi}{4}(1-t)]\big).\end{aligned}$$

Thus for fixed $0 < t < 1$, the function $\mathbb{R} \to \mathbb{R}$ defined by $s \mapsto f(\exp(isC') \cdot \ell_1(t))$ is given by

$$(e^{2s} \sin^2 \tfrac{\pi}{4}(1-t) - e^{-2s} \cos^2 \tfrac{\pi}{4}(1-t))(e^{-2s} \sin^2 \tfrac{\pi}{4}(1-t) - e^{2s} \cos^2 \tfrac{\pi}{4}(1-t))$$

and vanishes exactly twice, namely on $G \cdot w_1$ and on $G \cdot w_4$. Therefore $\exp(i\mathfrak{c}') \cdot \ell_1(t)$ never crosses the boundary of Ω and consequently the complex orbit $T^{\mathbb{C}} \cdot \ell_1(t)$ is entirely contained in Ω, as claimed. Moreover, for every fixed $s > 0$, one has

$$\lim_{n\to\infty} \exp(isC') \cdot \ell_1(1/n) = \ell_2(s) \in \Omega,$$
$$\lim_{n\to\infty} \exp(-isC') \cdot \ell_1(1/n) = \ell_4(s) \in \Omega.$$

Then the orbit-convexity of Ω in Ω^* implies that the sequence $\{\ell_1(1/n)\}_n$ has a limit point in Ω. On the other hand, in $G^{\mathbb{C}}/K^{\mathbb{C}}$ one has $\lim_n \ell_1(1/n) = z_2$, which is not in Ω. This yields a contradiction and proves that Ω is not Stein. The classification of all Stein G-invariant domains in $M^{\mathbb{C}}$ is now complete.

We conclude this section with a remark which is a consequence of Theorem 6.1 and is often used in the sequel.

Remark 6.4. Let D be a domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ with smooth boundary ∂D. It is well known that if D is not pseudoconvex at $z \in \partial D$, then no holomorphic function on D diverges in the vicinity of z. Let $\ell : I \to G^{\mathbb{C}}/K^{\mathbb{C}}$ be a slice for principal G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. By the classification of Stein, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ given in Theorem 6.1, the following facts hold.

(i) Assume that the Levi form of the orbits parametrized by ℓ is definite. Let $(c, d) \subset I$ be an interval with $0 \le c < d$ and $d \in I$. Then no holomorphic function on the invariant domain $G \cdot \ell((c, d))$ diverges in the vicinity of the boundary orbit $G \cdot \ell(d)$ (for instance, if $I = (0, 1)$ and $l = l_1$, then the domain

$D_1(d)$ is strictly pseudoconvex at every point of the boundary orbit $G\cdot\ell_1(d)$. Thus the domain $G\cdot\ell_1((c,d))$ is not pseudoconvex at any point of $G\cdot\ell_1(d)$).

(ii) Assume that the Levi form of the orbits parametrized by ℓ is indefinite. Let $(c,d)\subset I$ be an interval with $c\in I$. Then no holomorphic function on the invariant domain $G\cdot\ell((c,d))$ diverges in the vicinity of the boundary orbit $G\cdot\ell(c)$. Similarly, if $d\in I$, then no holomorphic function diverges in the vicinity of $G\cdot\ell(d)$.

7. Univalence over $G^{\mathbb{C}}/K^{\mathbb{C}}$

Let G be a connected, noncompact, real simple Lie group, let $K\subset G$ be a maximal compact subgroup, and let $G^{\mathbb{C}}$ be the universal complexification of G. Assume that the center Γ of G is finite and that G is not a covering of $\mathrm{SL}(2,\mathbb{R})$. In this section, we show that a holomorphically separable, G-equivariant Riemann domain $q:\Sigma\to G^{\mathbb{C}}/K^{C}$ is necessarily univalent if the rank of G/K is equal to one; see Theorem 7.6 and Remark 7.8.

In most cases the map q is injective on every G-orbit; see Section 5. So we are reduced to prove the injectivity of q over the global slices for the G-action defined by diagrams (4-3), (4-4), (4-9), and (4-10). Recall that the slices parametrizing principal G-orbits are diffeomorphic to open intervals of $\mathbb{R}$ and that a local diffeomorphism of a one-dimensional smooth manifold into the real line $\mathbb{R}$ is necessarily injective. As a consequence, q is injective on every connected component of Σ over a domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ consisting of principal orbits.

However, in order to ensure monodromy around the singular orbit $G\cdot z_2$ (see the diagrams in Section 4), it is necessary to combine the uniqueness property of path liftings for Riemann domains with the complex geometry of the G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Before proving the main result of this section, some preliminary lemmas are needed.

Let $\ell:I\to G^{\mathbb{C}}/K^{\mathbb{C}}$ be one of the slices for principal G-orbits defined in (4-5), (4-6), (4-11), (4-12) and (4-13). Define

$$\hat{I}:=\begin{cases}(0,1] & \text{if } I=(0,1),\\ I & \text{if } I=\mathbb{R}^{>0}.\end{cases}\tag{7-1}$$

Recall that $I=(0,1)$ only when $\ell=\ell_1$ or $\ell=\ell_3$. In those cases extend ℓ to $\hat{I}=(0,1]$ by defining

$$\ell_1(1):=eK^{\mathbb{C}}\quad\text{and}\quad\ell_3(1):=\exp(iA_3)K^{\mathbb{C}}.$$

We refer to $\ell:\hat{I}\to G^{\mathbb{C}}/K^{\mathbb{C}}$ as an *extended slice*. Note that the images of the extended slices ℓ_1 and ℓ_3 include the points z_1 and z_3, respectively.

Let $q:\Sigma\to G^{\mathbb{C}}/K^{\mathbb{C}}$ be a G-equivariant Riemann domain, and let $\ell:\hat{I}\to G^{\mathbb{C}}/K^{\mathbb{C}}$ be an extended slice. A *local lifting* of ℓ is a smooth path $\tilde{\ell}:J\to\Sigma$ defined on a nonempty interval J open in $\hat{I}$, and satisfying the condition $q\circ\tilde{\ell}=\ell$ on J. A local lifting $\tilde{\ell}:J\to\Sigma$ is *maximal* if it cannot be extended to a larger interval J' with $J\subsetneq J'\subset\hat{I}$.

Lemma 7.1. *Assume that G is embedded in its universal complexification $G^{\mathbb{C}}$ and is different from* $\mathrm{SL}(2,\mathbb{R})$ *and* $\mathrm{Spin}(3,1)$. *Let $q:\Sigma\to G^{\mathbb{C}}/K^{\mathbb{C}}$ be a Stein, G-equivariant Riemann domain, and let $\tilde{\ell}:J\to\Sigma$ be a maximal local lifting of an extended slice $\ell:\hat{I}\to G^{\mathbb{C}}/K^{\mathbb{C}}$.*

(i) *if the Levi form of the principal orbits parametrized by ℓ is definite, then the invariant domain $G\cdot\ell(J)$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ is Stein; see Theorem 6.1.*

(ii) *If the Levi form of the principal orbits parametrized by ℓ is indefinite, then J coincides with $\hat{I}$.*

Proof. (i) Consider first the case $\hat{I}=\mathbb{R}^{>0}$ (see diagram (4-3), Example 6.3 and Remark 4.8). By Theorem 6.1, we need to show that $J=(b,+\infty)$ for some $b\geq 0$. Assume by contradiction that $J=(b,d)$ with $0\leq b<d<\infty$. Since the lifting $\tilde{\ell}(J)$ is a one-dimensional real-analytic submanifold of Σ, the local diffeomorphism $q|\tilde{\ell}(J)$ is injective. By Proposition 5.7, the map q is injective on every G-orbit. Therefore the restriction $q|G\cdot\tilde{\ell}(J):G\cdot\tilde{\ell}(J)\to G\cdot\ell(J)$ is a biholomorphism.

By the maximality of $\tilde{\ell}$, when n grows, the sequence $\{\tilde{\ell}(d-1/n)\}_n$ leaves every given compact subset in Σ. Since Σ is Stein, there exists a holomorphic function $f\in\mathbb{O}(\Sigma)$ such that $\lim_{n\to\infty}|f(\tilde{\ell}(d-1/n))|=\infty$.

On the other hand, the push-forward of f by $q|G\cdot\tilde{\ell}(J)$ defines a holomorphic function in $\mathbb{O}(G\cdot\ell(J))$ that diverges in the vicinity of the boundary orbit $G\cdot\ell(d)$. This contradicts Remark 6.4(ii), implying that J is of the form (b,∞), as claimed.

Consider now the case $\hat{I}=(0,1]$. This only occurs for $\ell=\ell_1$ or, when the restricted root system of $\mathfrak{g}$ is reduced, for $\ell=\ell_3$; see the diagrams in Section 4 and [Geatti 2002, Proposition 5.6]. By Theorem 6.1, we need to show that $J=(a,1]$ for some $a\geq 0$. Assume by contradiction that $J=(a,d)$ with $0\leq a<d\leq 1$. The argument used in the previous case shows that $J=(a,1)$ and that there exists a holomorphic function $f\in\mathbb{O}(\Sigma)$ such that $\lim_{n\to\infty}|f(\tilde{\ell}(1-1/n))|=\infty$. Moreover, the push-forward of f by $q|G\cdot\tilde{\ell}(J)$ defines a holomorphic function $\bar{f}\in\mathbb{O}(G\cdot\ell(J))$, which diverges in the vicinity of the boundary orbit $G\cdot\ell(1)$. On the other hand, such an orbit is a totally real submanifold of $G\cdot\ell((a,1])$. Thus $\bar{f}$ extends to a holomorphic function on $G\cdot\ell((a,1])$. This yields a contradiction, implying that $J=(a,1]$, as desired.

(ii) Assume first that $\hat{I}=\mathbb{R}^{>0}$. Then Remark 6.4(ii) and an argument analogous to the proof of (i) show that $J=\hat{I}$. Consider then the case $\hat{I}=(0,1]$. This only occurs when the restricted root system of $\mathfrak{g}$ is nonreduced and $\ell = \ell_3$; see the diagrams in Section 4 and [Geatti 2002, Proposition 5.6]. An argument like the proof of (i) shows that if a lifting $\tilde{\ell}_3 : J \to \Sigma$ is maximal, then either $J = (0, 1]$ or $J = (0, 1)$.

To prove that $J = (0, 1]$, suppose by contradiction that $J = (0, 1)$. Consider a sequence $\{z_n\}$ in $G\cdot\ell_3(J)$ that converges to a point on the boundary orbit $G\cdot w_5$, say w_5. Since the Levi form of $G \cdot w_5$ is indefinite (see Remark 9.10), no holomorphic function on $G \cdot \ell_3(J)$ diverges on $\{z_n\}$. Note that the restriction

$$q|G \cdot \tilde{\ell}_3(J) : G \cdot \tilde{\ell}_3(J) \to G \cdot \ell_3(J)$$

is a biholomorphism. Hence no holomorphic function of $G \cdot \tilde{\ell}_3(J)$ diverges on the sequence $\{\zeta_n\}$ in Σ defined by $\zeta_n := (q|G \cdot \tilde{\ell}_3(J))^{-1}(z_n)$. By the Steinness of Σ, there exists a subsequence of $\{\zeta_n\}$ converging to a point η_5 in Σ. Since q is continuous, one has $q(\eta_5) = w_5$.

By the G-equivariance of q, the description of the slice representation at z_3 given in Remark 4.2, and Proposition 5.7, there exists a G-invariant neighborhood V of η_5 in Σ on which q is injective. Its image $q(V)$ intersects the slice ℓ_5 in $\ell_5((0, \epsilon))$ for some $\epsilon > 0$. By statement (i) of this lemma, the local lifting $s \mapsto (q|V)^{-1}(\ell_5(s))$, with $s \in (0, \epsilon)$, extends to a lifting $\tilde{\ell}_5 : \hat{I}_5 \to \Sigma$ of ℓ_5. Note that q maps the G-invariant domain $W := G \cdot (\tilde{\ell}_3(J) \cup \eta_5 \cup \tilde{\ell}_5(\hat{I}_5))$ in Σ biholomorphically onto the domain $q(W) = G \cdot (\ell_3(J) \cup w_5 \cup \ell_5(\hat{I}_5))$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Since $G \cdot \ell_3(1)$ is a totally real submanifold of $q(W) \cup G \cdot \ell_3(1)$ (see [Geatti 2002, Lemma 2.11 and Remark 2.13]), every holomorphic function on $q(W)$ extends to a holomorphic function on $q(W) \cup G \cdot \ell_3(1)$. As a consequence, no holomorphic function on W can diverge on the sequence $\{\tilde{\ell}_3(1 - 1/n)\}_n$ in Σ.

On the other hand, by the maximality of $\tilde{\ell}_3$, the sequence $\{\tilde{\ell}(1 - 1/n)\}_n$ leaves every given compact subset in Σ as n grows. Since Σ is Stein, there exists a holomorphic function $f \in \mathcal{O}(\Sigma)$ such that $\lim_{n\to\infty} |f(\tilde{\ell}_3(b - 1/n))| = \infty$. This yields a contradiction, implying that J necessarily coincides with $(0, 1]$. □

Let ℓ_1 and ℓ_3 be the slices parametrizing the principal orbits through the fundamental Cartan subset $\mathcal{A}$. Denote by $\mathcal{C} = \exp i\mathfrak{c} \cdot z_2$ the standard Cartan subspace with base point z_2, and define $\mathcal{C}^* := \mathcal{C} \setminus \{z_2\}$. Recall that in the reduced case, $\mathfrak{c} = \mathbb{R}(X + \theta(X))$ for some nonzero vector $X \in \mathfrak{g}^{\alpha}$, and $z_2 = \exp(iA_2)K^{\mathbb{C}}$ with $\alpha(A_2) = \pi/2$. In the nonreduced case, $\mathfrak{c} = \mathbb{R}(X + \theta(X))$ for some nonzero vector $X \in \mathfrak{g}^{2\alpha}$, and $z_2 = \exp(iA_2)K^{\mathbb{C}}$ with $\alpha(A_2) = \pi/4$. In both cases, $\exp \mathfrak{c}$ is a compact, one-dimensional, real torus in G, which we denote by T. Both T and its universal complexification $T^{\mathbb{C}} \cong \mathbb{C}^*$ act on $G^{\mathbb{C}}/K^{\mathbb{C}}$ by left translations.

In the next proposition, we single out two distinguished G-invariant domains Ω and Ω' in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing all $T^{\mathbb{C}}$-orbits through the slices $\ell_1(I_1)$ and $\ell_3(I_3)$, respectively.

Lemma 7.2. *Let G/K be a noncompact, rank-one, Riemannian symmetric space. Consider the domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ defined by*

$$\Omega := G \cdot (\ell_1(I_1) \cup w_1 \cup w_4 \cup \mathscr{C}^*).$$

Then for every point $z \in \ell_1(I_1)$, the complex orbit $T^{\mathbb{C}} \cdot z$ is contained in Ω.

Similarly, define

$$\Omega' := G \cdot (\ell_3(I_3) \cup w_2 \cup w_3 \cup \mathscr{C}^*).$$

Then for every point $z \in \ell_3(I_3)$, the complex orbit $T^{\mathbb{C}} \cdot z$ is contained in Ω'.

Proof. We first assume that $G = \mathrm{SO}_0(2, 1)$ and prove the statement by using the model $M^{\mathbb{C}}$ of $G^{\mathbb{C}}/K^{\mathbb{C}}$ constructed in Example 4.4. Let C be the generator of $\mathfrak{c}$ chosen there. Then, for $s \in \mathbb{R}$ and $t \in (0, 1)$, one has

$$\exp(isC) \cdot \ell_1(t) = \big(\sinh(2s) \sin \tfrac{\pi}{2}(1-t), i \cosh(2s) \sin \tfrac{\pi}{2}(1-t), \cos \tfrac{\pi}{2}(1-t)\big).$$

Since $z_2 = (0, i, 0)$ and the entries of the matrix group G are real, from the above expression one easily verifies that $\exp i\mathfrak{c} \cdot \ell_1(I_1) \cap G \cdot z_2 = \varnothing$. Consider then the G-invariant function $f(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - 1$ defined on $M^{\mathbb{C}}$. The function f vanishes on the real hypersurface $G \cdot (z_2 \bigcup_{j=1}^4 w_j)$, is negative on the sets $G \cdot \ell_j(I_j)$ for $j = 1, 3$, and is positive on the sets $G \cdot \ell_j(I_j)$ for $j = 2, 4$. Moreover, for every fixed $t_0 \in (0, 1)$, one sees that

$$f(\exp(isC) \cdot \ell_1(t_0)) = (\sinh^2 2s + \cosh^2 2s) \sin^2 \tfrac{\pi}{2}(1-t_0) - \cos^2 \tfrac{\pi}{2}(1-t_0) - 1$$

is strictly increasing as $|s| \to \infty$. Thus it vanishes exactly twice. As a consequence, the path $\exp(isC) \cdot \ell_1(t_0)$ crosses the hypersurface $f^{-1}(0) \setminus \{G \cdot z_2\}$ exactly twice, namely on the orbits $G \cdot w_1$ and $G \cdot w_2$. It follows that $\exp(isC) \cdot \ell_1(t_0) \in \Omega$, for every $s \in \mathbb{R}$. Thus the $T^{\mathbb{C}}$-orbit through $\ell_1(t_0)$ is entirely contained in Ω, as stated. An analogous argument proves the statement for the higher dimensional hyperquadrics. By Remark 4.1(ii), this settles the case when $\mathfrak{g}$ has a reduced restricted root system.

Consider now the case when the restricted root system of $\mathfrak{g}$ is nonreduced. We prove the statement by reducing to the two-dimensional case. Set $\hat{\mathfrak{g}} := \mathfrak{so}(2, 1)$ and fix a basis in $\hat{\mathfrak{g}}$ of the form $\{\widehat{X},\ \theta(\widehat{X}),\ \hat{A} = [\theta(\widehat{X}), \widehat{X}]\}$, where $\widehat{X}$ is a root vector in $\hat{\mathfrak{g}}^{\alpha}$ and $\alpha(\hat{A}) = \pi/2$. Define $\widehat{C} = \widehat{X} + \theta(\widehat{X})$. Choose a root vector $X \in \mathfrak{g}^{2\alpha}$ and normalize the triple $\{X, \theta(X), A = [\theta(X), X]\}$ in $\mathfrak{g}$ so that $\alpha(A) = \pi/4$. Such a triple generates a three-dimensional θ-stable subalgebra of $\mathfrak{g}$ isomorphic to $\hat{\mathfrak{g}}$. In particular, there exists an injective Lie algebra homomorphism $\varphi_* : \hat{\mathfrak{g}} \to \mathfrak{g}$ mapping $\widehat{X}$, $\hat{A}$, and $\theta(\widehat{X})$ into X, A, and $\theta(X)$, respectively. Clearly φ_* maps $\widehat{C} = \widehat{X} + \theta(\widehat{X})$ into $C = X + \theta(X)$ as well. Let $\widehat{K} = \mathrm{SO}(2)$ be the maximal compact subgroup of $\widehat{G} := \mathrm{SO}_0(2, 1)$, and let $\hat{\mathfrak{k}}$ be its Lie algebra. Note that $\hat{\mathfrak{k}}$ and $\mathfrak{k}$ are generated by $\widehat{C}$ and

C, respectively. One can check that the $\mathbb{C}$-linear extension $\hat{\mathfrak{g}}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ of φ_* induces a Lie group morphism $\varphi : \widehat{G}^{\mathbb{C}} \to G^{\mathbb{C}}$ mapping $\widehat{K}^{\mathbb{C}}$ to $K^{\mathbb{C}}$. As a consequence, one obtains a holomorphic map (denoted by the same symbol) $\varphi : \widehat{G}^{\mathbb{C}}/\widehat{K}^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$.

Let $\widehat{\Omega}$ be the domain

$$\widehat{\Omega} = \widehat{G} \cdot (\hat{\ell}_1(I_1) \cup \hat{w}_1 \cup \hat{\ell}_2(I_2) \cup \hat{w}_4 \cup \hat{\ell}_4(I_4))$$

in $\widehat{G}^{\mathbb{C}}/\widehat{K}^{\mathbb{C}}$, for which the statement has been proved above. We claim that $\varphi(\widehat{\Omega}) \subset \Omega$. The map φ is "equivariant" with respect to the action of $\widehat{G}$, that is $\varphi(g \cdot x) = \varphi(g) \cdot \varphi(x)$ for every $g \in \widehat{G}$ and $x \in \widehat{G}^{\mathbb{C}}/\widehat{K}^{\mathbb{C}}$. By the definition of φ_*, one has $\varphi(\exp(it\hat{A})) = \exp(itA)$ and $\varphi(\exp(it\widehat{C})) = \exp(itC)$. It follows that

$$\varphi(\hat{\ell}_1(I_1)) = \ell_1(I_1), \qquad \varphi(\hat{z}_2) = z_2, \qquad \varphi(\widehat{\mathscr{C}}) = \mathscr{C}.$$

We finish proving the claim by showing that $\varphi(\hat{w}_1) \in G \cdot w_1$ and $\varphi(\hat{w}_4) \in G \cdot w_4$ (possibly the orbit $G \cdot w_4$ and $G \cdot w_1$ coincide). By (4-1), there is a commutative diagram

$$\begin{array}{ccc} \widehat{G} \times_{\widehat{G}_{\hat{z}_2}} \widehat{V}_2 & \xrightarrow{\tilde{\varphi}} & G \times_{G_{z_2}} V_2 \\ \downarrow & & \downarrow \\ \widehat{G}^{\mathbb{C}}/\widehat{K}^{\mathbb{C}} & \xrightarrow{\varphi} & G^{\mathbb{C}}/K^{\mathbb{C}}. \end{array}$$

The vertical arrows correspond to the equivariant embeddings given in (4-1), and the map $\tilde{\varphi}$ is defined by $[\hat{g}, \widehat{X}] \to [\varphi(\hat{g}), \varphi_*(\widehat{X})]$. Since φ_* is an injective homomorphism, $\varphi(\hat{w}_1)$ does not lie on the singular orbit $G \cdot z_2$. Indeed in the twisted bundle $G \times_{G_{z_2}} V_2$ such an orbit corresponds to the set $\{[g, 0] : g \in G\}$. On the other hand, $\varphi(\hat{w}_1) \in \overline{G \cdot \ell_1(I_1)} \cap \overline{G \cdot \ell_2(I_2)}$. Therefore the image $\varphi(\hat{w}_1)$ necessarily lies on the orbit $G \cdot w_1$. Similarly one proves that $\varphi(\hat{w}_4) \in G \cdot w_4$. In conclusion, $\widehat{\Omega}$ is mapped by φ into Ω, as claimed.

Note that $\exp \mathfrak{c}^{\mathbb{C}} \cdot \ell_1(I_1) = \varphi(\exp \hat{\mathfrak{c}}^{\mathbb{C}} \cdot \hat{\ell}_1(I_1))$, and recall that in the 2-dimensional case we already showed that $\exp \hat{\mathfrak{c}}^{\mathbb{C}} \cdot \hat{\ell}_1(I_1) \subset \widehat{\Omega}$. Then, by the above claim, one has $T^{\mathbb{C}} \cdot \ell_1(z) \subset \Omega$ for every $z \in \ell_1(I_1)$, as required. The statement regarding the domain Ω' follows from similar arguments. $\square$

Lemma 7.3. *Assume that G is embedded in its universal complexification $G^{\mathbb{C}}$ and is different from the groups* $\mathrm{SL}(2, \mathbb{R})$ *and* $\mathrm{Spin}(3, 1)$. *Let $q : \Sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}$ be a Stein, G-equivariant Riemann domain.*

(i) *Let $\tilde{\ell}_1 : I_1 \to \Sigma$ be a lifting of the slice ℓ_1. Assume that the closure of $G \cdot \tilde{\ell}_1(I_1)$ in Σ contains points η_1 and η_4 mapped by q into the nonclosed orbits $G \cdot w_1$ and $G \cdot w_4$, respectively (possibly the orbits $G \cdot w_1$ and $G \cdot w_4$ coincide). Then the singular orbit $G \cdot z_2$ is contained in $q(\Sigma)$.*

(ii) *Let $\tilde{\ell}_3 : I_3 \to \Sigma$ be a lifting of the slice ℓ_3. Assume that the closure of $G \cdot \tilde{\ell}_3(I_3)$ in Σ contains points η_2 and η_3 mapped by q into the nonclosed orbits $G \cdot w_2$ and $G \cdot w_3$, respectively (possibly the orbits $G \cdot w_2$ and $G \cdot w_3$ coincide). Then the singular orbit $G \cdot z_2$ is contained in $q(\Sigma)$.*

Proof. (i) We begin by showing that Σ contains an open G-invariant set that is biholomorphic to the domain $\Omega = G \cdot (\ell_1(I_1) \cup w_1 \cup w_4 \cup \mathscr{C}^*)$ of Lemma 7.2. By the G-equivariance of q, by the description of the slice representation at z_2 given in Remark 4.2, and by Proposition 5.7, there exists a G-invariant neighborhood V of η_1 in Σ on which q is injective. Its image $q(V)$ intersects the slice ℓ_2 in $\ell_2((0, \epsilon))$ for some $\epsilon > 0$. By Lemma 7.1(i), the map $s \mapsto (q|V)^{-1}(\ell_2(s))$, with $s \in (0, \epsilon)$, extends to a lifting $\tilde{\ell}_2 : I_2 \to \Sigma$ of ℓ_2. A similar argument yields a lifting $\tilde{\ell}_4 : I_4 \to \Sigma$ of ℓ_4. Since q is injective on the set $\tilde{\ell}_1(I_1) \cup \eta_1 \cup \tilde{\ell}_2(I_2) \cup \eta_4 \cup \tilde{\ell}_4(I_4)$, as well as on every G-orbit (see Proposition 5.7), it is injective on the G-invariant subdomain of Σ given by

$$W := G \cdot (\tilde{\ell}_1(I_1) \cup \eta_1 \cup \tilde{\ell}_2(I_2) \cup \eta_4 \cup \tilde{\ell}_4(I_4)).$$

Note that $q(W) = \Omega$. In particular W is biholomorphic to Ω, as claimed.

Let $\mathscr{C} = \exp i\mathfrak{c} \cdot z_2$ be the standard Cartan subset in $G^{\mathbb{C}}/K^{\mathbb{C}}$ starting at z_2. Recall that $T := \exp \mathfrak{c}$ is a compact torus in G. By Heinzner's globalization theorem [1991, Section 6.6], the space Σ can be embedded in its $T^{\mathbb{C}}$-globalization Σ^* as a T-invariant, orbit-convex domain. By definition, this means that the intersection of Σ with an $\exp i\mathfrak{c}$-orbit in Σ^* is necessarily connected.

Consider now the induced local $T^{\mathbb{C}}$-orbit of a point $\zeta \in \tilde{\ell}_1(I_1)$ in Σ. Since $q|W$ is biholomorphic and G-equivariant by Lemma 7.2, such an orbit is in fact global. Let C be a generator of the abelian subalgebra $\mathfrak{c}$. For every fixed $s > 0$, one has

$$\lim_{n\to\infty} \exp(isC) \cdot \tilde{\ell}_1(1/n) = \tilde{\ell}_2(s) \in W,$$
$$\lim_{n\to\infty} \exp(-isC) \cdot \tilde{\ell}_1(1/n) = \tilde{\ell}_4(s) \in W.$$

By the orbit-convexity of Σ in its $T^{\mathbb{C}}$-globalization, the sequence $\{\tilde{\ell}_1(1/n)\}_n$ converges to a point $\zeta_2 \in \Sigma$. By the continuity of q, one has $q(\zeta_2) = z_2$. Therefore $z_2 \in q(\Sigma)$, as required.

Part (ii) is proved by showing that Σ contains an open subset biholomorphic to the domain Ω' of Lemma 7.2 and arguing as in the previous case. $\square$

Let G be a connected Lie group and $\widetilde{G} \to G = \widetilde{G}/\Gamma$ a covering of G. If X is a G-manifold, it can be regarded as a $\widetilde{G}$-manifold by letting Γ act trivially on it.

Lemma 7.4. *Let G be a connected, real Lie group, and let $\widetilde{G} \to G = \widetilde{G}/\Gamma$ be a finite covering of G. Let X be a complex G-manifold with the property that every*

Stein, G-equivariant Riemann domain over X is univalent. Let $q : \Sigma \to X$ be a Stein, $\widetilde{G}$-equivariant Riemann domain. Then

(i) *the image $q(\Sigma)$ is biholomorphic to the quotient Σ/Γ, and $q : \Sigma \to q(\Sigma)$ can be identified with the quotient map*;

(ii) *q is a $\widetilde{G}$-equivariant covering.*

In particular, $q(\Sigma)$ is Stein.

Proof. (i) Since Γ is a finite subgroup of $\widetilde{G}$, the quotient Σ/Γ can be regarded as the categorical quotient of Σ with respect to Γ. Then Σ/Γ is a Stein space, and the quotient map $\pi : \Sigma \to \Sigma/\Gamma$ is holomorphic; see Theorem 2.1. Moreover, since q is Γ-invariant, there exists a G-equivariant holomorphic map $\hat{q} : \Sigma/\Gamma \to X$ making the diagram

$$\begin{tikzcd} \Sigma \arrow[r, "\pi"] \arrow[d, "q"'] & \Sigma/\Gamma \arrow[dl, "\hat{q}"] \\ X & \end{tikzcd}$$

commute. Since $q = \hat{q} \circ \pi$ is locally biholomorphic, then π is also locally biholomorphic. In particular, Σ/Γ is a manifold and $\hat{q} : \Sigma/\Gamma \to X$ is a Stein, G-equivariant Riemann domain. By the assumption on X, the map $\hat{q}$ is injective, implying (i).

(ii) Without loss of generality, one may assume that Γ acts effectively on Σ. Then the statement follows by showing that Γ acts freely on Σ. Assume by contradiction that this is not the case. Then there exists $\gamma \in \Gamma$ whose fixed point set $\mathrm{Fix}(\gamma) := \{\zeta \in \Sigma : \gamma \cdot \zeta = \zeta\}$ is not empty. Since $\mathrm{Fix}(\gamma)$ is a proper analytic subset of Σ, it has no interior point. In particular there exist a $\zeta \in \mathrm{Fix}(\gamma)$ and a sequence $\{\zeta_n\}_n$ in the complement of $\mathrm{Fix}(\gamma)$ in Σ such that $\zeta_n \to \zeta$. Note that by the continuity of γ, one has $\gamma \cdot \zeta_n \to \gamma \cdot \zeta = \zeta$.

Let U be an open neighborhood of ζ on which π is injective. Then, for n large enough, both ζ_n and $\gamma \cdot \zeta_n$ lie in U. Since Γ acts trivially on Σ/Γ, it follows that $\pi(\zeta_n) = \gamma \cdot \pi(\zeta_n) = \pi(\gamma \cdot \zeta_n)$. On the other hand since $\zeta_n \notin \mathrm{Fix}(\gamma)$, one has $\gamma \cdot \zeta_n \neq \zeta_n$. This gives the desired contradiction. □

Recall the following consequence of the uniqueness of path-liftings on Riemann domains, which will be often used in the proof of the main theorem of this section.

Lemma 7.5. *Let $q : \Sigma \to Z$ be a Riemann domain, and let W be a domain of Σ such that the restriction $q|W : W \to Z$ is bijective. Then $W = \Sigma$.*

Next comes the main result of this section.

Theorem 7.6. *Let G/K be a noncompact, rank-one, Riemannian symmetric space. Assume that G is a connected, simple, real Lie group that is embedded in its universal complexification $G^{\mathbb{C}}$ and is different from* $\mathrm{SL}(2, \mathbb{R})$. *Then a holomorphically separable, G-equivariant Riemann domain $q : \Sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is univalent.*

Proof. Recall that Σ admits a G-equivariant holomorphic embedding into its envelope of holomorphy. Thus we may assume that Σ is Stein; see Section 2. We prove the theorem in the case when the G-orbit diagram of $G^{\mathbb{C}}/K^{\mathbb{C}}$ is of type (4-9), namely, for $\mathfrak{g} = \mathfrak{su}(n, 1)$. In all remaining cases but $G = \mathrm{Spin}(3, 1)$, which is discussed separately, the statement follows from the same arguments with fewer steps.

So we first assume that G is different from $\mathrm{Spin}(3, 1)$ and divide the proof in three subcases, depending on the image of Σ in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Finally we discuss the case $G = \mathrm{Spin}(3, 1)$.

Case (i): *The image $q(\Sigma)$ contains the singular orbit $G \cdot z_2$.* We begin by proving that there exists a G-invariant domain $V \subset \Sigma$ with the properties that q is injective on V and

$$q(V) = G \cdot \Big(\ell_1(1) \bigcup_{j=1}^{4} \big(\ell_j(I_j) \cup w_j\big) \cup z_2\Big).$$

The extended slices $\ell_j : \hat{I}_j \to G^{\mathbb{C}}/K^{\mathbb{C}}$ are defined in (7-1). Let ζ_2 be an element in $q^{-1}(z_2)$, and let U be an open neighborhood of ζ_2 in Σ on which the restriction $q|U$ is injective. Since the map q is open, the image $q(U)$ intersects the slices for principal orbits starting at z_2 in the sets $\ell_j((0, \epsilon))$ for $j = 1, \dots, 4$ and some $\epsilon > 0$. The image $q(U)$ also intersects all nonclosed G-orbits containing $G \cdot z_2$ in their closures. By Lemma 7.1, each extended slice ℓ_j admits a lifting $\tilde{\ell}_j : \hat{I}_j \to \Sigma$ such that $\tilde{\ell}_j(t) = (q|U)^{-1}\ell_j(t)$ for $t \in (0, \epsilon)$. For $j = 1, \dots, 4$, choose points $\eta_j \in (q|U)^{-1}(G \cdot w_j)$. Consider then the open G-invariant set in Σ given by

$$V := G \cdot \Big(\tilde{\ell}_1(1) \bigcup_{j=1}^{4} \big(\tilde{\ell}_j(I_j) \cup \eta_j\big) \cup \zeta_2\Big).$$

Since q is injective on each lifted slice $\tilde{\ell}_j$ and on all G-orbits (see Proposition 5.7), it is injective on V as well. Hence V is the open G-invariant domain in Σ with the required properties.

Consider a sequence $\{z_n\}$ in $G \cdot \ell_3(J)$ that converges to a point on the boundary orbit $G \cdot w_5$. Recall that the Levi form of $G \cdot w_5$ is indefinite; see Remark 9.10. Then, by arguing as in the proof of Lemma 7.1(ii), the domain V can be enlarged to an invariant domain W in Σ with the properties that the restriction $q|W$ is injective and $q(W) = G^{\mathbb{C}}/K^{\mathbb{C}}$. By Lemma 7.5, one has $W = \Sigma$, and the theorem follows.

Case (ii): *The image $q(\Sigma)$ does not contain the orbit $G \cdot z_2$, but contains a nonclosed G-orbit.* Assume for example that $w_1 \in q(\Sigma)$, and let $\eta_1 \in q^{-1}(w_1)$. By the G-equivariance of q, by the description of the slice representation at z_2 given in Remark 4.2, and by Proposition 5.7, there exists a G-invariant neighborhood V of η_1 in Σ on which q is injective. Its image $q(V)$ intersects the slices ℓ_1 and ℓ_2 in the sets $\ell_1((0,\epsilon))$ and $\ell_2((0,\epsilon))$ for some $\epsilon > 0$. Arguing as in the previous case, one can construct a G-invariant domain $V \subset \Sigma$ with the properties that q is injective on V and $q(V) = G \cdot (\ell_1(\hat{I}_1) \cup w_1 \cup \ell_2(\hat{I}_2))$.

If $V = \Sigma$ (this is possible by Theorem 6.1), then the map q is injective, as desired. If $V \neq \Sigma$, then there exists a point η in the closure of V in Σ that is mapped by q into one of the nonclosed orbits $G \cdot w_2$ or $G \cdot w_4$. Assume that $q(\eta)$ lies in $G \cdot w_4$. Then by Lemma 7.3(i), the image $q(\Sigma)$ necessarily contains $G \cdot z_2$, contradicting the current assumption.

If $q(\eta) \in G \cdot w_2$, then by iterating the procedure of lifting slices and orbits we can enlarge V to an invariant domain W in Σ on which q is injective and such that

$$q(W) = G \cdot (\ell_1(\hat{I}_1) \cup w_1 \cup \ell_2(\hat{I}_2) \cup w_2 \cup \ell_3(\hat{I}_3)).$$

In particular, W is biholomorphic to $q(W)$, which is not Stein by Theorem 6.1. Hence W is a proper subset of Σ, and there exists a point η in the closure of W in Σ whose image $q(\eta)$ lies either in $G \cdot w_3$ or in $G \cdot w_4$. In both cases, Lemma 7.3 implies that $q(\Sigma)$ contains $G \cdot z_2$, contradicting the current assumption. In conclusion, if $q(\Sigma)$ does not contain the singular orbit $G \cdot z_2$ but contains the nonclosed orbit $G \cdot w_1$, then q is injective. For the other nonclosed G-orbits, the theorem can be proved by arguing in a similar way.

Case (iii): *The image $q(\Sigma)$ contains no nonclosed G-orbits.* This assumption implies that the image $q(\Sigma)$ contains none of the singular orbits lying in the closure of a nonclosed G-orbit. More precisely, $q(\Sigma)$ contains neither $G \cdot z_2$ nor $G \cdot z_3$. Note that the hypersurfaces $G \cdot (z_2 \bigcup_{j=1}^{4} w_j)$ and $G \cdot (z_3 \cup w_5)$ disconnect $G^{\mathbb{C}}/K^{\mathbb{C}}$. Therefore there exists a slice $\ell = \ell_j$ for some $j = 1, \ldots, 5$ such that $q(\Sigma) = G \cdot \ell(J)$ for some interval $J \subset \hat{I}$ that is open in $\hat{I}$. Define $M := q^{-1}(\ell(J))$. One has that $\Sigma = G \cdot M$. Moreover, since q is injective on G-orbits (see Proposition 5.7) and every orbit in $q(\Sigma)$ intersects $\ell(J)$ in a single point, every G-orbit in Σ intersects M in a single point as well. As a consequence, the surjective map $\Pi : \Sigma \mapsto M$ given by $\zeta \mapsto G \cdot \zeta \cap M$ is well defined.

Claim. The map Π is continuous.

Proof of the claim. Let N be an open set in M. We prove the claim by showing that for every $m \in N$ and $\zeta \in \Pi^{-1}(m)$, there exists an open neighborhood of ζ in Σ which is contained in $\Pi^{-1}(N)$. By construction, $\zeta = g \cdot m$ for some $g \in G$. Let V be an open neighborhood of m in Σ on which q is injective. Choose an open interval $J' \subset J$ such that $q(m) \in \ell(J') \subset q(V)$. Note that $q(m)$ either sits on a principal G-orbit or on the singular orbit $G \cdot z_1 \cong G/K$. Let $\tilde{\ell}(J')$ be the lifting of $\ell(J')$ via the restriction $q|V$. By shrinking J' if necessary, one can find an open neighborhood U of the identity in G such that $U \cdot \ell(J')$ is open and contained in $q(V)$. This fact is clear if $q(m)$ lies on a principal G-orbit; see diagram (4-9). If $q(m)$ lies on the singular orbit $G \cdot z_1$, it follows from the equivariant embedding (4-1) at z_1 and the compactness of the isotropy subgroup $G_{z_1} \cong K$.

As a result, $U \cdot \tilde{\ell}(J') = (q|V)^{-1}(U \cdot \ell(J'))$ is an open neighborhood of m in Σ, and $gU \cdot \tilde{\ell}(J')$ is an open neighborhood of ζ contained in $\Pi^{-1}(N)$. Hence $\Pi^{-1}(N)$ is open in Σ, as wished (one can show that $M \cong \Sigma/G$ and that Π can be identified with the quotient map).

By the above claim, M is connected and is a one-dimensional real-analytic submanifold of Σ. It follows that q is injective on M. Moreover M and $q(M)$ are slices for the G-action in Σ and $q(\Sigma)$, respectively. Since q is injective on G-orbits, it is injective on Σ implying the theorem.

Case (iv): *The group G is* Spin(3, 1). Assume by contradiction that $q : \Sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is not univalent. Recall that the center of G acts trivially on $G^{\mathbb{C}}/K^{\mathbb{C}}$ and that by Cases (i)–(iii), the statement holds true for the group $\mathrm{SO}_0(3, 1)$. Then Lemma 7.4 applies to show that the restriction of q to every G-orbit is a double covering and the image $q(\Sigma)$ is Stein. On the other hand, by Theorem 6.1, all Stein G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ contain a singular orbit diffeomorphic to G/K. Since G/K is simply connected, this gives a contradiction. This proves the theorem. □

When $G = \mathrm{SL}(2, \mathbb{R})$, noninjective, Stein G-equivariant Riemann domains over $G^{\mathbb{C}}/K^{\mathbb{C}}$ do exist. Next we construct one such Riemann domain explicitly. It turns out that such an example is essentially the only possible one. Indeed by Lemma 7.4, if $q : \Sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein, G-equivariant Riemann domain that is not univalent, then the center $\Gamma = \{\pm I_2\}$ acts freely on Σ. Moreover, q is a G-equivariant covering onto its image $q(\Sigma)$ that turns out to be Stein. It follows that the restriction of q to every G-orbit is a double covering. Thus the singular orbits $G \cdot z_1$ and $G \cdot z_3$, which are simply connected, cannot lie in $q(\Sigma)$. Then, by Theorem 6.1, the image $q(\Sigma)$ coincides with a domain $S_i(b)$ for some $i = 1, 2$ and $b \geq 0$. For every $S_i(b)$ there is exactly one G-equivariant double covering. In the example below, we carry out its construction for $q(\Sigma) = S_1(0)$.

