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**TWO CLASSES OF PSEUDOSYMMETRIC CONTACT METRIC
3-MANIFOLDS**

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TWO CLASSES OF PSEUDOSYMMETRIC CONTACT METRIC 3-MANIFOLDS

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We classify the pseudosymmetric contact metric 3-manifolds that satisfy $\nabla_{\xi}\tau = 0$, and also the pseudosymmetric contact metric 3-manifolds of constant type satisfying $\nabla_{\xi}\tau = 2a\tau\phi$, where a is a smooth function.

1. Introduction

According to R. Deszcz [1992], a Riemannian manifold (M^m, g) is pseudosymmetric if the curvature tensor R satisfies the condition $R(X, Y) \cdot R = L\{(X \wedge Y) \cdot R\}$, where L is a smooth function, the endomorphism field $X \wedge Y$ is defined by

$$(1-1) \quad (X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y$$

for all vectors fields X, Y, Z on M , and the dot means that $R(X, Y)$ and $X \wedge Y$ act as derivations on R .

If L is constant, M said to be a pseudosymmetric manifold of constant type; if $L = 0$, then M is a semisymmetric manifold. Hence a pseudosymmetric manifold is a natural generalization of a semisymmetric manifold [Szabó 1982; 1985], which in turn is a generalization of a locally symmetric space, that is, one with $\nabla R = 0$; see [Takagi 1972].

Three-dimensional pseudosymmetric spaces of constant type have been studied by many researchers, beginning with O. Kowalski and M. Sekizawa [1996b; 1996a; 1997; 1998]. Later, N. Hashimoto and M. Sekizawa classified 3-dimensional, conformally flat pseudosymmetric spaces of constant type [2000], while G. Calvaruso gave the complete classification of conformally flat pseudosymmetric spaces of constant type for dimensions greater than two [2006]. J. T. Cho and J. Inoguchi studied pseudosymmetric contact homogeneous 3-manifolds [2005].

It is well known that in the geometry of a contact metric manifold, the tensors $\tau = L_{\xi}g$ and $\nabla_{\xi}\tau$, introduced by S. S. Chern and R. S. Hamilton [1985], play a fundamental role. The condition $\nabla_{\xi}\tau = 2a\tau\phi$, where a is a constant and $(\tau\phi)(X, Y)$ is

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interpreted as $\tau(\phi X, Y)$, is necessary for a contact metric 3-manifold to be homogeneous. We call a 3-dimensional contact metric manifold a 3- τ - a manifold if it satisfies $\nabla_{\xi}\tau = 2a\tau\phi$, where a is a smooth function; if $a = 0$, we call it a 3- τ manifold. The condition $\nabla_{\xi}\tau = 0$ appeared first in [Chern and Hamilton 1985] in the study of compact contact 3-manifolds, while Perrone [1990] proved that it is the critical point condition for the functional “integral of the scalar curvature” defined on the set of all metrics associated to the fixed contact form η . Moreover, this condition $\nabla_{\xi}\tau = 0$ is equivalent to the condition requiring equality of the sectional curvature of all planes at a given point and perpendicular to the contact distribution [Gouli-Andreou and Xenos 1998a].

This article studies contact metric 3-manifolds in which

- (i) M is a pseudosymmetric manifold and $\nabla_{\xi}\tau = 0$, where $\tau = L_{\xi}g$; or
- (ii) M is a pseudosymmetric manifold of constant type with $\nabla_{\xi}\tau = 2a\tau\phi$, where a is a smooth function on M .

2. Preliminaries

Let (M^m, g) for $m \geq 3$ be a connected Riemannian smooth manifold. We denote by ∇ the Levi-Civita connection of M^m and by R the corresponding Riemannian curvature tensor given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$.

A Riemannian manifold (M^m, g) for $m \geq 3$ is said to be *pseudosymmetric* in the sense of R. Deszcz [1992] if at every point of M the curvature tensor satisfies the equation

$$(2-1) \quad (R(X, Y) \cdot R)(X_1, X_2, X_3) = L\{((X \wedge Y) \cdot R)(X_1, X_2, X_3)\},$$

where

$$(2-2) \quad (R(X, Y) \cdot R)(X_1, X_2, X_3) = \\ R(X, Y)(R(X_1, X_2)X_3) - R(R(X, Y)X_1, X_2)X_3 \\ - R(X_1, R(X, Y)X_2)X_3 - R(X_1, X_2)(R(X, Y)X_3),$$

$$(2-3) \quad ((X \wedge Y) \cdot R)(X_1, X_2, X_3) = \\ (X \wedge Y)(R(X_1, X_2)X_3) - R((X \wedge Y)X_1, X_2)X_3 \\ - R(X_1, (X \wedge Y)X_2)X_3 - R(X_1, X_2)((X \wedge Y)X_3),$$

and $X \wedge Y$ is given by (1-1). In particular, if $L = 0$, then M is semisymmetric. For details and examples of pseudosymmetric manifolds, see [Belkhef et al. 2002] and [Deszcz 1992].