Example 7.7. Let $G = \mathrm{SL}(2, \mathbb{R})$. Consider the Stein domain $S_1(0)$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ defined in (6-1). Let

$$\ell_2 : \mathbb{R}^{>0} \to G^{\mathbb{C}}/K^{\mathbb{C}} \quad \text{and} \quad \ell_2(s) := \exp(isC)z_2$$

be the slice map defined in (4-6). The isotropy subgroup in G of every point $\ell_2(s)$ coincides with $\{\pm I_2\}$; see Remarks 5.4 and 4.1. It follows that $S_1(0) := G \cdot \ell_2(\mathbb{R}^{>0})$ is topologically equivalent to $\mathrm{SO}_0(2,1) \times \mathbb{R}^{>0}$. Define $\Sigma := G \times \mathbb{R}^{>0}$. Since G is a double covering of $\mathrm{SO}_0(2,1)$, the map

$$q : \Sigma \to S_1(0), \quad (g, s) \mapsto g\ell_2(s)$$

defines a double covering of $S_1(0)$. As a consequence, with the complex structure pulled back from $S_1(0)$, the manifold Σ is Stein; see [Stein 1956]. Also the map q is a holomorphic covering. In other words, $q : \Sigma \to S_1(0)$ defines a nonunivalent Stein, G-equivariant Riemann domain over $G^{\mathbb{C}}/K^{\mathbb{C}}$. □

Remark 7.8. By the results of Lemma 7.4, one can show that Theorem 7.6 also holds for G not embedded in $G^{\mathbb{C}}$, provided that the center Γ of G is finite and G is not a covering of $\mathrm{SL}(2,\mathbb{R})$ (see Case (iv) in the proof of Theorem 7.6). If G is a covering of $\mathrm{SL}(2,\mathbb{R})$, a construction similar to the one in Example 7.7 yields a nonunivalent, Stein G-equivariant Riemann domain over $G^{\mathbb{C}}/K^{\mathbb{C}}$. □

As an application of Theorem 7.6 and the classification of all Stein G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ given in Section 6, we now exhibit a family of Kobayashi hyperbolic G-invariant subdomains of $\mathrm{SU}(1,1)^{\mathbb{C}}/\mathrm{U}(1)^{\mathbb{C}}$ whose envelopes of holomorphy are not Kobayashi hyperbolic.

Example 7.9. Let $G = \mathrm{SU}(1,1)$, and let $W_{1,1} := D_1(0) \cup G \cdot w_1 \cup S_1(0)$ be the Stein G-invariant domain defined in Example 6.3. Recall that $W_{1,1}$ is biholomorphic to $\Delta \times \mathbb{C}$ via the map

$$F : \Delta \times \mathbb{C} \to W_{1,1}, \quad (u, v) \mapsto ([u : 1], [\bar{v} : 1 + \bar{u}\bar{v}]).$$

Consider its invariant subdomains given by

$$D_c := D_1(0) \cup G \cdot w_1 \cup G \cdot \ell_2(0, c) \quad \text{for } 0 < c < \infty.$$

Denote by $\tilde{f}$ the pull-back via F of the G-invariant function f defined in (6-2). Then

$$\tilde{f}(u, v) = -(1 - |u|^2)(|1 + uv|^2 - |v|^2),$$

and $D_{\mathfrak{c}}$ is biholomorphic to a sublevel set $B_R = \{\tilde{f} < R\}$ in $\Delta \times \mathbb{C}$ for some $R > 0$. Consider the holomorphic projection $\pi : B_R \to \Delta$ onto the first factor. An easy computation shows that, for $u \in \Delta$, the preimage $\pi^{-1}(u)$ is a disk in $\mathbb{C}$ of center $(\operatorname{Re} u, -\operatorname{Im} u)/(1 - |u|^2)$ and radius $(1 + R)/(1 - |u|^2)^2$. It follows that for every $u \in \Delta$ there exists a neighborhood U of u such that $\pi^{-1}(U)$ is Kobayashi hyperbolic. Then, by [Kobayashi 1998, Theorem 3.2.14], the domains B_r and D_c are Kobayashi hyperbolic as well.

Finally from Theorem 7.6 and Theorem 6.1, it follows that the envelope of holomorphy of D_c is given by $W_{1,1}$. In particular, it is not Kobayashi hyperbolic.

8. Univalence over $G^{\mathbb{C}}$

Let G be a connected, noncompact, real simple Lie group, let $K \subset G$ be a maximal compact subgroup, and let $G^{\mathbb{C}}$ be its universal complexification. In this section we prove a univalence result for $G\times K$-equivariant Riemann domains over $G^{\mathbb{C}}$ when the symmetric space G/K has rank one. We also discuss some examples.

Theorem 8.1. *Let G/K be a noncompact, rank-one, Riemannian symmetric space. Assume that G is a connected, simple, real Lie group that has finite center and is not a covering of* $\mathrm{SL}(2, \mathbb{R})$. *Then a holomorphically separable, $G\times K$-equivariant Riemann domain $p : Y \to G^{\mathbb{C}}$ is univalent.*

Proof. Recall that Y admits a $G\times K$-equivariant holomorphic embedding into its envelope of holomorphy. Thus we may assume that Y is Stein (see Section 2). Consider the induced Stein, G-equivariant Riemann domain $q : Y /\!/ K \to G^{\mathbb{C}}/K^{\mathbb{C}}$ constructed in Section 3. By Theorem 7.6 and Remark 7.8 the map q is injective. Then, by Corollary 3.3 the Riemann domain $p : Y \to G^{\mathbb{C}}$ is univalent, as wished. □

When G is either $\mathrm{SL}(2, \mathbb{R})$ or a nontrivial covering of $\mathrm{SL}(2, \mathbb{R})$, a construction similar to the one in Example 7.7 yields examples of nonunivalent, Stein, $G\times K$-equivariant Riemann domains over $G^{\mathbb{C}}$.

Example 8.2. Let $G = \mathrm{SL}(2, \mathbb{R})$, and let $S_1(0)$ be the Stein, G-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ defined in (6-1). As we observed in Example 7.7, the domain $S_1(0)$ is diffeomorphic to $\mathrm{SO}_0(2, 1) \times \mathbb{R}^{>0}$. Define $\Omega := \pi^{-1}(S_1(0))$, where $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is the canonical projection. Since π is holomorphic and both $S_1(0)$ and $G^{\mathbb{C}}$ are Stein, the domain Ω is Stein as well. Consider the slice $\ell_2 : \mathbb{R}^{>0} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ (see (4-6)) and its lifting to $G^{\mathbb{C}}$ defined by $\tilde{\ell}_2(s) := \exp(isC)\exp(iA_2)$. Define $Y := G \times \mathbb{R}^{>0} \times K^{\mathbb{C}}$. Note that the map

$$p : Y \to \Omega, \quad (g, s, k) \mapsto g\tilde{\ell}_2(s)k^{-1}$$

is a double covering. With the complex structure pulled back from Ω, the manifold Y is Stein; see [Stein 1956]. Also, the map p is holomorphic. Let $G \times K$ act on Y by $(l, h)\cdot(g, s, k) := (lg, s, hk)$ and on Ω by left and right translations. Then p defines a nonunivalent, Stein, $G\times K$-equivariant Riemann domain over $G^{\mathbb{C}}$.

Let $G = K \times N$ be the product of a compact Lie group and a simply connected nilpotent Lie group. Then a holomorphically separable, G-equivariant Riemann domain over $G^{\mathbb{C}}$ is necessarily univalent; see [Cœuré and Loeb 1986; Iannuzzi 1999; Casadio Tarabusi et al. 2000]. The above example shows that an analogous statement does not hold for a semisimple Lie group G. Next we exhibit a different counterexample for $G = \mathrm{SO}_0(2, 1)$, a group that meets the assumptions of Theorem 8.1. Such an example was pointed out to us by K. Oeljeklaus. We are not aware of similar constructions in higher dimension. That is, if the dimension of G/K is greater than two, univalence of holomorphically separable, G-equivariant Riemann domains over $G^{\mathbb{C}}$ seems to be an open question.

Example 8.3. Let $G = \mathrm{SO}_0(2,1)$. Then $G^{\mathbb{C}} = \mathrm{SO}(2,1,\mathbb{C})$ and $K^{\mathbb{C}} = \mathrm{SO}(2,\mathbb{C})$. Let $S_1(0)$ be the G-invariant Stein domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ defined in (6-1), and let $\Omega = \pi^{-1}(S_1(0))$, where $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is the canonical projection. As we already observed in Example 8.2, the domain Ω is a Stein, G-invariant domain in $G^{\mathbb{C}}$ which is diffeomorphic to $G \times \mathbb{R}^{>0} \times K^{\mathbb{C}}$. Denote by $\widetilde{K}^{\mathbb{C}}$ the universal covering of $K^{\mathbb{C}}$ and by $\psi : \widetilde{K}^{\mathbb{C}} \to K^{\mathbb{C}}$ the covering homomorphism. Let $Y := G \times \mathbb{R}^{>0} \times \widetilde{K}^{\mathbb{C}}$, and let G act on Y by left translations. Consider the slice $\ell_2 : \mathbb{R}^{>0} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ (see (4-6)) and its lifting to $G^{\mathbb{C}}$ given by $\tilde{\ell}_2(s) := \exp(isC)\exp(iA_2)$. Define a G-equivariant covering of Ω by

$$p : Y \to \Omega, \quad (g, s, k) \mapsto g\tilde{\ell}_2(s)\psi(k^{-1}).$$

With the complex structure pulled back from Ω, the manifold Y is Stein; see [Stein 1956]. Also the map p is holomorphic. In particular $p : Y \to \Omega$ defines a nonunivalent, Stein, G-equivariant Riemann domain over $G^{\mathbb{C}}$.

Remark. One can show that Ω is a holomorphically trivial $\mathbb{C}^*$-bundle over $S_1(0)$. Thus it is biholomorphic to $S_1(0) \times \mathbb{C}^*$, and consequently Y is biholomorphic to $S_1(0) \times \mathbb{C}$. After identifying $S_1(0)$ with $\mathrm{SO}_o(2,1) \times \mathbb{R}^{>0}$, one sees that the map $\mathrm{SO}_o(2,1) \times \mathbb{R}^{>0} \to G^{\mathbb{C}}$ given by $(g, s) \mapsto g\tilde{\ell}_2(s)$ defines a global C^∞-section of the holomorphic $\mathbb{C}^*$-bundle $\pi|_\Omega : \Omega \to S_1(0)$. Hence such bundle is differentiably trivial and, by the Oka principle, is also holomorphically trivial [Grauert 1958], as claimed. For completeness, we explicitly construct a trivialization on the model of $G^{\mathbb{C}}/K^{\mathbb{C}}$ discussed in Example 4.7 and Remark 4.8.

Let $G = \mathrm{SU}(1,1)$ and identify $G^{\mathbb{C}}/K^{\mathbb{C}}$ with $\mathbb{P}^1 \times \overline{\mathbb{P}}^1 \setminus \{\langle z, w\rangle_{1,1} = 0\}$. Note that $S_1(0)$ corresponds to the subset $\{([1:u], [\bar{v}:1]) : u, v \in \Delta,\ u \neq v\}$; see Example 6.3. Let D be the diagonal in $\Delta \times \Delta$. Then the injective holomorphic map

$$\Delta \times \Delta \setminus D \to \mathbb{P}^1 \times \overline{\mathbb{P}}^1 \setminus \{\langle z, w\rangle_{1,1} = 0\}, \quad (u, v) \mapsto ([1:u], [\bar{v}:1])$$

identifies $\Delta \times \Delta \setminus D$ with $S_1(0)$. The map

$$\Delta \times \Delta \setminus D \to G^{\mathbb{C}}, \quad (u, v) \mapsto \begin{pmatrix} 1 & 1/(u-v) \\ v & u/(u-v) \end{pmatrix} =: M(u, v)$$

defines a global holomorphic section of the $\mathbb{C}^*$-bundle $\pi|_\Omega : \Omega \to S_1(0)$, since one has $M(u, v) \cdot ([0:1], [0:1]) = ([1:u], [\bar{v}:1])$. As a consequence the map

$$(\Delta \times \Delta) \setminus D \times \mathbb{C}^* \to \Omega, \quad (u, v, \lambda) \to M(u, v)\,\mathrm{diag}(\lambda^{-1}, \lambda)$$

defines a biholomorphism from $S_1(0) \times \mathbb{C}^*$ onto Ω.

9. Appendix: The Levi form of nonclosed hypersurface orbits

In this section we outline the computation of the Levi form of nonclosed hypersurface G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. We used the results in Section 6 to complete the classification of Stein G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Recall that every real hypersurface S in a complex manifold inherits a CR-structure of hypersurface type. Let J denote the complex structure of the ambient manifold. For every $x \in S$, the tangent space to S at x decomposes as $TS_x = T_{\mathbb{C}}S_x \oplus NS_x$, where $T_{\mathbb{C}}S_x = TS_x \cap J(TS_x)$ is a complex subspace of TS_x, called the *complex tangent space*, and NS_x is a one-dimensional *real* subspace. Denote by $TS = T_{\mathbb{C}}S \oplus NS$ the tangent bundle of S. The subbundle $(T_{\mathbb{C}}S)^{\mathbb{C}} \subset TS^{\mathbb{C}}$ of the complexified tangent bundle $TS^{\mathbb{C}}$ decomposes as $HS \oplus AS$, where HS and AS denote its $(1,0)$ and $(0,1)$ components, respectively. Let Z be a tangent vector in $T_{\mathbb{C}}S_x$ and $\widehat{Z}$ an arbitrary extension of Z to a local section of $T_{\mathbb{C}}S$. Then the vector fields

$$\tfrac{1}{2}(\widehat{Z} - iJ\widehat{Z}) \quad \text{and} \quad \tfrac{1}{2}(\widehat{Z} + iJ\widehat{Z})$$

define local sections of the bundles HS and AS, respectively. The Levi form of S at z is the hermitian form $L_x : T_{\mathbb{C}}S_x \times T_{\mathbb{C}}S_x \to (TS_x)^{\mathbb{C}}/(T_{\mathbb{C}}S_x)^{\mathbb{C}}$ defined by

$$L_x(Z, W,) := \frac{i}{4}[\widehat{Z} - iJ\widehat{Z}, \widehat{W} + iJ\widehat{W}]_x \mod (T_{\mathbb{C}}S)^{\mathbb{C}}.$$

In the hypersurface case, $(TS_x)^{\mathbb{C}}/(T_{\mathbb{C}}S_x)^{\mathbb{C}}$ is a one-dimensional complex vector space. When Z varies in $T_{\mathbb{C}}S_x$, the image of the quadratic form $L_x(Z, Z,)$ is contained in its real part, which can be identified with $NS_x \cong \mathbb{R}$. We say that the Levi form of S is *definite* if $\{L_x(Z, Z)\}$ is a halfline in NS_x, that it is *indefinite* if $\{L_x(Z, Z)\}$ coincides with NS_x, and that it is *identically zero* if $\{L_x(Z, Z)\} = \{0\}$; for more details, see [Boggess 1991].

9.1. ***Nonclosed orbits with a totally real singular orbit in their closure.*** We first consider nonclosed G-orbits that contain in their closure the orbit of a point $z = \exp iAK^{\mathbb{C}} \in \mathcal{A}_0$, satisfying the condition $\alpha(A) = \pi/2$, with α a simple restricted root; see (4-2) and (4-7). The singular orbit $G \cdot z$ is diffeomorphic to a rank-one, pseudo-Riemannian symmetric space G/H, embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as a totally real submanifold of maximal dimension. Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau_z)$ be the corresponding symmetric algebra. Nonclosed G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing $G \cdot z$ in their closure are in one-to-one correspondence with the nilpotent Ad_H-orbits in $\mathfrak{q}$; see (4-1) and Remark 4.2.

Let X be an element in $\mathfrak{q}$, and let $x = \exp iX \cdot z$ be the corresponding point in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Denote by S the G-orbit of x. Denote by $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ the canonical projection and by π_* its differential. Then the tangent space to S at x is generated by the vector fields induced on $G^{\mathbb{C}}/K^{\mathbb{C}}$ by the one-parameter subgroups in G, via the map

$$(9\text{-}1)\qquad {}^*:\mathfrak{g}\to T(G^{\mathbb{C}}/K^{\mathbb{C}})_x,\quad X\mapsto X^*:=(\pi_*)_x\Big(\frac{d}{dt}\Big|_{t=0}\exp tX\Big).$$

Observe that $T(G^{\mathbb{C}}/K^{\mathbb{C}})_z\cong\mathfrak{q}^{\mathbb{C}}$ and $T(G^{\mathbb{C}}/K^{\mathbb{C}})_x\cong\mathrm{Ad}_x\,\mathfrak{q}^{\mathbb{C}}$. Hence the vector X^* is the $\mathrm{Ad}_x\,\mathfrak{q}^{\mathbb{C}}$-component of X in the decomposition $\mathfrak{g}^{\mathbb{C}}=\mathrm{Ad}_x\,\mathfrak{h}^{\mathbb{C}}\oplus\mathrm{Ad}_x\,\mathfrak{q}^{\mathbb{C}}$.

To explicitly determine base points for such nonclosed orbits and their tangent spaces, we decompose $\mathfrak{g}$ by an appropriate restricted root system. Fix a maximal abelian subalgebra $\mathfrak{b}\subset\mathfrak{h}\cap\mathfrak{p}$. Because $\mathfrak{g}$ is of real rank one, $\dim\mathfrak{b}=1$ and $Z_{\mathfrak{g}}(\mathfrak{b})=\mathfrak{b}\oplus Z_{\mathfrak{k}}(\mathfrak{b})$. Let $\Delta_{\mathfrak{b}}$ be the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{b}$, and let $\mathfrak{g}=\mathfrak{g}^0\oplus\mathfrak{g}^{\pm\lambda}\oplus\mathfrak{g}^{\pm2\lambda}$ and $\mathfrak{g}^0=Z_{\mathfrak{g}}(\mathfrak{b})$ be the corresponding restricted root decomposition. Every root space $\mathfrak{g}^{\lambda}$ is τ_z-stable. For every $\mu\in\Delta_{\mathfrak{b}}\cup\{0\}$, we indicate by $\mathfrak{g}^{\mu}_{\mathfrak{h}}$ and $\mathfrak{g}^{\mu}_{\mathfrak{q}}$ the intersections of $\mathfrak{g}^{\mu}$ with $\mathfrak{h}$ and $\mathfrak{q}$, respectively. In particular, we have a combined decomposition

$$(9\text{-}2)\quad \mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q},\quad\text{where } \mathfrak{h}=\mathfrak{g}^0_{\mathfrak{h}\cap\mathfrak{k}}\oplus\mathfrak{g}^{\pm\lambda}_{\mathfrak{h}}\oplus\mathfrak{g}^{\pm2\lambda}_{\mathfrak{h}}\oplus\mathfrak{b}\text{ and }\mathfrak{q}=\mathfrak{g}^0_{\mathfrak{q}\cap\mathfrak{k}}\oplus\mathfrak{g}^{\pm\lambda}_{\mathfrak{q}}\oplus\mathfrak{g}^{\pm2\lambda}_{\mathfrak{q}}.$$

Here $\mathfrak{g}^0_{\mathfrak{h}\cap\mathfrak{k}}$ and $\mathfrak{g}^0_{\mathfrak{q}\cap\mathfrak{k}}$ denote the intersections of $Z_{\mathfrak{k}}(\mathfrak{b})$ with $\mathfrak{h}$ and $\mathfrak{q}$, respectively. Note that, by the real rank one condition, $\mathfrak{g}^0_{\mathfrak{q}\cap\mathfrak{k}}$ coincides with $\mathfrak{g}^0_{\mathfrak{q}}$. If the restricted roots system $\Delta_{\mathfrak{b}}$ is reduced, then $\mathfrak{g}^{\pm2\lambda}=\{0\}$.

Lemma 9.1. *Let $\mathfrak{g}$ be a simple real Lie algebra of real rank one with reduced restricted root system* (*that is*, $\mathfrak{g}=\mathfrak{so}(n,1)$). *Then the following facts hold*:

(i) $\dim\mathfrak{g}^{\pm\lambda}_{\mathfrak{q}}=1$.

(ii) $[\mathfrak{g}^{\lambda}_{\mathfrak{q}},\mathfrak{g}^{-\lambda}_{\mathfrak{h}}]=\mathfrak{g}^0_{\mathfrak{q}}$ *and* $[\mathfrak{g}^{-\lambda}_{\mathfrak{q}},\mathfrak{g}^{\lambda}_{\mathfrak{h}}]=\mathfrak{g}^0_{\mathfrak{q}}$.

Proof. Observe that $\theta\mathfrak{g}^{\lambda}_{\mathfrak{q}}=\mathfrak{g}^{-\lambda}_{\mathfrak{q}}$. Hence $\mathfrak{g}_{\mathfrak{q}}[\lambda]:=\mathfrak{g}^{\lambda}_{\mathfrak{q}}\oplus\mathfrak{g}^{-\lambda}_{\mathfrak{q}}$ is a θ-stable subspace of $\mathfrak{q}$ and $\dim\mathfrak{g}_{\mathfrak{q}}[\lambda]\cap\mathfrak{p}=\dim\mathfrak{g}^{\lambda}_{\mathfrak{q}}$. Since $\mathfrak{g}^0_{\mathfrak{q}}\subset\mathfrak{k}$ and $\dim\mathfrak{p}\cap\mathfrak{q}=1$ (see the proof of Lemma 4.3(ii), statement (i) holds. Statement (ii) can be verified directly. □

Lemma 9.2. *Let $\mathfrak{g}$ be a real simple Lie algebra of real rank one with nonreduced restricted root system* (*that is*, $\mathfrak{g}=\mathfrak{su}(n,1)$, $\mathfrak{sp}(n,1)$, *or* $\mathfrak{f}_4^*$). *Then the following facts hold*:

(i) *The root spaces $\mathfrak{g}^{\pm2\lambda}$ are contained in $\mathfrak{h}$. Therefore $\mathfrak{g}^{\pm2\lambda}_{\mathfrak{q}}=\{0\}$.*

(ii) $\dim\mathfrak{g}^{\pm\lambda}_{\mathfrak{q}}>1$.

(iii) *Fix $X^0_{\lambda}\in\mathfrak{g}^{\lambda}_{\mathfrak{q}}$ and denote by $(\mathfrak{g}^{\lambda}_{\mathfrak{q}})_0$ a complement of $\mathbb{R}X^0_{\lambda}$ in $\mathfrak{g}^{\lambda}_{\mathfrak{q}}$; denote by $(\mathfrak{g}^{-\lambda}_{\mathfrak{q}})_0$ a complement of $\mathbb{R}\theta X^0_{\lambda}$ in $\mathfrak{g}^{-\lambda}_{\mathfrak{q}}$. Then*

$$[X^0_{\lambda},\mathfrak{g}^0_{\mathfrak{h}\cap\mathfrak{k}}]=(\mathfrak{g}^{\lambda}_{\mathfrak{q}})_0,\quad [X^0_{\lambda},\mathfrak{g}^{-\lambda}_{\mathfrak{h}}]=\mathfrak{g}^0_{\mathfrak{q}},\quad [X^0_{\lambda},\mathfrak{g}^{-2\lambda}_{\mathfrak{h}}]=(\mathfrak{g}^{-\lambda}_{\mathfrak{q}})_0.$$

Proof. Real rank one Lie algebras with a nonreduced restricted root system are equal-rank. Hence the root system Δ of $\mathfrak{g}^{\mathbb{C}}$, with respect to a maximally split Cartan subalgebra of $\mathfrak{g}$ extending $\mathfrak{b}$, has a real root. Since $\dim \mathfrak{g}^{2\lambda}$ is odd, the restriction of such a root to $\mathfrak{b}$ coincides with the restricted root 2λ; see [Helgason 1978, page 584]. Further, by [Geatti 2002, Remark 2.13], the subalgebra $\mathfrak{h}$ is a noncompact real form of $\mathrm{Ad}_z\,\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{k}^{\mathbb{C}}$ with respect to the conjugation $\sigma\tau_z|\,\mathrm{Ad}_z\,\mathfrak{k}^{\mathbb{C}}$. Precisely, if $\mathfrak{g} = \mathfrak{su}(n,1)$, $\mathfrak{sp}(n,1)$, or $\mathfrak{f}_4^*$, then $\mathfrak{h}$ is given by $\mathfrak{u}(n-1,1) \oplus \mathfrak{u}(1)$, $\mathfrak{sp}(n-1,1) \oplus \mathfrak{sp}(1)$, or $\mathfrak{so}(8,1)$, respectively. Since $\mathfrak{h}$ is equal-rank, the root spaces $\mathfrak{g}^{\pm 2\lambda}$ have nontrivial intersection with $\mathfrak{h}$. Statements (i) and (ii) then follow by looking at the dimensions of the restricted root spaces of $\mathfrak{h}$ and $\mathfrak{g}$; (see [Helgason 1978, page 532]). Statement (iii) can be verified directly. □

Reference points for nonclosed G-orbits. Let $\mathscr{C} = \exp i\mathfrak{c}\cdot z$ be the standard Cartan subset in $G^{\mathbb{C}}/K^{\mathbb{C}}$ with base point z. Recall that $\mathfrak{c} = \mathbb{R}(X+\theta(X))$, where X is a nonzero vector in $\mathfrak{g}^{\alpha}$ (here $\mathfrak{g}^{\alpha}$ is a restricted root space with respect to the adjoint action of $\mathfrak{a} \subset \mathfrak{p}$, as in Section 4). Normalize the triple $\{X, \theta(X), A := [\theta(X), X]\}$ so that $\alpha(A) = 2$. Define $B := X - \theta(X)$ and $\mathfrak{b} := \mathbb{R}(X - \theta(X))$. One easily verifies that $\mathfrak{b}$ is a maximal abelian subalgebra in $\mathfrak{h} \cap \mathfrak{p}$. If the restricted root system $\Delta_{\mathfrak{b}}$ is reduced, then

$$X^0_\lambda = \tfrac{1}{2}(A - (X + \theta X)) \quad \text{and} \quad X^0_{-\lambda} = \tfrac{1}{2}(A + (X + \theta X)) \tag{9-3}$$

are generators of the one-dimensional spaces $\mathfrak{g}^{\lambda}_{\mathfrak{q}}$ and $\mathfrak{g}^{-\lambda}_{\mathfrak{q}}$, respectively. They satisfy the relations

$$[B, X^0_\lambda] = 2X^0_\lambda, \quad [B, X^0_{-\lambda}] = -2X^0_{-\lambda}, \quad [X^0_\lambda, X^0_{-\lambda}] = B, \quad \theta X^0_\lambda = -X^0_{-\lambda}.$$

The vectors X^0_λ, $X^0_{-\lambda}$, $-X^0_\lambda$ and $-X_{-\lambda^0}$ are a complete set of representatives of the nilpotent Ad_H-orbits in $\mathfrak{q}$. The corresponding points in $G^{\mathbb{C}}/K^{\mathbb{C}}$,

$$x_0 = \exp iX^0_\lambda \cdot z, \quad x_1 = \exp iX^0_{-\lambda}\cdot z, \quad y_0 = \exp(-iX^0_\lambda)\cdot z, \quad y_1 = \exp(-iX^0_{-\lambda})\cdot z$$

lie on nonclosed G-orbits containing the singular orbit $G\cdot z$ in their closures. In the orbit diagram (4-3), the G-orbits of x_0, x_1, y_0, y_1 are represented by w_3, w_2, w_1, w_4, respectively. If $\dim G/K > 2$, the points x_0 and x_1 lie on the same G-orbit and likewise the points y_0 and y_1; see diagram (4-4). When the restricted root system $\Delta_{\mathfrak{b}}$ is nonreduced, all points $x = \exp iX_\lambda \cdot z$ with $X_\lambda \in \mathfrak{g}^{\lambda}_{\mathfrak{q}}$ and $y = \exp iX_{-\lambda}\cdot z$ with $X_{-\lambda} \in \mathfrak{g}^{-\lambda}_{\mathfrak{q}}$ lie on the same G-orbit. They are represented by w_5 in the orbit diagrams (4-9) and (4-10).

Remark 9.3. When the restricted root system $\Delta_{\mathfrak{b}}$ is reduced, the points x_0 and x_1 lie on the boundary of the Stein domain $D_2(0)$. The points y_0 and y_1 lie on the boundary of the Stein domain $D_1(0)$; see (6-1).

The tangent space to the G-orbit of x_0. Denote by S the G-orbit of the point $x_0 = \exp iX^0_\lambda \cdot z$ with $X^0_\lambda \in \mathfrak{g}^\lambda_\mathfrak{q}$. In the next lemma, we determine the generators of the tangent space to S at x_0, namely, the vectors $X^* \in TS_{x_0}$ for X ranging in the root spaces $\mathfrak{g}^\mu$ for $\mu \in \Delta_\mathfrak{b}$; see (9-2).

Lemma 9.4. *We have the following table.*

X	$\mathfrak{g}^\mu$	X^*, *if* $X \in \mathfrak{g}^\mu$
$Y_{2\lambda}$	$\mathfrak{g}^{2\lambda}_\mathfrak{h}$	0
Y_λ	$\mathfrak{g}^{\lambda}_\mathfrak{h}$	0
X_λ	$\mathfrak{g}^{\lambda}_\mathfrak{q}$	$\mathrm{Ad}_{x_0} X_\lambda.$
B	$\mathfrak{b}$	$i\lambda(B)\,\mathrm{Ad}_{x_0} X^0_\lambda$
W_0	$\mathfrak{g}^0_{\mathfrak{h}\cap\mathfrak{k}}$	$-i\,\mathrm{Ad}_{x_0}[X^0_\lambda, W_0]$
Z_0	$\mathfrak{g}^0_\mathfrak{q}$	$\mathrm{Ad}_{x_0} Z_0$
$Y_{-\lambda}$	$\mathfrak{g}^{-\lambda}_\mathfrak{h}$	$-i\,\mathrm{Ad}_{x_0}[X^0_\lambda, Y_{-\lambda}]$
$X_{-\lambda}$	$\mathfrak{g}^{-\lambda}_\mathfrak{q}$	$\mathrm{Ad}_{x_0} X_{-\lambda} - \frac{1}{2}\mathrm{Ad}_{x_0}[X^0_\lambda, [X^0_\lambda, X_{-\lambda}]]$
$Y_{-2\lambda}$	$\mathfrak{g}^{-2\lambda}_\mathfrak{h}$	$-i\,\mathrm{Ad}_{x_0}[X^0_\lambda, Y_{-2\lambda}] + \frac{i}{6}\mathrm{Ad}_{x_0}[X^0_\lambda, [X^0_\lambda, [X^0_\lambda, Y_{-2\lambda}]]]$

Proof. All rows are obtained by combining the formula $\mathrm{Ad}_{\exp iX} Y = \exp \mathrm{ad}_{iX} Y$ with the bracket relations among root vectors. We omit the computations, which are long but straightforward. □

Fix $\theta X^0_\lambda \in \mathfrak{g}^{-\lambda}_\mathfrak{q}$, and denote by $(\mathfrak{g}^{-\lambda}_\mathfrak{q})_0$ a complementary subspace to $\mathbb{R}\theta X^0_\lambda$ in $\mathfrak{g}^{-\lambda}_\mathfrak{q}$. By Lemma 9.2(iii) and Lemma 9.4, the tangent space to S at x_0 is given by $TS_{x_0} = T_\mathbb{C} S_{x_0} \oplus NS_{x_0}$, where

(9-4) $$T_\mathbb{C} S_{x_0} = \mathrm{Ad}_{x_0}(\mathfrak{g}^0_\mathfrak{q})^\mathbb{C} \oplus \mathrm{Ad}_{x_0}(\mathfrak{g}^\lambda_\mathfrak{q})^\mathbb{C} \oplus \mathrm{Ad}_{x_0}(\mathfrak{g}^{-\lambda}_\mathfrak{q})^\mathbb{C}_0 \quad \text{and} \quad NS_{x_0} = \mathbb{R}\,\mathrm{Ad}_{x_0}\theta X^0_\lambda.$$

Note that if $\Delta_\mathfrak{b}$ is reduced, one has $(\mathfrak{g}^{-\lambda}_\mathfrak{q})_0 = \{0\}$ by Lemma 9.1(i).

Remark 9.5. There exists a basis of $\mathfrak{g}$ such that the above decomposition of TS_{x_0} is orthogonal with respect to the Killing form B of $\mathfrak{g}^\mathbb{C}$. If the restricted root system $\Delta_\mathfrak{b}$ is nonreduced, one can construct it starting from a basis of $\mathfrak{g}^\mathbb{C}/\mathfrak{s}^\mathbb{C}$ consisting of root vectors with respect to a maximally split Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ extending $\mathfrak{b}$. In the reduced case, this is immediate by Lemma 9.1(i).

The Levi form of the G-orbit of x_0. The same arguments used in [Geatti 2002, Section 4] yield the following formulas for the Levi form of S at x_0. Let Z and W be vectors in $T_\mathbb{C} S_{x_0}$. Then

(9-5) $$L_{x_0}(Z, W) = \frac{1}{2}[(\,\cdot\,)^{-1} JW, Z] - \frac{i}{2}[(\,\cdot\,)^{-1} W, Z] \mod (T_\mathbb{C} S_{x_0})^\mathbb{C},$$

where $(\,\cdot\,)^{-1}JW$ and $(\,\cdot\,)^{-1}W$ are arbitrary elements in the preimages of JW and W by the map defined in (9-1). In the next lemma we compute the Levi form of S at x_0. Fix $F^0_{-\lambda} := \mathrm{Ad}_{x_0}\theta X^0_\lambda$ as a generator of NS_{x_0}.

Lemma 9.6. (i) *Let $X_{-\lambda} \in (\mathfrak{g}_\mathfrak{q}^{-\lambda})_0$. Set $F_{-\lambda} := \mathrm{Ad}_{x_0} X_{-\lambda}$. Then*

$$L_{x_0}(F_{-\lambda}, F_{-\lambda}) = -\tfrac{1}{6}\,\mathrm{Ad}_{x_0}[[X^0_\lambda, X_{-\lambda}], X_{-\lambda}] = pF^0_{-\lambda} \bmod (T_\mathbb{C} S_{x_0})^\mathbb{C}, \text{ where } p \geq 0.$$

(ii) *Let $Z_0 \in \mathfrak{g}^0_\mathfrak{q}$. Write $Z_0 = [X^0_\lambda, Y_{-\lambda}]$ for some $Y_{-\lambda} \in \mathfrak{g}_\mathfrak{h}^{-\lambda}$ (see Lemma 9.2), and set $F_0 := \mathrm{Ad}_{x_0} Z_0$. Then*

$$L_{x_0}(F_0, F_0) = -\tfrac{1}{2}\,\mathrm{Ad}_{x_0}[Y_{-\lambda}, Z_0] = nF^0_{-\lambda} \bmod (T_\mathbb{C} S_{x_0})^\mathbb{C}, \text{ where } n \leq 0.$$

(iii) *Let $X_\lambda \in \mathfrak{g}^\lambda_\mathfrak{q}$, and set $F_\lambda := \mathrm{Ad}_{x_0} X_\lambda$. Then $L_{x_0}(F_\lambda, F_\lambda) = 0$.*

(iv) *Let $X_{-\lambda} \in (\mathfrak{g}_\mathfrak{q}^{-\lambda})_0$ and $Z_0 \in \mathfrak{g}^0_\mathfrak{q}$. Set $F_{-\lambda} := \mathrm{Ad}_{x_0} X_{-\lambda}$ and $F_0 := \mathrm{Ad}_{x_0} Z_0$. Then $L_{x_0}(F_{-\lambda}, F_0) = 0$.*

(v) *Let $X_{-\lambda} \in (\mathfrak{g}_\mathfrak{q}^{-\lambda})_0$ and $X_\lambda \in \mathfrak{g}^\lambda_\mathfrak{q}$. Set $F_{-\lambda} := \mathrm{Ad}_{x_0} X_{-\lambda}$ and $F_\lambda := \mathrm{Ad}_{x_0} X_\lambda$. Then $L_{x_0}(F_{-\lambda}, F_\lambda) = aF^0_{-\lambda}$, where $a \in \mathbb{C}$.*

(vi) *Let $X_\lambda \in \mathfrak{g}^\lambda_\mathfrak{q}$ and $Z_0 \in \mathfrak{g}^0_\mathfrak{q}$. Set $F_\lambda := \mathrm{Ad}_{x_0} X_\lambda$ and $F_0 := \mathrm{Ad}_{x_0} Z_0$. Then $L_{x_0}(F_0, F_\lambda) = 0$.*

Proof. By way of example, we prove the first two statements. The remaining ones follow similarly, and the details are omitted.

(i) Let $F_{-\lambda} = \mathrm{Ad}_{x_0} X_{-\lambda}$. In order to compute the brackets (9-5), we invert the relations in Lemma 9.4 and decompose the results in $\mathfrak{g}^\mathbb{C} = \mathrm{Ad}_{x_0}\mathfrak{h}^\mathbb{C} \oplus \mathrm{Ad}_{x_0}\mathfrak{q}^\mathbb{C}$. Write $X_{-\lambda} = [X^0_\lambda, Y_{-2\lambda}]$ for some $Y_{-2\lambda} \in \mathfrak{g}_\mathfrak{h}^{2\lambda}$; see Lemma 9.2. Then

$$\begin{aligned}(\,\cdot\,)^{-1}JF_{-\lambda} &= -Y_{-2\lambda} + \tfrac{1}{6}\,\mathrm{ad}^2_{X^0_\lambda}(Y_{-2\lambda})\\ &= -\mathrm{Ad}_{x_0} Y_{-2\lambda} + i\,\mathrm{Ad}_{x_0}\mathrm{ad}_{X^0_\lambda}(Y_{-2\lambda}) + \tfrac{1}{2}\,\mathrm{Ad}_{x_0}\mathrm{ad}^2_{X^0_\lambda}(Y_{-2\lambda})\\ &\quad - \tfrac{i}{6}\,\mathrm{Ad}_{x_0}\mathrm{ad}^3_{X^0_\lambda}(Y_{-2\lambda}) + \tfrac{3}{8}\,\mathrm{Ad}_{x_0}\mathrm{ad}^4_{X^0_\lambda}(Y_{-2\lambda})\end{aligned}$$

and

$$\begin{aligned}(\,\cdot\,)^{-1}F_{-\lambda} &= \mathrm{ad}_{X^0_\lambda}(Y_{-2\lambda}) + \tfrac{1}{2}\,\mathrm{ad}^3_{X^0_\lambda}(Y_{-2\lambda})\\ &= \mathrm{Ad}_{x_0}\mathrm{ad}_{X^0_\lambda}(Y_{-2\lambda}) - i\,\mathrm{Ad}_{x_0}\mathrm{ad}^2_{X^0_\lambda}(Y_{-2\lambda}) + \tfrac{i}{6}\,\mathrm{Ad}_{x_0}\mathrm{ad}^4_{X^0_\lambda}(Y_{-2\lambda}).\end{aligned}$$

By formulas (9-5), we obtain

$$\begin{aligned}L_{x_0}(F_{-\lambda}, F_{-\lambda}) &= -\tfrac{1}{6}\,\mathrm{Ad}_{x_0}[\mathrm{ad}^2_{X^0_\lambda}(Y_{-2\lambda}), \mathrm{ad}_{X^0_\lambda}(Y_{-2\lambda})]\\ &= -\tfrac{1}{6}\,\mathrm{Ad}_{x_0}[[X^0_\lambda, X_{-\lambda}], X_{-\lambda}] \quad \bmod (T_\mathbb{C} S_{x_0})^\mathbb{C}.\end{aligned}$$

To complete the proof of the statement, set $F^0_\lambda := \mathrm{Ad}_{x_0} X^0_\lambda$, and note that due to Remark 9.5, the component $pF^0_{-\lambda}$ of the above brackets in NS_{x_0} is given by

$$B(L_{x_0}(F_{-\lambda}, F_{-\lambda}), F^0_\lambda) = pB(F^0_{-\lambda}, F^0_\lambda).$$

Since $B(F^0_{-\lambda}, F^0_\lambda) = B(X^0_{-\lambda}, \theta X^0_{-\lambda})$ is negative, the real number p has the same sign as

$$B([[X^0_\lambda, X_{-\lambda}], X_{-\lambda}], X^0_\lambda) = -B([X^0_\lambda, X_{-\lambda}], [X^0_\lambda, X_{-\lambda}]).$$

By Lemmas 9.1 and 9.2, the brackets $[X^0_\lambda, X_{-\lambda}]$ lie in $\mathfrak{k}$, so

$$B([X^0_\lambda, X_{-\lambda}], [X^0_\lambda, X_{-\lambda}])$$

is nonpositive. It follows that $p \geq 0$, as claimed.

(ii) Write $Z_0 = [X^0_\lambda, Y_{-\lambda}]$ for some $Y_{-\lambda} \in \mathfrak{g}_{\mathfrak{h}}^{-\lambda}$; see Lemmas 9.1 and 9.2. By computations similar to the above ones, we have

$$(\,\cdot\,)^{-1} J F_0 = -Y_{-\lambda} = -\mathrm{Ad}_{x_0} Y_{-\lambda} + i\,\mathrm{Ad}_{x_0}\mathrm{ad}_{X^0_\lambda}(Y_{-\lambda}) + \tfrac{1}{2}\mathrm{Ad}_{x_0}\mathrm{ad}^2_{X^0_\lambda}(Y_{-\lambda}),$$

$$(\,\cdot\,)^{-1} F_0 = Z_0 = \mathrm{Ad}_{x_0} Z_0 - i\,\mathrm{Ad}_{x_0}\mathrm{ad}_{X^0_\lambda}(Z_0)$$

and

$$L_{x_0}(F_0, F_0) = -\tfrac{1}{2}\mathrm{Ad}_{x_0}[Y_{-\lambda}, [X^0_\lambda, Y_{-\lambda}]] = -\tfrac{1}{2}\mathrm{Ad}_{x_0}[Y_{-\lambda}, Z_0] \mod T_{\mathbb{C}}S_{x_0})^{\mathbb{C}}.$$

To complete the proof of (ii), observe that $n = B(L(F_0, F_0), F^0_\lambda)/B(F^0_{-\lambda}, F^0_\lambda)$ has the same sign as $B([Y_{-\lambda}, [X^0_\lambda, Y_{-\lambda}]], X^0_\lambda) = B([X^0_\lambda, Y_{-\lambda}], [X^0_\lambda, Y_{-\lambda}])$. Since $[X^0_\lambda, Y_{-\lambda}]$ lies in $\mathfrak{k}$, the above expression is nonpositive and $n \leq 0$, as claimed. □

Proposition 9.7. *Let S be the G-orbit of the point $x_0 = \exp iX^0_\lambda \cdot z$.*

If the restricted root system $\Delta_{\mathfrak{b}}$ is reduced, then the Levi form of the orbit S is definite provided that $\dim G/K > 2$. It is identically zero if $\dim G/K = 2$.

If the restricted root system $\Delta_{\mathfrak{b}}$ is nonreduced, then the Levi form of the orbit S is indefinite.

Proof. If the restricted root system $\Delta_{\mathfrak{b}}$ is reduced, then only (ii), (iii), and (iv) of Lemma 9.6 apply. By Lemma 9.6(ii), for every $F_0 \in \mathrm{Ad}_{x_0}(\mathfrak{g}^0_{\mathfrak{q}})^{\mathbb{C}}$, the real numbers $B(L(F_0, F_0), F^0_\lambda)$ all have the same sign. In other words, the restriction of the Levi form to $\mathrm{Ad}_{x_0}(\mathfrak{g}^0_{\mathfrak{q}})^{\mathbb{C}} \subset T_{\mathbb{C}}S_{x_0}$ is either definite or identically zero. It is identically zero when $\mathrm{ad}_{X^0_\lambda} : \mathfrak{g}_{\mathfrak{h}}^{-\lambda} \to \mathfrak{g}^0_{\mathfrak{q}}$ is the zero-map. This happens if and only if $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and $\dim G/K = 2$.

If the restricted root system $\Delta_{\mathfrak{b}}$ is nonreduced, then $\dim G/K > 2$. In this case, the restriction of the Levi form to $\mathrm{Ad}_{x_0}(\mathfrak{g}^0_{\mathfrak{q}})^{\mathbb{C}} \subset T_{\mathbb{C}}S_{x_0}$ is definite. Also, by Lemma 9.6(i) and Lemma 9.2(iii), the restriction of the Levi form to $\mathrm{Ad}_{x_0}(\mathfrak{g}^0_{\mathfrak{q}})^{\mathbb{C}} \subset T_{\mathbb{C}}S_{x_0}$ is definite with opposite sign. As a result, the Levi form of S is indefinite. □

The Levi form of the G-orbit y_0. By the same methods, one can compute the tangent space and the Levi form of the G-orbit S of the point $y_0 = \exp(-iX^0_\lambda)\cdot z$. As we already remarked, the orbits $G\cdot x_0$ and $G\cdot y_0$ are distinct only when the restricted root system of $\mathfrak{g}$ is reduced. So we focus on this case. For the tangent space to S at y_0, one has $TS_{y_0} = T_{\mathbb{C}}S_{y_0} \oplus NS_{y_0}$, where

$$T_{\mathbb{C}}S_{y_0} = \mathrm{Ad}_{y_0}(\mathfrak{g}^0_{\mathfrak{q}})^{\mathbb{C}} \oplus \mathrm{Ad}_{y_0}(\mathfrak{g}^\lambda_{\mathfrak{q}})^{\mathbb{C}} \quad \text{and} \quad NS_{y_0} = \mathbb{R}\,\mathrm{Ad}_{y_0}\theta X^0_\lambda.$$

Fix $F^0_{-\lambda} := \mathrm{Ad}_{y_0}\theta X^0_\lambda$ as a generator of NS_{y_0}. For the Levi form, one has the following results.

Lemma 9.8. (i) *Let $Z_0 \in \mathfrak{g}^0_{\mathfrak{q}}$. Write $Z_0 = [X^0_\lambda, Y_{-\lambda}]$, for some $Y_{-\lambda} \in \mathfrak{g}^{-\lambda}_{\mathfrak{h}}$ (see Lemma 9.2), and set $F_0 := \mathrm{Ad}_{y_0} Z_0$. Then*

$$L_{y_0}(F_0, F_0) = \tfrac{1}{2}\mathrm{Ad}_{y_0}[Y_{-\lambda}, Z_0] = sF^0_{-\lambda} \mod (T_{\mathbb{C}}S_{y_0})^{\mathbb{C}}, \quad \text{where } s \geq 0.$$

(ii) *Let $X_\lambda \in \mathfrak{g}^\lambda_{\mathfrak{q}}$, and set $F_\lambda := \mathrm{Ad}_{y_0} X_\lambda$. Then $L_{y_0}(F_\lambda, F_\lambda) = 0$.*

(iii) *Let $Z_0 \in \mathfrak{g}^0_{\mathfrak{q}}$ and $X_\lambda \in \mathfrak{g}^\lambda_{\mathfrak{q}}$. Set $F_0 := \mathrm{Ad}_{y_0} Z_0$ and $F_\lambda := \mathrm{Ad}_{y_0} X_\lambda$. Then $L_{y_0}(F_0, F_\lambda) = 0$.*

Proposition 9.9. *Let S be the G-orbit of the point y_0.*

If the restricted root system $\Delta_{\mathfrak{b}}$ is reduced, then the Levi form of the orbit S is definite provided that $\dim G/K > 2$. It is identically zero if $\dim G/K = 2$.

Remark 9.10. By Propositions 9.7 and 9.9, if the restricted root system $\Delta_{\mathfrak{b}}$ is reduced, then the Levi form of the orbits represented by w_1 and w_2 in diagram (4-4) is definite. This is consistent with the fact that these orbits lie in the boundary of the Stein domains $D_1(0)$ and $D_2(0)$, respectively; see Theorem 6.1. If $\dim G/K = 2$, all orbits represented by $w_1, \dots, w_4$ in diagram (4-3) are Levi flat. We refer to Example 6.3 for a classification of G-invariant Stein domains bounded by such orbits. If the restricted root system $\Delta_{\mathfrak{b}}$ is nonreduced, then the Levi form of the orbit represented by w_5 in diagrams (4-9) and (4-10) is indefinite. As a consequence, this orbit cannot lie in the boundary of a Stein G-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

9.2. ***Nonclosed orbits with a CR singular orbit in their closure.*** We consider now nonclosed G-orbits containing in their closure the orbit of a point $z = gK^{\mathbb{C}} = \exp iAK^{\mathbb{C}} \in \mathcal{A}_0$, satisfying the condition $\alpha(A) = \pi/4$, with α a simple restricted root; see (4-7). In this case the singular orbit $G\cdot z$ has dimension greater than $\dim G/K$. Recall from Section 4.2 that the isotropy subgroup H' of z in G is contained in $G' := Z_G(g^4)$ and that G'/H' is diffeomorphic to a rank-one, pseudo-Riemannian symmetric space, totally real in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Let $(\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{q}', \tau_z)$ be the associated symmetric algebra. Nonclosed G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing $G\cdot z$ in their closure are in one-to-one correspondence with the nilpotent $\mathrm{Ad}_{H'}$-orbits in $\mathfrak{q}'$; see (4-1) and Remark 4.2.

To explicitly determine reference points for such nonclosed orbits and their tangent spaces, we decompose $\mathfrak{g}$ and $\mathfrak{g}'$ by an appropriate restricted root system. Let $\mathscr{C}' = \exp i\mathfrak{c}' \cdot z$ be the standard Cartan subset with base point z. Recall that $\mathfrak{c}' = \mathbb{R}(X+\theta(X))$, where X is a nonzero vector in $\mathfrak{g}^{2\alpha}$. In particular, $\mathfrak{c}'$ is contained in $\mathfrak{g}'$; see (4-8). Define $\mathfrak{b}' = \mathbb{R}(X-\theta(X))$. Then $\mathfrak{b}'$ is a maximal abelian subalgebra in $\mathfrak{h}' \cap \mathfrak{p}$ and the restricted root decompositions of $\mathfrak{g}$ with respect to $\mathfrak{b}'$ is given by

$$\mathfrak{g} = Z_{\mathfrak{k}}(\mathfrak{b}') \oplus \mathfrak{b}' \oplus \mathfrak{g}^{\pm 2\lambda} \oplus \mathfrak{g}^{\pm\lambda}.$$

In order to determine how the above root decomposition restricts to the subalgebra $\mathfrak{g}'$, observe that in general $\mathfrak{g}'$ is not simple, but is the direct sum of a copy of $\mathfrak{so}(m, 1)$ with $m = \dim \mathfrak{g}^{2\alpha} + 1$ (even) and a compact subalgebra $\mathfrak{l}$ entirely contained in $\mathfrak{h}'$, that is,

$$\mathfrak{g}' = \mathfrak{l} \oplus \mathfrak{so}(m, 1) \quad \text{and} \quad \mathfrak{h}' = \mathfrak{l} \oplus \mathfrak{so}(m-1, 1).$$

Observe also that all real rank one Lie algebras with a nonreduced restricted root system are equal-rank. Hence the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ with respect to a maximally split Cartan subalgebra of $\mathfrak{g}$ extending $\mathfrak{b}'$ contains a real root. Since $\mathfrak{g}^{2\lambda}$ is odd-dimensional (see Table 4.0), the restriction of this real root to $\mathfrak{b}'$ coincides the restricted root 2λ; see [Helgason 1978, page 584]. Since $\mathfrak{g}'$ has a reduced restricted root system (see (4-8)) and because $\mathfrak{so}(m, 1)$ with m even is equal-rank, we have $\mathfrak{g}' \cap \mathfrak{g}^{2\lambda} \neq \{0\}$. It follows that the restricted root decomposition of $\mathfrak{g}'$ with respect to $\mathfrak{b}'$ is given by

$$\mathfrak{g}' = Z_{\mathfrak{k}}(\mathfrak{b}') \oplus \mathfrak{b}' \oplus \mathfrak{g}^{\pm 2\lambda}. \tag{9-6}$$

Let

$$\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{q}', \quad \text{with } \mathfrak{h}' = \mathfrak{g}^0_{\mathfrak{h}'\cap\mathfrak{k}} \oplus \mathfrak{g}^{\pm 2\lambda}_{\mathfrak{h}'} \oplus \mathfrak{b}' \quad \text{and} \quad \mathfrak{q}' = \mathfrak{g}^0_{\mathfrak{q}'} \oplus \mathfrak{g}^{\pm 2\lambda}_{\mathfrak{q}'},$$

be the combined decomposition of $\mathfrak{g}'$. Note that $\mathfrak{g}'$ has real rank one as well. Therefore $\mathfrak{g}^0_{\mathfrak{q}'} \subset \mathfrak{k}$ and an analogue of Lemma 9.1 holds for $\mathfrak{g}'$. Set $\mathfrak{g}[\lambda] := \mathfrak{g}^{\lambda} \oplus \mathfrak{g}^{-\lambda}$ and $\mathfrak{g}[\alpha] := \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}$ (here α is a restricted root in $\Delta_{\mathfrak{a}}$, as in Section 4).

Lemma 9.11. *The following facts hold*:

(i) $\dim \mathfrak{g}^{\pm 2\lambda}_{\mathfrak{q}'} = 1$.

(ii) $[\mathfrak{g}^{2\lambda}_{\mathfrak{q}'}, \mathfrak{g}^{-2\lambda}_{\mathfrak{h}'}] = \mathfrak{g}^0_{\mathfrak{q}'}$ *and* $[\mathfrak{g}^{-2\lambda}_{\mathfrak{q}'}, \mathfrak{g}^{2\lambda}_{\mathfrak{h}'}] = \mathfrak{g}^0_{\mathfrak{q}'}$.

(iii) *the decomposition* $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}[\alpha]$ *is* $\mathrm{ad}_{\mathfrak{b}'}$*-stable. In particular* $\mathfrak{g}[\alpha] = \mathfrak{g}[\lambda]$.

Proof. Statement (i) follows from the fact that $\dim \mathfrak{q}' \cap \mathfrak{p} = 1$ (see the proof of Lemma 4.5(ii), while (ii) can be checked directly.

To prove (iii), note that $\mathrm{ad}_{\mathfrak{b}'}\,\mathfrak{g}' \subset \mathfrak{g}'$. Moreover, $\mathrm{ad}_{\mathfrak{b}'}(\mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}) \subset (\mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha})$. By (4-8) and (9-6) it follows that the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}[\alpha]$ is $\mathrm{ad}_{\mathfrak{b}'}$-stable and $\mathfrak{g}[\alpha] = \mathfrak{g}[\lambda]$. □

Reference points for nonclosed G-orbits. Reference points for nonclosed orbits containing $G \cdot z$ in their closures can be obtained by applying the methods of the previous section to the symmetric space G'/H'; see (9-3). In this case take $X \in \mathfrak{g}^{2\alpha}$, θX and $A := [\theta X, X]$, normalized so that $2\alpha(A) = 2$. Then

$$X^0_{2\lambda} = \tfrac{1}{2}(A - (X + \theta X)) \quad \text{and} \quad X^0_{-2\lambda} = \tfrac{1}{2}(A + (X + \theta X)) \tag{9-7}$$

are generators of $\mathfrak{g}^{2\lambda}_{\mathfrak{q}'}$ and $\mathfrak{g}^{-2\lambda}_{\mathfrak{q}'}$, respectively, and the points

$$x_0 = \exp i X^0_{2\lambda} \cdot z, \qquad x_1 = \exp i X^0_{-2\lambda} \cdot z,$$
$$y_0 = \exp(-i X^0_{2\lambda}) \cdot z, \qquad y_1 = \exp(-i X^0_{-2\lambda}) \cdot z$$

lie on nonclosed G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing the singular orbit $G \cdot z$ in their closures. If the orbit diagram is of type (4-9), there are four such orbits, represented by w_3, w_2, w_1 and w_4, respectively. If the orbit diagram is of type (4-10), the points x_0 and x_1 lie on the same G-orbit, represented by w_2. Similarly, the points y_0 and y_1 lie on the same G-orbit represented by w_1. The G-orbits of y_0 and y_1 lie on the boundary of the Stein domain $D_1(0)$; see Theorem 6.1.

The tangent space to the G-orbit of x_0. Denote by S the G-orbit of the point $x_0 = \exp i X^0_{2\lambda} \cdot z$. To compute the tangent space TS_{x_0}, observe that at the point z

$$T(G \cdot z)_z = \mathfrak{q}' \oplus V_z, \quad \text{and} \quad T(G^{\mathbb{C}}/K^{\mathbb{C}})_z = \mathrm{Ad}_z\, \mathfrak{p}^{\mathbb{C}} = (\mathfrak{q}')^{\mathbb{C}} \oplus V_z, \tag{9-8}$$

where $\mathfrak{q}' = T(G' \cdot z)_z$ and $V_z = \mathrm{Ad}_z\, \mathfrak{g}[\alpha]^{\mathbb{C}}_{\mathfrak{p}}$ is a complex subspace of $\mathfrak{g}[\alpha]^{\mathbb{C}}$; see [Geatti 2002, Proposition 3.2]. It follows that

$$TS_{x_0} \subset \mathrm{Ad}_{x_0}(\mathfrak{q}')^{\mathbb{C}} \oplus \mathrm{Ad}_{x_0} V_z. \tag{9-9}$$

To determine generators for TS_{x_0}, fix a maximally split Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ extending $\mathfrak{b}'$ and entirely contained in $\mathfrak{h}'$ (one can check that in all cases under consideration $\mathfrak{h}'$ has the same rank as $\mathfrak{g}$ and such a Cartan subalgebra indeed exists). Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{s}^{\mathbb{C}} \bigoplus_{\beta \in \Delta} \mathfrak{g}^{\beta}$$

be the corresponding root decomposition of $\mathfrak{g}^{\mathbb{C}}$, and let $\{Z_\beta\}_{\beta \in \Delta}$ be a complex basis of $\mathfrak{g}^{\mathbb{C}}/\mathfrak{s}^{\mathbb{C}}$ consisting of root vectors $Z_\beta \in \mathfrak{g}^{\beta}$. Choose compatible orderings of $\Delta_{\mathfrak{b}'}$ and Δ (that is, a root $\beta \in \Delta$ is positive if its restriction to $\mathfrak{b}'$ is). Fix $\lambda \in \Delta_{\mathfrak{b}'}$ (either a positive or a negative short restricted root), and denote by Δ_λ the set of roots in Δ that, when restricted to $\mathfrak{b}'$, are equal to λ. The set Δ_λ consists of pairs of complex roots

$$\beta_1, \bar{\beta}_1, \ldots, \beta_m, \bar{\beta}_m, \quad \text{where } m = \tfrac{1}{2}\dim \mathfrak{g}^\lambda,$$

all with the same real part, equal to λ. For $\beta_i, \bar{\beta}_i \in \Delta_\lambda$, choose root vectors $Z_{\beta_i} \in \mathfrak{g}^{\beta_i}$ and $\sigma Z_{\beta_i} \in \mathfrak{g}^{\bar{\beta}_i}$. Then the vectors defined as

$$X^i_\lambda = Z_{\beta_i} + \sigma Z_{\beta_i} \quad \text{and} \quad Y^i_\lambda = -i(Z_{\beta_i} - \sigma Z_{\beta_i}) \quad \text{for } i = 1, \ldots, m,$$

belong to $\mathfrak{g}$ and form a basis of the restricted root space $\mathfrak{g}^\lambda$.

Lemma 9.12. *The following facts hold*:

(i) *For all $i = 1, \ldots, m$, one has $\tau_z Z_{\beta_i} = -Z_{\beta_i}$ and $i\tau_z X^i_\lambda = Y^i_\lambda$.*

(ii) *For every $i = 1, \ldots, m$, the brackets $[X^i_\lambda, i\tau_z X^i_\lambda]$ lie in $\mathfrak{g}^{2\lambda}_{\mathfrak{q}'}$. For at least one index i, such brackets are nonzero.*

(iii) *For all $i, j = 1, \ldots, m$ with $i \neq j$, the brackets $[X^i_\lambda, i\tau_z X^j_\lambda]$ have no components in $\mathfrak{g}^{2\lambda}_{\mathfrak{q}'}$.*

Proof. (i) Since the Cartan subalgebra $\mathfrak{s}$ lies in $\mathfrak{h}'$, it is pointwise fixed by τ_z. As a consequence, all root spaces $\mathfrak{g}^\beta$ with $\beta \in \Delta$ are τ_z-stable. The inclusion $V_z \subset \mathrm{Ad}_z\, \mathfrak{p}^{\mathbb{C}}$ (see (9-8)) implies that $\tau_z Z_{\beta_i} = -Z_{\beta_i}$ for $i = 1, \ldots, m$. Since $\sigma\tau_z = -\tau_z\sigma$ on $V_z \subset \mathfrak{g}[\alpha]^{\mathbb{C}}$, one has $i\tau_z X^i_\lambda = Y^i_\lambda$, as desired.

(ii) By the definitions of X^i_λ and Y^i_λ, one has

$$[X^i_\lambda, i\tau_z X^i_\lambda] = [X^i_\lambda, Y^i_\lambda] = 2i[Z_{\beta_i}, \sigma Z_{\beta_i}] \in \mathfrak{g}^{2\lambda}.$$

By (i) and the fact that $\tau_z\sigma = -\sigma\tau_z$ on $\mathfrak{g}[\lambda]^{\mathbb{C}} = \mathfrak{g}[\alpha]^{\mathbb{C}}$, one also has

$$\tau_z(2i[Z_{\beta_i}, \sigma Z_{\beta_i}]) = -2i[Z_{\beta_i}, \sigma Z_{\beta_i}].$$

This implies that $[X^i_\lambda, i\tau_z X^i_\lambda]$ lies in $\mathfrak{g}^{2\lambda}_{\mathfrak{q}'}$, as claimed. To prove the second part (ii), consider the set $\Delta_{2\lambda}$ consisting of the roots in Δ that, when restricted to $\mathfrak{b}'$, coincide with 2λ. Since $\Delta_{2\lambda}$ contains a real root in Δ and such a root is not simple (see Satake diagrams in [Helgason 1978, page 532]), there exist $\beta, \bar{\beta} \in \Delta_\lambda$ such that $\beta + \bar{\beta} = 2\lambda$. This shows that at least one of the brackets $[X^i_\lambda, i\tau_z X^i_\lambda]$ has a nonzero component in $\mathfrak{g}^{2\lambda}$.

(iii) Let β_i, β_j be roots in Δ_λ, with $\beta_j \neq \beta_i, \bar{\beta}_i$. If either $\beta_i + \beta_j$ or $\beta_i + \bar{\beta}_j$ is a root in Δ, then it is a root in $\Delta_{2\lambda}$, with nonzero imaginary part. Since the root spaces relative to the real root in $\Delta_{2\lambda}$ are contained in $(\mathfrak{g}^{2\lambda}_{\mathfrak{q}'})^{\mathbb{C}}$ and $\dim(\mathfrak{g}^{2\lambda}_{\mathfrak{q}'})^{\mathbb{C}} = 1$ (see Lemma 9.2), root spaces relative to the remaining roots in $\Delta_{2\lambda}$ are necessarily contained in $(\mathfrak{g}^{2\lambda}_{\mathfrak{h}'})^{\mathbb{C}}$. Hence the statement follows. □

For $\lambda \in \Delta^+_{\mathfrak{b}'}$, fix bases of $\mathfrak{g}^\lambda$ and $\mathfrak{g}^{-\lambda}$ of the form

$$X^1_\lambda, i\tau_z X^1_\lambda, \dots, X^m_\lambda, i\tau_z X^m_\lambda \quad \text{and} \quad X^1_{-\lambda}, i\tau_z X^1_{-\lambda}, \dots, X^m_{-\lambda}, i\tau_z X^m_{-\lambda}, \tag{9-10}$$

respectively. For i, $j = 1, \dots, m$, define

$$w_i := \tfrac{1}{2}\operatorname{Ad}_{x_0}(X^i_\lambda - \tau_z X^i_\lambda) \quad \text{and} \quad v_j := \tfrac{1}{2}\operatorname{Ad}_{x_0}(X^j_{-\lambda} - \tau_z X^j_{-\lambda}).$$

In the next lemma, we compute the images of the vectors in (9-10) under the map $^* : \mathfrak{g} \to TS_{x_0}$ defined in (9-1). We omit the straightforward proof.

Lemma 9.13. *The images of the vectors in (9-10) under the map (9-1) are*

(i) $(X^i_\lambda)^* = w_i$;

(ii) $(i\tau_z X^i_\lambda)^* = -iw_i$;

(iii) $(X^j_{-\lambda})^* = v_j - iw'$, *where* $w' = \operatorname{Ad}_{x_0}[X^0_{2\lambda}, X^j_{-\lambda}]$; *and*

(iv) $(i\tau_z X^j_{-\lambda})^* = -iv_j - iw''$, *where* $w'' = \operatorname{Ad}_{x_0}[X^0_{2\lambda}, i\tau_z X^j_{-\lambda}]$.

Let $W^+_{x_0}$ be the complex subspace of W_{x_0} spanned by the vectors $\{w_1, \dots, w_m\}$, and let $W^-_{x_0}$ be the one spanned by $\{v_1, \dots, v_m\}$. By (9-9), the results of Section 9.1 applied to the symmetric space G'/H' and Lemma 9.13, the tangent space to S at x_0 is given by $TS_{x_0} = T_{\mathbb{C}}S_{x_0} \oplus NS_{x_0}$, where

$$T_{\mathbb{C}}S_{x_0} = T_{\mathbb{C}}(G'\cdot x_0)_{x_0} \oplus W^+_{x_0} \oplus W^-_{x_0} \quad \text{and} \quad NS_{x_0} = \mathbb{R}\operatorname{Ad}_{x_0}\theta X^0_{2\lambda}. \tag{9-11}$$

Fix $F^0_{-2\lambda} := \operatorname{Ad}_{x_0}\theta X^0_{2\lambda}$ as a generator of NS_{x_0}.

Lemma 9.14. *The following facts hold.*

(i) *The decomposition of $T_{\mathbb{C}}S_{x_0}$ given in (9-11) is orthogonal with respect to the Levi form.*

(ii) *Let $W \in W^+_{x_0}$. Then $L_{x_0}(W, W) = 0$.*

(iii) *Let $W \in W^-_{x_0}$. Then $L_{x_0}(W, W) = bF^0_{-2\lambda}$, with $b \geq 0$.*

(iv) *Let $Z \in T_{\mathbb{C}}(G'\cdot x_0)_{x_0}$. Then $L_{x_0}(Z, Z) = nF^0_{-2\lambda}$, with $n \leq 0$.*

Proof. (i) Let $Z \in T_{\mathbb{C}}(G'\cdot x_0)_{x_0}$ and $W \in W_{x_0}$. To show that $L(Z, W) \equiv L(W, Z) \equiv 0$, observe that both $(\,\cdot\,)^{-1}JZ$ and $(\,\cdot\,)^{-1}Z$ belong to $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{q}'$ and can be written

$$(\,\cdot\,)^{-1}JZ = \operatorname{Ad}_{x_0} X_0 + \operatorname{Ad}_{x_0} X_{2\lambda} + \operatorname{Ad}_{x_0} X_{-2\lambda},$$
$$(\,\cdot\,)^{-1}Z = \operatorname{Ad}_{x_0} Y_0 + \operatorname{Ad}_{x_0} Y_{2\lambda} + \operatorname{Ad}_{x_0} Y_{-2\lambda},$$

according to the $\operatorname{ad}_{\mathfrak{b}'}$-root decomposition of $\mathfrak{g}'$ given in (9-6). Similarly, by (9-8), the vector $W \in W^+_{x_0} \oplus W^+_{x_0} = \operatorname{Ad}_{x_0}\operatorname{Ad}_z \mathfrak{g}[\lambda]^{\mathbb{C}}_{\mathfrak{p}}$ can be written as

$$W = \operatorname{Ad}_{x_0}\operatorname{Ad}_z P_\lambda + i\operatorname{Ad}_{x_0}\operatorname{Ad}_z Q_\lambda,$$

where

$$\mathrm{Ad}_z P_\lambda = U_\lambda + iV_{-\lambda} - \theta U_\lambda + i\theta V_{-\lambda} \quad \text{and} \quad \mathrm{Ad}_z Q_\lambda = U'_\lambda + iV'_{-\lambda} - \theta U'_\lambda + i\theta V'_{-\lambda},$$

with $U_\lambda, U'_\lambda \in \mathfrak{g}^\lambda$ and $V_{-\lambda}, V'_{-\lambda} \in \mathfrak{g}^{-\lambda}$. One can verify that none of the brackets in (9-5) has a component in $\mathrm{Ad}_{x_0} \mathfrak{g}_{\mathfrak{q}'}^{-2\lambda}$, and $L_{x_0}(Z, W) \equiv 0$, as required.

Let $w_i \in W_{x_0}^+$ and $v_j \in W_{x_0}^-$. Then, modulo $(T_{\mathbb{C}} S_{x_0})^{\mathbb{C}}$, the Levi form is given by

$$2L_{x_0}(w_i, v_j) \equiv -\tfrac{1}{2}\,\mathrm{Ad}_{x_0}[i\tau_z X_\lambda^i, (X_{-\lambda}^j - \tau_z X_{-\lambda}^j)] - \tfrac{i}{2}\,\mathrm{Ad}_{x_0}[X_\lambda^i, (X_{-\lambda}^j - \tau_z X_{-\lambda}^j)].$$

In particular, $L_{x_0}(w_i, v_j) = 0$ for all $i, j = 1, \ldots, m$. This proves (i).

In the same way, one shows $L(w_i, w_j) = 0$ for all $w_i, w_j \in W_{x_0}^+$, proving (ii).

(iii) Similar calculations and Lemma 9.12(iii) imply that $L_{x_0}(v_i, v_j) = 0$ for all $v_i, v_j \in W_{x_0}^-$ with $i \neq j$. When $i = j$, one has

$$L_{x_0}(v_i, v_i) = \mathrm{Ad}_{x_0}[X_{-\lambda}^i, i\tau_z X_{-\lambda}^i] = \mathrm{Ad}_{x_0}\, i[Z_{-\beta_i}, \sigma Z_{-\beta_i}] = b_i F_{-2\lambda}^0 \quad \text{for } b_i \in \mathbb{R}.$$

In order to prove that $b_i \geq 0$ observe that, by Lemma 9.11(iii), one can write $X_{-\lambda}^i = X_\alpha^i + X_{-\alpha}^i$ for appropriate $X_\alpha^i \in \mathfrak{g}^\alpha$ and $X_{-\alpha}^i \in \mathfrak{g}^{-\alpha}$. Since $z = \exp iAK^{\mathbb{C}}$, with $A \in \mathfrak{a}$ and $\alpha(A) = \pi/4$, one also has $i\tau_z X_{-\lambda}^i = \theta X_\alpha^i - \theta X_{-\alpha}^i$ and

$$[X_{-\lambda}^i, i\tau_z X_{-\lambda}^i] = ([X_\alpha^i, \theta X_\alpha^i] - [X_{-\alpha}^i, \theta X_{-\alpha}^i]) - ([X_\alpha^i, \theta X_{-\alpha}^i] + [X_{-\alpha}^i, \theta X_\alpha^i]),$$

which lies in $\mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a})$. By [Geatti 2002, Lemma 5.1(i)], the first two terms of the above sum can be written as

$$[X_\alpha^i, \theta X_\alpha^i] = B(X_\alpha^i, \theta X_\alpha^i) A_\alpha \quad \text{and} \quad [\theta X_{-\alpha}^i, \theta(\theta X_{-\alpha}^i)] = B(X_{-\alpha}^i, \theta X_{-\alpha}^i) A_\alpha,$$

where A_α is an element in $\mathfrak{a}$ satisfying the condition $\alpha(A_\alpha) > 0$. By the normalization of the reference points chosen in (9-7), one has $\theta X_{2\lambda}^0 = -X_{2\lambda}^0$. Hence $L_{x_0}(v_i, v_i) = b_i\, \mathrm{Ad}_{x_0}\, \theta X_{2\lambda}^0$ for some real number $b_i \geq 0$, as claimed. This concludes the proof of (iii).

(iv) Recall that the symmetric space G'/H' has a reduced restricted root system and that the Lie algebra $\mathfrak{g}'$ is given by (9-6). Then the Levi form on $T_{\mathbb{C}}(G' \cdot x_0)_{x_0}$ can be computed by the methods of Section 9.1. By (9-4), one has

$$T_{\mathbb{C}}(G' \cdot x_0)_{x_0} = \mathrm{Ad}_{x_0}(\mathfrak{g}_{\mathfrak{q}'}^0)^{\mathbb{C}} \oplus \mathrm{Ad}_{x_0}(\mathfrak{g}_{\mathfrak{q}'}^{2\lambda})^{\mathbb{C}} \quad \text{and} \quad N(G' \cdot x_0)_{x_0} = \mathbb{R}\,\mathrm{Ad}_{x_0}\, \theta X_{-2\lambda}^0.$$

Let $F_0 \in \mathrm{Ad}_{x_0}(\mathfrak{g}_{\mathfrak{q}'}^0)^{\mathbb{C}}$ and $F_{2\lambda} \in \mathrm{Ad}_{x_0}(\mathfrak{g}_{\mathfrak{q}'}^{2\lambda})^{\mathbb{C}}$. Then by Lemma 9.6 one has

$$L_{x_0}(F_{2\lambda}, F_{2\lambda}) = L_{x_0}(F_0, F_{2\lambda}) = 0 \quad \text{and} \quad L_{x_0}(F_0, F_0) = nF_{-2\lambda}^0 \quad \text{where } n \leq 0. \qquad \square$$

The next proposition is a direct consequence of Lemmas 9.12 and 9.14.

Proposition 9.15. *Let S be the G-orbit of x_0. The Levi form of S at x_0 is indefinite if $\mathfrak{g} = \mathfrak{sp}(n, 1)$ or $\mathfrak{g} = \mathfrak{f}_4^*$. It is definite if $\mathfrak{g} = \mathfrak{su}(n, 1)$.*

Proof. By Lemma 9.12 and Lemma 9.14(iii) the Levi form L_{x_0} is definite on $W^-_{x_0}$. If $\mathfrak{g} = \mathfrak{su}(n, 1)$, then $\dim G'/H' = 1$, and the Levi form is identically zero on $T_{\mathbb{C}}(G' \cdot x_0)_{x_0}$. As a result, in this case the Levi form L_{x_0} is definite.

If $\mathfrak{g} = \mathfrak{sp}(n, 1)$ or $\mathfrak{g} = \mathfrak{f}_4^*$, then $\dim G'/H' > 2$, and the Levi form L_{x_0} on $T_{\mathbb{C}}(G' \cdot x_0)_{x_0}$ is definite of sign opposite to that on $W^-_{x_0}$; see Proposition 9.7 and Lemma 9.14. As a result, L_{x_0} is indefinite, as claimed. □

The Levi form of the G-orbit of y_0. By the same methods, one can compute the tangent space and the Levi form of the G-orbit S of the point $y_0 = \exp i(-X^0_{2\lambda}) \cdot z$. The tangent space to S at y_0 is given by $TS_{y_0} = T_{\mathbb{C}} S_{y_0} \oplus NS_{y_0}$, where

$$T_{\mathbb{C}} S_{y_0} = T_{\mathbb{C}}(G' \cdot y_0)_{y_0} \oplus W^+_{y_0} \oplus W^-_{y_0} \quad \text{and} \quad NS_{y_0} = \mathbb{R}\,\mathrm{Ad}_{y_0}\,\theta X^0_{2\lambda}. \tag{9-12}$$

Fix $F^0_{-2\lambda} := \mathrm{Ad}_{y_0}\,\theta X^0_{2\lambda}$ as a generator of NS_{y_0}.

Lemma 9.16. *The following facts hold.*

(i) *The decomposition of $T_{\mathbb{C}} S_{y_0}$ given in (9-12) is orthogonal with respect to the Levi form.*

(ii) *Let $W \in W^+_{y_0}$. Then $L_{y_0}(W, W) \equiv 0$.*

(iii) *Let $W \in W^-_{y_0}$. Then $L_{y_0}(W, W) = bF^0_{-2\lambda}$, with $b \geq 0$.*

(iv) *Let $Z \in T_{\mathbb{C}}(G' \cdot y_0)_{y_0}$. Then $L_{y_0}(Z, Z) = pF^0_{-2\lambda}$, with $p \geq 0$.*

Proof. The proof is like the proof of Lemma 9.14. One can check that the Levi form is not identically zero on $W^-_{y_0}$ and has the same signature as on $W^-_{x_0}$. Part (iv) follows from Lemma 9.8. □

Proposition 9.17. *Let S be the G-orbit of y_0. The Levi form of S at y_0 is definite.*

Proof. The proposition follows from Lemma 9.16 and the fact that the Levi form L_{y_0} on $W^-_{y_0}$ is not identically zero. □

Remark 9.18. If the restricted root system $\Delta_{\mathfrak{b}}$ is nonreduced, then Proposition 9.17 says that the Levi form of the orbits represented by w_1 and w_4 in diagrams (4-9) and (4-10) is definite. This is consistent with the fact that these orbits lie in the boundary of the Stein domain $D_1(0)$; see Theorem 6.1. When $\mathfrak{g} = \mathfrak{su}(n, 1)$, by Proposition 9.15, the same is true for the Levi form of the orbits represented by w_2 and w_3 in diagram (4-9). We refer to Example 6.3 for a classification of the G-invariant Stein domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ bounded by these orbits. Proposition 9.15 also says that the Levi form of the orbit represented by w_2 in diagram (4-10) is indefinite. Hence this orbit cannot lie in the boundary of a Stein G-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

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References

[Akhiezer and Gindikin 1990] D. N. Akhiezer and S. G. Gindikin, "On Stein extensions of real symmetric spaces", *Math. Ann.* **286** (1990), 1–12. MR 91a:32047 Zbl 0681.32022

[Boggess 1991] A. Boggess, *CR manifolds and the tangential Cauchy–Riemann complex*, CRC Press, Boca Raton, FL, 1991. MR 94e:32035 Zbl 0760.32001

[Burns et al. 2003] D. Burns, S. Halverscheid, and R. Hind, "The geometry of Grauert tubes and complexification of symmetric spaces", *Duke Math. J.* **118**:3 (2003), 465–491. MR 2004g:32025 Zbl 1044.53039

[Casadio Tarabusi et al. 2000] E. Casadio Tarabusi, A. Iannuzzi, and S. Trapani, "Globalizations, fiber bundles, and envelopes of holomorphy", *Math. Z.* **233**:3 (2000), 535–551. MR 2001f:32035 Zbl 0961.32021

[Cœuré and Loeb 1986] G. Cœuré and J.-J. Loeb, "Univalence de certaines enveloppes d'holomorphie", *C. R. Acad. Sci. Paris Sér. I Math.* **302**:2 (1986), 59–61. MR 87k:32024 Zbl 0588.32014

[Fels et al. 2006] G. Fels, A. Huckleberry, and J. A. Wolf, *Cycle spaces of flag domains: A complex geometric viewpoint*, Progress in Mathematics **245**, Birkhäuser, Boston, 2006. MR 2006h:32018 Zbl 1084.22011

[Geatti 2002] L. Geatti, "Invariant domains in the complexification of a noncompact Riemannian symmetric space", *J. Algebra* **251**:2 (2002), 619–685. MR 2003i:22010 Zbl 1018.32030

[Geatti 2006] L. Geatti, "A remark on the orbit structure of complexified symmetric spaces", preprint, 2006.

[Gindikin and Krötz 2002] S. Gindikin and B. Krötz, "Invariant Stein domains in Stein symmetric spaces and a nonlinear complex convexity theorem", *Int. Math. Res. Not.* **18** (2002), 959–971. MR 2003d:32026 Zbl 1011.32018

[Grauert 1958] H. Grauert, "Analytische Faserungen über holomorph-vollständigen Räumen", *Math. Ann.* **135** (1958), 263–273. MR 20 #4661 Zbl 0081.07401

[Heinzner 1991] P. Heinzner, "Geometric invariant theory on Stein spaces", *Math. Ann.* **289** (1991), 631–662. MR 92j:32116 Zbl 0728.32010

[Heinzner 1993] P. Heinzner, "Equivariant holomorphic extensions of real analytic manifolds", *Bull. Soc. Math. France* **121**:3 (1993), 445–463. MR 94i:32050 Zbl 0794.32022

[Heinzner and Iannuzzi 1997] P. Heinzner and A. Iannuzzi, "Integration of local actions on holomorphic fiber spaces", *Nagoya Math. J.* **146** (1997), 31–53. MR 98k:32047 Zbl 0879.55006

[Helgason 1978] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics **80**, Academic Press, New York, 1978. MR 80k:53081 Zbl 0451.53038

[Hochschild 1965] G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, 1965. MR 34 #7696 Zbl 0131.02702

[Iannuzzi 1999] A. Iannuzzi, "Characterizations of G-tube domains", *Manuscripta Math.* **98** (1999), 425–445. MR 2000e:32034 Zbl 0935.32002

[Knapp 1996] A. W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics **140**, Birkhäuser, Boston, 1996. MR 98b:22002 Zbl 0862.22006

[Kobayashi 1998] S. Kobayashi, *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften **318**, Springer, Berlin, 1998. MR 99m:32026 Zbl 0917.32019

[Krötz and Stanton 2005] B. Krötz and R. J. Stanton, "Holomorphic extensions of representations, II: Geometry and harmonic analysis", *Geom. Funct. Anal.* **15**:1 (2005), 190–245. MR 2006d:43010 Zbl 1078.22009

[Luna 1975] D. Luna, "Sur certaines opérations différentiables des groupes de Lie", *Amer. J. Math.* **97** (1975), 172–181. MR 51 #527 Zbl 0334.57022

[Matsuki 1997] T. Matsuki, "Double coset decompositions of reductive Lie groups arising from two involutions", *J. Algebra* **197**:1 (1997), 49–91. MR 99a:22012 Zbl 0887.22009

[Neeb 1999] K.-H. Neeb, "On the complex geometry of invariant domains in complexified symmetric spaces", *Ann. Inst. Fourier* (*Grenoble*) **49**:1 (1999), 177–225. MR 2000i:32040 Zbl 0921.22003

[Palais 1957] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. **22**, American Mathematical Society, Providence, R.I., 1957. MR 22 #12162 Zbl 0178.26502

[Rossi 1963] H. Rossi, "On envelopes of holomorphy", *Comm. Pure Appl. Math.* **16** (1963), 9–17. MR 26 #6436 Zbl 0113.06001

[Snow 1982] D. M. Snow, "Reductive group actions on Stein spaces", *Math. Ann.* **259**:1 (1982), 79–97. MR 83f:32026 Zbl 0509.32021

[Stein 1956] K. Stein, "Überlagerungen holomorph-vollständiger komplexer Räume", *Arch. Math.* **7** (1956), 354–361. MR 18,933a Zbl 0072.08002

[Steinberg 1968] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. **80**, American Mathematical Society, Providence, R.I., 1968. MR 37 #6288 Zbl 0164.02902

[Wolf 1984] J. A. Wolf, *Spaces of constant curvature*, 5th ed., Publish or Perish, Houston, TX, 1984. MR 88k:53002

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AN INVARIANT SUPERTRACE FOR THE CATEGORY OF REPRESENTATIONS OF LIE SUPERALGEBRAS OF TYPE I

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In this paper we give a renormalization of the supertrace on the category of representations of Lie superalgebras of type I, by a kind of modified superdimension. The genuine superdimensions and supertraces are generically zero. However, these modified superdimensions are nonzero and lead to a kind of supertrace which is nontrivial and invariant. As an application we show that this new supertrace gives rise to a nonzero bilinear form on a space of invariant tensors of a Lie superalgebra of type I. The results of this paper are completely classical results in the theory of Lie superalgebras but surprisingly we cannot prove them without using quantum algebra and low-dimensional topology.