A contact manifold is a differentiable manifold M^{2n+1} together with a global 1-form η (a *contact form*) such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Since $d\eta$ is of rank $2n$, there exists a unique vector field ξ on M^{2n+1} (the *Reeb* or *characteristic*

vector field of the contact structure η) satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for all X . The distribution defined by the subspace $X \in T_p M : \eta(X) = 0$ for $p \in M$ is called a *contact distribution*. Every contact manifold has an underlying *almost contact structure* (η, ϕ, ξ) , where ϕ is a global tensor field of type $(1, 1)$, such that

$$(2-4) \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \phi^2 = -I + \eta \otimes \xi.$$

A Riemannian metric g (the *associated metric*) can be defined such that

$$(2-5) \quad \eta(X) = g(X, \xi) \quad \text{and} \quad d\eta(X, Y) = g(X, \phi Y)$$

for all vector fields X and Y on M^{2n+1} . We note that g and ϕ are not unique for a given contact form η , but g and ϕ are canonically related to each other. We refer to $(M^{2n+1}, \eta, \xi, \phi, g)$ as a *contact metric structure*.

We denote by S the Ricci tensor of type $(0, 2)$, by Q the corresponding Ricci operator satisfying $g(QX, Y) = S(X, Y)$, and by $r = \text{Tr } Q$ the scalar curvature. We also define the tensor fields l, h and τ by the relations

$$(2-6) \quad l = R(\cdot, \xi)\xi, \quad h = \frac{1}{2}L_\xi\phi, \quad \tau = L_\xi g,$$

where L is the Lie differentiation. On every contact metric manifold M^{2n+1} , we have the important formulas

$$(2-7) \quad h\xi = l\xi = 0, \quad \eta \circ h = 0, \quad \text{Tr } h = \text{Tr } h\phi = 0, \quad h\phi = -\phi h,$$

$$(2-8) \quad hX = \lambda X \quad \text{implies} \quad h\phi X = -\lambda\phi X,$$

$$(2-9) \quad \nabla_\xi\phi = 0, \quad \nabla_X\xi = -\phi X - \phi hX, \quad \text{Tr } l = g(Q\xi, \xi) = 2n - \text{Tr } h^2,$$

$$(2-10) \quad \tau = 2g(\phi\cdot, h\cdot), \quad \nabla_\xi\tau = 2g(\phi\cdot, \nabla_\xi h\cdot).$$

A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field, that is, for which $L_\xi g = 0$, is called a K-contact manifold. A contact metric manifold is K-contact if and only if $\tau = 0$ (or equivalently $h = 0$).

If we take the product $M^{2n+1} \times \mathbb{R}$, then the contact structure on M^{2n+1} gives rise to an almost complex structure J on $M^{2n+1} \times \mathbb{R}$ given by

$$J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt}).$$

If this structure is integrable, then the contact structure is said to be normal and M^{2n+1} is called Sasakian. A contact metric manifold is Sasakian if and only if $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$ for all vectors fields X, Y on the manifold. If $\dim M^{2n+1} = 3$ then a K-contact manifold is Sasakian and the converse also holds. More details on contact manifolds are found in [Blair 2002].

Let (M, ϕ, ξ, η, g) be a contact metric 3-manifold. Let U be the open subset of points $p \in M$ such that $h \neq 0$ in a neighborhood of p . Let U_0 be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p . That h is a smooth function

on M implies $U \cup U_0$ is an open and dense subset of M , so any property satisfied in $U_0 \cup U$ is also satisfied in M . For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p (this we call a ϕ -basis). On U , we put $he = \lambda e$, where λ is a nonvanishing smooth function assumed to be positive. From (2-8) we have $h\phi e = -\lambda\phi e$.

Lemma 2.1 [Calvaruso and Perrone 2002; Gouli-Andreou and Xenos 1998a]. *On the open set U we have*

$$\begin{aligned}
 \nabla_\xi e &= a\phi e, & \nabla_e e &= b\phi e, & \nabla_{\phi e} e &= -c\phi e + (\lambda - 1)\xi, \\
 \nabla_\xi \phi e &= -ae, & \nabla_e \phi e &= -be + (1 + \lambda)\xi, & \nabla_{\phi e} \phi e &= ce, \\
 \nabla_\xi \xi &= 0, & \nabla_e \xi &= -(1 + \lambda)\phi e, & \nabla_{\phi e} \xi &= (1 - \lambda)e, \\
 (2-11) \quad \nabla_\xi h &= -2ah\phi + (\xi \cdot \lambda)s
 \end{aligned}$$

where a is a smooth function,

$$\begin{aligned}
 (2-12) \quad b &= \frac{1}{2\lambda}((\phi e \cdot \lambda) + A) \quad \text{with } A = \eta(Qe) = S(\xi, e), \\
 c &= \frac{1}{2\lambda}((e \cdot \lambda) + B) \quad \text{with } B = \eta(Q\phi e) = S(\xi, \phi e),
 \end{aligned}$$

and s is the type $(1, 1)$ tensor field defined by $s\xi = 0$, $se = e$ and $s\phi e = -\phi e$.