Introduction

The theory of quantum groups and classical representation theory of Lie algebras has been widely and productively used in low-dimensional topology. There are few examples of low-dimensional topology or quantum groups being used to produce results in the classical theory of Lie algebras. Good examples of such work include the theory of crystal bases [Kashiwara 1990] and the use of the Kontsevich integral to give a new proof of the multiplicativity of the Duflo–Kirillov map $S(\mathfrak{g}) \to U(\mathfrak{g})$ for metrized Lie (super-)algebras $\mathfrak{g}$ [Bar-Natan et al. 2003]. In this paper we use low-dimensional topology and quantum groups to define a nontrivial kind of supertrace on the category of representations of a Lie superalgebra of type I. The genuine supertrace is generically zero on such a category Proposition 2.2.

In [Geer and Patureau-Mirand 2006; Geer et al. 2007], we give a renormalization of the Reshetikhin–Turaev quantum invariants, by modified quantum dimensions. In the case of simple Lie algebras these modified quantum dimensions are proportional to the genuine quantum dimensions. For Lie superalgebras of type I the genuine quantum dimensions are generically zero but the modified quantum dimensions are nonzero and lead to nontrivial link invariants. In this case the

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modified quantum dimension of a quantized module is given by an explicit formula which is determined by the underlying Lie superalgebra module. In this paper we take the classical limit of the modified quantum dimension to obtain a modified superdimension. Then we use this modified superdimension to renormalize the supertrace and define a nontrivial bilinear form on a space of invariant tensor.

Our proof that the modified supertrace is well defined and has the desired properties is as follows. We first formulate the desired statements at the level of the Lie superalgebra. Then we "deform" these statements to the quantum level and use low-dimensional topology to prove these "deformed" statements. Taking the classical limit we recover the original statements. To make this proof precise we use the Etingof–Kazhdan theory of quantization.

1. Preliminaries

1.1. ***The category* SV *of superspaces.*** A *superspace* is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar 0} \oplus V_{\bar 1}$ over $\mathbb{C}$. We denote the parity of an homogeneous element $x \in V$ by $\bar x \in \mathbb{Z}_2$. We say x is even if it lies in $V_{\bar 0}$, and odd if it lies in $V_{\bar 1}$). We now recall some basic features and conventions concerning the category of superspaces.

The theory of superspaces follows the rule "whenever you permute two odd elements in an expression, put a $-$ sign". With this in mind, many concepts of linear algebra have super analogs. These analogs have new and different properties which are relevant to this paper. Let us discuss some of these differences.

In all the following, elements of superspaces are generally assumed to be homogeneous and thus their parity is well defined. The definitions must be generalized by linearity for nonhomogeneous elements.

The category SV of superspaces is a category whose objects are superspaces. The morphisms in SV between two objects U and V denoted by $\mathrm{Hom}_{\mathbb{C}}(U, V)$ is the superspace of linear maps with the parity given by

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(U, V)_{\bar 0} &= \mathrm{Hom}_{\mathbb{C}}(U_{\bar 0}, V_{\bar 0}) \oplus \mathrm{Hom}_{\mathbb{C}}(U_{\bar 1}, V_{\bar 1}), \\ \mathrm{Hom}_{\mathbb{C}}(U, V)_{\bar 1} &= \mathrm{Hom}_{\mathbb{C}}(U_{\bar 0}, V_{\bar 1}) \oplus \mathrm{Hom}_{\mathbb{C}}(U_{\bar 1}, V_{\bar 0}). \end{aligned}$$

This category is "supermonoidal" with the super version of the operator $\otimes$ (we denote by $\underline{\otimes}$ the usual tensor product in the category Vect):

For two objects U, V of SV their tensor product is the vector space $U \,\underline{\otimes}\, V$ with the $\mathbb{Z}_2$-grading given by

$$\begin{aligned} (U \otimes V)_{\bar 0} &= U_{\bar 0} \,\underline{\otimes}\, V_{\bar 0} \,\oplus\, U_{\bar 1} \,\underline{\otimes}\, V_{\bar 1}, \\ (U \otimes V)_{\bar 1} &= U_{\bar 0} \,\underline{\otimes}\, V_{\bar 1} \,\oplus\, U_{\bar 1} \,\underline{\otimes}\, V_{\bar 0}. \end{aligned}$$

and for morphisms $f \in \mathrm{Hom}_{\mathbb{C}}(U, U')$ and $g \in \mathrm{Hom}_{\mathbb{C}}(V, V')$, $f \otimes g$ is given by

$$f \otimes g = \begin{cases} f \underline{\otimes} g \text{ on } U_{\bar{0}} \underline{\otimes} V, \\ (-1)^{\bar{g}} f \underline{\otimes} g \text{ on } U_{\bar{1}} \underline{\otimes} V. \end{cases}$$

Thus $(f \otimes g)(x \otimes y) = (-1)^{\bar{g}.\bar{x}} f(x) \otimes g(y)$. When U and V are finite-dimensional, this tensor product realizes an isomorphism:

$$\text{(1)} \qquad \mathrm{Hom}_{\mathbb{C}}(U, U') \otimes \mathrm{Hom}_{\mathbb{C}}(V, V') \simeq \mathrm{Hom}_{\mathbb{C}}(U \otimes V, U' \otimes V').$$

Let $\mathrm{SV}_{\bar{0}}$ be the subcategory of SV with the same objects but only even morphisms (i.e., $\mathrm{Hom}_{\mathrm{SV}_{\bar{0}}}(U, V) = \mathrm{Hom}_{\mathbb{C}}(U, V)_{\bar{0}}$). The tensor product $\otimes$ restricted to $\mathrm{SV}_{\bar{0}}$ is the usual bifunctor of Vect with an appropriate grading on objects. Moreover, $\mathrm{SV}_{\bar{0}}$ is a symmetric monoidal category with symmetry isomorphisms $\tau_{U,V} : U \otimes V \simeq V \otimes U$ given by the superpermutation $\tau_{U,V}(u \otimes v) = (-1)^{\bar{u}.\bar{v}} v \otimes u$. The category SV is not a symmetric monoidal category because in general there are morphisms f and g with the property that $(\mathrm{Id} \otimes g) \circ (f \otimes \mathrm{Id}) \neq (f \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes g)$.

For a superspace U, the *superdual* U^* is defined as the superspace $\mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$. The tensor product gives the canonical isomorphism

$$U^* \otimes V^* = \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(U \otimes V, \mathbb{C} \otimes \mathbb{C}) = (U \otimes V)^*.$$

If $f \in \mathrm{Hom}_{\mathbb{C}}(U, V)$, the *supertranspose* of f is the linear map $f^* \in \mathrm{Hom}_{\mathbb{C}}(V^*, U^*)$ given by

$$f^*(\phi) = (-1)^{\bar{f}.\bar{\phi}} \phi \circ f$$

for $\phi \in V^*$. Then, if f, g are composable morphisms of SV, we have

$$(f \circ g)^* = (-1)^{\bar{f}.\bar{g}} g^* \circ f^*.$$

By convention the dual is a left dual:

- (left duality) $\mathrm{ev}_V \in \mathrm{Hom}_{\mathbb{C}}(V^* \otimes V, \mathbb{C})$ is simply the contraction $\langle \phi, x \rangle = \phi(x)$.
- (right duality) $\mathrm{ev}'_V \in \mathrm{Hom}_{\mathbb{C}}(V \otimes V^*, \mathbb{C})$ is given by $\langle x, \phi \rangle = (-1)^{\bar{x}.\bar{\phi}} \phi(x)$

This defines a canonical isomorphism $V \to V^{**}$ when V is finite dimensional. Again here, when restricted to $\mathrm{SV}_{\bar{0}}$ the $*$ became a functor, namely, the usual contravariant duality functor with some grading information.

The category* $\mathfrak{g}$-Mod *of* $\mathfrak{g}$*-modules. A *Lie superalgebra* is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a superbracket $[\,,\,] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ that preserves the $\mathbb{Z}_2$-grading, is superantisymmetric ($[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$), and satisfies the super-Jacobi identity [Kac 1977]. In this section, we assume that $\mathfrak{g}$ is a classical Lie superalgebra which means that the Lie algebra $\mathfrak{g}_{\bar{0}}$ is reductive.

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is a Hopf superalgebra, that is, $U(\mathfrak{g})$ is a Hopf algebra object in $\mathrm{SV}_{\bar{0}}$. Let $\mathfrak{g}$-Mod be the category where objects are finite-dimensional superspaces with a structure of regular $\mathfrak{g}$-modules (that is, regular $U(\mathfrak{g})$-modules). By regular we mean that elements of the center of the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$ act as diagonalisable endomorphisms of a $\mathfrak{g}$-module. It is equivalent to require that V is semisimple as a $\mathfrak{g}_{\bar{0}}$-module (i.e., V splits as a direct sum of irreducible $\mathfrak{g}_{\bar{0}}$-modules). The morphisms of $\mathfrak{g}$-Mod are the morphisms f of SV that are ("super") $\mathfrak{g}$-linear:

$$f(x.v) = (-1)^{\bar{x}.\bar{f}} x.f(v) \quad \text{for all } x \in \mathfrak{g} \text{ and } v \in V.$$

If U and V are two $\mathfrak{g}$-modules we denote by $\mathrm{Hom}_{\mathfrak{g}}(U, V)$ the superspace of $\mathfrak{g}$-module morphisms. The superspace $\mathrm{Hom}_{\mathfrak{g}}(U, V)$ should not be confused with $\mathrm{Hom}_{\mathbb{C}}(U, V)$ (where U and V are viewed as superspaces) which is naturally equipped with a $\mathfrak{g}$-module structure.

The structure of Hopf superalgebra on $U(\mathfrak{g})$ gives the tensor product of two $\mathfrak{g}$-modules a natural structure of $\mathfrak{g}$-modules and the tensor product of two $\mathfrak{g}$-linear morphisms is $\mathfrak{g}$-linear. Similarly, if V is an object of $\mathfrak{g}$-Mod then the superspace V^* is a $\mathfrak{g}$-module whose action is induced from the antipodal map of $U(\mathfrak{g})$. $\mathrm{Hom}_{\mathfrak{g}}(U, V)$ is canonically isomorphic to the superspace of invariant elements of $V \otimes U^*$ and so

$$\mathrm{Hom}_{\mathfrak{g}}(U, V) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V \otimes U^*). \tag{2}$$

Let $\mathfrak{g}$-$\mathrm{Mod}_{\bar{0}}$ be the category whose objects are the objects of $\mathfrak{g}$-Mod and whose morphisms are morphisms of $\mathrm{SV}_{\bar{0}}$ which are $\mathfrak{g}$-linear. Then as above $\mathfrak{g}$-$\mathrm{Mod}_{\bar{0}}$ becomes a symmetric monoidal category with duality. Note that in general $\mathfrak{g}$-Mod is not such a category. This is the reason we require that the morphisms α and β in the definition of $\mathcal{I}$ Proposition 1.2 are in $\mathfrak{g}$-$\mathrm{Mod}_{\bar{0}}$. In other words, the proof of Theorem 1 requires that we work in the category $\mathfrak{g}$-$\mathrm{Mod}_{\bar{0}}$.

1.2. ***Lie superalgebras of type I.*** In this subsection we recall notations and properties related to Lie superalgebras of type I.

Throughout the rest of the paper, let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra of type I, so $\mathfrak{g}$ is equal to $\mathfrak{sl}(m|n)$ or $\mathfrak{osp}(2|2n)$. We will assume that $m \neq n$. Let $\mathfrak{b}$ be the distinguished Borel subsuperalgebra of $\mathfrak{g}$. Then $\mathfrak{b}$ can be written as the direct sum of a Cartan subsuperalgebra $\mathfrak{h}$ and a positive nilpotent subsuperalgebra $\mathfrak{n}_+$. Moreover, $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let W be the Weyl group of the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$.

Let $\Delta_{\bar{0}}^+$ be the even positive roots and $\Delta_{\bar{1}}^+$) the odd ones. Let $\rho_{\bar{0}}$ and $\rho_{\bar{1}}$ denote the half-sum of all even and odd positive roots, respectively. Set $\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$. A positive root is called *simple* if it cannot be decomposed into a sum of two positive roots.

A Cartan matrix associated to a Lie superalgebra is a pair consisting of a $r \times r$ matrix $A = (a_{ij})$ and a set $\tau \subset \{1, \dots, r\}$ determining the parity of the generators.

Let (A, τ) be the Cartan matrix arising from $\mathfrak{g}$ and the distinguished Borel subsuperalgebra $\mathfrak{b}$. Here the set $\tau = \{s\}$ consists of only one element because of our choice of Borel subalgebra $\mathfrak{b}$.

By Proposition 1.5 of [Kac 1978] there exists $e_i \in \mathfrak{n}_+$, $f_i \in \mathfrak{n}_-$ and $h_i \in \mathfrak{h}$ for $i = 1, \dots, r$ such that the Lie superalgebra $\mathfrak{g}$ is generated by e_i, f_i, h_i where

$$[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j.$$

Note that these generators also satisfy the Serre relations and higher order Serre type relations [Yamane 1994].

There exist $d_1, \dots, d_r$ in $\{\pm 1, \pm 2\}$ such that the matrix $(d_i a_{ij})$ is symmetric. Let $\langle\, .\, ,\, .\, \rangle$ be the symmetric nondegenerate form on $\mathfrak{h}$ determined by $\langle h_i, h_j \rangle = d_j^{-1} a_{ij}$. This form gives an identification of $\mathfrak{h}$ and $\mathfrak{h}^*$. Moreover, the form $\langle\, .\, ,\, .\, \rangle$ induces a W-invariant bilinear form on $\mathfrak{h}^*$, which we will also denote by $\langle\, .\, ,\, .\, \rangle$.

1.3. *Irreducible $\mathfrak{g}$-modules.* Modules over Lie superalgebras of type I are different in nature than modules over semisimple Lie algebras. For example, each Lie superalgebra of type I has one parameter families of irreducible modules. Any module in such a family has superdimension zero and so the supertrace of an endomorphism of such a module is zero — see Equality (3).

There is a super analog of the Schur's Lemma [Kac 1978]:

Lemma 1.1 (Schur's Lemma). *Let V be a superspace, $\mathcal{M}$ an irreducible family of operators from $\mathrm{End}_{\mathrm{SV}}(V)$, and $C(\mathcal{M}) = \{a \in \mathrm{End}_{\mathrm{SV}}(V) : [a, m] = 0, \forall m \in \mathcal{M}\}$. Then either*

- *$C(\mathcal{M})$ is generated by Id_V, or*
- *$C(\mathcal{M})$ is generated by Id_V and s where s is an odd endomorphism of V such that $s^2 = \mathrm{Id}_V$ (and in particular $\dim V_{\bar{0}} = \dim V_{\bar{1}}$).*

For $\mathfrak{g}$-Mod the situation is simplified: for any $\mathfrak{g}$-module V of $\mathfrak{g}$-Mod,

$$V \text{ is irreducible } \Rightarrow \mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}\,\mathrm{Id}_V\,. \tag{3}$$

This follows from the fact that any module of $\mathfrak{g}$-Mod is regular and thus it is a weight module (a direct sum of its weight spaces) and any irreducible module of $\mathfrak{g}$-Mod is a highest weight module with unique (up to a scalar) highest weight vector. Kac [1977] gives a construction of these irreducible modules: Let $\lambda \in \mathfrak{h}^*$ be a linear functional on $\mathfrak{h}$. Kac defined an irreducible highest weight $\mathfrak{g}$-module $V(\lambda)$ of weight λ with a highest weight vector v_0 having the property that $h.v_0 = \lambda(h) v_0$ for all $h \in \mathfrak{h}$ and $\mathfrak{n}_+ v_0 = 0$. Let $a_i = \lambda(h_i)$. Kac showed that $V(\lambda)$ is finite-dimensional if and only if $a_i \in \mathbb{N}$ for $i \neq s$. Therefore, a_s can be an arbitrary complex number. Irreducible finite-dimensional $\mathfrak{g}$-modules are divided into two classes: typical and atypical.

There are many equivalent definitions for a weight module to be typical [Kac 1978]. Here we say that $V(\lambda)$ is typical if it splits in any finite-dimensional regular $\mathfrak{g}$-module (i.e., if it is a submodule or a factor-module of a finite-dimensional regular $\mathfrak{g}$-module then it is a direct summand). By Theorem 1 of [Kac 1978] this is equivalent to requiring that

$$\langle \lambda+\rho,\, \alpha\rangle \neq 0 \tag{4}$$

for all $\alpha \in \Delta_{\bar{1}}^{+}$. If $V(\lambda)$ is (a)typical we will say the weight λ is (a)typical.

In Section 2 we construct a trace on the "ideal" generated by typical modules. With this in mind let us recall some properties of these modules. The space of typical weights is dense in the space of weights corresponding to finite-dimensional modules. In particular, if $a_i \in \mathbb{N}$ for $1 \leq i \leq r$ and $i \neq s$ then there are only finitely many atypical weights with $a_i = \lambda(h_i)$. Furthermore, if λ is atypical then $a_s = \lambda(h_s) \in \mathbb{Z}$. Thus, the name typical is fitting.

For any object V of $\mathfrak{g}$-Mod whose $\mathbb{Z}_2$ grading is given by $V = V_{\bar{0}} \oplus V_{\bar{1}}$ let $\operatorname{sdim}(V) = \dim(V_{\bar{0}}) - \dim(V_{\bar{1}})$ be the superdimension of V. By Proposition 2.10 of [Kac 1978], if V is a typical $\mathfrak{g}$-module then $\operatorname{sdim}(V) = 0$. This vanishing can make other mathematical objects trivial. For example, the supertrace on endomorphisms of a typical module Proposition 2.2 and quantum invariants of links arising from Lie superalgebras [Geer and Patureau-Mirand 2006].

Fix a typical module V_0. Let $\mathcal{I}_{V_0}$ be the set of objects V of $\mathfrak{g}$-Mod such that there exists an object W of $\mathfrak{g}$-Mod and even $\mathfrak{g}$-linear morphisms $\alpha : V \to V_0 \otimes W$ and $\beta : V_0 \otimes W \to V$ with $\beta \circ \alpha = \mathrm{Id}_V$.

Proposition 1.2. *The definition of $\mathcal{I}_{V_0}$ does not depend on the choice of V_0, i.e., $\mathcal{I}_{V_0} = \mathcal{I}_{V_1}$ for any two typical modules V_0 and V_1.*

The set $\mathcal{I}_{V_0}$ is an ideal in the sense that for any $V, V' \in \mathcal{I}_{V_0}$ and $W \in \mathfrak{g}$-Mod we have $V \otimes W \in \mathcal{I}_{V_0}$ and $V \oplus V' \in \mathcal{I}_{V_0}$.

We define $\mathcal{I}$ to be the set $\mathcal{I}_V$ where V is any typical module, which is well defined by the proposition.

Proof. We will prove the first statement; the second follows easily from the definition of $\mathcal{I}_{V_0}$. First, $W \in \mathcal{I}_V$ if and only if $\mathcal{I}_W \subset \mathcal{I}_V$. We will use this fact in the remainder of the proof.

As mentioned above irreducible finite-dimensional $\mathfrak{g}$-modules are in one to one correspondence with $\mathbb{N}^{r-1} \times \mathbb{C}$. We will denote $V_\alpha^{\bar{c}}$ as the module corresponding to $(\bar{c}, \alpha) \in \mathbb{N}^{r-1} \times \mathbb{C}$. Let $V_\alpha^{\bar{0}}$ and $V_\beta^{\bar{c}}$ be typical modules. From the character formula for typical modules we know that $V_\beta^{\bar{c}}$ is a submodule of $V_\alpha^{\bar{0}} \otimes V_{\beta-\alpha}^{\bar{c}}$. Since typical modules always split we have $V_\beta^{\bar{c}} \in \mathcal{I}_{V_\alpha^{\bar{0}}}$ and so $\mathcal{I}_{V_\beta^{\bar{c}}} \subset \mathcal{I}_{V_\alpha^{\bar{0}}}$.

On the other hand, from the discussion in the previous paragraph we have $\mathrm{Hom}_{\mathfrak{g}}(V_\alpha^{\bar 0}\otimes V_{\beta-\alpha}^{\bar c}, V_\beta^{\bar c})\neq 0$, implying $\mathrm{Hom}_{\mathfrak{g}}(V_\alpha^{\bar 0}, V_\beta^{\bar c}\otimes(V_{\beta-\alpha}^{\bar c})^*)\neq 0$. Therefore, as $V_\alpha^{\bar 0}$ is typical, $V_\alpha^{\bar 0}\in\mathcal{I}_{V_\beta^{\bar c}}$ and so $\mathcal{I}_{V_\alpha^{\bar 0}}\subset\mathcal{I}_{V_\beta^{\bar c}}$. □

2. A trace

In this section we define a nonzero supertrace on $\mathrm{End}_{\mathfrak{g}}(V)$ for $V\in\mathcal{I}$. First, let us prove that the usual supertrace on $\mathrm{End}_{\mathfrak{g}}(V)$ is zero.

Let V be a superspace and let $\{v_i\}$ be a basis of V with homogeneous vectors. Let $\{v_i^*\}$ be the dual basis of V^*. We have that $\bar v_i^* = \bar v_i = \bar v_i.\bar v_i^*$. Define the supertrace on $\mathrm{End}_{\mathbb{C}}(V)$ to be the function $\mathrm{str}_V:\mathrm{End}_{\mathbb{C}}(V)\to\mathbb{C}$ given by $f\mapsto \sum_i(-1)^{\bar v_i}v_i^*(f(v_i))$. Then str has the property that if $f\in\mathrm{Hom}_{\mathbb{C}}(V,W)$ and $g\in\mathrm{Hom}_{\mathbb{C}}(W,V)$ then $\mathrm{str}_W(f\circ g)=(-1)^{\bar f\cdot\bar g}\,\mathrm{str}_V(g\circ f)$.

Let us define the partial supertrace that is a generalization of the supertrace. For this, we first define the evaluation and coevaluation morphisms $\mathrm{ev}_V: V\otimes V^*\to\mathbb{C}$ and $\mathrm{coev}_V:\mathbb{C}\to V\otimes V^*$ given by $v\otimes f\mapsto(-1)^{\bar f\bar v}f(v)$ and $1\mapsto\sum_i v_i\otimes v_i^*$, respectively.

Definition 2.1. *Let U and V be superspaces and $f\in\mathrm{End}_{\mathbb{C}}(U\otimes V)$. Then we call the partial supertrace of f the endomorphism*

$$\mathrm{ptr}(f)=(\mathrm{Id}_U\otimes\mathrm{ev}_V)\circ(f\otimes\mathrm{Id}_{V^*})\circ(\mathrm{Id}_U\otimes\mathrm{coev}_V)\in\mathrm{End}_{\mathbb{C}}(U).$$

For f as in Definition 2.1 we have $\mathrm{str}_{U\otimes V}(f)=\mathrm{str}_U(\mathrm{ptr}(f))$. In addition, if $f\in\mathrm{End}_{\mathfrak{g}}(U\otimes V)$ then $\mathrm{ptr}(f)\in\mathrm{End}_{\mathfrak{g}}(U)$.

Let V be an element of $\mathcal{I}=\mathcal{I}_{V_0}$ and $f\in\mathrm{End}_{\mathfrak{g}}(V)$. Choose morphisms $\alpha: V_0\otimes W\to V$ and $\beta: V\to V_0\otimes W$ such that $\alpha\circ\beta=\mathrm{Id}_V$. Then $\mathrm{ptr}(\beta\circ f\circ\alpha)$ is an invariant map of V_0 and so $\mathrm{ptr}(\beta\circ f\circ\alpha)=\mathfrak{c}\,\mathrm{Id}_{V_0}$ for some $\mathfrak{c}\in\mathbb{C}$. We define the bracket of the triple (f,α,β) to be $\langle f;\alpha;\beta\rangle=\mathfrak{c}$.

Proposition 2.2. *Let $V\in\mathcal{I}$ and $f\in\mathrm{End}_{\mathfrak{g}}(V)$ then* $\mathrm{str}_V(f)=0$.

Proof. Using the notation above, we have

$$\mathrm{str}_V(f)=\mathrm{str}_V(f\circ\alpha\circ\beta)=\mathrm{str}_{V_0\otimes W}(\beta\circ f\circ\alpha)=\mathrm{str}_{V_0}(\mathrm{ptr}(\beta\circ f\circ\alpha)).$$

But $\mathrm{ptr}(\beta\circ f\circ\alpha)=\langle f;\alpha;\beta\rangle\,\mathrm{Id}_{V_0}$ so

$$\mathrm{str}_V(f)=\mathrm{str}_{V_0}(\langle f;\alpha;\beta\rangle\,\mathrm{Id}_{V_0})=\langle f;\alpha;\beta\rangle\,\mathrm{sdim}(V_0)=0$$

as the superdimension of V_0 is zero. □

Definition 2.3. *Let* $\mathsf{d}:\{$*typical modules*$\}\to\mathbb{C}$ *be the function defined by*

$$\mathsf{d}(V(\lambda))=\prod_{\alpha\in\Delta_{\bar 0}^+}\frac{\langle\lambda+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}\Big/\prod_{\alpha\in\Delta_{\bar 1}^+}\langle\lambda+\rho,\ \alpha\rangle.$$

Note that Equation (4) implies that d is well defined. As an example, if $\mathfrak{g} = \mathfrak{sl}(n|1)$ with $n \geq 2$, and $\lambda = (0, \ldots, 0|a)$ with $a \in \mathbb{C} \setminus \{0, -1, \ldots, 1-n\}$, we have $\mathsf{d}(V(\lambda)) = \prod_{i=0}^{n-1} 1/(a+i)$.

Theorem 1. *Let $V \in \mathcal{I}$ and $f \in \operatorname{End}_{\mathfrak{g}}(V)$. Choose a typical module V_0, and morphisms $\alpha \in \operatorname{Hom}_{\mathfrak{g}}(V_0 \otimes W, V)_{\bar{0}}$ and $\beta \in \operatorname{Hom}_{\mathfrak{g}}(V, V_0 \otimes W)_{\bar{0}}$ such that $\alpha \circ \beta = \operatorname{Id}_V$. Then*

$$\operatorname{str}'_V(f) = \mathsf{d}(V_0)\langle f; \alpha; \beta\rangle$$

depends only on f; it does not depend on the choice of V_0, α or β. Furthermore, str' is a trace in the following sense: for any $V, V' \in \mathcal{I}$ and any $\mathfrak{g}$-module U,

(a) $\operatorname{str}'_V : \operatorname{End}_{\mathfrak{g}}(V) \to \mathbb{C}$ *is linear.*

(b) $\operatorname{str}'_{V'}(f \circ g) = (-1)^{\bar{g}\bar{f}} \operatorname{str}'_V(g \circ f)$ *for any* $f \in \operatorname{Hom}_{\mathfrak{g}}(V, V')$, $g \in \operatorname{Hom}_{\mathfrak{g}}(V', V)$.

(c) $\operatorname{str}'_{V\otimes U}(f \otimes g) = \operatorname{str}'_V(f)\ \operatorname{str}_U(g)$ *for any* $f \in \operatorname{End}_{\mathfrak{g}}(V)$ *and any* $g \in \operatorname{End}_{\mathfrak{g}}(U)$, *in particular* $\operatorname{str}'(f \otimes g) = \operatorname{str}(g) = 0$ *if* $U \in \mathcal{I}$.

(d) $\operatorname{str}'_{V\otimes U}(f) = \operatorname{str}'_V(\operatorname{ptr}(f))$ *for any* $f \in \operatorname{End}_{\mathfrak{g}}(V \otimes U)$.

The proof of Theorem 1 will be given in Section 4. Let us now make a few comments about this theorem. First, remark that property (d) implies property (c). Next, property (d) implies a kind of invariance for str'. Let us make this statement more precise.

Let U, U' be $\mathfrak{g}$-modules and V, V' be in $\mathcal{I}$. The following spaces of morphisms are canonically isomorphic:

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{g}}(\operatorname{Hom}_{\mathbb{C}}(U', V'), \operatorname{Hom}_{\mathbb{C}}(U, V)) &\cong \operatorname{Hom}_{\mathfrak{g}}(U \otimes V', V \otimes U') \\ &\cong \operatorname{Hom}_{\mathfrak{g}}(V' \otimes U, U' \otimes V) \cong \operatorname{Hom}_{\mathfrak{g}}(\operatorname{Hom}_{\mathbb{C}}(V, U), \operatorname{Hom}_{\mathbb{C}}(V', U')).\end{aligned}$$

Let $\Psi \in \operatorname{Hom}_{\mathfrak{g}}(\operatorname{Hom}_{\mathbb{C}}(U', V'), \operatorname{Hom}_{\mathbb{C}}(U, V))$ and let h, $h^{\#}$, $\Psi^{\#}$ be the corresponding morphisms in the other three spaces, respectively. We have $h^{\#} = \tau \circ h \circ \tau$ where τ is the superpermutation. Also, if $f \in \operatorname{Hom}_{\mathbb{C}}(U', V')$ and $g \in \operatorname{Hom}_{\mathbb{C}}(V, U)$ then $\Psi(f) = \operatorname{ptr}(h \circ (\operatorname{Id}_U \otimes f))$ and $\Psi^{\#}(g) = \operatorname{ptr}(h^{\#} \circ (\operatorname{Id}_{V'} \otimes g))$ (here we use a generalization of the partial trace $\operatorname{ptr} : \operatorname{Hom}(A \otimes C, B \otimes C) \to \operatorname{Hom}(A, B)$). Thus, applying property (d), we get that

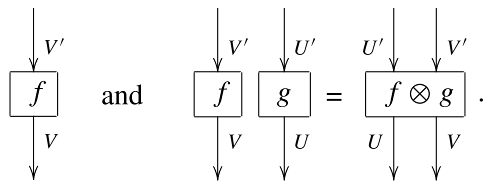
$$\operatorname{str}'\left(\Psi(f) \circ g\right) = (-1)^{\bar{\Psi}.\bar{f}} \operatorname{str}'\left(f \circ \Psi^{\#}(g)\right). \tag{5}$$

Indeed,

$$\begin{aligned}\operatorname{str}'\left(\Psi(f) \circ g\right) &= \operatorname{str}'\left(\operatorname{ptr}(h \circ (\operatorname{Id}_U \otimes f)) \circ g\right) = (-1)^{\bar{g}.\bar{f}} \operatorname{str}'\left(\operatorname{ptr}(h \circ (g \otimes f))\right) \\ &= (-1)^{\bar{g}.\bar{f}} \operatorname{str}'\left(h \circ (g \otimes f)\right) = \operatorname{str}'\left(h^{\#} \circ (f \otimes g)\right) \\ &= \operatorname{str}'\left(\operatorname{ptr}(h^{\#} \circ (f \otimes g))\right) = (-1)^{\bar{g}.\bar{f}} \operatorname{str}'\left(\operatorname{ptr}(h^{\#} \circ (\operatorname{Id}_{V'} \otimes g)) \circ f\right) \\ &= (-1)^{\bar{\Psi}.\bar{f}} \operatorname{str}'\left(f \circ \Psi^{\#}(g)\right).\end{aligned}$$

The results of this section can be stated in the language of symmetric monoidal category with duality or more generally ribbon categories. We will not make this formalism precise, however we will end this section by giving the following graphs which we hope will shed light on the above results. For more details on ribbon categories see [Turaev 1994].

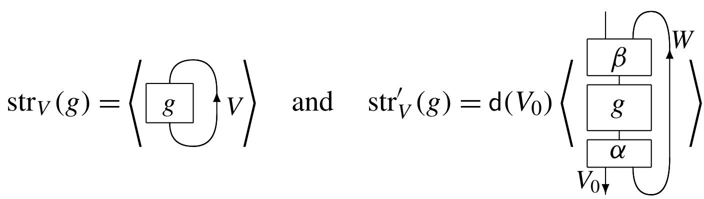
Here we will represent morphisms with ribbon graphs, which are read from bottom to top. The tensor product of two morphisms is represented by setting the two corresponding graphs next to each other. For example, if $f : V \to V'$ and $g : U \to U'$ are even morphism of $\mathfrak{g}$-Mod then we represent f and $f \otimes g$ by

(6) 

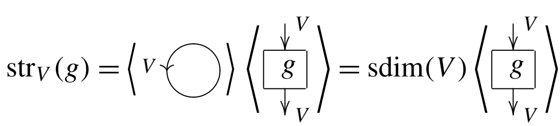
Let the graphs  and  represent the morphisms $\mathrm{ev}_V : V \otimes V^* \to \mathbb{C}$ and $\mathrm{coev}_V : \mathbb{C} \to V \otimes V^*$, respectively.

Let $g : V \to V$ be an even invariant morphism of a $\mathfrak{g}$-module V and let G be a ribbon graph representing g, as in Equation (6). If V is simple then the morphism g is a scalar times the identity, which we denote by $\langle g \rangle = \langle G \rangle$.

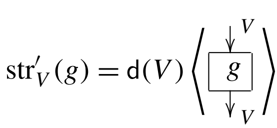
The elements $\mathrm{str}_V(g)$ and $\mathrm{str}'_V(g)$ can be represented by

(7) 

where we require $V \in \mathcal{I}$ in the second case. When V is simple the supertrace can be rewritten as

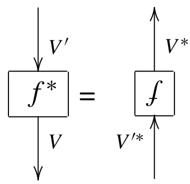


where $\mathrm{sdim}(V) = 0$ if V is typical. Also, when V is a typical module the str' becomes



Thus, the function d can be thought of as a nonzero replacement of the usual superdimension. Moreover, d can be thought of as the classical analogue of the modified quantum dimensions defined in [Geer et al. 2007].

If $f : V \to V'$ is an even invariant morphism let $f^* : (V')^* \to V^*$ be the "supertranspose" of f. We can represent f^* by



We will use the "supertranspose" in the next section.

3. Invariant tensors

In this section we define a nontrivial bilinear form on a space of invariant tensors of $\mathfrak{g}$. The standard bilinear form on $\mathfrak{g}$ is zero on this space of tensors.

Let V be an object of $\mathfrak{g}$-Mod and let $T(V) = \oplus_i T(V)_i$ be the tensor algebra of V, where $T(V)_i$ is the space $V^{\otimes i}$. Let $T(V)^{\mathfrak{g}}$ be the invariant tensors of $T(V)$.

Lemma 3.1. *All invariant tensors of* $T(\mathfrak{g})$ *are even.*

Proof. We will prove the lemma for $\mathfrak{g} = \mathfrak{sl}(m|n)$, the prove for $\mathfrak{osp}(2|2n)$ is similar. We can identify $\mathfrak{sl}(m|n)$ with the Lie superalgebra of supertrace zero $(m+n) \times (m+n)$ matrices. This standard representation is obtained by sending e_i to the elementary matrix $E_{i,i+1}$, f_i to $E_{i+1,i}$, h_i to $E_{i,i} - E_{i+1,i+1}$ if $i \neq m$ and h_m to $E_{m,m} + E_{m+1,m+1}$. The Cartan subalgebra $\mathfrak{h}$ with basis $\{h_i\}$ is contained in the space of diagonal matrices X. The space X^* has a canonical basis $\{\epsilon_1, \dots \epsilon_{m+n}\}$ which is dual to the basis formed by the matrices $\{E_{i,i}\}$. Set $\delta_i = \epsilon_{i+m}$, then $\mathfrak{h}$ is the kernel of the supertrace $\mathrm{str} = \sum \epsilon_i - \sum \delta_j$. Therefore, $\mathfrak{h}^*$ is the quotient of X^* by the supertrace.

Let $\Lambda \subset \mathfrak{h}$ be the root lattice generated by the positive roots. Let $f : \Lambda \to \mathbb{Z}$ be the linear function determined by $\epsilon_i \mapsto n$ and $\delta_j \mapsto m$ (note that $\mathrm{str} \mapsto 0$). By definition the simple positive even roots $\epsilon_i - \epsilon_j$ and $\delta_i - \delta_j$ map to zero and the simple positive odd roots $\epsilon_i - \delta_j$ map to $-(m-n)$. Therefore, the image of f is $(m-n)\mathbb{Z}$ and f induces a linear map $\bar{f} : \Lambda \to \mathbb{Z}/2\mathbb{Z}$ given by $\alpha \mapsto \frac{f(\alpha)}{m-n}$ modulo 2. The map $\bar{f}$ in turn induces a map on the weight vectors of $T(\mathfrak{g})$ (which we also denote by $\bar{f}$) that satisfies $\bar{f}(x \otimes y) = \bar{f}(x) + \bar{f}(y)$ for $x, y \in T(\mathfrak{g})$. Note that $\bar{f}$ gives the parity of a weight vector of $T(\mathfrak{g})$.

Let t be an element of $T(\mathfrak{g})_k$ with weight $a_1\epsilon_1 + \cdots + a_m\epsilon_m + b_1\delta_1 + \cdots + b_n\delta_n$. If t is in $(T(\mathfrak{g})_k)^{\mathfrak{g}}$ then the Cartan subalgebra acts by zero and so the weight of t is zero: $a_i = b_j = 0$ for all i and j. But from above we know that the parity of t is equal to $\bar{f}(t) = \big(n \sum a_i + m \sum b_j\big)/(m-n)$ modulo 2, which is zero if t is in $(T(\mathfrak{g})_k)^{\mathfrak{g}}$. Thus, all the invariant tensors of $T(\mathfrak{g})$ are even. $\square$

From Propositions 2.5.3 and 2.5.5 of [Kac 1977] there exists a unique (up to constant factor) nondegenerate supersymmetric invariant even bilinear form $(,)$ on $\mathfrak{g}$. Let $b : \mathfrak{g} \to \mathfrak{g}^*$ be the isomorphism given by the assignment $x \mapsto (x, \cdot)$.

We extend this bilinear form to $T(\mathfrak{g})$ by

$$(x_1 x_2 \dots x_k, x'_1 x'_2 \dots x'_l) = \delta_{kl} \prod_{i=1}^{k} (-1)^{\sum_{i<j} \bar{x}_j \bar{x}'_i} (x_i, x'_i)$$

where $x_i, x'_j \in \mathfrak{g}$. Since $(,)$ is nondegenerate on $\mathfrak{g}$, this extension is a nondegenerate bilinear form on $T(\mathfrak{g})$. Moreover, since $(. , .)$ is supersymmetric on $\mathfrak{g}$ and $(x, x') = 0$ for all $x, x' \in \mathfrak{g}$ such that $\bar{x} \neq \bar{x}'$, we have that the extension is supersymmetric on $T(\mathfrak{g})$.

For $t \in (T(\mathfrak{g})_N)^{\mathfrak{g}} \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, T(\mathfrak{g})_N)$ we have $t^* \in \mathrm{Hom}_{\mathfrak{g}}(T(\mathfrak{g}^*)_N, \mathbb{C})$, where $*$ is the "supertranspose". Using this notation the bilinear form is given by $(t, t') = \langle t^* \circ b^{\otimes N} \circ t' \rangle$. Here and after, if $g \in \mathrm{End}_{\mathbb{C}}(\mathbb{C})$, we denote by $\langle g \rangle$ the scalar $g(1)$.

Recall the definition of the coevaluation morphism coev_V given in Section 2.

Definition 3.2. *For $N \in \mathbb{N}$ define*

$$\mathcal{I}\mathcal{T}_N = \{ f(\mathrm{coev}_V(1)) : f \in \mathrm{Hom}_{\mathfrak{g}}(V \otimes V^*, \mathfrak{g}^{\otimes n}) \textit{ for some } V \in \mathcal{I} \}$$

and $\mathcal{I}\mathcal{T} = \oplus_N \mathcal{I}\mathcal{T}_N$.

Let $t \in \mathcal{I}\mathcal{T}_N$ and $t' \in (T(\mathfrak{g})_N)^{\mathfrak{g}}$. We will now show that (t, t') can be written in terms of the supertrace. We regard t, t' as elements of $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathfrak{g}^{\otimes N})$. As $t = f(\mathrm{coev}_V)$ for some $f \in \mathrm{Hom}_{\mathfrak{g}}(V \otimes V^*, \mathfrak{g}^{\otimes N})$ where $V \in \mathcal{I}$, we have $t^* = \mathrm{coev}_V^* \circ f^*$ and

$$(t, t') = \langle \mathrm{coev}_V^* \circ f^* \circ b^{\otimes N} \circ t' \rangle.$$

The morphism $f^* \circ b^{\otimes N} \circ t' \in \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V^* \otimes V) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V \otimes V^*)$ can be identified with a $\mathfrak{g}$-linear endomorphism of V which we denote by $[f^* \circ b^{\otimes N} \circ t']$. Thus, we have $(t, t') = \mathrm{str}_V([f^* \circ b^{\otimes N} \circ t'])$, which is zero by Proposition 2.2. The above discussion can be summarized in the following lemma.

Lemma 3.3. *If $t \in \mathcal{I}\mathcal{T}_N$ and $t' \in (T(\mathfrak{g})_N)^{\mathfrak{g}}$ then $(t, t') = \mathrm{str}_V([f^* \circ b^{\otimes N} \circ t'])$ which is zero.*

Proposition 3.4. *The sets $\mathcal{I}\mathcal{T}_N$ are vector spaces. Moreover, $\mathcal{I}\mathcal{T} = \oplus_N \mathcal{I}\mathcal{T}_N$ is a two sided ideal of $T(\mathfrak{g})^{\mathfrak{g}}$ which is in the kernel of the restriction of $(. , .)$ to the space of invariant tensor $T(\mathfrak{g})^{\mathfrak{g}}$.*

Proof. We will first show that $\mathcal{I}\mathcal{T}_N$ is a vector space. Let $t_1, t_2 \in \mathcal{I}\mathcal{T}_N$ and $\lambda \in \mathbb{C}$. Then $t_i = f_i(\mathrm{coev}_{V_i}(1))$ for some f_i and V_i. Set $V = V_1 \oplus V_2$. Let $f : V \otimes V^* \to \mathfrak{g}^{\otimes N}$ be the invariant map given by

$$f((v_1 \oplus v_2) \otimes (\varphi_1 \oplus \varphi_2)) = f_1(v_1 \otimes \varphi_1) + \lambda f_2(v_2 \otimes \varphi_2).$$

Then $f(\mathrm{coev}_V(1)) = t_1 + \lambda t_2$. Thus, $\mathcal{IT}_N$ is a vector space.

Now we will show that $\mathcal{IT}$ is an ideal. Let $t' \in (\mathfrak{g}^{\otimes M})^{\mathfrak{g}}$ and let t_1 be as above. Let $g : V_1 \otimes V_1^* \to \mathfrak{g}^{\otimes(M+N)}$ be the invariant map given by

$$g(v_1 \otimes \varphi_1) = t' \otimes f_1(v_1 \otimes \varphi_1).$$

Then $g(\mathrm{coev}_{V_1}(1)) = t' \otimes t_1$ and so $t' \otimes t_1 \in \mathcal{IT}_{M+N}$.

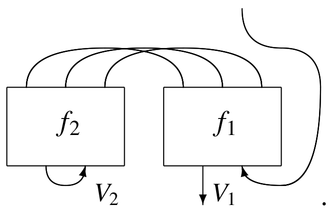
The last statement of the proposition follows from Lemma 3.3. □

Next we define a bilinear form on $\mathcal{IT}$. The following definition is motivated by Lemma 3.3 and justified by Theorem 2.

Definition 3.5. *For $t_1 \in \mathcal{IT}_N$ and $t_2 \in \mathcal{IT}_M$ with $t_i = f_i(\mathrm{coev}_{V_i})$, define*

$$(t_1, t_2)' = \delta_{M,N}\, \mathrm{str}'_{V_1}\left([f_1^* \circ b^{\otimes N} \circ t_2]\right)$$

We can represent $[f_1^* \circ b^{\otimes N} \circ t_2]$ by the following picture, where $M = N = 3$ for simplicity:



.

It is tempting to think that the above construction could work for $t_1 \in \mathcal{IT}$ and any $t_2 \in T(\mathfrak{g})$ but this is false because there are examples of $t_2 \in T(\mathfrak{g})$ for which the above scalar depends not only of t_1 but also of f_1.

To simplify notation we will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using the isomorphism b but will no longer write b.

Theorem 2. $(\,.\,,\,.\,)'$ *is a well defined symmetric bilinear form on $\mathcal{IT}$ satisfying $(G(t_1), t_2)' = (t_1, G^*(t_2))'$ for any $t_1 \in \mathcal{IT}_M$, $t_2 \in \mathcal{IT}_N$, $G \in \mathrm{Hom}_{\mathfrak{g}}(T(\mathfrak{g})_M, T(\mathfrak{g})_N)$. In particular, the symmetric group S_N acts orthogonally on $\mathcal{IT}_N$.*

Proof. Let t_1 and t_2 be elements of $\mathcal{IT}_N$ with $t_i = f_i(\mathrm{coev}_{V_i})$. We need to show that the definition of $(t_1, t_2)'$ is independent of f_1, f_2, V_1, and V_2.

Using the canonical isomorphism (2), we can make the identifications

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}(V_2 \otimes V_2^*, V_1 \otimes V_1^*) &\cong \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V_1 \otimes V_1^* \otimes V_2^* \otimes V_2)\\ &\cong \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V_1 \otimes V_2 \otimes V_1^* \otimes V_2^*) \cong \mathrm{End}_{\mathfrak{g}}(V_1 \otimes V_2).\end{aligned}$$

Therefore, below we will consider $f_1^* \circ f_2$ as an element of $\mathrm{End}_{\mathfrak{g}}(V_1 \otimes V_2)$. Notice that for fixed $t_1 = f_1(\mathrm{coev}_{V_1})$ the map $\mathcal{IT}_N \to \mathbb{C}$ given by

$$t \mapsto \mathrm{str}'_{V_1}(f_1^* \circ t)$$

is well defined and linear. Then from Theorem 1(d) we have $\mathrm{str}'_{V_1}(f_1^* \circ t_2) = \mathrm{str}'_{V_1 \otimes V_2}(f_1^* \circ f_2) = \mathrm{str}'_{V_1 \otimes V_2}(f_2^* \circ f_1) = \mathrm{str}'_{V_2}(f_2^* \circ t_1)$, which does not depend on f_1 or V_1. Thus, $(\,.\,,\,.\,)'$ is a well defined symmetric bilinear form.

For the last statement of the theorem,

$$(G(t_1), t_2)' = \mathrm{str}'_{V_1 \otimes V_2}(f_1^* \circ G^* \circ f_2) = (t_1, G^*(t_2))'. \qquad \square$$

4. Proof of Theorem 1

The proof of Theorem 1 uses quantized Lie superalgebras and low-dimensional topology. In particular, we have the following general plan: (1) start with the desired statement at the level of $\mathfrak{g}$-Mod, (2) translate these statements to the quantum level, (3) use properties of invariants of ribbon graphs to prove these statements and (4) take the classical limit to obtain the proof of the original statements. With this in mind we will begin this section by recalling some properties about the Drinfeld–Jimbo type quantization of $\mathfrak{g}$.

Let h be an indeterminate and set $q = \mathrm{e}^{h/2}$. We use the notation $q^z = \mathrm{e}^{zh/2}$ for $z \in \mathbb{C}$. Let $U_h^{DJ}(\mathfrak{g})$ be the Drinfeld–Jimbo type quantization of $\mathfrak{g}$ defined in [Yamane 1994]. The quantization $U_h^{DJ}(\mathfrak{g})$ is a braided $\mathbb{C}[[h]]$-Hopf superalgebra given by generators and relations. As we will explain now $U_h^{DJ}(\mathfrak{g})$ is related to a quasi-Hopf superalgebra.

For each Lie algebra Drinfeld defined a quasi-Hopf quantized universal enveloping algebra:

$$(U(\mathfrak{g})[[h]], \Delta_0, \epsilon_0, \Phi_{KZ}).$$

The morphisms Δ_0 and ϵ_0 are the standard coproduct and counit of $U(\mathfrak{g})[[h]]$. The element Φ_{KZ} is the KZ-associator. Let $A_\mathfrak{g}$ be the analogous topologically free quasi-Hopf superalgebra (for more details see [Geer 2006]).

Let $U_h^{DJ}(\mathfrak{g})$-$\mathrm{Mod}_{\mathrm{fr}}$ and $A_\mathfrak{g}$-$\mathrm{Mod}_{\mathrm{fr}}$ be the tensor categories of topologically free $U_h^{DJ}(\mathfrak{g})$-modules and $A_\mathfrak{g}$-modules of finite rank, respectively (that is, those of the form $V[[h]]$, where V is a finite-dimensional $\mathfrak{g}$-module). We say a module $V[[h]]$ in $U_h^{DJ}(\mathfrak{g})$-$\mathrm{Mod}_{\mathrm{fr}}$ is typical if V is a typical $\mathfrak{g}$-module.

In [Geer 2006] the first author proves that there exists a functor $G : A_\mathfrak{g}\text{-}\mathrm{Mod}_{\mathrm{fr}} \to U_h^{DJ}(\mathfrak{g})\text{-}\mathrm{Mod}_{\mathrm{fr}}$ which is an equivalence of tensor categories. There is a natural tensor functor $G' : \mathfrak{g}\text{-Mod} \to A_\mathfrak{g}\text{-}\mathrm{Mod}_{\mathrm{fr}}$ given by $V \mapsto V[[h]]$ and $f \mapsto G'(f)$ where the action of $\mathfrak{g}$ on V extends to an action of $U(\mathfrak{g})[[h]]$ on $V[[h]]$ be linearity and $G'(f)(\sum v_i h^i) = \sum f(v_i) h^i$. We have the commutative diagram of functors

$$\begin{tikzcd} A_\mathfrak{g}\text{-}\mathrm{Mod}_{\mathrm{fr}} \arrow[rr, "G"] & & U_h^{DJ}(\mathfrak{g})\text{-}\mathrm{Mod}_{\mathrm{fr}} \arrow[dl, "\text{classical limit}"] \\ & \mathfrak{g}\text{-Mod} \arrow[ul, "G'"] & \end{tikzcd} \tag{8}$$

where the down left arrow is the classical limit given by taking the limit as h goes to zero. For any object V and morphism g of $\mathfrak{g}$-Mod let us denote $G \circ G'(V)$ and $G \circ G'(g)$ by $\widetilde{V}$ and $\widetilde{g}$, respectively. Here the functor $G \circ G'$ composed with the classical limit is the identity functor: $V \equiv \widetilde{V} \bmod h$ and $g \equiv \widetilde{g} \bmod h$.

In [Geer and Patureau-Mirand 2006] we defined an invariant of framed colored links. Let us now recall the basic construction and some properties of this invariant. Here we say that a link or more generally a tangle is *colored* if each of its components are assigned an object of $U_h^{DJ}(\mathfrak{g})$-Mod$_{\mathrm{fr}}$.

Let F be the usual Reshetikhin–Turaev functor from the category of framed colored tangles to the category of $U_h^{DJ}(\mathfrak{g})$-Mod$_{\mathrm{fr}}$. In [Geer and Patureau-Mirand 2006] a function from the set of typical $U_h^{DJ}(\mathfrak{g})$-module to the ring $\mathbb{C}[[h]][h^{-1}]$ is defined. As remarked in that article, this function can be multiplied by $h^{|\Delta^+_{\bar 1}|}$ to obtain a function which takes values in $\mathbb{C}[[h]]$. Let us denote this function by d_{h}.

Lemma 4.1.

$$\mathsf{d}_{\mathsf{h}}(\widetilde{V}(\lambda)) = h^{|\Delta^+_{\bar 1}|} \frac{\displaystyle\prod_{\alpha\in\Delta^+_{\bar 0}} \frac{q^{\langle\lambda+\rho,\alpha\rangle} - q^{-\langle\lambda+\rho,\alpha\rangle}}{q^{\langle\rho,\alpha\rangle} - q^{-\langle\rho,\alpha\rangle}}}{\displaystyle\prod_{\alpha\in\Delta^+_{\bar 1}} (q^{\langle\lambda+\rho,\alpha\rangle} - q^{-\langle\lambda+\rho,\alpha\rangle})}.$$

In particular, $\mathsf{d}(V(\lambda))$ *is equal to* $\mathsf{d}_{\mathsf{h}}(\widetilde{V}(\lambda))$ *mod* h.

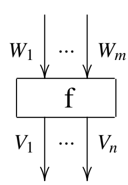
Proof. The proof follows from the formulas for $h^{-|\Delta^+_{\bar 1}|}\,\mathsf{d}_{\mathsf{h}}$ given in the Appendix of [Geer and Patureau-Mirand 2006] and from the definition of d. $\square$

Suppose L is a framed colored link such that by cutting some component of L one obtains a framed colored $(1,1)$-tangle $T_{V(\lambda)}$ such that the open string is colored by the deformed typical module $\widetilde{V}(\lambda)$ of highest weight λ. Then $F(T_{V(\lambda)}) = x.\mathrm{Id}_{\widetilde{V}(\lambda)}$, for some x in $\mathbb{C}[[h]]$. Set $\langle T_{V(\lambda)}\rangle = x$. In [Geer and Patureau-Mirand 2006] it is shown that the assignment

$$L \mapsto \mathsf{d}_{\mathsf{h}}(\widetilde{V}(\lambda))\langle T_{V(\lambda)}\rangle$$

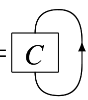
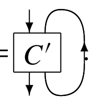
is a well defined colored framed link invariant denoted by F'. In particular, $F'(L)$ is independent of $\widetilde{V}(\lambda)$, $T_{V(\lambda)}$ and where L is cut.

An even morphism $f : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_m$ in the category $U_h^{DJ}(\mathfrak{g})$-Mod$_{\mathrm{fr}}$ can be represented by



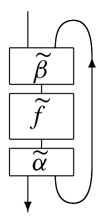
Such a box is called a coupon, which we denote by $C^{W_1,\dots,W_m}_{V_1,\dots,V_n}(f)$. Here we will say a *ribbon graph* is a framed tangle with coupons and colors coming from the category $U_h^{DJ}(\mathfrak{g})$-$\mathrm{Mod}_{\mathrm{fr}}$. In [Geer et al. 2007] it is shown that the construction of F' can be extended to ribbon graphs having at least one component colored by a typical $U_h^{DJ}(\mathfrak{g})$-module.

The invariant F' can also be extended to ribbon graphs having at least one component colored by a deformed module in $\mathcal{I}$ [Geer et al. 2007]. We will now describe this extension in the following situation. Let C (C') be a $(1,1)$-tangle (resp. $(2,2)$-tangle) ribbon graph such that the input(s) and output(s) are equal. Let L_C be the closed ribbon graph obtained from closing the coupon C. Let $T_{C'}$ be the $(1,1)$-tangle ribbon graph obtained from closing right most component. The ribbon graphs L_C and $T_{C'}$ can be represented by

$L_C =$  $T_{C'} =$ 

These pictures represent respectively the trace and the partial trace of the morphisms in the coupon.

Let $V \in \mathcal{I}$ and let $\alpha : V_0 \times W \to V$ and $\beta : V \to V_0 \otimes W$ be morphisms in $\mathfrak{g}$-Mod such that $\alpha \circ \beta = \mathrm{Id}_V$. Let $f \in \mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}}$ and let $T(f;\alpha;\beta)$ be the $(1,1)$-tangle ribbon graph $T_{C^{\tilde{V}_0\otimes\tilde{W}}_{\tilde{V}}(\tilde{\beta})\circ C^{\tilde{V}}_{\tilde{V}}(\tilde{f})\circ C^{\tilde{V}}_{\tilde{V}_0\otimes\tilde{W}}(\tilde{\alpha})}$. That is,

$T(f;\alpha;\beta) =$ 

Then we define

$$F'(L_{C^{\tilde{V}}_{\tilde{V}}(\tilde{f})}) = \mathsf{d}_{\mathsf{h}}(\widetilde{V_0})\langle T(f;\alpha;\beta)\rangle.$$

In [Geer and Patureau-Mirand 2006; Geer et al. 2007] it is shown that F' is well defined. Now we are ready to prove the main theorem of the paper.

Proof of Theorem 1. Let V_1 be a typical $\mathfrak{g}$-module. Then $\mathcal{I} = \mathcal{I}_{V_0} = \mathcal{I}_{V_1}$. Choose $\alpha_i : V_i \times W_i \to V$ and $\beta_i : V \to V_i \otimes W$ such that $\alpha_i \circ \beta_i = \mathrm{Id}_V$, for $i = 0, 1$. If $f \in \mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}}$ then $\langle f;\alpha_0;\beta_0\rangle = \langle f;\alpha_1;\beta_1\rangle = 0$ as $\mathrm{ptr}(\beta\circ f\circ\alpha) = \langle f;\alpha;\beta\rangle\,\mathrm{Id}_{V_0}$ and $\beta\circ f\circ\alpha$ is odd. Therefore, we can assume that $f \in \mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}}$ (that is, f is a morphism in the symmetric monoidal category $\mathfrak{g}$-$\mathrm{Mod}_{\bar{0}}$). We will show that

$$\mathsf{d}(V_0)\langle f;\alpha_0;\beta_0\rangle = \mathsf{d}(V_1)\langle f;\alpha_1;\beta_1\rangle. \tag{9}$$

By definition of the ribbon category $U_h^{DJ}(\mathfrak{g})$-Mod$_{\mathrm{fr}}$ we have equality between $\langle f;\alpha_i;\beta_i\rangle$ and $\langle T(f;\alpha_i;\beta_i)\rangle \bmod h$, for $i=0,1$. Combining this with Lemma 4.1 we obtain $\mathsf{d}(V_i)\langle f;\alpha_i;\beta_i\rangle$ is equal to $\mathsf{d}_{\mathsf{h}}(\widetilde{V}_i)\langle T(f;\alpha_i;\beta_i)\rangle \bmod h$, for $i=0,1$. Finally, by [Geer et al. 2007], the extension of F' to ribbon graphs is well defined. In particular, we have $\mathsf{d}_{\mathsf{h}}(\widetilde{V}_0)\langle T(f;\alpha_0;\beta_0)\rangle=\mathsf{d}_{\mathsf{h}}(\widetilde{V}_1)\langle T(f;\alpha_1;\beta_1)\rangle$. Thus, Equation (9) holds and $\mathrm{str}'_V(f)$ only depends on f.

Now we prove the remaining statements of the theorem. The function str'_V is linear because $F(C_{\widetilde{V}}^{\widetilde{V}}(a\widetilde{f}+b\widetilde{g}))=aF(C_{\widetilde{V}}^{\widetilde{V}}(\widetilde{f}))+bF(C_{\widetilde{V}}^{\widetilde{V}}(\widetilde{g}))$ for $f,g\in\mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}}$ and $a,b\in\mathbb{C}$. Part (c) follows from the property that $F'(L\sqcup L')=F'(L)F(L')$ for any two links L and L' [Geer et al. 2007]. The proof of (d) follows from the behavior of F' with respect to cabling [Geer et al. 2007].

To prove part (b) we need to be careful because coupons must be labeled by even morphisms, but the morphisms in the statement of (b) can be odd. If V is an object of $\mathfrak{g}$-Mod then denote V^- as the $\mathfrak{g}$-module obtained from V by taking the opposite parity. Then V and V^- are isomorphic by an odd isomorphism $\sigma_V:V\to V^-$, which changes the parity.

Lemma 4.2. *Let $\gamma\in\mathrm{End}_{U_h^{DJ}(\mathfrak{g})}(\widetilde{W}\otimes\widetilde{V})_{\bar{0}}$ and set $\eta=(\mathrm{Id}\otimes\widetilde{\sigma}_V)\gamma(\mathrm{Id}\otimes\widetilde{\sigma}_V)$. Then*

$$F\Big(T_{C^{\widetilde{W}\otimes\widetilde{V}}_{\widetilde{W}\otimes\widetilde{V}}(\gamma)}\Big)=-F\Big(T_{C^{\widetilde{W}\otimes\widetilde{V}^-}_{\widetilde{W}\otimes\widetilde{V}^-}(\eta)}\Big).$$

Proof. Let $\{w_i\}_{i=1}^q$ and $\{v_j\}_{j=1}^p$ be bases of the $\mathfrak{g}$-modules V and W, respectively. Then $\{v_j\}_{j=1}^p$, $\{\sigma_V(v_j)\}_{j=1}^p$ and $\{w_i\}_{i=1}^q$ are bases for the $U_h^{DJ}(\mathfrak{g})$-modules $\widetilde{V}$, $\widetilde{V}^-$ and $\widetilde{W}$, respectively.

Let γ_{ij}^{kl} be the elements of $\mathbb{C}[\![h]\!]$ defined by

$$\gamma(w_i\otimes v_j)=\sum_{k=1}^{q}\sum_{l=1}^{p}\gamma_{ij}^{kl}w_k\otimes v_l.$$

A direct calculation shows that

$$F\Big(T_{C^{\widetilde{W}\otimes\widetilde{V}}_{\widetilde{W}\otimes\widetilde{V}}(\gamma)}\Big)(w_i)=\sum_{k=1}^{q}\sum_{l=1}^{p}(-1)^{\bar{v}_j}\gamma_{ij}^{kj}w_k,\tag{10}$$

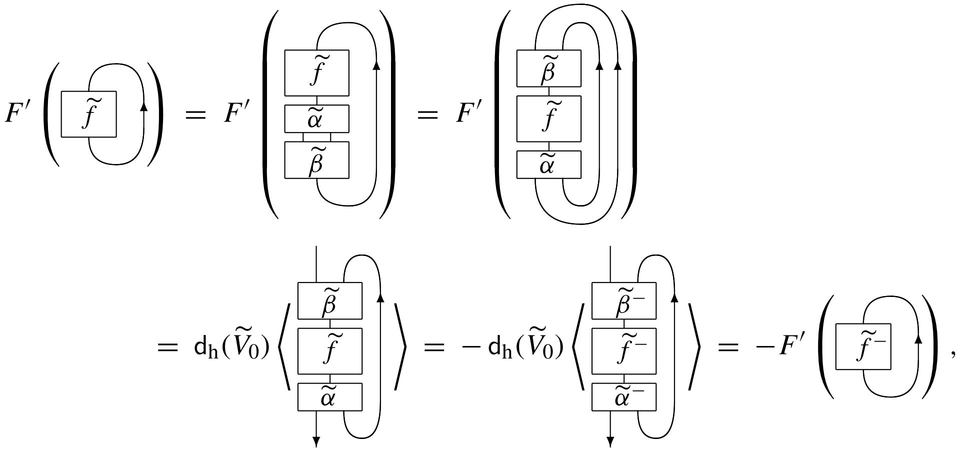
$$F\Big(T_{C^{\widetilde{W}\otimes\widetilde{V}^-}_{\widetilde{W}\otimes\widetilde{V}^-}(\eta)}\Big)(w_i)=\sum_{k=1}^{q}\sum_{l,j=1}^{p}(-1)^{\bar{w}_i+\bar{w}_k}(-1)^{(\bar{1}+\bar{v}_l)(\bar{1}+\bar{v}_j)}\delta_{lj}\gamma_{ij}^{kl}w_k,\tag{11}$$

where $\delta_{lj}(-1)^{(\bar{1}+\bar{v}_l)(\bar{1}+\bar{v}_j)}=(-1)^{\bar{1}+\bar{v}_j}$ and $\bar{w}_i=\bar{w}_k$ since η is an even morphism. Therefore, the right sides of (10) and (11) are the negative of each other and the lemma follows. $\square$

Lemma 4.3. *For $V \in \mathscr{I}$ and $f \in \mathrm{End}_{\mathfrak{g}}(V)_{\bar 0}$ we have*

$$F'\big(L_{C^{\widetilde V}_{\widetilde V}(\widetilde f)}\big) = -F'\big(L_{C^{\widetilde V^-}_{\widetilde V^-}(\widetilde\sigma\circ\widetilde f\circ\widetilde\sigma)}\big).$$

Proof. Let $\alpha \in \mathrm{End}_{\mathfrak{g}}(V_0\otimes W, V)_{\bar 0}$ and $\beta \in \mathrm{End}_{\mathfrak{g}}(V, V_0\otimes W)_{\bar 0}$ such that $\mathrm{Id}_V = \alpha\circ\beta$. Then for $\alpha^- = (\mathrm{Id}_{V_0}\otimes\sigma_W)\circ\alpha\circ\sigma_V \in \mathrm{End}_{\mathfrak{g}}(V_0\otimes W^-, V^-)_{\bar 0}$ and $\beta^- = \sigma_V\circ\beta\circ(\mathrm{Id}_{V_0}\otimes\sigma_W) \in \mathrm{End}_{\mathfrak{g}}(V^-, V_0\otimes W^-)_{\bar 0}$, we have $\mathrm{Id}_{V^-} = \alpha^-\circ\beta^-$. Now, we also denote $\widetilde f^- = \widetilde\sigma_V\circ\widetilde f\circ\widetilde\sigma_V^{-1} \in \mathrm{End}_{\mathfrak{g}}(V^-)_{\bar 0}$ and it is convenient to give a pictorial proof:



where the fourth equality comes from Lemma 4.2. □

Now we are ready to prove part (b). Let $f : V \to V'$ and $g : V' \to V$ be morphisms of $\mathfrak{g}$-Mod such that $f\circ g$ is even. If f and g are both even then part (b) follows from the fact that the closure of $C^{\widetilde V'}_{\widetilde V}(\widetilde f)\circ C^{\widetilde V}_{\widetilde V'}(\widetilde g)$ is isotopic to closure of $C^{\widetilde V}_{\widetilde V'}(\widetilde g)\circ C^{\widetilde V'}_{\widetilde V}(\widetilde f)$. If f and g are both odd then (b) follows from the following lemma.

Lemma 4.4. *If f and g are both odd then*

$$F'\big(L_{C^{\widetilde V}_{\widetilde V}(\widetilde f\circ\widetilde g)}\big) = -F'\big(L_{C^{\widetilde V'}_{\widetilde V'}(\widetilde g\circ\widetilde f)}\big).$$

Proof. From Lemma 4.3 we have

$$F'\big(L_{C^{\widetilde V}_{\widetilde V}(f\circ g)}\big) = -F'\big(L_{C^{\widetilde V^-}_{\widetilde V^-}(\widetilde\sigma\circ\widetilde f\circ\widetilde g\circ\widetilde\sigma)}\big). \tag{12}$$

Now since $\widetilde\sigma\circ\widetilde f$ and $\widetilde g\circ\widetilde\sigma$ are even, the right side of Equation (12) is equal to

$$\begin{aligned}-F'\big(L_{C^{\widetilde V^-}_{\widetilde V}(\widetilde\sigma\circ\widetilde f)\circ C^{\widetilde V}_{\widetilde V^-}(\widetilde g\circ\widetilde\sigma)}\big) &= -F'\big(L_{C^{\widetilde V}_{\widetilde V^-}(\widetilde g\circ\widetilde\sigma)\circ C^{\widetilde V^-}_{\widetilde V}(\widetilde\sigma\circ\widetilde f)}\big)\\ &= -F'\big(L_{C^{\widetilde V}_{\widetilde V}(\widetilde g\circ\widetilde\sigma\widetilde\sigma\circ\widetilde f)}\big) = -F'\big(L_{C^{\widetilde V}_{\widetilde V}(\widetilde g\circ\widetilde f)}\big).\end{aligned}$$

Thus we have proved the lemma. □

This finishes the proof of part (b) and the theorem. □

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References

[Bar-Natan et al. 2003] D. Bar-Natan, T. T. Q. Le, and D. P. Thurston, "Two applications of elementary knot theory to Lie algebras and Vassiliev invariants", *Geom. Topol.* **7** (2003), 1–31. MR 2004f:57017 Zbl 1032.57008

[Geer 2006] N. Geer, "Etingof–Kazhdan quantization of Lie superbialgebras", *Adv. Math.* **207**:1 (2006), 1–38. MR 2007g:17019 Zbl 05078165

[Geer and Patureau-Mirand 2006] N. Geer and B. Patureau-Mirand, "Multivariable link invariants arising from Lie superalgebras of type I", preprint, 2006. arXiv math.GT/0609034

[Geer et al. 2007] N. Geer, B. Patureau-Mirand, and V. Turaev, "Modified quantum dimensions and re-normalized link invariants", preprint, 2007. arXiv math.QA/0711.4229

[Kac 1977] V. G. Kac, "Lie superalgebras", *Advances in Math.* **26**:1 (1977), 8–96. MR 58 #5803 Zbl 0366.17012

[Kac 1978] V. Kac, "Representations of classical Lie superalgebras", pp. 597–626 in *Differential geometrical methods in mathematical physics, II* (Bonn, 1977), edited by K. Bleuler et al., Lecture Notes in Math. **676**, Springer, Berlin, 1978. MR 80f:17006 Zbl 0388.17002

[Kashiwara 1990] M. Kashiwara, "Crystalizing the q-analogue of universal enveloping algebras", *Comm. Math. Phys.* **133**:2 (1990), 249–260. MR 92b:17018 Zbl 0724.17009

[Turaev 1994] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, Studies in Mathematics **18**, de Gruyter, Berlin, 1994. MR 95k:57014 Zbl 0812.57003

[Yamane 1994] H. Yamane, "Quantized enveloping algebras associated with simple Lie superalgebras and their universal R-matrices", *Publ. Res. Inst. Math. Sci.* **30**:1 (1994), 15–87. MR 95d:17017 Zbl 0821.17005

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THE ANALOGUE OF BÜCHI'S PROBLEM FOR CUBES IN RINGS OF POLYNOMIALS

THANASES PHEIDAS AND XAVIER VIDAUX

Let F be a field of characteristic zero. We give the following answer to a generalization of a problem of Büchi over $F[t]$: A sequence of 92 or more cubes in $F[t]$, not all constant, with constant third difference equal to 6, consists of cubes of successive elements $x,\ x+1,\ \dots$, for some $x \in F[t]$. We use this, in conjunction to the negative answer to Hilbert's tenth problem for $F[t]$, to show that the solvability of systems of degree-one equations, where some of the variables are assumed to be cubes and (or) nonconstant, is an unsolvable problem over $F[t]$.

1. Introduction

Büchi asked the following question, known as the n-squares problem:

> *Is there a positive integer M such that any sequence of at least M integer squares*, *with constant second difference* 2, *is equal to a sequence of squares of successive integers*?

He intended to apply a possible positive answer to obtain a result in logic (we discuss this below). The question was made public in [Lipshitz 1990]. P. Vojta [2000] proved that a positive answer to the analogous question for rational numbers is implied by a conjecture of S. Lang, or by a positive answer to a weaker question of E. Bombieri; he also answered in the affirmative the analogue of Büchi's question for meromorphic functions defined on $\mathbb{C}$ or for function fields of curves of characteristic zero. (For rings of functions, one naturally demands that the elements of the sequence be nonconstant.) Further discussion can be found in [Mazur 1994]. The original n-squares problem is still open.

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D. Hensley [1983] proved that the analogue of Büchi's problem in finite fields $\mathbb{F}_p$, where p is a prime number, has a positive answer with $M = p$. He also gave a nice "heuristic proof" of the conjecture and various lower and upper bounds on the heights of the terms of a Büchi sequence. D. Buell [1987] characterized all the nontrivial integer sequences of length four (we call a sequence of squares of successive numbers *trivial*). R. G. E. Pinch [1993] proved, under a certain condition on the size, that a family of four-term sequences cannot be extended to five-term sequences. J. Browkin and J. Brzeziński [2006] showed that there exist infinitely many nontrivial five- and six-term sequences (originally, Büchi asked the question for five-term sequences), but for certain constants distinct from 2. It is not known to us whether there exist any nontrivial five-term sequences of integers when the constant is 2 as in the original problem.

Vojta's conditional result claims finiteness of the set of eight-term nontrivial sequences of integers. His result actually does not give a value for M, but only implies that there exists an $M \geq 8$ such that Büchi's original problem has a positive answer. We must apologize for citing Vojta's result wrongly in our previous works, where we claimed that his conjectural result was for $M = 8$.

In [Pheidas and Vidaux 2005] we generalized Büchi's question as follows:

Question 1.1. *Let $k > 1$ be an integer. Is there a positive integer M such that any sequence $y = (y_0, \ldots, y_{M-1})$ of k-th powers of integers with constant k-th difference equal to $k!$ is necessarily a sequence of k-th powers of successive integers*? (*That is*, *one such that $y_n = (x+n)^k$ for a fixed integer x and $n \in \{0, \ldots, M-1\}$*.)

Except for Vojta's results mentioned above and those of [Pheidas and Vidaux 2006], the question is open for all k and for any global field in place of the integers. (Recall that in the case of function fields we restrict our attention to sequences of *nonconstant* functions.)

In the present paper we prove a positive answer to the analogue of the Question in the case $k = 3$ and for a polynomial ring $F[t]$ in place of the integers, where F is a field of characteristic zero. We prove:

Theorem 1.2. *Let F be a field of characteristic* 0 *and t a transcendental element over F. Assume that $x_0, \ldots, x_{M-1} \in F[t]$, that at least one of the x_n is nonconstant and that M is not less than* 92*. If the third difference of the sequence $(x_0^3, \ldots, x_{M-1}^3)$ is constant and equal to* 6, *which is to say*, *if*

$$x_{n+3}^3 - 3x_{n+2}^3 + 3x_{n+1}^3 - x_n^3 = 6 \quad \text{for } n = 0, \ldots, M-4, \tag{1}$$

then, *for some $x \in F[t]$ and for any $n = 0, \ldots, M-1$*, *we have*

$$x_n^3 = (x+n)^3.$$

Here is a consequence of this theorem to mathematical logic:

Theorem 1.3. *Let F be a field of zero characteristic and let t be a variable. Let $L_{3,T}$ be the language $\{0, 1, +, P_3, T\}$. Interpret the unary predicate P_3 as '$P_3(y)$ if and only if y is a cube (third power) in $F[t]$', interpret the unary predicate T as '$T(x)$ if and only if x is a nonconstant polynomial' and interpret 0, 1 and $+$ as usual. Let $L_{3,t}$ be the language $\{0, 1, +, P_3, R\}$ where R is a constant-symbol for the function which sends any x to tx (and the remaining symbols are interpreted as above).*

(a) *Multiplication in $F[t]$ is positive-existentially definable in each of the languages $L_{3,T}$ and $L_{3,t}$.*

(b) *The positive-existential theory of $F[t]$ in the language $L_{3,T}$ is undecidable.*

(c) *The positive-existential theory of $F[t]$ in the language $L_{3,t}$ is undecidable.*

This strengthens a result of J. Denef [1978] — an analogue of Hilbert's tenth problem for rings of polynomials in the variable t, in the language $\{+, \cdot, 0, 1, t\}$ (see expositions in [Matiyasevich 1970; Davis 1973; Pheidas and Zahidi 2000; Poonen 2003; Shlapentokh 2000]). It also strengthens the similar result in [Pheidas and Zahidi 1999] referring to the language $\{+, \cdot,$ 'x is nonconstant' $, 0, 1\}$.

Here is an immediate consequence of Theorem 1.3:

Corollary 1.4 (Undecidability of simultaneous representation by cubic forms). *There is no algorithm (Turing machine) that solves the following problem:*

> *Let A and B be two matrices with integer entries and with dimensions $m \times n$ and $m \times 1$, respectively. Assume that $x_1, \dots, x_n$ are variables and X is the column matrix of the x_i^3. Assume that $f_j(Y_1, \dots Y_n)$ are polynomials of the variables $Y_1, \dots Y_n$ of degree 1, for $j = 1, \dots, n$. Determine whether the system of equations*
>
> $$A \cdot X = B$$
>
> *has a solution with $x_1, \dots x_n \in F[t]$ with the property that for each $j, f_j(x_1^3, \dots, x_n^3) \notin F$.*

It would be desirable to be able to prove the similar statement having in place of the conditions $f_j(x_1^3, \dots, x_n^3) \notin F$ conditions only of the form $x_i \notin F$, or, even, 'some of the x_i are nonconstant'. But for the moment we cannot prove any of these. The proofs of 1.3 and 1.4 (at the end of the paper) show also that the analogous statements (omitting the conditions for nonconstancy) are equivalent over domains such as $\mathbb{Z}$ and $\mathbb{Q}$. It follows that the analogues of Corollary 1.4 over $\mathbb{Z}$ and over $\mathbb{Q}$ are open problems.

Open problems. It is natural to ask about the truth of the statements of Theorem 1.2 and 1.3 for domains other than polynomials. Some examples are:

(a) The ring of holomorphic and the field of meromorphic functions (on the complex plane or a p-adic plane);

(b) A polynomial ring $F[t]$ in any characteristic other than 3;

(c) The ring of algebraic functions of the variable t, integral over $F[t]$ (this would strengthen the result of A. Shlapentokh [Shlapentokh 1992]);

(d) Fields of rational functions in any characteristic other than 3;

(e) Fields of algebraic functions in any characteristic other than 3 (this would strengthen, for example, results of K. Eisentraeger and A. Shlapentokh [2007] (see also [Shlapentokh 2002; 2006]) and of K. Zahidi [2000]);

(f) $\mathbb{Z}$ and $\mathbb{Q}$ (and, in general, global fields).

Outline of the proof. We compute an invariant ν of the sequence which in the end turns out to be an x as in Theorem 1.2. We observe that Equation (1) is equivalent to $x_n^3 = a+nb+(\nu+n)^3$, where a and b are invariants. Differentiating the terms of this equation, combining with the initial one and using an argument involving heights (degrees) we show that a certain invariant of the sequence is equal to 0 (Lemmas 2.7 and 2.9). In this way we obtain a dependence of a on b and ν. Iterating the procedure we obtain b as a function of ν. In consequence the pairs of nontrivial solutions (x_m, x_n) are shown to be on certain curves over F, of genus greater than 0, which is impossible for nonconstant x_n and x_m. We obtain a number of degenerate cases which we have to rule out before we conclude with Theorem 1.2.

Our method can presumably be applied to the analogous problem for $k > 3$ (with k as in Question 1.1) but the number and nature of degenerate cases seems to increase in a way that we have not been able to systematize to this point. Because of the fact that we use derivatives our proof does not transfer to the analogous problem over the integers or the rationals. □

Remark. Very recently, H. Pasten [2008] proved a strong version of Büchi's problem for squares over polynomial rings. His result gives new evidence that the analogous problem for any (fixed) power could have a positive answer.

2. Büchi's problem for cubes in polynomial rings

From now on we will fix a solution $(x_0, \dots, x_{M-1})$ of the system (1) and write $u_n = x_n^3$, so

$$u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n = 6 \quad \text{for } n = 0, \dots, M-4. \tag{2}$$

We call the sequence $u = (u_0, \dots, u_{M-1})$ *trivial* if it is a sequence of cubes of successive elements; that is, if there is $x \in F[t]$ such that $u_n = (x+n)^3$ for all n.

Without loss of generality we can suppose that the field F is algebraically closed. From now on we make the following assumption:

The field F is algebraically closed of characteristic zero, and at least one x_n is not in F.

Lemma 2.1. *The system (2) is equivalent to*

$$2u_n = n(n-1)u_2 - 2n(n-2)u_1 + (n-2)(n-1)u_0 + 2(n-2)(n-1)n \tag{3}$$

for $n = 0, \dots, M-1$, and more generally,

$$\begin{aligned} 2u_n = (k-n)(k-n-1)u_{k+1} - 2(k-n-1)(k-n+1)u_k \\ +(k-n)(k-n+1)u_{k-1} - 2(k-n-1)(k-n)(k-n+1) \end{aligned} \tag{4}$$

for any $k = 1, \dots, M-2$.

Proof. A brute-force proof by induction on n is possible, but we will present here a shorter one due to the referee. Since the sequence (w_n) defined by

$$w_n = \tfrac{1}{2}n(n-1)u_2 - n(n-2)u_1 + \tfrac{1}{2}(n-2)(n-1)u_0 + (n-2)(n-1)n$$

is a polynomial in n with leading coefficient n^3, its third difference is the constant sequence (6). Therefore, (w_n) satisfies Equation (2). Since $w_i = u_i$ for $i = 0, 1, 2$, the sequences (w_n) and (u_n) have the same three first terms, hence are equal. This proves that the system (3) holds. The system (4) holds by a similar argument. □

Lemma 2.2. *For any pairwise distinct indices $m, n, q \in \{0, \dots, M-1\}$, the expression*

$$\nu_{m,n,q} = -\frac{1}{3}\left(\frac{(q-n)u_m + (m-q)u_n + (n-m)u_q}{(q-n)(m-q)(n-m)} + m + n + q\right) \tag{5}$$

does not depend on m, n and q.

Proof. Replace u_m, u_n and u_q by the expressions given by (3). □

For any m, n and q, we will be writing ν instead of $\nu_{m,n,q}$. We will call ν the *ν-invariant of the sequence u*. Since

$$3\nu = \tfrac{1}{2}(u_2 - 2u_1 + u_0 - 6),$$

the ν-invariant of the trivial solution of Büchi's problem (when $x_2 = x_0 + 2$ and $x_1 = x_0 + 1$) is x_0. To measure how far a solution u of (2) is from being trivial, we will introduce the new variables

$$a = u_0 - \nu^3 \quad \text{and} \quad b = (u_1 - u_0) - \left((\nu+1)^3 - \nu^3\right).$$

We find

$$u_n = a + nb + (\nu + n)^3 \tag{6}$$

(using the expression for $\nu_{0,n,1}$). Note that (x_n) is the trivial solution if and only if $a = b = 0$.

Definition 2.3. For $x \in F[t] \setminus \{0\}$, we denote by $\deg x$ the degree of x, while $\deg 0 := -\infty$. We denote by e the maximum of the degrees of the u_n for $n = 0, \dots, M-1$ (hence $e > 0$). If the degree e is divisible by 3 we write $d = \frac{1}{3}e$. In particular, if $u_n = x_n^3$ for each n, then d is the maximum of the degrees of the x_n.

Corollary 2.4. *One of the following is true*:

(a) *Each u_n has degree e.*

(b) *There is an index l such that for each $n \neq l$ we have* $\deg u_n = e$ *and* $\deg u_l < e$.

(c) *There are indices $l_1 \neq l_2$ such that for each $n \neq l_i$, $i = 1, 2$, we have* $\deg u_n = e$ *and* $\deg u_{l_i} < e$, $i = 1, 2$.

Proof. Assume we are not in cases (a) or (b). Let $l_1 \neq l_2$ such that $\deg u_{l_i} < e$ and let k be an index such that $\deg u_k = e$. By Lemma 2.2 we have

$$3\nu = \nu_{k,l_1,l_2} = -\frac{(l_2-l_1)u_k + (k-l_2)u_{l_1} + (l_1-k)u_{l_2}}{(l_2-l_1)(k-l_2)(l_1-k)} - k - l_1 - l_2$$

hence $\deg \nu = \deg u_k = e$. So for any index $n \neq l_1, l_2$ we have

$$3\nu = -\frac{(l_2-l_1)u_n + (n-l_2)u_{l_1} + (l_1-n)u_{l_2}}{(l_2-l_1)(n-l_2)(l_1-n)} - n - l_1 - l_2,$$

which implies $\deg u_n = \deg \nu = e$. □

Corollary 2.5. *If m, n, q and r are pairwise distinct indices of the sequence u, then u_m, u_n, u_q and u_r are coprime* (*the four polynomials do not have any common divisor*).

Proof. We have

$$\begin{aligned} 3\nu = 3\nu_{m,n,q} &= -\frac{(q-n)u_m + (m-q)u_n + (n-m)u_q}{(q-n)(m-q)(n-m)} - m - n - q \\ = 3\nu_{m,n,r} &= -\frac{(r-n)u_m + (m-r)u_n + (n-m)u_r}{(r-n)(m-r)(n-m)} - m - n - r. \end{aligned}$$

Suppose that there is a nonconstant polynomial P dividing u_m, u_n, u_q and u_r. P has a zero in F. Computing the last two quantities of the latter relations at that zero we obtain $m+n+q = m+n+r$, hence $q = r$, which contradicts our hypothesis. □

Definition 2.6. Recalling Corollary 2.4, we let l_1 and l_2 be two indices such that $\deg u_{l_1} \leq e$, $\deg u_{l_2} \leq e$, and

$$\deg u_n = e \quad \text{for all } n \text{ other than } l_1 \text{ and } l_2.$$

Lemma 2.7. *Let $\{r_1, \dots, r_m\} \subseteq \{0, \dots, M-1\}$ be a set of m distinct indices. If Q is a nonzero polynomial in $F[t]$ divisible by each x_{r_k} for $k = 1, \dots, m$, then the degree of Q is at least $\frac{1}{3}(m-2)/d$. In particular, if we choose $M \geq 92$ and $m = M$ then the degree of Q is at least $30d$.*

Proof. Set $R=\{r_1,\dots,r_m\}$. For all $n\in R$, let $P_n\in F[t]$ be such that $Q=x_nP_n$. Since Q is not the zero polynomial, for each $n\in R$, neither x_n nor P_n is the zero polynomial. We write μ for the least common multiple of the elements of the set $\{x_n\mid n\in R\}$. Hence μ divides Q and it is enough to show that the degree of μ is at least $\frac{1}{3}(m-2)/d$.

We claim that the product $\prod_{n\in R}x_n$ divides μ^3. Let P be an arbitrary prime of $F[t]$ which divides μ. Write $\operatorname{ord}_P(x)$ for the order of $x\in F[t]$ at P. It suffices to show that

$$\operatorname{ord}_P\Bigl(\prod_{n\in R}x_n\Bigr)\le 3\operatorname{ord}_P(\mu).$$

If P does not divide any x_n, the result is obvious. So assume that P divides x_{k_1} for some index k_1 that we choose so that $\operatorname{ord}_P(x_{k_1})$ is maximum:

$$\operatorname{ord}_P(x_{k_1})=\operatorname{ord}_P(\mu).$$

By Corollary 2.5, P divides either precisely one x_n, or precisely two, or precisely three. Let x_{k_i}, $i=1,\dots,j$, be the polynomials divisible by P in case j. In order to treat the three cases simultaneously, let x_{k_2} and x_{k_3} be such that P does not divide any x_n with $n\ne k_1,k_2,k_3$. If we choose the indices so that $\operatorname{ord}_P(x_{k_1})\ge\operatorname{ord}_P(x_{k_2})\ge\operatorname{ord}_P(x_{k_3})$, we obtain, as required,

$$\operatorname{ord}_P\Bigl(\prod_{n\in R}x_n\Bigr)=\operatorname{ord}_P(x_{k_1})+\operatorname{ord}_P(x_{k_2})+\operatorname{ord}_P(x_{k_3})\le 3\operatorname{ord}_P(x_{k_1})=3\operatorname{ord}_P(\mu),$$

It follows from the claim that

$$\sum_{n\in R}\deg x_n\le 3\deg\mu,$$

and by Corollary 2.4 we obtain

$$(m-2)d\le\sum_{n\in R}\deg x_n,$$

where the -2 corresponds to the indices l_1 and l_2 from Definition 2.6. □

Notation 2.8. We write

$$A=-\nu''a'+\nu'a''+6\nu'^3\nu,\qquad B=\nu''b'-\nu'b''-6\nu'^3,$$

and if $B\ne 0$

$$q=\frac{A}{B}.$$

Observe that if $B\nu'\ne 0$ we can write

$$q=\frac{(a'/\nu')'+6\nu\nu'}{-(b'/\nu')'-6\nu'}. \tag{7}$$

Lemma 2.9. *Only the following mutually exclusive two cases can occur*:

<u>*Case* 1</u>: $\nu' = 0$.

<u>*Case* 2</u>: $B \neq 0$, $\nu' \neq 0$ *and we have*

$$a + bq + (\nu + q)^3 = 0, \tag{8}$$

$$a' + b'q + 3\nu'(\nu + q)^2 = 0. \tag{9}$$

Proof. By differentiating twice the sides of (6) we get

$$u_n' = a' + nb' + 3\nu'(\nu + n)^2 \tag{10}$$

and

$$u_n'' = a'' + nb'' + 6\nu'^2(\nu + n) + 3\nu''(\nu + n)^2. \tag{11}$$

By plugging into (10) the expression for $3(\nu + n)^2$ that results from (11) we obtain

$$\nu'' u_n' = \nu'' a' + n\nu'' b' + \nu'\big(u_n'' - a'' - nb'' - 6\nu'^2(\nu + n)\big),$$

which we can rewrite as

$$nB = A + U_n, \tag{12}$$

where

$$U_n = \nu'' u_n' - \nu' u_n''.$$

Multiplying (6) by B^3 and (10) by B^2 we get

$$\begin{aligned} B^3 u_n &= aB^3 + nbB^3 + (\nu B + nB)^3, \\ B^2 u_n' &= a'B^2 + nb'B^2 + 3\nu'(\nu B + nB)^2; \end{aligned}$$

hence, replacing the expression of nB from (12),

$$\begin{aligned} B^3 u_n &= aB^3 + (A + U_n)bB^2 + (\nu B + A + U_n)^3, \\ B^2 u_n' &= a'B^2 + (A + U_n)b'B + 3\nu'(\nu B + A + U_n)^2. \end{aligned}$$

Separating terms that depend on n from ones that don't, in both equations, we get

$$\begin{aligned} B^3 u_n - U_n\big(bB^2 + 3(\nu B + A)^2 + 3(\nu B + A)U_n + U_n^2\big) \\ = aB^3 + AbB^2 + (\nu B + A)^3, \end{aligned} \tag{13}$$

$$B^2 u_n' - U_n\big(b'B + 6\nu'(\nu B + A) + 3\nu' U_n\big) = a'B^2 + Ab'B + 3\nu'(\nu B + A)^2. \tag{14}$$

We give names to the right-hand sides of these two equations:

$$\Delta = aB^3 + AbB^2 + (\nu B + A)^3, \quad \Gamma = a'B^2 + Ab'B + 3\nu'(\nu B + A)^2.$$

We now use Lemma 2.7 to prove that $\Delta = \Gamma = 0$. Since $u_n = x_n^3$, its first and second derivatives, u_n' and u_n'', are each a multiple of x_n, hence $U_n = \nu'' u_n' - \nu' u_n''$ is a multiple of x_n. Therefore, Δ and Γ are both multiples of x_n for each $n \in \{0, \ldots, M-1\}$. Let us compute an upper bound for the degrees of Δ and Γ. Recalling Definition 2.6 we see that the degree of u_n is not more than e, hence that of ν is not more than e and we have $\deg a \le 3e$, $\deg b \le 2e$, and

$$\deg A \le 4e-3, \quad \deg B \le 3e-3, \quad \deg U_n \le 2e-3 \quad \text{and} \quad \deg(\nu B + A) \le 4e-3.$$

Therefore, computing the degrees of the left-hand sides of (13) and (14), we find

$$\begin{aligned} \deg \Delta &\le 10e - 9 = 30d - 9 < 30d, \\ \deg \Gamma &\le \ \ 7e - 7 = 21d - 7 < 30d. \end{aligned}$$

We deduce from Lemma 2.7 that we have $\Delta = 0$ and $\Gamma = 0$.

If B is not zero then ν' is not zero and we have

$$\frac{\Delta}{B^3} = a + \frac{A}{B} b + \left(\nu + \frac{A}{B}\right)^3 = 0, \qquad \frac{\Gamma}{B^2} = a' + \frac{A}{B} b' + 3\nu' \left(\nu + \frac{A}{B}\right)^2 = 0,$$

which proves (8) and (9).

We next assume that $B = 0$ and prove that $\nu' = 0$. We know from Equation (12) that $A + U_n = 0$ for all n. Since U_n is a multiple of x_n, and $\deg U_n \le 2e - 3 = 6d - 3$, we deduce from Lemma 2.7 that U_n is zero. From Corollary 2.4, we know that at most two of the u_n may be constant, namely u_{l_1} and u_{l_2}. For all $n \in \{0, \ldots, M-1\}$ distinct from l_1 and l_2, we may write

$$\frac{U_n}{u_n'^{\,2}} = \frac{\nu'' u_n' - \nu' u_n''}{u_n'^{\,2}} = \left(\frac{\nu'}{u_n'}\right)'$$

and deduce that for those n, the quotient ν'/u_n' must be a constant in F, say c_n. So we have $c_n u_n' = \nu'$ for at least $M-2$ distinct values of n, so for at least 90 distinct values of n. We conclude by Lemma 2.7: since

$$\deg \nu' \le e - 1 = 3d - 1 < \frac{90-2}{3} d,$$

we have $\nu' = 0$. □

We will need the following proposition, whose proof comes from the theory of elliptic curves (see, for example, [Husemöller 2004, Definition (6.2), page 17] or [Silverman 1986, Hurwitz's Theorem, II.5]). The main observation that concerns us here is that a nonsingular cubic curve is of genus 1.

Proposition 2.10. *Let $\mu, \xi \in F$.*

(a) *The curve with affine equation $Y^3 = \mu X^3 + \xi$ has genus 1 provided that $\mu\xi \neq 0$.*

(b) *The curve with affine equation* $Y^2+\mu Y+\xi = X^3$ *has genus* 1 *provided that* $\mu^2 \neq 4\xi$.

Remark. The general strategy from now on will be the following: we will provide relations among a, b and ν that will produce equations that will define curves as in Proposition 2.10, where the coefficients μ and ξ will depend on one or various indices n. These curves will have rational parametrization by polynomials made up of products of various x_n's; hence they will define curves of genus 0 (for all the indices considered). Proposition 2.10 will then tell us that this can happen for very few values of n (as long as any of x_n or x_0 is nonconstant, and in particular, if n is different from l_1 and l_2). So we will have space to choose the indices such that one of the curves considered is of genus 1, while it admits a rational parametrization, and this will give us a contradiction. The only case that will survive is that in which for all n we have $x_n^3 = (\nu+n)^3$, which will prove Theorem 1.2.

Lemma 2.11. *Case* 1 *is impossible*, *that is*, ν' *can not be zero.*

Proof. We will show first that if ν is constant then so is a, and then that ν and a cannot be both constant.

Assume that $\nu' = 0$ and $a' \neq 0$. So we have $a' = u_0'$ from the definition of a, and

$$u_n' = a' + nb', \quad u_n'' = a'' + nb''$$

from (6). This leads to $u_n' b'' = a'b'' + nb''b' = a'b'' + (u_n'' - a'')b'$, that is,

$$u_n' b'' - u_n'' b' = a'b'' - a''b'.$$

Since x_n divides u_n' and u_n'' and the degree of $u_n' b'' - u_n'' b'$ is no more than $3e-3$, we deduce from Lemma 2.7 that

$$a'b'' - a''b' = 0.$$

Since $a' \neq 0$, we can write

$$\left(\frac{b'}{a'}\right)' = 0,$$

so $b = ra + s$ for some constants $r, s \in F$. By (6), we have

$$x_n^3 = u_n = a + nb + (\nu+n)^3 = a + n(ra+s) + (\nu+n)^3 = (1+nr)a + ns + (\nu+n)^3$$

for each n; hence, recalling the definition of a,

$$x_n^3 = (1+nr)x_0^3 + ns + (\nu+n)^3 - (1+nr)\nu^3.$$

Thus, for each n such that x_n is nonconstant (hence for at least 90 distinct values of n), the curve

$$Y^3 = (1+nr)X^3 + ns + (\nu+n)^3 - (1+nr)\nu^3$$

is a curve over F that admits the parametrization $(X, Y) = (x_0, x_n)$ by nonconstant rational functions, hence is a curve of genus 0. According to Proposition 2.10 this implies that $(1+nr)\big(ns+(\nu+n)^3-(1+nr)\nu^3\big) = 0$, which cannot happen for more than four values of n. This gives us a contradiction.

Now we prove that ν and a cannot be both constant. Recall that

$$x_1^3 = a+b+(\nu+1)^3,$$

hence

$$b = x_1^3 - a - (\nu+1)^3.$$

Therefore, for each n, we have

$$x_n^3 = a+n\big(x_1^3-a-(\nu+1)^3\big)+(\nu+n)^3 = nx_1^3+(1-n)a-n(\nu+1)^3+(\nu+n)^3.$$

If both ν and a are constant, the curve

$$Y^3 = nX^3+(1-n)a-n(\nu+1)^3+(\nu+n)^3$$

is a curve over F that admits the parametrization $(X, Y) = (x_1, x_n)$ by nonconstant rational functions, hence is a curve of genus 0. As in the previous paragraph we conclude that this cannot happen for more than four values of n. □

Lemma 2.12. *In Case* 2 *of Lemma 2.9 there are two mutually exclusive subcases*:

Case 2.1: *For all n we have* $x_n^3 = (\nu+n)^3$ (*that is*, *the trivial solution*).

Case 2.2: $q' = 0$.

Proof. According to Case 2, we assume that $B \neq 0$ and $\nu' \neq 0$. Observe that if (x_n) is the trivial solution then $a = b = 0$ and $q = -\nu$, hence $q' = -\nu' \neq 0$.

Suppose q' is not zero. By differentiating (8) we get

$$a'+b'q+bq'+3(\nu'+q')(\nu+q)^2 = 0,$$

and subtracting (9), we obtain $bq'+3q'(\nu+q)^2 = 0$, that is,

$$b = -3(\nu+q)^2. \tag{15}$$

Recall that

$$q = \frac{(a'/\nu')'+6\nu\nu'}{-(b'/\nu')'-6\nu'}.$$

We write $\alpha = a'/\nu'$ and $\beta = b'/\nu'$. We obtain

$$q = -\frac{\alpha'+6\nu\nu'}{\beta'+6\nu'},$$

hence

$$-\alpha' = q(\beta'+6\nu')+6\nu\nu'. \tag{16}$$

On the other hand, dividing by ν' in Equation (9) we obtain

$$\alpha+\beta q+3(\nu+q)^2=0$$

which, by differentiating, gives

$$-\alpha'=\beta' q+\beta q'+6(\nu'+q')(\nu+q),$$

hence

$$-\alpha'=\beta' q+\beta q'+6(\nu'\nu+\nu' q+q'\nu+q'q).$$

Substituting the expression for α' from Equation (16) we obtain

$$q(\beta'+6\nu')+6\nu\nu'=\beta' q+\beta q'+6(\nu'\nu+\nu' q+q'\nu+q'q);$$

hence, simplifying the $q\beta'$, $\nu\nu'$, and $q\nu'$,

$$0=\beta q'+6(q'\nu+q'q),$$

hence

$$\beta=-6(\nu+q),$$

or again

$$b'=-6\nu'(\nu+q).$$

From Equation (15) we obtain

$$b'=-6(\nu'+q')(\nu+q),$$

hence $\nu+q=0$. Therefore, Equation (15) implies $b=0$, and Equation (8) implies $a=0$. By Equation (6), we get

$$u_n=(\nu+n)^3.$$

This proves the lemma. □

Lemma 2.13. *Case* 2.2 *of the previous lemma is impossible*, *that is*, $q'\neq 0$.

Proof. By Equations (6) and (8) we have

$$u_n=(n-q)b+(\nu+n)^3-(\nu+q)^3,$$

therefore

$$u_n=(n-q)b+3\nu^2(n-q)+3\nu(n^2-q^2)+n^3-q^3,$$

so, for all n distinct from q,

$$\frac{u_n}{n-q}=b+3\nu^2+3\nu(n+q)+n^2+qn+q^2$$

hence

$$\frac{u_n}{n-q}=b+3\nu^2+3q\nu+q^2+n(3\nu+q)+n^2.$$

If we write

$$w_n = y_n^3 = \frac{u_n}{n-q}, \quad \alpha = b + 3\nu^2 + 3q\nu + q^2, \quad \beta = 3\nu + q,$$

then we have

$$w_n = \alpha + \beta n + n^2, \tag{17}$$

and, differentiating both sides,

$$w_n' = \alpha' + \beta' n. \tag{18}$$

Multipying (17) by β'^2 and substituting $\beta' n$ from (18) we get

$$\beta'^2 w_n = \beta'^2 \alpha + \beta' \beta (w_n' - \alpha') + (w_n' - \alpha')^2$$

hence

$$\beta'^2 w_n - \beta' \beta w_n' - w_n'^2 + 2\alpha' w_n' = \beta'^2 \alpha - \beta' \beta \alpha' + \alpha'^2. \tag{19}$$

We intend to apply Lemma 2.7.

For the sake of contradiction, in the rest of the proof we assume that q is constant. So, each y_n is a polynomial of the same degree as x_n, and by Corollary 2.5, any four distinct y_n are coprime. Also, we have $\deg \alpha \le 2e$, $\deg \beta \le e$ and $\deg w_n \le e$. Hence, the left-hand side of (19) has degree $\le 3e - 2 = 9d - 2$. Observe that w_n is a cube and is divisible by x_n^3. Hence the left-hand side of (19) is divisible by x_n. So we can apply Lemma 2.7 and conclude that

$$\beta'^2 \alpha - \beta' \beta \alpha' + \alpha'^2 = 0. \tag{20}$$

Recall that $\nu' \neq 0$, so $\beta' \neq 0$. Hence (20) can be written as

$$\left(\frac{\alpha'}{\beta'}\right)^2 - \beta \frac{\alpha'}{\beta'} + \alpha = 0.$$

Therefore, for some $\gamma \in F(t)$, we have

$$\beta^2 - 4\alpha = \gamma^2 \tag{21}$$

and

$$\frac{\alpha'}{\beta'} = \tfrac{1}{2}(\beta + \varepsilon\gamma) \tag{22}$$

for some $\varepsilon \in \{-1, 1\}$.

Substituting the value of α from (21) into (22) we obtain

$$\gamma(\beta' + \varepsilon\gamma') = 0. \tag{23}$$

Thus we have two cases, according to whether $\beta' = -\varepsilon\gamma'$ or $\gamma = 0$.

Case 2.2.1: We assume $\beta' = -\varepsilon\gamma'$. From Equation (22) we obtain

$$\alpha' = c\beta'$$

for some $c \in F$; substituting this expression for α' in (20) we obtain

$$\alpha = c\beta - c^2.$$

Therefore, by (17),

$$y_n^3 = (n+c)\beta + n^2 - c^2.$$

So, for any indices m and n, we have

$$y_m^3 y_n^3 = \big((m+c)\beta + m^2 - c^2\big)\big((n+c)\beta + n^2 - c^2\big),$$

hence

$$\lambda_{m,n}^3 y_m^3 y_n^3 = \beta^2 + \mu_{m,n}\beta + \xi_{m,n} \tag{24}$$

where

$$\lambda_{m,n}^3 = \frac{1}{(m+c)(n+c)}, \quad \mu_{m,n} = \frac{(m+c)(n^2-c^2)+(n+c)(m^2-c^2)}{(m+c)(n+c)}$$

and

$$\xi_{m,n} = \frac{(m^2-c^2)(n^2-c^2)}{(m+c)(n+c)}$$

provided that $(m+c)(n+c) \neq 0$. It is obvious that we can choose $m, n \leq M-1$ so that $(m+c)(n+c)(\mu_{m,n}^2 - 4\xi_{m,n}) \neq 0$. So, by Proposition 2.10, the curve

$$Y^3 = X^2 + \mu_{m,n}X + \xi_{m,n} \tag{25}$$

is of genus 1. But by Equation (24) the latter is a curve over F that admits the parametrization $(X, Y) = (\beta, \lambda_{m,n}y_m y_n)$ by nonconstant rational functions (recall that $\beta \notin F$), hence is a curve of genus 0, a contradiction that proves that Case 2.2.1 is impossible.

Case 2.2.2: We assume that $\gamma = 0$. From (21) we obtain $4\alpha = \beta^2$, while (17) becomes

$$4y_n^3 = (\beta + 2n)^2.$$

Hence y_n is a square: $y_n = z_n^2$ for some $z_n \in F[t]$. So we have

$$2z_n^3 = \varepsilon_n(\beta + 2n),$$

where $\varepsilon_n = \pm 1$, and we may assume $\varepsilon_n = 1$ for all n by changing z_n by $-z_n$ if necessary. Hence, for each m and n distinct from q, l_1 and l_2, the curve

$$4X^3 = Y^2 + 2(m+n)Y + 4mn$$

admits the parametrization $(X, Y) = (z_m z_n, \beta)$ by nonconstant rational functions, hence is of genus 0. By Proposition 2.10, we have

$$4(m+n)^2 = 16m^2n^2.$$

As long as m has been chosen, this can happen for at most two choices of n. So we get a contradiction and conclude that Case 2.2.2 is impossible. □

Proof of Theorem 1.2. By Lemmas 2.9, 2.11, 2.12 and 2.13, the only possible case is Case 2.1 of Lemma 2.12, that is, $x_n^3 = (\nu+n)^3$ for each n. □

Proof of Theorem 1.3. (a) By Theorem 1.2, the formula

$$\begin{aligned}&\phi(x,z,w):\ \exists y_0 \ldots \exists y_{91}\\ &\qquad x=y_0 \wedge z=y_1 \wedge w=y_2 \wedge \bigwedge_{n=0}^{91} P_3(y_n) \wedge \bigwedge_{n=0}^{88} y_{n+3}-3y_{n+2}+3y_{n+1}-y_n=6\end{aligned}$$

is equivalent over $F[t]$ to:

> *Either x, z, w are constant polynomials or $x = \nu^3$ and $z = (\nu+1)^3$ and $w = (\nu+2)^3$ for some $\nu \in F[z]$.*

Therefore, the formula

$$\psi(\nu,u):\ \exists x \exists z \exists w\ \ \psi(x,z,w) \wedge 6\nu+6=(w-z)-(z-x) \wedge z-x=3u+3\nu+1$$

is equivalent over $F[t]$ to:

> *Either $\nu, u \in F$ or $u = \nu^2$.*

Both ϕ and ψ are formulas in the intersection of the languages $L_{3,t}$ and $L_{3,T}$.

Let us prove that the formula

$$\psi_1(\nu,u):\ \exists g \exists h\ \ \psi(\nu,u) \wedge \psi(\nu+t,g) \wedge \psi(\nu-t,h) \wedge g+h=2u+2t^2$$

is satisfied in $F[t]$ if and only if

$$u = \nu^2.$$

One the one hand, if $u = \nu^2$ then we can choose $g = (\nu+t)^2$ and $h = (\nu-t)^2$. On the other hand, if $\psi_1(\nu,u)$ is satisfied in $F[t]$, then either $u = \nu^2$ and we are done, or $u, \nu \in F$, in which case $\nu+t, \nu-t \notin F$, hence $g = (\nu+t)^2$ and $h = (\nu-t)^2$, hence $2u+2t^2 = g+h = 2\nu^2+2t^2$ implies $u = \nu^2$.

Observe that ψ_1 is equivalent to a positive-existential $L_{3,t}$-formula. Similarly, the formula

$$\begin{aligned}&\psi_2(\nu,u):\ \exists f \exists g \exists h \exists z\\ &\qquad T(f) \wedge \psi(f,z) \wedge \psi(\nu,u) \wedge \psi(\nu+f,g) \wedge \psi(\nu-f,h) \wedge g+h=2u+2z\end{aligned}$$

is equivalent to

$$u = \nu^2.$$

Observe that ψ_2 is equivalent to a positive-existential $L_{3,T}$-formula.

Therefore squaring over $F[t]$ is positive-existentially definable in each of the languages $L_{3,t}$ and $L_{3,T}$, hence so is multiplication (for details see L. Lipshitz [Lipshitz 1990]).

Statements (b) and (c) follow from (a) and the fact that the positive-existential theory of $F[t]$ in the language $\{0, 1, +, \cdot, T\}$ (resp. $\{0, 1, +, \cdot, t\}$) is undecidable [Pheidas and Zahidi 1999; Denef 1978]. □

Proof of Corollary 1.4. Any positive-existential $L_{3,T}$-sentence is equivalent to a disjunction of sentences each of which claims the solvability of a system of linear equations with integer coefficients, together with conditions stating that certain of the variables are cubes plus conditions which state that certain linear polynomials of the variables are nonconstant ($\notin F$). Now observe that for any x we have

$$6x+6=(x+2)^3-2(x+1)^3+x^3.$$

Hence we can substitute each variable x, which is not assumed to be necessarily a cube, by the expression

$$\tfrac{1}{6}z_1^3-\tfrac{1}{3}z_2^3+\tfrac{1}{6}z_3^3-\tfrac{1}{6},$$

where the z_j are new variables. Hence any positive-existential $L_{3,T}$-sentence is equivalent to a disjunction of sentences of form as in the Corollary. Consequently, if the satisfiability problem for such sentences were decidable, so would be the decidability problem for positive-existential sentences of $L_{3,T}$, which would contradict Theorem 1.3. □

References

[Browkin and Brzeziński 2006] J. Browkin and J. Brzeziński, "On sequences of squares with constant second differences", *Canad. Math. Bulletin* **49**:4 (2006), 481–491. MR 2007h:11136 Zbl 05177830

[Buell 1987] D. A. Buell, "Integer squares with constant second difference", *Math. Comp.* **49**:180 (1987), 635–644. MR 88j:11090 Zbl 0627.10009

[Davis 1973] M. Davis, "Hilbert's tenth problem is unsolvable", *Amer. Math. Monthly* **80** (1973), 233–269. MR 47 #6465 Zbl 0277.02008

[Denef 1978] J. Denef, "The Diophantine problem for polynomial rings and fields of rational functions", *Trans. Amer. Math. Soc.* **242** (1978), 391–399. MR 58 #10809 Zbl 0399.10048

[Eisentraeger and Shlapentokh 2007] K. Eisentraeger and A. Shlapentokh, "Undecidability in function fields of positive characteristic", preprint, 2007. arXiv 0709.1739v2

[Hensley 1983] D. Hensley, "Sequences of squares with second difference of two and a conjecture of Büchi", unpublished, 1983.

[Husemöller 2004] D. Husemöller, *Elliptic curves*, 2nd ed., Graduate Texts in Mathematics **111**, Springer, New York, 2004. MR 2005a:11078 Zbl 1040.11043

[Lipshitz 1990] L. Lipshitz, "Quadratic forms, the five square problem, and diophantine equations", pp. 677–680 in *The collected works of J. Richard Büchi*, edited by S. MacLane and D. Siefkes, Springer, Berlin, 1990.

[Matiyasevich 1970] Y. V. Matiyasevich, "Enumerable sets are diophantine", *Dokl. Akad. Nauk SSSR* **19** (1970), 279–282. In Russian; translated in Sov. Math. Dokl. **11** (1970), 354-358. Zbl 0212.33401

[Mazur 1994] B. Mazur, "Questions of decidability and undecidability in number theory", *J. Symbolic Logic* **59**:2 (1994), 353–371. MR 96c:03091 Zbl 0814.11059

[Pasten 2008] H. Pasten, "Extension of Büchi's problem for polynomials", preprint, Departamento de Matemática de la Universidad de Concepción, Chile, 2008.

[Pheidas and Vidaux 2005] T. Pheidas and X. Vidaux, "Extensions of Büchi's problem: questions of decidability for addition and kth powers", *Fund. Math.* **185**:2 (2005), 171–194. MR 2006f:03064 Zbl 1079.03005

[Pheidas and Vidaux 2006] T. Pheidas and X. Vidaux, "The analogue of Büchi's problem for rational functions", *J. London Math. Soc.* (2) **74**:3 (2006), 545–565. MR 2007k:03097 Zbl 1109.03032

[Pheidas and Zahidi 1999] T. Pheidas and K. Zahidi, "Undecidable existential theories of polynomial rings and function fields", *Comm. Algebra* **27**:10 (1999), 4993–5010. MR 2000f:03125 Zbl 0934.03014

[Pheidas and Zahidi 2000] T. Pheidas and K. Zahidi, "Undecidability of existential theories of rings and fields: a survey", pp. 49–105 in *Hilbert's tenth problem: relations with arithmetic and algebraic geometry* (Ghent, 1999), edited by J. Denef et al., Contemp. Math. **270**, Amer. Math. Soc., Providence, RI, 2000. MR 2002a:03085 Zbl 0981.03013

[Pinch 1993] R. G. E. Pinch, "Squares in quadratic progression", *Math. Comp.* **60**:202 (1993), 841–845. MR 93h:11029 Zbl 0779.11013

[Poonen 2003] B. Poonen, "Hilbert's Tenth Problem over rings of number-theoretic interest", preprint, 2003, Available at http://math.mit.edu/~poonen/papers/aws2003.pdf.

[Shlapentokh 1992] A. Shlapentokh, "Hilbert's tenth problem for rings of algebraic functions of characteristic 0", *J. Number Theory* **40**:2 (1992), 218–236. MR 94d:11018 Zbl 0746.03008

[Shlapentokh 2000] A. Shlapentokh, "Hilbert's tenth problem over number fields, a survey", pp. 107–137 in *Hilbert's tenth problem: relations with arithmetic and algebraic geometry* (Ghent, 1999), edited by J. Denef et al., Contemp. Math. **270**, Amer. Math. Soc., Providence, RI, 2000. MR 2002a:03085 Zbl 0994.03001

[Shlapentokh 2002] A. Shlapentokh, "On Diophantine definability and decidability in some rings of algebraic functions of characteristic 0", *J. Symbolic Logic* **67**:2 (2002), 759–786. MR 2003d:03070 Zbl 1011.03027

[Shlapentokh 2006] A. Shlapentokh, *Hilbert's tenth problem: diophantine classes and other extensions to global fields*, Cambridge U. Press, Cambridge, 2006. Zbl 05129217

[Silverman 1986] J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics **106**, Springer, New York, 1986. MR 87g:11070 Zbl 0585.14026

[Vojta 2000] P. Vojta, "Diagonal quadratic forms and Hilbert's tenth problem", pp. 261–274 in *Hilbert's tenth problem: relations with arithmetic and algebraic geometry* (Ghent, 1999), edited by J. Denef et al., Contemp. Math. **270**, Amer. Math. Soc., Providence, RI, 2000. MR 2001k:11260 Zbl 0995.11070

[Zahidi 2000] K. Zahidi, "The existential theory of real hyperelliptic function fields", *J. Algebra* **233**:1 (2000), 65–86. MR 2001i:11150 Zbl 0985.11062

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A NEW GENERAL CONJUGATE BAILEY PAIR

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We introduce a new general conjugate Bailey pair which bridges the gap between Bailey and Slater's work and the work done recently by Andrews and Warnaar. With this new general pair we are able to find many useful conjugate Bailey pairs similar to those of Andrews and Warnaar. Using our new pairs we show results related to the sums of triangular numbers, indefinite quadratic forms and partition identities. We close with a brief discussion of the many other paths that can and will be taken in the future.

1. Introduction

The Bailey transform. In [1948], W. N. Bailey introduced a new proof of the Rogers–Ramanujan identities

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}=\frac{1}{(q,q^4;q^5)_\infty}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty},$$

where we use the standard hypergeometric q-series notation [Gasper and Rahman 2004, page xvi]: For $|q|<1$,

$$\begin{aligned}(a)_k&=(a;q)_k=(1-a)(1-aq)\cdots(1-aq^{k-1})=\prod_{i=0}^{k-1}(1-aq^i),\\(a)_\infty&=(a;q)_\infty=\lim_{k\to\infty}(a;q)_k=\prod_{i=0}^{\infty}(1-aq^i),\\(a_1;q)_k(a_2;q)_k&\cdots(a_n;q)_k=(a_1,a_2,\ldots,a_n;q)_k.\end{aligned}$$

Bailey also included more Rogers–Ramanujan-type identities, which he had found using a similar method of proof. Two years later Bailey formulated this method into what is now known as the Bailey transform:

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Theorem 1.1 (The Bailey transform). *If*

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \quad \textit{and} \quad \gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \quad \textit{then} \quad \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n,$$

subject to conditions on the four sequences α_n, β_n, γ_n and δ_n which make all the infinite series absolutely convergent.

The main result of the Bailey transform is dependent on two relations. The first relation defines a *Bailey pair*, (α_n, β_n), and the second defines a *conjugate Bailey pair*, (δ_n, γ_n). With the introduction of the Bailey transform, Bailey included many general pairs of both types. We recall the following conjugate Bailey pair.

Corollary 1.2 [Bailey 1948]. *If we let $u_n = 1/(q)_n$ and $v_n = 1/(aq)_n$ in the Bailey transform, then we have the conjugate Bailey pair*

$$\delta_n = (\rho_1)_n (\rho_2)_n \left(\frac{aq}{\rho_1 \rho_2} \right)^n$$

and

$$\gamma_n = \frac{(aq/\rho_1)_\infty (aq/\rho_2)_\infty}{(aq)_\infty (aq/\rho_1\rho_2)_\infty} \cdot \frac{(\rho_1)_n (\rho_2)_n}{(aq/\rho_1)_n (aq/\rho_2)_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n.$$

The above conjugate Bailey pair was then used with the Bailey transform to show multiple Rogers–Ramanujan-type identities. The same conjugate Bailey pair was then used three years later by Slater [1952] to prove her list of around 130 new and known Rogers–Ramanujan-type identities.

1.1. ***The bilateral Bailey transforms.*** Andrews and Warnaar [2007] recently introduced a handful of new conjugate Bailey pairs. We state the pairs they found in the following variations of the Bailey transform (see their paper for proofs):

Theorem 1.3 (Symmetric bilateral Bailey transform). *If*

$$\beta_n = \sum_{r=-n}^{n} \alpha_r u_{n-r} v_{n+r} \;\; \textit{and} \;\; \gamma_n = \sum_{r \ge |n|} \delta_r u_{r-n} v_{r+n}, \;\; \textit{then} \;\; \sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n,$$

subject to conditions on the four sequences which make all of the relevant infinite series absolutely convergent.

Theorem 1.4 (Asymmetric bilateral Bailey transform). *Let $m = \max\{n, -n-1\}$. If*

$$\beta_n = \sum_{r=-n-1}^{n} \alpha_r u_{n-r} v_{n+r+1} \; \textit{and} \; \gamma_n = \sum_{r=m}^{\infty} \delta_r u_{r-n} v_{r+n+1}, \;\; \textit{then} \;\; \sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n,$$

subject to conditions on the four sequences which make all of the relevant infinite series absolutely convergent.

We can now introduce the conjugate Bailey pairs of Andrews and Warnaar.

Theorem 1.5 [Andrews and Warnaar 2007]. *Let $u_n = v_n = 1/(q^2;q^2)_n$ in the symmetric bilateral Bailey transform. Then we have the conjugate Bailey pairs*

$$\delta_n = \frac{(q^2;q^2)_{2n}}{(-q;q)_{2n+1}} q^n, \qquad \gamma_n = q^{-n^2} \sum_{j \geq |n|} q^{j^2+j}$$

and

$$\delta_n = (q)_{2n} q^n, \qquad \gamma_n = q^{-2n^2} \sum_{j \geq 2|n|} q^{j(j+1)/2}.$$

Theorem 1.6 [Andrews and Warnaar 2007]. *Let $u_n = v_n = 1/(q^2;q^2)_n$ in the asymmetric bilateral Bailey transform. Then we have the conjugate Bailey pairs*

$$\delta_n = \frac{(q^2;q^2)_{2n+1}}{(-q;q)_{2n+2}} q^n, \qquad \gamma_n = q^{-n(n+1)} \sum_{j \geq m} q^{j(j+2)}$$

and

$$\delta_n = (q)_{2n+1} q^n, \qquad \gamma_n = q^{-2n(n+1)} \sum_{j \geq 2m} q^{j(j+3)/2},$$

where $m = \max\{n, -n-1\}$.

One of the more striking observations of these new pairs is the existence of a restricted sum in γ_n, a characteristic not commonly seen in previous conjugate Bailey pairs. Andrews and Warnaar were then able to apply these pairs to show many results both new and known relating to false and partial theta series.

1.2. ***Bridging gaps and contributions.*** This work below will bridge the previously unknown gap between the work done by Bailey and Slater and the work done by Andrews and Warnaar. Section 2 introduces a new conjugate Bailey pair that encompasses the pairs used by Bailey and Slater and those used by Andrews and Warnaar.

In Section 3 we use our general theorem to define specific new conjugate Bailey pairs. In Sections 4–6 we touch on some of the many applications that are obtainable with these new conjugate Bailey pairs.

2. A general conjugate Bailey pair

In this section we introduce a new general conjugate Bailey pair. As we show in Section 3, its special cases tie together the conjugate Bailey pairs of Andrews and Warnaar as well as the conjugate Bailey pair used by Bailey and Slater. In finding our generalization, we are able to find many other new conjugate Bailey pairs similar to those of Andrews and Warnaar. Their applications are seen in later sections.

The following theorem is our main result regarding conjugate Bailey pairs. We present a very general conjugate Bailey pair and its proof. For the purpose of the proof, we define an ${}_{n+1}\phi_n$ *basic hypergeometric series* as (see [Gasper and Rahman 2004, page 4])

$$ {}_{n+1}\phi_n\left[\begin{matrix} a_1, \dots, a_{n+1} \\ b_1, \dots, b_n \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_{n+1}; q)_k}{(q, b_1, \dots, b_n; q)_k} z^k. $$

Theorem 2.1. *Let $u_n = 1/(q)_n$ and $v_n = 1/(fq)_n$. Then*

$$ \delta_n = \frac{(efq^2/abc, efq/a; q)_\infty}{(efq^2/ab, efq^2/ac; q)_\infty} \cdot \frac{(a, b, c; q)_n}{(eq; q)_n} \left(\frac{efq^2}{abc}\right)^n $$

is a conjugate Bailey pair, and

$$ \begin{aligned} \gamma_n = {} & \frac{(efq/a, a; q)_n}{(fq, fq/a; q)_n} \left(-\frac{1}{a}\right)^n q^{-n(n-1)/2} \\ & \times \sum_{j \ge n} \frac{(efq^{n+1}/a, fq/a, b, c; q)_j (eq/a; q)_{j-n}}{(efq^2/ab, efq^2/ac, fq^{n+1}, eq; q)_j (q; q)_{j-n}} (1 - efq^{2j+1}/a) \left(-\frac{ef}{bc}\right)^j q^{j(j+3)/2}. \end{aligned} $$

Proof. Our proof is an application of Watson's ${}_8\phi_7$ transformation:

$$ \begin{aligned} \gamma_n &= \sum_{j \ge n} \frac{\delta_j}{(q; q)_{j-n} (fq; q)_{j+n}} \\ &= \frac{(efq^2/abc, efq/a; q)_\infty}{(efq^2/ab, efq^2/ac; q)_\infty} \sum_{j \ge n} \frac{(a, b, c; q)_j}{(eq; q)_j (q; q)_{j-n} (fq; q)_{j+n}} \left(\frac{efq^2}{abc}\right)^j \\ &= \frac{(efq^2/abc, efq/a; q)_\infty}{(efq^2/ab, efq^2/ac; q)_\infty} \frac{(a, b, c; q)_n}{(eq; q)_n (fq; q)_{2n}} \left(\frac{efq^2}{abc}\right)^n {}_3\phi_2\left[\begin{matrix} aq^n, bq^n, cq^n \\ eq^{n+1}, fq^{2n+1} \end{matrix}; q, \frac{efq^2}{abc}\right] \\ &= \frac{(efq^2/abc, efq/a; q)_\infty}{(efq^2/ab, efq^2/ac; q)_\infty} \frac{(a, b, c; q)_n}{(eq; q)_n (fq; q)_{2n}} \left(\frac{efq^2}{abc}\right)^n \frac{(efq^{n+2}/ab, efq^{n+2}/ac; q)_\infty}{(efq^{2n+2}/a, efq^2/abc; q)_\infty} \\ &\qquad \times \lim_{d \mapsto \infty} {}_8\phi_7 X, \end{aligned} $$

where

$$ X = \left[\begin{matrix} \frac{efq^{2n+1}}{a}, \sqrt{\frac{efq^{2n+3}}{a}}, -\sqrt{\frac{efq^{2n+3}}{a}}, \frac{fq^{n+1}}{a}, \frac{eq}{a}, bq^n, cq^n, d \\ \sqrt{\frac{efq^{2n+1}}{a}}, -\sqrt{\frac{efq^{2n+1}}{a}}, eq^{n+1}, fq^{2n+1}, \frac{efq^{n+2}}{ab}, \frac{efq^{n+2}}{ac}, 0 \end{matrix}; q, \frac{efq^{2+n}}{bcd}\right]. $$

In the above, the last equality follows from [Gasper and Rahman 2004, Equation (III.17)] with $a = efq^{2n+1}/a$, $b = fq^{n+1}/a$, $c = eq/a$, $d = q^{-k}$, $e = bq^n$ and $f = cq^n$, followed by $k \to \infty$. We note that allowing $d = q^{-k}$ ensures the termination of our series. After some simplification we see that this is

$$
\begin{aligned}
\gamma_n &= \frac{(efq/a;q)_{2n+1}}{(efq^2/ab, efq^2/ac;q)_n}\cdot\frac{(a,b,c;q)_n}{(eq;q)_n(fq;q)_{2n}}\Big(\frac{efq^2}{abc}\Big)^n \\
&\quad\times\lim_{d\mapsto\infty}\sum_{j=0}^{\infty}\frac{\Big(1-\frac{efq^{2n+2j+1}}{a}\Big)\Big(\frac{efq^{2n+1}}{a},\frac{fq^{n+1}}{a},\frac{eq}{a},bq^n,cq^n,d;q\Big)_j}{\Big(1-\frac{efq^{2n+1}}{a}\Big)\Big(eq^{n+1},fq^{2n+1},\frac{efq^{n+2}}{ab},\frac{efq^{n+2}}{ac},q;q\Big)_j}\Big(\frac{efq^{2+n}}{bcd}\Big)^j \\
&= \frac{(efq/a,a;q)_n}{(fq,fq/a;q)_n}\Big(-\frac{1}{a}\Big)^n q^{-n(n-1)/2} \\
&\quad\times\sum_{j\geq n}\frac{(efq^{n+1}/a,fq/a,b,c;q)_j(eq/a;q)_{j-n}}{(efq^2/ab,efq^2/ac,fq^{n+1},eq;q)_j(q;q)_{j-n}}(1-efq^{2j+1}/a)\Big(-\frac{ef}{bc}\Big)^j q^{j(j+3)/2},
\end{aligned}
$$

the desired expression. □

We note that our conjugate Bailey pair presented above has the form

$$\gamma_n = C_n\sum_{j=n}^{\infty}D_j\frac{(eq/a)_{j-n}(efq/a)_{j+n}}{(q)_{j-n}(fq)_{j+n}}.$$

Definition 2.2 [Andrews 2001]. Two sequences $\alpha_n(A,K)$ and $\beta_n(A,K)$ form a *WP-Bailey pair* if

$$\beta_n(A,K) = \sum_{j=0}^{n}\frac{(K/A)_{n-j}(K)_{n+j}}{(q)_{n-j}(Aq)_{n+j}}\alpha_j(A,K).$$

Andrews uses this definition to define the WP-Bailey chain. In the same spirit, we can define our own WP-conjugate Bailey pair:

Definition 2.3. We say that two sequences $\delta_n(A,K)$ and $\gamma_n(A,K)$ form a *WP-conjugate Bailey pair* if

$$\gamma_n(A,K) = \sum_{j=n}^{\infty}\frac{(K/A)_{n-j}(K)_{n+j}}{(q)_{n-j}(Aq)_{n+j}}\delta_j(A,K).$$

We can then see that Theorem 2.1 satisfies such a definition if we choose the two sequences $(\gamma_n/C_n, D_n)$ with $A = f$ and $K = efq/a$. We will not explore the realm of *WP-conjugate Bailey chains* here, but we certainly foresee its appearance in subsequent work.

3. Specific conjugate Bailey pairs

The conjugate Bailey pair of Corollary 1.2, used by Bailey [1948] and Slater [1952] in their work, is a special case of our theorem. We can see this by allowing $a = eq$, followed by some simple change of variables. We also note that the special case

$a = eq$ in Theorem 2.1 not only simplifies our pair but also completely eliminates the restricted sum from γ_n. To recover the pairs found by Andrews and Warnaar, we explore another option that will simplify the term $(eq/a;q)_{j-n}/(q;q)_{j-n}$ in our γ_n. We can accomplish this by allowing $e \to a$. We will consider two corollaries of our main result:

Corollary 3.1. *Let $u_n = v_n = 1/(q^2;q^2)_n$ in the Bailey transform. Then we have the conjugate Bailey pair*

$$\gamma_n = \frac{(a;q^2)_n}{(q^2/a;q^2)_n}\Big(-\frac{1}{a}\Big)^n q^{-n(n-1)} \times \sum_{j\ge n} \frac{(q^2/a,b,c;q^2)_j}{(q^4/b,q^4/c,aq^2;q^2)_j}(1-q^{4j+2})\Big(-\frac{a}{bc}\Big)^j q^{j(j+3)}$$

and

$$\delta_n = \frac{(q^4/bc,q^2;q^2)_\infty}{(q^4/b,q^4/c;q^2)_\infty} \cdot \frac{(a,b,c;q^2)_n}{(aq^2;q^2)_n}\Big(\frac{q^4}{bc}\Big)^n.$$

Proof. Let $e \to a$ and $f = 1$ and $q \to q^2$ in Theorem 2.1. □

Corollary 3.2. *Let $u_n = 1/(q^2;q^2)_n$ and $v_n = 1/(q^4;q^2)_n$ in the Bailey transform. Then we have the conjugate Bailey pair*

$$\gamma_n = \frac{(a;q^2)_n}{(q^4/a;q^2)_n}\Big(-\frac{1}{a}\Big)^n q^{-n(n-1)} \times \sum_{j\ge n} \frac{(q^4/a,b,c;q^2)_j}{(q^6/b,q^6/c,aq^2;q^2)_j}(1-q^{4j+4})\Big(-\frac{a}{bc}\Big)^j q^{j(j+5)}$$

and

$$\delta_n = \frac{(q^6/bc,q^4;q^2)_\infty}{(q^6/b,q^6/c;q^2)_\infty} \cdot \frac{(a,b,c;q^2)_n}{(aq^2;q^2)_n}\Big(\frac{q^6}{bc}\Big)^n.$$

Proof. Let $e \to a$ and $f = q$ and $q \to q^2$ in Theorem 2.1. □

We now have the proper tools to prove Andrews and Warnaar's results.

Proof of Theorem 1.5. When using the symmetric bilateral Bailey transform with $u_n = v_n = 1/(q^2;q^2)_n$, we have $\gamma_n = \gamma_{-n}$, so that we can assume $n > 0$. But then the relation between γ and δ in Theorem 1.3 is the same as that of Theorem 1.1. It is left to show that the conjugate Bailey pairs due to Andrews and Warnaar are special cases of Corollary 3.1. We see this by considering $a = -q$, $b = q$ and $c = q^2$ in Corollary 3.1 and $b = q$, $c = q^2$ and $a \to 0$ in Corollary 3.1. □

Proof of Theorem 1.6. When using the asymmetric bilateral Bailey transform with $u_n = v_n = 1/(q^2;q^2)_n$, we see that

$$\gamma_n = \sum_{j\ge m} \frac{\delta_j}{(q^2;q^2)_{j-n}(q^2;q^2)_{j+n+1}} = \sum_{j\ge m} \frac{\delta_j}{(q^2;q^2)_{j-m}(q^2;q^2)_{j+m+1}},$$

where $m=\max\{n,-n-1\}$. If we consider $a=-q^2,\ \ b=q^2$ and $c=q^3$ in Corollary 3.2, we see that

$$\delta_n=\frac{(q,q^4;q^2)_\infty}{(q^4,q^3;q^2)_\infty}\cdot\frac{(-q^2,q^2,q^3;q^2)_n}{(-q^4;q^2)_n}\Big(\frac{q^6}{q^5}\Big)^n=\frac{(1+q^2)}{(1+q^{2n+2})}(q)_{2n+1}q^n$$

and γ_n is equal to

$$\frac{(-q^2;q^2)_n}{(-q^2;q^2)_n}\Big(\frac{1}{q^2}\Big)^n q^{-n(n-1)}\sum_{j\geq n}\frac{(-q^2,q^2,q^3;q^2)_j}{(q^4,q^3,-q^4;q^2)_j}(1-q^{4j+4})\Big(\frac{q^2}{q^5}\Big)^j q^{j(j+5)}$$
$$=(1-q^4)q^{-n(n+1)}\sum_{j\geq n}q^{j(j+2)}.$$

Thus,

$$q^{-n(n+1)}\sum_{j\geq n}q^{j(j+2)}=\sum_{j\geq n}\frac{(q;q)_{2j+1}q^j}{(1+q^{2j+2})(q^2;q^2)_{j-n}(q^2;q^2)_{j+n+1}}.$$

Since $n(n+1)=m(m+1)$ when $m=\max\{n,-n-1\}$, we are done.

We can prove the second pair in the same way using $b=q^2,\ \ c=q^3$ and $a\to 0$ in Corollary 3.2. □

We now introduce some new special cases of our general conjugate Bailey pair; these will be used in later sections. All of the conjugate Bailey pairs we introduce below are with respect to the symmetric bilateral Bailey transform and $u_n=v_n$. This ensures that if γ_n is a conjugate Bailey pair with respect to the Bailey transform, then $\gamma'_n=\gamma_{|n|}$ is a conjugate Bailey pair with respect to the symmetric bilateral Bailey transform. Thus, each conjugate Bailey pair that satisfies Corollary 3.1 is a pair in the symmetric bilateral Bailey transform as well as the Bailey transform.

Corollary 3.3. *Let $u_n=v_n=1/(q^2;q^2)_n$ in the symmetric bilateral Bailey transform. Then we have the conjugate Bailey pairs listed in Table 1.*

Proof. Each pair follows from a choice of a, b and c in Corollary 3.1.

For (1), we take $a=b=c=q$, followed by $q\to -q$.

For (2), we take $a=-b=-c=q$, followed by $q\to -q$.

For (3), we take $a=b=q,\ \ c=-q^2$, followed by $q\to -q$.

For (4), we take $a=-q,\ \ b=q^3$ and $c\to\infty$.

For (5), we take $a=-b=q,\ \ c\to\infty$, followed by $q\to -q$.

For (6), we take $b=c=q,\ \ a\to 0$.

For (7), we take $b=q,\ \ a\to 0,\ \ c\to\infty$.

For (8), we take $b=q^3,\ \ a\to 0,\ \ c\to\infty$.

For (9), we take $b=q,\ \ c=-q^2,\ \ a\to 0$, followed by $q\to -q$.

For (10), we take $a=q,\ \ b=q^2$ and $c\to\infty$, followed by $q\to -q$. □

	δ_n	γ_n
(1)	$\dfrac{(q^2;q^2)_\infty^2}{(-q;q^2)_\infty^2}\cdot\dfrac{(-q;q^2)_n^2}{(1+q^{2n+1})}q^{2n}$	$q^{-n^2}\sum\limits_{j\geq\lvert n\rvert}\dfrac{(1-q^{2j+1})}{(1+q^{2j+1})^2}q^{j(j+2)}$
(2)	$\dfrac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\cdot\dfrac{(q;q^2)_n^2}{(1+q^{2n+1})}q^{2n}$	$q^{-n^2}\sum\limits_{j\geq\lvert n\rvert}\dfrac{q^{j(j+2)}}{(1-q^{2j+1})}$
(3)	$\dfrac{(q)_\infty}{(-q)_\infty}\cdot\dfrac{(-q)_{2n}}{(1+q^{2n+1})}q^n$	$q^{-n^2}\sum\limits_{j\geq\lvert n\rvert}\dfrac{(1-q^{2j+1})}{(1+q^{2j+1})}q^{j(j+1)}$
(4)	$\dfrac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\cdot\dfrac{(q;q^2)_{n+1}}{(1+q^{2n+1})}(-1)^nq^{n^2}$	$q^{-n^2}\sum\limits_{j\geq\lvert n\rvert}(1-q^{2j+1})^2(-1)^jq^{2j^2}$
(5)	$\dfrac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\cdot\dfrac{(q;q^2)_n}{(1+q^{2n+1})}(-1)^nq^{n(n+2)}$	$q^{-n^2}\sum\limits_{j\geq\lvert n\rvert}(-1)^jq^{2j(j+1)}$
(6)	$\dfrac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}(q;q^2)_n^2q^{2n}$	$q^{-2n^2}\sum\limits_{j\geq\lvert n\rvert}\dfrac{(1+q^{2j+1})}{(1-q^{2j+1})}q^{2j(j+1)}$
(7)	$\dfrac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(q;q^2)_n(-1)^nq^{n(n+2)}$	$q^{-2n^2}\sum\limits_{j\geq\lvert n\rvert}(1+q^{2j+1})(-1)^jq^{j(3j+2)}$
(8)	$\dfrac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(q;q^2)_{n+1}(-1)^nq^{n^2}$	$q^{-2n^2}\sum\limits_{j\geq\lvert n\rvert}(1-q^{2j+1})(1-q^{4j+2})(-1)^jq^{3j^2}$
(9)	$\dfrac{(q)_\infty}{(-q)_\infty}(-q)_{2n}q^n$	$q^{-2n^2}\sum\limits_{j\geq 2\lvert n\rvert}(-1)^jq^{j(j+1)/2}$
(10)	$\dfrac{(q^2;q^2)_n}{(1+q^{2n+1})}(-1)^nq^{n(n+1)}$	$q^{-n^2}\sum\limits_{j\geq 2\lvert n\rvert}(-q)^{j(j+1)/2}$

Table 1

Corollary 3.4. *Let $u_n = v_n = 1/(q)_n$ in the symmetric bilateral Bailey transform. Then we have conjugate Bailey pairs listed in Table 2.*

Proof. Again, each pair follows from a choice of a, b and c in Corollary 3.1.

For (11), we take $b = -c = q,\ a \to 0$, followed by $q^2 \to q$.

For (12), we take $b = -q^2,\ a \to 0$ and $c \to \infty$, followed by $q^2 \to q$.

For (13), we take $b = q^2,\ a \to 0$ and $c \to \infty$, followed by $q^2 \to q$.

For (14), we take $a \to 0$ and $b, c \to \infty$, followed by $q^2 \to q$. □

4. Sums of triangular numbers

We turn our attention to Gauss's formula

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

	δ_n	γ_n
(11)	$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(q;q^2)_n(-1)^n q^n$	$q^{-n^2}\sum_{j\geq\|n\|}(-1)^j q^{j(j+1)}$
(12)	$\frac{(q)_\infty}{(-q)_\infty}(-q)_n q^{n(n+1)/2}$	$q^{-n^2}\sum_{j\geq\|n\|}(1-q^{2j+1})q^{j(3j+1)/2}$
(13)	$(q)_n(-1)^n q^{n(n+1)/2}$	$q^{-n^2}\sum_{j\geq\|n\|}(1-q^{2j+1})(-1)^j q^{j(3j+1)/2}$
(14)	$(q)_\infty q^{n(n+1)}$	$q^{-n^2}\sum_{j\geq\|n\|}(-1)^j q^{j(j+1)/2}$

Table 2

and remark that $n(n+1)/2$ is the n-th triangular number. With this identity, we are able to make some interesting remarks about generating functions related to the sums of the triangular numbers. We define the function $\psi(q)=\sum_{n=0}^{\infty} q^{n(n+1)/2}$, a classical theta function studied by Ramanujan [Berndt 1991, page 100, (5.1)]. We can then see that $\psi^k(q)=\sum_{n=0}^{\infty} t_k(n)q^n$, where $t_k(n)$ counts the number of representations of n as the sum of k triangular numbers. We note that order is important, unlike with partitions: for example, $t_3(5)=3$ since $5=3+1+1=1+3+1=1+1+3$. While (15)–(18) and (20) can be found in previous work, none have used the conjugate Bailey pair approach presented below, and no other method has been able to encompass so many results.

Corollary 4.1.

$$\psi^2(q)=\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{j(j+1)}}{(1-q^{2j+1})}, \tag{15}$$

$$\psi^2(q^2)=\sum_{j=0}^{\infty}\frac{q^j}{1+q^{2j+1}}, \tag{16}$$

$$\psi^4(q)=\sum_{j=-\infty}^{\infty}\frac{q^j}{(1-q^{2j+1})^2}, \tag{17}$$

$$\psi^3(q)=\sum_{j\geq|n|}\frac{(1+q^{2j+1})}{(1-q^{2j+1})}q^{2j(j+1)-n(2n-1)}, \tag{18}$$

$$\psi^2(q)=\sum_{j\geq|n|}(-1)^{j+n}q^{j(3j+2)-n(3n+1)}(1+q^{2j+1}), \tag{19}$$

$$\psi^2(q)=\sum_{j\geq|n|}(-1)^{j+n}q^{j(j+1)-n^2}. \tag{20}$$

We note that (16) can be found in [Berndt 1991, page 139, Example (iv)], (17) can be found in [Dickson 1966, page 285], (18) can be found in [Andrews 1986a, page 114, (1.5)], and (20) can be found in [Andrews 1984, page 452, (1.4)] with $q \to q^2$ and $z = 1/q$.

Proof. For Equation (15), we consider the Bailey pair

$$\alpha_n = q^{n^2}, \quad \beta_n = \frac{(-q;q^2)_n^2}{(q^2;q^2)_{2n}} \tag{21}$$

found in [Andrews 1998, page 49, Example 1]. Combining this Bailey pair with the conjugate Bailey pair (3) with $q \to -q$ and $u_n = v_n = 1/(q^2;q^2)_n$, we get

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1-q^{2n+1})} &= \sum_{n=-\infty}^{\infty} (-1)^n \sum_{j \geq |n|} \frac{(1+q^{2j+1})}{(1-q^{2j+1})} q^{j(j+1)} \\ &= \frac{(-q;-q)_\infty}{(1-q)(q;-q)_\infty} {}_2\phi_1\left(\begin{matrix} q, -q \\ q^3 \end{matrix}; q^2, -q\right) = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2},\end{aligned}$$

where our last equality is due to [Gasper and Rahman 2004, III.2, page 359].

For Equation (16), we consider the Bailey pair

$$\alpha_n = q^n, \quad \beta_n = \frac{q^{-n}}{(q)_{2n}} \tag{22}$$

found in [Slater 1951, F(3)] with the conjugate Bailey pair (2) with $q \to -q$ and $u_n = v_n = 1/(q^2;q^2)_n$ to get

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{q^j}{1+q^{2j+1}} &= \sum_{n=-\infty}^{\infty} (-1)^n q^{-n(n-1)} \sum_{j \geq |n|} \frac{(-1)^j q^{j(j+1)}}{1+q^{2j+1}} \\ &= \frac{(q^2;q^2)_\infty^2}{(1-q)(-q;q^2)_\infty^2} {}_2\phi_1\left(\begin{matrix} -q, -q \\ q^3 \end{matrix}; q^2, q\right) \\ &= \frac{(q^2;q^2)_\infty^2}{(-q;q^2)_\infty^2} \cdot \frac{(-q^2;q^2)_\infty^2}{(q;q^2)_\infty^2},\end{aligned}$$

where our first equality is due to the identity

$$\sum_{n=-j}^{j} (-1)^n q^{-n(n-1)} = (-1)^j q^{-j(j+1)}, \tag{23}$$

which can be easily proved with induction; our last equality is due to [Gasper and Rahman 2004, III.2, page 359]. We then see that our infinite product is $\psi^2(q^2)$.

For Equation (17), we consider the Bailey pair (22) with the conjugate Bailey pair (1) with $u_n = v_n = 1/(q^2;q^2)_n$. We then apply (23) and [Gasper and Rahman 2004, III.2, page 359] as with our previous result.

For Equation (18), we consider the Bailey pair (22) with the conjugate Bailey pair (6) with $u_n = v_n = 1/(q^2;q^2)_n$ to get

$$\sum_{j\ge|n|}\frac{(1+q^{2j+1})}{(1-q^{2j+1})}q^{2j(j+1)-n(2n-1)} = \sum_{n=-\infty}^{\infty} q^{-n(2n-1)}\sum_{j\ge|n|}\frac{(1+q^{2j+1})}{(1-q^{2j+1})}q^{2j(j+1)}$$
$$= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2}{(q)_{2n}}q^n$$
$$= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\cdot\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

where the last equality is due to q-binomial theorem [Andrews 1998, page 17, Theorem 2.1]

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}t^n = \frac{(at)_\infty}{(t)_\infty}. \tag{24}$$

For Equation (19), we consider the Bailey pair

$$\alpha_n = (-1)^n q^{-n(n+1)}, \qquad \beta_n = \frac{(-1)^n q^{-n(n+1)}}{(q^2;q^2)_n}$$

found in the fourth row of the second table in [Slater 1951, page 468] with the conjugate Bailey pair (8) with $u_n = v_n = 1/(q^2;q^2)_n$. We then apply the q-binomial theorem as with our previous result.

For Equation (20), we consider the Bailey pair

$$\alpha_n = (-1)^n, \qquad \beta_n = \frac{(-1)^n}{(q^2;q^2)_n} \tag{25}$$

found in the seventh row of the second table in [Slater 1951, page 468] with the conjugate Bailey pair (10) with $u_n = v_n = 1/(q)_n$. We then apply the q-binomial theorem as with our two previous results. □

5. Indefinite quadratic forms

In the previous section we noticed that our new general conjugate Bailey pair is very capable of producing results of the form (18)–(20). In this section we take a deeper look into double series involving an indefinite quadratic form. By [1959], E. Hecke had studied many of these forms in detail, and among these was

$$\sum_{j\ge2|n|}(-1)^{j+n}q^{j(j+1)/2-n(3n-1)/2} = (q)_\infty^2, \tag{26}$$

which was originally discovered by L. J. Rogers [1894].

It is with little difficulty that we can show the above identity and others with our new conjugate Bailey pairs. Noting that our γ_n is already a restricted sum of the type we are looking for, all that is left is to find a suitable Bailey pair to match it with. The following section will discuss some new results as well as tying in some identities due to Andrews [1986a] and Andrews, Dyson, and Hickerson [1988].

Corollary 5.1. *Identity (26) is true.*

Proof. We consider the Bailey pair

$$\alpha_{2n} = (-1)^n q^{n(n+1)}, \ \alpha_{2n+1} = 0, \quad \beta_n = \frac{q^{n(n-1)/2}}{(q)_n (q;q^2)_n}, \tag{27}$$

which is found in [Slater 1951, C(5)], with the conjugate Bailey pair (10) in which $u_n = v_n = 1/(q)_n$. Applying (24) and allowing $q^2 \to q$ yields our final result. □

In [1986a], Andrews uses complicated Bailey pairs with the implementation of Bailey chains, as well as some clever algebra to prove identities such as the following.

Corollary 5.2.

$$\sum_{j \geq |n|} (-1)^n q^{j(3j+1)/2 - n^2} (1 - q^{2j+1}) = \frac{(q)_\infty^2}{(-q)_\infty}.$$

Proof. We consider the Bailey pair (25) with the conjugate Bailey pair (11) with $u_n = v_n = 1/(q)_n$. Our result then follows with the application of (24). □

In [1988], Andrews, Dyson and Hickerson then adapted the method used in [Andrews 1986a] to prove similar identities involving the rank of a partition. The rank of a partition is the excess of the largest part over the number of parts. The main motivation for their paper was the function

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n}, \tag{28}$$

which can be found in [Andrews 1986b]. We note that $\sigma(q)$ is the generating function for strict partitions with odd rank subtracted from those with even rank. We find the following corollary in [Andrews et al. 1988, page 392, Equation (1.5)].

Corollary 5.3.

$$\sigma(q) = \sum_{j \geq |n|} (-1)^{n+j} q^{j(3j+1)/2 - n^2} (1 - q^{2j+1}).$$

Proof. We consider the Bailey pair (25) with the conjugate Bailey pair (12) with $u_n = v_n = 1/(q)_n$. □

Andrews, Dyson and Hickerson, in [1988, page 404], define generating a function similar to $\sigma(q)$, as follows. For $n \geq 1$, consider partitions of n into odd parts, with the property that if k occurs as a part, then all positive odd parts less than k must also occur (*without odd gaps*). Let $S^*(n)$ be the excess of the number of such partitions with largest part congruent to 3 modulo 4 over the number with largest part congruent to 1 modulo 4. They then show [1988, page 404, Equation 5.2] that

$$\sum_{n\geq 1} S^*(n)q^n = \sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} = \sum_{n\geq 1}(-1)^n q^{n(3n+1)}(1+q^{2n}) \sum_{j=0}^{2n-1} q^{-j(j+1)/2}.$$

It is with minimal work that we can show an equivalent formula.

Corollary 5.4.

$$\sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} = \sum_{j\geq |n|}(-1)^j q^{j(3j+1)-n(2n-1)}(1-q^{4n+2}).$$

Proof. We consider the Bailey pair (22) with the conjugate Bailey pair (12) with $u_n = v_n = 1/(q^2;q^2)_n$. Applying (24) yields our result. $\square$

The following identity can be found in [Andrews 1984, page 457, (3.16)],

Corollary 5.5.

$$\sum_{j\geq 2|n|}(-1)^{j+n} q^{j(j+1)/2-n^2} = (q)_\infty (q^2;q^2)_\infty.$$

Proof. We consider the Bailey pair (25) with the conjugate Bailey pair (13) with $u_n = v_n = 1/(q)_n$. $\square$

The following corollaries are new indefinite quadratic forms.

Corollary 5.6.

$$\sum_{j\geq |n|}(-1)^j q^{2j(j+1)-n(2n-1)} = \frac{(q^4;q^4)_\infty^2}{(-q;q^2)_\infty}, \tag{29}$$

$$\sum_{j\geq |n|}(-1)^j q^{j(3j+2)-n(2n-1)}(1+q^{2j+1}) = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty}, \tag{30}$$

$$\sum_{j\geq |n|}(-1)^j q^{3j^2-n(2n-1)}(1-q^{2j+1})^2(1+q^{2j+1}) = -q\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty}, \tag{31}$$

$$\sum_{j\geq 2|n|}(-1)^{j+n} q^{j(j+1)/2-2n^2} = \frac{(q)_\infty^2}{(-q;q^2)_\infty}. \tag{32}$$

Proof. For (29) we consider the Bailey pair (22) with the conjugate Bailey pair (10) with $q \to q^2$ and $u_n = v_n = 1/(q^2;q^2)_n$. The result then follows from (24).

For (30) we consider the Bailey pair (22) with the conjugate Bailey pair (8) with $u_n = v_n = 1/(q^2;q^2)_n$. Our result follows from applying (24).

For (31) we consider the Bailey pair (22) with the conjugate Bailey pair (9) with $u_n = v_n = 1/(q^2;q^2)_n$. Our result follows from applying (24):

$$\begin{aligned}\sum_{j\geq|n|}(-1)^j q^{3j^2-n(2n-1)}(1-q^{2j+1})^2(1+q^{2j+1})\\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)}}{(q^2;q^2)_n}(1-q^{2n+1})\\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)}}{(q^2;q^2)_n} - q\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n}\right) = -q\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty}.\end{aligned}$$

For (32) we consider the Bailey pair (25) with the conjugate Bailey pair (14) with $u_n = v_n = 1/(q^2;q^2)_n$. Our result follows from applying (24). □

Corollary 5.7.

$$2\sum_{j\geq 2|n|}(-1)^j q^{j(j+1)-n(2n-1)} = (q^2;q^2)_\infty\left((-q)_\infty + (q)_\infty\right), \tag{33}$$

$$2\sum_{j\geq 2|n|}(-1)^n q^{j(3j+2)-2n(3n-1)}(1-q^{2j+1}) = (q^4;q^4)_\infty + (q)_\infty^2(-q^2;q^2)_\infty, \tag{34}$$

$$\begin{aligned}2\sum_{j\geq 2|n|}(-1)^n q^{3j^2-2n(3n-1)}(1+q^{2j+1})^2(1-q^{2j+1}) = (2+q)(q^4;q^4)_\infty\\ + q(q)_\infty^2(-q^2;q^2)_\infty.\end{aligned} \tag{35}$$

Proof. For (33), we consider the Bailey pair (22) with the conjugate Bailey pair (13) with $u_n = v_n = 1/(q^2;q^2)_n$. Our result follows from applying (24):

$$\begin{aligned}\sum_{j\geq 2|n|}(-1)^j q^{j(j+1)-n(2n-1)} &= (q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(q)_{2n}}\\ &= \frac{(q^2;q^2)_\infty}{2}\sum_{n=0}^{\infty}\frac{(1+(-1)^n)q^{n(n+1)/2}}{(q)_n}\\ &= \frac{(q^2;q^2)_\infty\left((q)_\infty+(-q)_\infty\right)}{2}.\end{aligned}$$

For (34), we consider the Bailey pair (27) with the conjugate Bailey pair (8) with $q \to -q$ and $u_n = v_n = 1/(q^2;q^2)_n$. We then apply (24) as before.

For (35), we consider the Bailey pair (27) with the conjugate Bailey pair (9) with $q \to -q$ and $u_n = v_n = 1/(q^2;q^2)_n$. Our result follows from applying (24):

$$\begin{aligned}\sum_{j\geq 2|n|} &(-1)^n q^{3j^2-2n(3n-1)}(1+q^{2j+1})^2(1-q^{2j+1})\\ &= \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})q^{n(2n-1)}}{(q)_{2n}}\\ &= (q)_\infty(-q^2;q^2)_\infty\left(\sum_{n=0}^{\infty}\frac{q^{n(2n-1)}}{(q)_{2n}} + q\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(q)_{2n}}\right)\\ &= \left(1+\frac{q}{2}\right)(q^4;q^4)_\infty + \frac{(q)_\infty^2(-q^2;q^2)_\infty}{2}. \qquad \square\end{aligned}$$

6. Applications to partitions

In this section we present some partition identities. We define a *partition* as a finite nonincreasing sequence of positive integers, $\lambda = (\lambda_1, \ldots, \lambda_k)$. We refer to each λ_i as a part of the partition. We say that λ is a partition of n, or $|\lambda| = n$, if the sum of its parts is equal to n. For example, there are 7 partitions of 5:

$$(5),\quad (4,1),\quad (3,2),\quad (3,1,1),\quad (2,2,1),\quad (2,1,1,1),\quad (1,1,1,1,1).$$

We say that a partition λ is *without gaps* if for every positive integer $k < \lambda_1$, there exists i such that $\lambda_i = k$. As in the last section, we can generalize this notion to *without odd/even gaps* by restricting k to odd/even positive integers.

Theorem 6.1. *Let $a(n)$ be the number of partitions of n with largest part odd and without odd gaps. Let $b(n)$ be the number of partitions of n into parts congruent to $\pm1, \pm3, \pm5, \pm7, \pm8, \pm9 \mod 20$. Then $a(n) = b(n-1)$ for $n \geq 1$.*

Proof. We consider the Bailey pair [Slater 1951, C(1)]

$$\alpha_{2n} = (-1)^n q^{2n(3n+1)},\quad \alpha_{2n+1} = 0,\quad \beta_n = 1/(q^2;q^4)_n(q^2;q^2)_n \tag{36}$$

with the conjugate Bailey pair (5) (note that $q \to -q$) with $u_n = v_n = 1/(q^2;q^2)_n$ to get

$$\begin{aligned}\frac{(q)_\infty}{(q^2;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q)_{2n+1}} &= \sum_{n=-\infty}^{\infty}(-1)^n q^{2n(n+1)}\sum_{j\geq 2|n|}(-1)^j q^{2j(j+1)}\\ &= \sum_{j=-\infty}^{\infty}(-1)^j q^{2j(5j+3)} = (q^4, q^{16}, q^{20}; q^{20})_\infty,\end{aligned}$$

where the last equality follows from the Jacobi triple product; see [Andrews 1998, Theorem 2.8]. It is left to note that

$$q\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q)_{2n+1}}=\sum_{n=0}^{\infty}\frac{q^{1+3+\cdots+(2n+1)}}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} \quad \text{and}$$

$$\frac{(q^2;q^4)_\infty(q^4,q^{16},q^{20};q^{20})_\infty}{(q)_\infty}=\frac{1}{(q,q^3,q^5,q^7,q^8,q^9,q^{11},q^{12},q^{13},q^{15},q^{17},q^{19};q^{20})_\infty}. \qquad \square$$

We can also define a weight function $\omega(\lambda)$ for a partition. If we say that $a(n)$ counts the number of partitions of n with respect to the weight function $\omega(\lambda)$ then $a(n)=\sum_{|\lambda|=n}\omega(\lambda)$.

Theorem 6.2. *Let $g(n)$ be the number of partitions of n without even gaps and with corresponding weight function $\omega(\lambda)=(-1)^{k_1}2^{k_2}$, where k_1 is the multiplicity of the largest part plus the number of even parts less than the largest part and k_2 is the number of different odd parts less than the largest part. Then*

$$g(n)=\begin{cases}(-1)^j(2j+1) & \text{if } n=j(2j+1) \text{ for } j\in\mathbb{N},\\ (-1)^{j+1}(2j+1) & \text{if } n=(j+1)(2j+1) \text{ for } j\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases}$$

Example 6.3. Consider $n=8$ and $n=10$:

$\|\lambda\|=8$	$\omega(\lambda)$	$\|\lambda\|=10$	$\omega(\lambda)$
(4, 2, 2)	−1	(4, 4, 2)	−1
(4, 2, 1, 1)	2	(4, 3, 2, 1)	4
(3, 3, 2)	−1	(4, 2, 2, 2)	1
(3, 2, 2, 1)	−2	(4, 2, 2, 1, 1)	−2
(3, 2, 1, 1, 1)	2	(4, 2, 1, 1, 1, 1)	2
(2, 2, 2, 2)	−1	(3, 3, 2, 2)	1
(2, 2, 2, 1, 1)	2	(3, 3, 2, 1, 1)	−2
(2, 2, 1, 1, 1, 1)	−2	(3, 2, 2, 2, 1)	2
(2, 1, 1, 1, 1, 1, 1)	2	(3, 2, 2, 1, 1, 1)	−2
(1, 1, 1, 1, 1, 1, 1, 1)	−2	(3, 2, 1, 1, 1, 1, 1)	2
	1	(2, 2, 2, 2, 2)	−1
		(2, 2, 2, 2, 1, 1)	2
		(2, 2, 2, 1, 1, 1, 1)	−2
		(2, 2, 1, 1, 1, 1, 1, 1)	2
		(2, 1, 1, 1, 1, 1, 1, 1, 1)	−2
		(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	1
	0		5

Proof. We consider the Bailey pair (21) and the conjugate Bailey pair (15) with $u_n = v_n = 1/(q^2;q^2)_n$ to get

$$\sum_{j=0}^{\infty}(2j+1)(1-q^{2j+1})(-1)^j q^{j(2j+1)} = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n(-1)^n q^{n(n+1)}}{(q;-q)_{2n}(1+q^{2n+1})}.$$

We then note that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n(n+1)}}{(q;-q)_{2n}(1+q^{2n+1})}\\ &=\sum_{n=0}^{\infty}(1+2q+2q^2+\cdots)(1+2q^3+2q^6+\cdots)\cdots(1+2q^{2n-1}+2q^{4n-2}+\cdots)\\ &\qquad\qquad\times(1-q^{2n+1}+q^{4n+2}-\cdots)\cdot\frac{(-1)^n q^{2+4+\cdots+2n}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}.\qquad\square\end{aligned}$$

Theorem 6.4. *Let $a(n)$ denote the number of partitions of n without even gaps and having an even number of parts, minus the number of partitions of n without even gaps and having an odd number of parts. Then $a(0) = 1$ and for $n > 0$,*

$$a(n) = \begin{cases} 1 & \text{if } n = j(5j+3)/2 \text{ for } j \in \mathbb{N},\\ -1 & \text{if } n = j(5j-3)/2 \text{ for } j \in \mathbb{N},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Consider the Bailey pair

$$\alpha_n = (-1)^n q^{n(3n+1)/2}, \qquad \beta_n = \frac{(q;q^2)_n}{(q^2;q^2)_{2n}}.$$

This follows from specializing [Bailey 1948, page 5, Section 6, (ii)] with $a = 1$, $b \to \infty$ and with x replaced by q, together with the conjugate Bailey pair (15) with $u_n = v_n = 1/(q^2;q^2)_n$. Then we get

$$\begin{aligned}\sum_{j=0}^{\infty}(1-q^{2j+1})q^{j(5j+3)/2} &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(-q)_{2n+1}}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{2+4+\cdots+2n}}{(1+q)(1+q^2)\cdots(1+q^{2n+1})}.\qquad\square\end{aligned}$$

7. Conclusions and future work

This work hopes to bridge the gap between the work done by Bailey and Slater and the work done recently by Andrews and Warnaar. We can now clearly see how our new conjugate Bailey pairs relate to those introduced over 50 years ago by Bailey.

We recall the reference to WP Bailey pairs and chains. As with the large amount of work done on Bailey chains, it is hoped and anticipated that these chains do appear for conjugate Bailey pairs as well.

We note that we have focused on applications to triangular numbers, indefinite quadratic forms and partitions, but we are not limited to these. Future work may address weighted q-series identities, generalized Lambert series, Ramanujan-type identities and a more thorough treatment of partition identities. It should also be noted that a combinatorial proof of any of the partition identities would also be desirable as it might provide more insight into similar identities.

We have only defined 15 new conjugate Bailey pairs from Corollary 3.1, though infinitely many exist. Using Corollary 3.2, we could also define analogous conjugate Bailey pairs with respect to the asymmetric bilateral Bailey transform, which would then lead to new results similar to those presented above.

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References

[Andrews 1984] G. E. Andrews, "Hecke modular forms and the Kac–Peterson identities", *Trans. Amer. Math. Soc.* **283**:2 (1984), 451–458. MR 85e:11031 Zbl 0545.10016

[Andrews 1986a] G. E. Andrews, "The fifth and seventh order mock theta functions", *Trans. Amer. Math. Soc.* **293**:1 (1986), 113–134. MR 87f:33011 Zbl 0593.10018

[Andrews 1986b] G. E. Andrews, "Ramanujan's "lost" notebook, V: Euler's partition identity", *Adv. in Math.* **61**:2 (1986), 156–164. MR 87i:11137 Zbl 0601.10007

[Andrews 1998] G. E. Andrews, *The theory of partitions*, Cambridge University Press, Cambridge, 1998. MR 99c:11126 Zbl 0996.11002

[Andrews 2001] G. E. Andrews, "Bailey's transform, lemma, chains and tree", pp. 1–22 in *Special functions 2000: Current perspective and future directions* (Tempe, AZ), edited by J. Bustoz et al., NATO Sci. Ser. II Math. Phys. Chem. **30**, Kluwer, Dordrecht, 2001. MR 2004i:33027 Zbl 1005.33005

[Andrews and Warnaar 2007] G. E. Andrews and S. O. Warnaar, "The Bailey transform and false theta functions", *Ramanujan J.* **14**:1 (2007), 173–188. MR 2008a:33019 Zbl 1116.33017

[Andrews et al. 1988] G. E. Andrews, F. J. Dyson, and D. Hickerson, "Partitions and indefinite quadratic forms", *Invent. Math.* **91**:3 (1988), 391–407. MR 89f:11071

[Bailey 1948] W. N. Bailey, "Identities of the Rogers–Ramanujan type", *Proc. London Math. Soc.* (2) **50** (1948), 1–10. MR 9,585b Zbl 0031.39203

[Berndt 1991] B. C. Berndt, *Ramanujan's notebooks, III*, Springer, New York, 1991. MR 92j:01069 Zbl 0733.11001

[Dickson 1966] L. E. Dickson, *History of the theory of numbers, II: Diophantine analysis*, Chelsea, New York, 1966. MR 39 #6807b Zbl 0958.11500

[Gasper and Rahman 2004] G. Gasper and M. Rahman, *Basic hypergeometric series*, 2nd ed., Encyclopedia of Mathematics and its Applications **96**, Cambridge University Press, Cambridge, 2004. MR 2006d:33028 Zbl 1129.33005

[Hecke 1959] E. Hecke, "Über einen Zusammenhang zwischen elliptischen Modulfunktionen und indefiniten quadratischen Formen", pp. 418–427 in *Mathematische Werke*, Vandenhoeck und Ruprecht, Göttingen, 1959. MR 21 #3303 Zbl 0092.00102

[Rogers 1894] L. J. Rogers, "Second memoir on the expansion of certain infinite products", *Proc. London Math. Soc.* **25** (1894), 318–343.

[Slater 1951] L. J. Slater, "A new proof of Rogers's transformations of infinite series", *Proc. London Math. Soc.* (2) **53** (1951), 460–475. MR 13,227g Zbl 0044.06102

[Slater 1952] L. J. Slater, "Further identies of the Rogers–Ramanujan type", *Proc. London Math. Soc.* (2) **54** (1952), 147–167. MR 14,138e Zbl 0046.27204

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THE K-ORBIT OF A NORMAL ELEMENT IN A COMPLEX SEMISIMPLE LIE ALGEBRA

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Given a complex semisimple Lie algebra $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$, we consider the converse question of Kostant's convexity theorem for a normal $x \in \mathfrak{g}$. Let $\pi : \mathfrak{g} \to \mathfrak{h}$ be the orthogonal projection under the Killing form onto the Cartan subalgebra $\mathfrak{h} := \mathfrak{t} + i\mathfrak{t}$ where $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{k}$. If $\pi(\mathrm{Ad}(K)x)$ is convex, then there is $k \in K$ such that each simple component of $\mathrm{Ad}(k)x$ can be rotated into the corresponding component of $\mathfrak{t}$. The result also extends a theorem of Au-Yeung and Tsing on the generalized numerical range.

1. Introduction

Let $A \in \mathbb{C}_{n\times n}$. Consider the set

$$\mathcal{W}(A) := \{\mathrm{diag}\,(UAU^{-1}) : U \in \mathrm{U}(n)\},$$

where $\mathrm{U}(n)$ denotes the unitary group. It is the image of the projection of the orbit

$$O(A) := \{UAU^{-1} : U \in \mathrm{U}(n)\}$$

onto the set of diagonal matrices. The following two results concern the geometric shape of $\mathcal{W}(A)$.

Theorem 1.1 (Schur–Horn [Schur 1923; Horn 1954]). *If $A \in \mathbb{C}_{n\times n}$ is Hermitian with eigenvalues $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, then*

$$\mathcal{W}(A) = \mathrm{conv}\, S_n\lambda,$$

where conv $S_n\lambda$ *is the convex hull of the orbit of λ under the action of the full symmetric group S_n.*

For general $A \in \mathbb{C}_{n\times n}$, $\mathcal{W}(A)$ is not convex. Indeed Tsing [1981] proved that $\mathcal{W}(A)$ is star-shaped with respect to the star center $\frac{1}{n}(\mathrm{tr}\, A)(1, \dots, 1)$.

Theorem 1.2 (Au-Yeung and Sing [1977]). *Let $A \in \mathbb{C}_{n\times n}$ be normal. If $\mathcal{W}(A)$ is convex, then the eigenvalues of A are collinear, that is, there exist $\alpha, \beta \in \mathbb{C}$ such that $\alpha A + \beta I$ is Hermitian.*

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So Theorem 1.2 may be viewed as the converse to Theorem 1.1 as one restricts the attention on normal matrices. We remark that if $A \in \mathbb{C}_{n\times n}$ has zero trace, then $\alpha A + \beta I$ being Hermitian means that $e^{i\gamma} A$ is Hermitian for some $\gamma \in \mathbb{R}$. The following result of Au-Yeung and Tsing is stronger than Theorem 1.2. It affirmatively answers the conjecture of Marcus [1979] about the (stronger) converse of the result of Westwick [1975] on the convexity of c-numerical range. Bebiano and Da Providência [1996] gave another proof of Theorem 1.3.

Theorem 1.3 (Au-Yeung and Tsing [1983]). *Let $A \in \mathbb{C}_{n\times n}$ be normal. If*

$$W_{A^*}(A) := \{\operatorname{tr} A^* U A U^{-1} : U \in \mathrm{U}(n)\}$$

is convex, then A has collinear eigenvalues.

The above results can be reduced to the case $\operatorname{tr} A = 0$, that is, the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We may write $A = \hat{A} + \frac{1}{n}(\operatorname{tr} A) I_n$, where $\hat{A} := A - \frac{1}{n}(\operatorname{tr} A) I_n$ has zero trace. Then

$$\mathcal{W}(A) = \mathcal{W}(\hat{A}) + \frac{\operatorname{tr} A}{n}(1, \dots, 1),$$

$$W_{A^*}(A) = W_{\hat{A}^*}(\hat{A}) + \frac{|\operatorname{tr} A|^2}{n^2}.$$

We will extend Theorems 1.2 and 1.3 in the context of semisimple Lie algebras.

2. Main results

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{k}$ be a real compact form of $\mathfrak{g}$. Let G be a complex Lie group with Lie algebra $\mathfrak{g}$. It has a finite center so K (the analytic group of $\mathfrak{k}$) is compact. As a real K-module, $\mathfrak{g}$ is just the direct sum of two copies of the adjoint module $\mathfrak{k}$ of K: $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ (direct sum), that is, Cartan decomposition of $\mathfrak{g}$. Denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Given $x \in \mathfrak{g}$, consider the orbit of x under the adjoint action of K

$$K \cdot x := \{\operatorname{Ad}(k)x : k \in K\}.$$

The orbit $K \cdot x$ depends on $\operatorname{Ad}_G K$ which is the analytic subgroup of the adjoint group $\operatorname{Int}(\mathfrak{g}) \subset \operatorname{Aut}(\mathfrak{g})$ corresponding to $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{k})$. Thus $K \cdot x$ is independent of the choice of G. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. The complexification $\mathfrak{h} := \mathfrak{t} + i\mathfrak{t}$ (direct sum) is a Cartan subalgebra of $\mathfrak{g}$. The rank of $\mathfrak{g}$ is $\dim_{\mathbb{C}} \mathfrak{h}$, denoted by $\operatorname{rank} \mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$, where Δ denotes the set of all nonzero roots. Denote by $B(\cdot\,,\cdot)$ the Killing form of $\mathfrak{g}$. As $B(\cdot\,,\cdot)$ is a nondegenerate bilinear form, it induces a vector space isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ sending

$x \to \varphi_x$, where $\varphi_x(y) = B(x, y)$ for all $y \in \mathfrak{g}$. Denote the inverse by $\varphi \to H_\varphi \in \mathfrak{g}$ ($\varphi \in \mathfrak{g}^*$), where $B(H_\varphi, y) = \varphi(y)$ for all $y \in \mathfrak{g}$. Let

$$\mathfrak{h}_{\mathbb{R}} := \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha$$

so that $B(\cdot, \cdot)$ is a real inner product on $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} + i\mathfrak{h}_{\mathbb{R}}$ (direct sum). Hence rank $\mathfrak{g} = \dim_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$. Moreover $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}$ [Helgason 1978, p. 259]. Notice that $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ [Helgason 1978, p. 166] whenever $\alpha + \beta \neq 0$ ($\mathfrak{g}_0 = \mathfrak{h}$) so the sum

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

is orthogonal under the Killing form. Thus we have the orthogonal projection $\pi : \mathfrak{g} \to \mathfrak{h}$ under $B(\cdot, \cdot)$. For $x \in \mathfrak{g}$, we consider $\pi(K \cdot x)$, that is, the projection of $K \cdot x$ onto $\mathfrak{h}$. When $x \in \mathfrak{k}$, $K \cdot x \subset \mathfrak{k}$ so $\pi(K \cdot x) \subset \mathfrak{t}$.

Kostant [1973] generalized Theorem 1.1 in the context of real semisimple Lie algebras. The following statement is for complex semisimple case. When $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, it is reduced to Theorem 1.1.

Theorem 2.1 (Kostant [1973]). *If $x \in \mathfrak{k}$, then $\pi(K \cdot x) \subset \mathfrak{t}$ is convex and equals to* conv $W x_{\mathfrak{t}}$, *where $x_{\mathfrak{t}} \in K \cdot x \cap \mathfrak{t}$ and W is the Weyl group, that is, $W = N(T)/T$, the normalizer of T modulo T.*

Let θ be the Cartan involution of $\mathfrak{g}$ if $\mathfrak{g}$ is viewed as a real Lie algebra, that is, $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $x + y \mapsto x - y$ if $x \in \mathfrak{k}$ and $y \in i\mathfrak{k}$. In other words, $\mathfrak{k}$ is the $+1$ eigenspace of θ and $i\mathfrak{k}$ is the -1 eigenspace of θ. Though θ is not an automorphism of $\mathfrak{g}$ over $\mathbb{C}$ (since $\theta(cx) = \bar{c}\,\theta x$ for $c \in \mathbb{C}$ and $x \in \mathfrak{g}$), it respects the bracket, that is,

$$\theta[x, y] = [\theta x, \theta y], \qquad x, y \in \mathfrak{g}.$$

Moreover $\mathrm{Ad}(k)$ and θ commute for all $k \in K$. Since $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ and $\mathfrak{k}$ is compact,

$$B_\theta(x, y) := -B(x, \theta y)$$

is an inner product on $\mathfrak{g}$ over $\mathbb{C}$. Let

$$\|x\|_\theta := B_\theta^{1/2}(x, x)$$

be the induced norm on $\mathfrak{g}$. The projection $\pi : \mathfrak{g} \to \mathfrak{h}$ under $B(\cdot, \cdot)$ coincides with that under $B_\theta(\cdot, \cdot)$ since $\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$, for $\alpha \in \Delta$.

An element $x \in \mathfrak{g}$ is said to be *normal* if $[x, \theta x] = 0$, where θ is the Cartan involution. When $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the Cartan decomposition is the usual Hermitian decomposition, $K = \mathrm{SU}(n)$ and $\theta(z) = -z^*$, $z \in \mathfrak{sl}_n(\mathbb{C})$. When $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{k} = \mathfrak{su}(n)$, normality reduces to the usual notion of normality of a matrix.

We want to know when $\pi(K\cdot x)$ is convex, that is, the converse question of Theorem 2.1 when we restrict ourselves to normal $x\in\mathfrak{g}$. Djoković and Tam [2003] proved that $B_\theta(K\cdot x, y)\subset\mathbb{C}$ is star shaped with respect the origin for each $y\in\mathfrak{g}$, if $x\in\mathfrak{g}$ is normal. In particular $B_\theta(K\cdot x, x)$ is star shaped. We also want to know when $B_\theta(K\cdot x, x)$ is convex. It turns out their answers coincide as suggested by Theorems 1.2 and 1.3. Indeed it is equivalent to say that $B_\theta(K\cdot x, y)$ is convex for all $y\in\mathfrak{g}$ in the following theorem.

Theorem 2.2. *Let $\mathfrak{g}=\mathfrak{g}_1+\cdots+\mathfrak{g}_\ell$ be a complex semisimple Lie algebra with simple components $\mathfrak{g}_1,\dots,\mathfrak{g}_\ell$. Let $x=x_1+\cdots+x_\ell\in\mathfrak{g}$ be normal, where $x_i\in\mathfrak{g}_i$, $i=1,\dots,\ell$. The following statements are equivalent*:

(1) $\pi(K\cdot x)$ *is convex.*

(2) $B_\theta(K\cdot x, x)$ *is convex.*

(3) $B_\theta(K\cdot x, x)$ *is a closed line segment in* $\mathbb{R}$.

(4) $K_j\cdot e^{i\theta_j}x_j\cap\mathfrak{t}_j$ *is nonempty for some* $\theta_j\in[0,2\pi]$, $j=1,\dots,\ell$.

(5) $B_\theta(K\cdot x, y)$ *is convex for all* $y\in\mathfrak{g}$.

Remark 2.3. Normality of $x\in\mathfrak{g}$ is necessary. When $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{C})$ and $K=\mathrm{SU}(n)$, it is known that $B_\theta(K\cdot x, y)$ is convex for all $y\in\mathfrak{sl}_n(\mathbb{C})$ if $x\in\mathfrak{sl}_n(\mathbb{C})$ and the matrix rank of x is 1 (not necessarily normal), according to a result of Tsing [1984]. For example, if

$$x=\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus 0_{n-2}\,,$$

then $B_\theta(K\cdot x, y)$ is convex for all $y\in\mathfrak{sl}_n(\mathbb{C})$. However statement (3) in Theorem 2.2 does not hold.

We first establish some results in order to prove Theorem 2.2.

A line L is called a *support* of $B_\theta(K\cdot x, x)\subset\mathbb{C}$ at $\xi\in\partial B_\theta(K\cdot x, x)$ if $B_\theta(K\cdot x, x)$ lies in one of the closed half planes determined by L. A point $\xi\in B_\theta(K\cdot x, x)$ is called an *extreme point* of $B_\theta(K\cdot x, x)$ if ξ does not belong to any open line segment lying in $B_\theta(K\cdot x, x)$. It is clear that extreme points belong to $\partial B_\theta(K\cdot x, x)$. An extreme point $\xi\in B_\theta(K\cdot x, x)$ is called a *sharp point* if $B_\theta(K\cdot x, x)$ has more than one support line at ξ. Clearly a sharp point ξ of $B_\theta(K\cdot x, x)$ is an extreme point. The definitions are valid for convex sets in $\mathbb{C}$. The notions of extreme point and sharp point of a convex polygon in $\mathbb{C}$ coincide. We remark that $B_\theta(K\cdot x, x)$ is not necessarily a convex polygon.

Proposition 2.4. *Let $x\in\mathfrak{g}$ be normal.*

(a) $B_\theta(K\cdot x, x)\subset\mathbb{C}$ *is symmetric about the real axis.*

(b) *$B_\theta(K \cdot x, x) \subset \mathbb{C}$ is contained in the convex polygon*

$$B_\theta(\mathrm{conv}\, Wx_1 + i\,\mathrm{conv}\, Wx_2, x),$$

where $x = x_1 + ix_2$, $x_1, x_2 \in \mathfrak{k}$. Both sets contain the point $B_\theta(x, x) \geq 0$ which has the largest magnitude. Thus $B_\theta(x, x)$ is a sharp point of both $B_\theta(K \cdot x, x)$ and $B_\theta(\mathrm{conv}\, Wx_1 + i\,\mathrm{conv}\, Wx_2, x)$.

Proof. Since θ and $\mathrm{Ad}(k)$ ($k \in K$) commute, for $x, y \in \mathfrak{g}$,

$$B_\theta(\mathrm{Ad}(k)x, \mathrm{Ad}(k)y) = -B(\mathrm{Ad}(k)x, \mathrm{Ad}(k)\theta y) = B_\theta(x, y)$$

and hence $\mathrm{Ad}(k) : \mathfrak{g} \to \mathfrak{g}$ is an isometry with respect to $B_\theta(\cdot\,,\cdot)$.

(a) Let $x \in \mathfrak{g}$ be normal. Clearly

$$\overline{B_\theta(\mathrm{Ad}(k)x, x)} = B_\theta(x, \mathrm{Ad}(k)x) = B_\theta(\mathrm{Ad}(k^{-1})x, x).$$

Hence (a) is established.

(b) Since $x = x_1 + ix_2 \in \mathfrak{g}$ ($x_1, x_2 \in \mathfrak{k}$) is normal, $K \cdot x$ intersects $\mathfrak{h}$ [Djoković and Tam 2003, Lemma 3.3.14]. So we may assume that $x_1, x_2 \in \mathfrak{t}$. By Theorem 2.1

$$\begin{aligned}
\pi(K \cdot x) &= \pi(K \cdot (x_1 + ix_2)) \\
&\subset \pi(K \cdot x_1 + iK \cdot x_2) \\
&= \pi(K \cdot x_1) + i\pi(K \cdot x_2) \\
&= \mathrm{conv}\, Wx_1 + i\,\mathrm{conv}\, Wx_2,
\end{aligned}$$

where the sum $\mathrm{conv}\, Wx_1 + i\,\mathrm{conv}\, Wx_2$ is a convex polytope in $\mathfrak{h}$. Since $\pi : \mathfrak{g} \to \mathfrak{h}$ is also an orthogonal projection with respect to $B_\theta(\cdot\,,\cdot)$,

$$B_\theta(K \cdot x, x) = B_\theta(\pi(K \cdot x), x)$$

is contained in the convex polygon $B_\theta(\mathrm{conv}\, Wx_1 + i\,\mathrm{conv}\, Wx_2, x)$. Let

$$y \in \mathrm{conv}\, Wx_1 \subset \mathfrak{t} \quad \text{and} \quad z \in \mathrm{conv}\, Wx_2 \subset \mathfrak{t}.$$

Since $\mathfrak{h}_\mathbb{R} := \sum_{\alpha\in\Delta} \mathbb{R}H_\alpha = i\mathfrak{t}$ [Helgason 1978, p. 259] and $\alpha(H) \in \mathbb{R}$ for each $H \in \mathfrak{h}_\mathbb{R}$, $\alpha \in \Delta$, $\alpha(y), \alpha(x_1) \in i\mathbb{R}$ and $\alpha(iz), \alpha(ix_2) \in \mathbb{R}$. Hence

$$\alpha(\theta x) = -\overline{\alpha(x)}$$

so

$$\|x\|_\theta^2 = B_\theta(x, x) = \sum_{\alpha\in\Delta} |\alpha(x)|^2.$$

Moreover

$$\|y + iz\|_\theta^2 = \sum_{\alpha\in\Delta} \alpha(y + iz)\,\overline{\alpha(y + iz)} = \sum_{\alpha\in\Delta} (|\alpha(y)|^2 + |\alpha(iz)|^2) = \|y\|_\theta^2 + \|iz\|_\theta^2.$$

By Cauchy–Schwarz's inequality

$$|B_\theta(y+iz,x)|^2 \le \|y+iz\|_\theta^2\|x\|_\theta^2 = (\|y\|_\theta^2+\|iz\|_\theta^2)\|x\|_\theta^2. \tag{2-1}$$

Using triangle inequality, we have

$$\|y\|_\theta^2 \le \|x_1\|_\theta^2, \quad \|iz\|_\theta^2 \le \|ix_2\|_\theta^2, \tag{2-2}$$

since the elements in W are isometries. By (2-1) and (2-2)

$$|B_\theta(y+iz,x)|^2 \le B_\theta^2(x,x). \qquad \square$$

Remark 2.5. Given $x \in \mathfrak{h}$, $Wx \subset K\cdot x$ and thus $Wx \subset \pi(K\cdot x)$. We do not know whether $\pi(K\cdot x) \subset \operatorname{conv} Wx$ or not though it is true when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

Lemma 2.6. *Let $x \in \mathfrak{g}$ be normal. Then $B_\theta(K\cdot x,x)$ is convex if and only if it is a closed interval in $\mathbb{R}$.*

Proof. One implication is trivial. Suppose $B_\theta(K\cdot x,x)$ is convex and we may assume $x \ne 0$. By Proposition 2.4

$$\xi := B_\theta(x,x) = \|x\|_\theta$$

is a sharp point of $B_\theta(K\cdot x,x)$. There are two supporting lines passing through ξ and one is the reflection of the other by Proposition 2.4 (a). Clearly $B_\theta(K\cdot x,x)$ is inside the cone determined by the two lines. Let L be the upper supporting line for definiteness. So $B_\theta(K\cdot x,x)$ is in the lower half plane determined by L.

By [Djoković and Tam 2003, Lemma 3.14] we may assume that $x = x_1+ix_2 \in \mathfrak{h}$, $x_1,x_2 \in \mathfrak{t}$. Let $\xi_j := B_\theta(\mathrm{Ad}(k_j)x,x)$ ($k_j \in K$) be on the upper boundary of $B_\theta(K\cdot x,x)$ so that $|\xi-\xi_j| < 1/j$ but $\xi_j \ne \xi$, $j = 1,2,\dots$. Since K is compact, there is a convergent subsequence $\{k_{j_m}\}_{m=1}^\infty$ of $\{k_j\}_{j=1}^\infty$. Let $\lim_{m\to\infty} k_{j_m} = k_0 \in K$. So

$$B_\theta(\mathrm{Ad}(k_0)x,x) = \xi = B_\theta(x,x) = \|\mathrm{Ad}(k_0)x\|_\theta\|x\|_\theta$$

since $\mathrm{Ad}(k_0)$ is an isometry. By the equality case of Cauchy–Schwarz's inequality, $\mathrm{Ad}(k_0)x = x$. Thus

$$B_\theta(\mathrm{Ad}(k_j)x,x) = B_\theta(\mathrm{Ad}(k_j)x,\mathrm{Ad}(k_0)x) = B_\theta(\mathrm{Ad}(k_0^{-1}k_j)x,x).$$

We may replace k_{j_m} by $k_0^{-1}k_{j_m} \to e$ (the identity) or simply assume that $k_0 = e$. The exponential map is an analytic diffeomorphism between an open neighborhood of $0 \in \mathfrak{k}$ and an open neighborhood of $e \in K$. So for each sufficiently large m, there is $s_{j_m} \in \mathfrak{k}$ such that

$$\exp s_{j_m} = k_{j_m} \to e.$$

Since $x \in \mathfrak{h}$,

$$\begin{aligned}\xi_{j_m} &= B_\theta(\mathrm{Ad}(e^{s_{j_m}})x, x) = B_\theta(e^{\mathrm{ad}\, s_{j_m}} x, x) \\ &= B_\theta(x, x) + B_\theta(\mathrm{ad}\, s_{j_m} x, x) + \frac{1}{2} B_\theta((\mathrm{ad}\, s_{j_m})^2 x, x) \\ &\quad + \sum_{k=3}^{\infty} \frac{1}{k!} B_\theta((\mathrm{ad}\, s_{j_m})^k x, x).\end{aligned} \tag{2-3}$$

The first term of (2-3) is just ξ. The second term is

$$-B(\mathrm{ad}\,(s_{j_m})x, \theta x) = -B([s_{j_m}, x], \theta x) = -B(s_{j_m}, [x, \theta x]) = 0,$$

because $[x, \theta x] = 0$. Since the elements in $\mathrm{ad}\, \mathfrak{k}$ are skew Hermitian with respect to $B_\theta(\cdot\,, \cdot)$, the third term is

$$B_\theta((\mathrm{ad}\, s_{j_m})^2 x, x) = -B_\theta(\mathrm{ad}\, s_{j_m} x, \mathrm{ad}\,(s_{j_m})x) = -\|\mathrm{ad}\,(s_{j_m})x\|_\theta^2.$$

Taking the absolute value of the last term of (2-3), we have

$$\begin{aligned}\left|\sum_{k=3}^{\infty} \frac{1}{k!} B_\theta((\mathrm{ad}\, s_{j_m})^k x, x)\right| &= \left|\sum_{k=3}^{\infty} \frac{1}{k!} B_\theta(\mathrm{ad}\, s_{j_m} \circ (\mathrm{ad}\, s_{j_m})^{k-2} \circ \mathrm{ad}\,(s_{j_m})x, x)\right| \\ &= \sum_{k=3}^{\infty} \frac{1}{k!} |B_\theta((\mathrm{ad}\, s_{j_m})^{k-2} \circ \mathrm{ad}\,(s_{j_m})x, \mathrm{ad}\,(s_{j_m})x)| \\ &\le \sum_{k=3}^{\infty} \frac{1}{k!} \|(\mathrm{ad}\, s_{j_m})^{k-2} \mathrm{ad}\,(s_{j_m})x\|_\theta \|\mathrm{ad}\,(s_{j_m})x\|_\theta \\ &\le \sum_{k=3}^{\infty} \frac{1}{k!} \|(\mathrm{ad}\, s_{j_m})^{k-2}\|_\theta \|\mathrm{ad}\,(s_{j_m})x\|_\theta^2 \\ &\le (e^{\|\mathrm{ad}\, s_{j_m}\|} - 1)\|\mathrm{ad}\,(s_{j_m})x\|_\theta^2\,,\end{aligned}$$

where

$$\|\mathrm{ad}\, s_{j_m}\| := \max_{y \in \mathfrak{g}, \|y\|_\theta = 1} \|\mathrm{ad}\,(s_{j_m})y\|_\theta$$

is the operator norm of $\mathrm{ad}\, s_{j_m} : \mathfrak{g} \to \mathfrak{g}$ with respect to $\|\cdot\|_\theta$. Notice that $\mathrm{ad}\,(s_{j_m})x \neq 0$ otherwise $\xi = \xi_{j_m}$ from (2-3). Since $s_{j_m} \to 0$ ($x \neq 0$, $s_{j_m} \neq 0$),

$$\lim_{m \to \infty} \frac{\left|\sum_{k=3}^{\infty} \frac{1}{k!} B_\theta((\mathrm{ad}\, s_{j_m})^k x, x)\right|}{\|\mathrm{ad}\,(s_{j_m})x\|_\theta^2} = 0.$$

Consequently we have

$$\lim_{m \to \infty} \frac{\xi - \xi_{j_m}}{\|\mathrm{ad}\,(s_{j_m})x\|_\theta^2} = \frac{1}{2}. \tag{2-4}$$

Since $B_\theta(K\cdot x, x)$ is convex, there is $\xi' \in L\cap \partial B_\theta(K\cdot x, x)$ so that the line segment $[\xi, \xi'] \subset \partial B_\theta(K\cdot x, x)$. For sufficiently large m, $\xi_{j_m} \in [\xi, \xi']$, thus the limit on the left-hand side of (2-4) must be a positive multiple of $\xi - \xi'$. So $\xi' \in \mathbb{R}$ and thus $L \subset \mathbb{R}$. Therefore the compact connected set $B_\theta(K\cdot x, x)$ is a closed interval in $\mathbb{R}$. □

Proposition 2.7. *Let $\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_\ell$ be a complex semisimple Lie algebra with simple components $\mathfrak{g}_1, \dots, \mathfrak{g}_\ell$. Let $x, y \in \mathfrak{h} := \mathfrak{t} + i\mathfrak{t}$. Write $x = x_1 + \cdots + x_\ell$ and $y = y_1 + \cdots + y_\ell$, where $x_i, y_i \in \mathfrak{h}_i$, $i = 1, \dots, \ell$. Suppose that x_i, y_i are nonzero for all $i = 1, \dots, \ell$. Then the following statements are equivalent.*

(1) *$B_\theta(K\cdot x, y)$ is a (closed) line segment in $\mathbb{C}$.*

(2) *$B_\theta(W\cdot x, y)$ is on a line segment in $\mathbb{C}$, where W is the Weyl group.*

(3) *$K_j \cdot e^{i\theta_j} x_j \cap \mathfrak{t}_j$ and $K_j \cdot e^{i\rho_j} y_j \cap \mathfrak{t}_j$ are nonempty for some $\theta_j, \rho_j \in [0, 2\pi]$, $j = 1, \dots, \ell$, and $\kappa := \theta_j - \rho_j$ is a constant for all $j = 1, \dots, \ell$.*

Proof. (1) ⇒ (2) is trivial.

(3) ⇒ (1): We may assume that $e^{i\theta_j} x_j \in \mathfrak{t}_j$ and $e^{i\rho_j} x_j \in \mathfrak{t}_j$ since

$$B_\theta(K\cdot x, y) = B_\theta(K\cdot x, K\cdot y).$$

Now

$$\begin{aligned} B_\theta(K\cdot x, y) &= B_\theta(K_1\cdot x_1, y_1) + \cdots + B_\theta(K_\ell \cdot x_\ell, y_\ell) \\ &= e^{-i\kappa} \sum_{j=1}^{\ell} B_\theta(K_1 \cdot e^{i\theta_j} x_j, e^{i\rho_j} y_j) \end{aligned}$$

and each summand $B_\theta(K_1 \cdot e^{i\theta_j} x_j, e^{i\rho_j} y_j) \subset \mathbb{R}$.

(2) ⇒ (3): Suppose $B_\theta(Wx, y)$ is a (closed) line segment. By rotation on x or y we may assume that $B_\theta(Wx, y) \subset \mathbb{R}$. Since

$$B_\theta(Wx, y) = B_\theta(W_1 x_1, y_1) + \cdots + B_\theta(W_\ell x_\ell, y_\ell),$$

each $B_\theta(W_j x_j, y_j)$ is a real line segment, $j = 1, \dots, \ell$. So it suffices to consider simple $\mathfrak{g}_j$. To simplify notations, from now on we drop the index j from $\mathfrak{g}_j$, $\mathfrak{k}_j$, $\mathfrak{t}_j$, $\mathfrak{h}_j$, x_j, r_j and so on, or simply assume that $\mathfrak{g}$ is simple.

Notice that

$$\tau_{H_\beta}(H_\alpha) = H_\alpha - \frac{2B(H_\alpha, H_\beta)}{B(H_\beta, H_\beta)} H_\beta, \quad \alpha, \beta \in \Delta.$$

As a finite reflection group, the Weyl group W is generated by the reflections τ_{H_β}, $\beta \in \Delta$, and

$$B_\theta(Wx, \tau_{H_\beta} y) = B_\theta(Wx, y) \subset \mathbb{R}$$

so for all $\omega \in W$ and $\beta \in \Delta$,

$$B_\theta(\omega x, \tau_{H_\beta} y) = B_\theta\Big(\omega x, y - \frac{2\beta(y)}{\|\beta\|_\beta^2} H_\beta\Big)$$
$$= B_\theta(\omega x, y) - \frac{2\overline{\beta(y)}}{\|\beta\|_\beta^2} B_\theta(\omega x, H_\beta).$$

Hence for all $\beta \in \Delta$,

$$\frac{2\overline{\beta(y)}}{\|\beta\|_\theta^2} B_\theta(Wx, H_\beta) \subset \mathbb{R}$$

so either (a) $B_\theta(H_\beta, y) = \beta(y) = 0$ for all $\beta \in \Delta$, or (b) for some $\beta \in \Delta$ (depends on y), $\beta(y) \neq 0$, that is, $e^{i\gamma} B_\theta(Wx, H_\beta) \subset \mathbb{R}$ for some $\gamma \in \mathbb{R}$.

Since $\mathfrak{h} = \sum_{\beta\in\Delta} \mathbb{C} H_\beta$ and B is nondegenerate on $\mathfrak{h}$, (a) would not occur because we assume that $y \neq 0$. So (b) occurs, that is, $B_\theta(We^{i\gamma}x, H_\beta) \subset \mathbb{R}$. But then

$$B_\theta(WH_\beta, e^{i\gamma}x) = \overline{B_\theta(We^{i\gamma}x, H_\beta)} \subset \mathbb{R}.$$

Similarly for all $\alpha \in \Delta$,

$$\frac{2\overline{\alpha(e^{i\gamma}x)}}{\|H_\alpha\|_\theta^2} B_\theta(WH_\beta, H_\alpha) \subset \mathbb{R}.$$

Now $B_\theta(WH_\beta, H_\alpha) \subset \mathbb{R}$ since $H_\alpha, H_\beta \in \mathfrak{h}_\mathbb{R} = i\mathfrak{t}$. By contragradience the Weyl group permutes the roots. If $\omega \in W$ then $\omega H_\beta = H_{\omega\cdot\beta}$. We claim that

$$B_\theta(\omega H_\beta, H_\alpha) \neq 0, \text{ for some } \omega \in W.$$

It is because that the Weyl group acts simply transitively on each subset of roots of the same length [Helgason 1978, p. 523]. If H_α and H_β are of the same length, then $\omega H_\beta = H_\alpha$ for some $\omega \in W$ and $B_\theta(\omega H_\beta, H_\alpha) = \|H_\alpha\|_\theta^2 > 0$. Hence the claim follows immediately. When $\mathfrak{g} = \mathfrak{a}_n, \mathfrak{d}_n, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, all the roots are of the same length [Helgason 1978, p. 462–474]. Notice that

$$\mathfrak{b}_n : \Delta = \{\pm e_i \pm e_j : 1 \le i \neq j \le n\} \cup \{\pm e_i : 1 \le i \le n\},$$
$$\mathfrak{c}_n : \Delta = \{\pm e_i \pm e_j : 1 \le i \neq j \le n\} \cup \{\pm 2e_i : 1 \le i \le n\}, \text{ and}$$
$$\mathfrak{f}_4 : \Delta = \pm\{e_i\ (i = 1, \dots, 4);\ e_i \pm e_j\ (1 \le i < j \le 4);\ \tfrac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

For each case, the root length squares are either 1 or 2 and the claim is clearly true for them. Finally when $\mathfrak{g} = \mathfrak{g}_2$, the root length squares are either 2 or 6,

$$\Delta = \pm\{e_1 - e_2, e_2 - e_3, e_1 - e_3, 2e_1 - e_2 - e_3, 2e_2 - e_1 - e_3, 2e_3 - e_1 - e_2\}$$

and the claim is also true. As a result $\alpha(e^{i\gamma}x) \in \mathbb{R}$ for all $\alpha \in \Delta$ so $e^{i\gamma}x \in \mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}$. Similarly we have the same conclusion for y. Then clearly $\theta_j - \rho_j$ is a constant, $j = 1, \ldots, \ell$. □

Proof of Theorem 2.2. We first show that the first four statements are equivalent.

(1) ⇒ (2): We may assume that $x \in \mathfrak{h}$. We have $B_\theta(K \cdot x, x) = B_\theta(\pi(K \cdot x), x)$ and it is convex since $\pi(K \cdot x)$ is convex.

(2) ⇔ (3): Lemma 2.6.

(3) ⇒ (4): The case $x = 0$ is trivial. For $x \neq 0$, we may assume that each component $x_j \neq 0$ in the expression $x = x_1 + \cdots + x_\ell \in \mathfrak{g}$. Then apply Proposition 2.7.

(4) ⇒ (1): By Theorem 2.1.

(5) ⇒ (2): obvious.

(4) ⇒ (5): Let $y = y_1 + \cdots + y_\ell \in \mathfrak{g}_1 + \cdots + \mathfrak{g}_\ell$. Then

$$B_\theta(K \cdot x, y) = B_\theta(K_1 \cdot x_1, y_1) + \cdots + B_\theta(K_1 \cdot x_\ell, y_\ell).$$

By (4) there exist $k_j \in K_j$ and $\theta_j \in \mathbb{R}$ so that $t_j := e^{i\theta_j}\mathrm{Ad}(k_j)x_j \in \mathfrak{t}_j$ for each $j = 1, \ldots, \ell$. Write

$$y_j = y_j^{(1)} + iy_j^{(2)},$$

for $y_j^{(1)}, y_j^{(2)} \in \mathfrak{k}$. So

$$\begin{aligned} B_\theta(K_j \cdot x_j, y_j) &= e^{-i\theta_j} B_\theta(K \cdot t_j, y_j) \\ &= e^{-i\theta_j} \left\{ B\big(\mathrm{Ad}(k_j)t_j, y_j^{(1)}\big) + iB\big(\mathrm{Ad}(k_j)t_j, y_j^{(2)}\big) : k_j \in K_j \right\} \end{aligned}$$

which is convex by a result of Tam [2002]. Hence $B_\theta(K \cdot x, y)$ is a sum of convex sets and thus convex. □

Remark 2.8. The second author conjectured (see [Tam 2001, Conjecture 4.1]) that for a normal $x \in \mathfrak{g}$ (semisimple), if $B_\theta(K \cdot x, x)$ is convex, then there is $\gamma \in \mathbb{R}$ such that $e^{i\gamma}x \in \mathfrak{t}$. It is not true in view of Theorem 2.2. Consider the semisimple $\mathfrak{g} := \mathfrak{a}_1 \times \mathfrak{a}_1$. To be concrete, let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$ with $K = \mathrm{SU}(2) \oplus \mathrm{SU}(2)$. Consider the normal $x = \mathrm{diag}\,(x_1, -x_1) \oplus \mathrm{diag}\,(x_2, -x_2)$, where $x_1, x_2 \in \mathbb{C}$. Then for $k = k_1 \oplus k_2 \in K$,

$$\begin{aligned} \mathrm{tr}\, kxk^{-1}x^* = {}& \mathrm{tr}\, k_1 \mathrm{diag}\,(x_1, -x_1)k_1^{-1}\mathrm{diag}\,(\bar{x}_1, -\bar{x}_1) \\ &+ \mathrm{tr}\, k_2 \mathrm{diag}\,(x_2, -x_2)k_2^{-1}\mathrm{diag}\,(\bar{x}_2, -\bar{x}_2). \end{aligned}$$

By Theorem 1.3 the set

$$\left\{ \mathrm{tr}\, k_i \mathrm{diag}\,(x_i, -x_i)k_1^{-1}\mathrm{diag}\,(\bar{x}_i, -\bar{x}_i) : k_i \in \mathrm{SU}(n) \right\}$$

is convex, $i = 1, 2$, so $\{\mathrm{tr}\, kxk^{-1}x^* : k \in K\}$ is the sum of two convex sets and thus is convex. However, x_1, x_2 need not be collinear with 0.

By Proposition 2.4 (a) $B_\theta(K \cdot x, x)$ is symmetric about the real axis. For some simple Lie algebras, more symmetry occurs for $B_\theta(K \cdot x, x)$ if $x \in \mathfrak{g}$ is normal. Indeed the symmetry is also true for $B_\theta(K \cdot x, y)$ for each $y \in \mathfrak{g}$.

Proposition 2.9. *Let $\mathfrak{g}$ be simple and of type $\mathfrak{b}_\ell$, $\mathfrak{c}_\ell$, $\mathfrak{d}_\ell$ (ℓ even), $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$. Let $x \in \mathfrak{g}$ be normal. The sets $\pi(K \cdot x) \subset \mathfrak{h}$ and $B_\theta(K \cdot x, y) \subset \mathbb{C}$ are symmetric about the origin for each $y \in \mathfrak{g}$.*

Proof. We may assume that $x \in \mathfrak{h}$. The Weyl group W contains -1 [Helgason 1978, p. 523] so the desired result follows. □

It is known [Djoković and Tam 2003] that if $x \in \mathfrak{g}$ is normal, then $B_\theta(K \cdot x, y)$ is star-shaped with respect to the center 0 for each $y \in \mathfrak{g}$.

We do not know whether $\pi(K \cdot x)$ is star shaped or not and the following conjectures [Tam 2001] are still open.

Conjecture 2.10. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If $x, y \in \mathfrak{g}$, then $B_\theta(K \cdot x, y)$ is star-shaped with respect to the star center* 0.

Conjecture 2.11. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If $x \in \mathfrak{g}$, then $\pi(K \cdot x)$ is star-shaped with respect to the star center* 0.

We remark that these conjectures can be reduced to the simple cases. The cases $\mathfrak{a}_\ell$ ($\ell \geq 1$), $\mathfrak{d}_\ell$ ($\ell \geq 2$), $\mathfrak{e}_6$, $\mathfrak{e}_7$ for Conjecture 2.10 are true [Cheung and Tsing 1996; Djoković and Tam 2003].

Added in proof

The authors very recently proved Conjecture 2.11 affirmatively.

References

[Au-Yeung and Sing 1977] Y. H. Au-Yeung and F. Y. Sing, "A remark on the generalized numerical range of a normal matrix", *Glasgow Math. J.* **18**:2 (1977), 179–180. MR 56 #8594 Zbl 0359.47001

[Au-Yeung and Tsing 1983] Y. H. Au-Yeung and N.-K. Tsing, "A conjecture of Marcus on the generalized numerical range", *Linear and Multilinear Algebra* **14**:3 (1983), 235–239. MR 85e:15029 Zbl 0521.15016

[Bebiano and Da Providência 1996] N. Bebiano and J. Da Providência, "Another proof of a conjecture of Marcus on the c-numerical range", *Linear and Multilinear Algebra* **41**:1 (1996), 35–40. MR 97g:15029 Zbl 0871.15028

[Cheung and Tsing 1996] W.-S. Cheung and N.-K. Tsing, "The C-numerical range of matrices is star-shaped", *Linear and Multilinear Algebra* **41**:3 (1996), 245–250. MR 97k:15060 Zbl 0876.15021

[Djoković and Tam 2003] D. Z. Djoković and T.-Y. Tam, "Some questions about semisimple Lie groups originating in matrix theory", *Canad. Math. Bull.* **46**:3 (2003), 332–343. MR 2004g:22008 Zbl 1047.22013

[Helgason 1978] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics **80**, Academic Press, New York, 1978. MR 80k:53081 Zbl 0451.53038

[Horn 1954] A. Horn, "Doubly stochastic matrices and the diagonal of a rotation matrix", *Amer. J. Math.* **76** (1954), 620–630. MR 16,105c Zbl 0055.24601

[Kostant 1973] B. Kostant, "On convexity, the Weyl group and the Iwasawa decomposition", *Ann. Sci. École Norm. Sup.* (4) **6** (1973), 413–455 (1974). MR 51 #806 Zbl 0293.22019

[Marcus 1979] M. Marcus, "Some combinatorial aspects of numerical range", *Ann. New York Acad. Sci.* **319** (1979), 368–376. MR 81g:15028 Zbl 0483.15017

[Schur 1923] I. Schur, "Über eine Klasse von Mittelbildungen mit Anwendungen auf der Determinantentheorie", *Sitzungsber. Berl. Math. Ges.* **22** (1923), 9–20. JFM 49.0054.01

[Tam 2001] T.-Y. Tam, "On the shape of numerical ranges associated with Lie groups", *Taiwanese J. Math.* **5**:3 (2001), 497–506. MR 2002f:15043 Zbl 0986.15022

[Tam 2002] T.-Y. Tam, "Convexity of generalized numerical range associated with a compact Lie group", *J. Aust. Math. Soc.* **72**:1 (2002), 57–66. MR 2002g:15056 Zbl 1007.15021

[Tsing 1981] N. K. Tsing, "On the shape of the generalized numerical ranges", *Linear and Multilinear Algebra* **10**:3 (1981), 173–182. MR 82m:15026 Zbl 0471.47002

[Tsing 1984] N.-K. Tsing, "The constrained bilinear form and the C-numerical range", *Linear Algebra Appl.* **56** (1984), 195–206. MR 85b:15033 Zbl 0528.15007

[Westwick 1975] R. Westwick, "A theorem on numerical range", *Linear and Multilinear Algebra* **2** (1975), 311–315. MR 51 #11132 Zbl 0303.47001

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