From Lemma 2.1 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$, we can prove that

$$\begin{aligned}
 (2-13) \quad [e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi, \\
 [e, \xi] &= \nabla_e \xi - \nabla_\xi e = -(a + \lambda + 1)\phi e, \\
 [\phi e, \xi] &= \nabla_{\phi e} \xi - \nabla_\xi \phi e = (a - \lambda + 1)e,
 \end{aligned}$$

and from (1-1) we estimate

$$\begin{aligned}
 (2-14) \quad (e \wedge \phi e)e &= -\phi e, & (e \wedge \xi)e &= -\xi, & (\phi e \wedge \xi)\xi &= \phi e, \\
 (e \wedge \phi e)\phi e &= e, & (e \wedge \xi)\xi &= e, & (\phi e \wedge \xi)\phi e &= -\xi,
 \end{aligned}$$

while $(X \wedge Y)Z = 0$ whenever $X \neq Y \neq Z \neq X$ and $X, Y, Z \in \{e, \phi e, \xi\}$.

By direct computations, we calculate the nonvanishing independent components of the type $(1, 3)$ Riemannian curvature tensor field R :

$$\begin{aligned}
 (2-15) \quad R(\xi, e)\xi &= -Ie - Z\phi e, & R(e, \phi e)e &= -C\phi e - B\xi, \\
 R(\xi, \phi e)\xi &= -Ze - D\phi e, & R(\xi, e)\phi e &= -Ke + Z\xi, \\
 R(e, \phi e)\xi &= Be - A\phi e, & R(\xi, \phi e)\phi e &= He + D\xi, \\
 R(\xi, e)e &= K\phi e + I\xi, & R(e, \phi e)\phi e &= Ce + A\xi, \\
 R(\xi, \phi e)e &= -H\phi e + Z\xi,
 \end{aligned}$$

where

$$\begin{aligned}
 (2-16) \quad C &= -b^2 - c^2 + \lambda^2 - 1 + 2a + (e \cdot c) + (\phi e \cdot b), & Z &= \xi \cdot \lambda, \\
 H &= b(\lambda - a - 1) + (\xi \cdot c) + (\phi e \cdot a), & I &= -2a\lambda - \lambda^2 + 1, \\
 K &= c(\lambda + a + 1) + (\xi \cdot b) - (e \cdot a), & D &= 2a\lambda - \lambda^2 + 1.
 \end{aligned}$$

Setting $X = e$, $Y = \phi e$ and $Z = \xi$ in the Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ and using (2-13), we get

$$\begin{aligned}
 (2-17) \quad b(a + \lambda + 1) - (\xi \cdot c) - (\phi e \cdot \lambda) - (\phi e \cdot a) &= 0, \\
 c(a - \lambda + 1) + (\xi \cdot b) + (e \cdot \lambda) - (e \cdot a) &= 0,
 \end{aligned}$$

or equivalently $A = H$ and $B = K$.

We give the components of the Ricci operator Q with respect to a ϕ -basis:

$$\begin{aligned}
 (2-18) \quad Qe &= (\frac{1}{2}r - 1 + \lambda^2 - 2a\lambda)e + Z\phi e + A\xi, \\
 Q\phi e &= Ze + (\frac{1}{2}r - 1 + \lambda^2 + 2a\lambda)\phi e + B\xi, \\
 Q\xi &= Ae + B\phi e + 2(1 - \lambda^2)\xi,
 \end{aligned}$$

where

$$(2-19) \quad r = \text{Tr } Q = 2(1 - \lambda^2 - b^2 - c^2 + 2a + (e \cdot c) + (\phi e \cdot b)).$$

The relations (2-16) and (2-19) yield

$$(2-20) \quad C = -b^2 - c^2 + \lambda^2 - 1 + 2a + (e \cdot c) + (\phi e \cdot b) = 2\lambda^2 - 2 + r/2$$

Definition 2.2 [Gouli-Andreou et al. 2008]. Let M^3 be a 3-dimensional contact metric manifold. Let $h = \lambda h^+ - \lambda h^-$ be the spectral decomposition of h on U . If

$$\nabla_{h^- X} h^- X = [\xi, h^+ X]$$

for all vector fields X on M^3 and all points of an open subset W of U , and if $h = 0$ on the points of M^3 that do not belong to W , then the manifold is said to be a *semi-K contact* manifold.

From Lemma 2.1 and the relations (2-13), the condition above for $X = e$ leads to $[\xi, e] = 0$; for $X = \phi e$ it leads to $\nabla_{\phi e} \phi e = 0$. Hence on a semi-K contact manifold we have $a + \lambda + 1 = c = 0$. If we apply the deformation $e \rightarrow \phi e$, $\phi e \rightarrow e$, $\xi \rightarrow -\xi$, $\lambda \rightarrow -\lambda$, $b \rightarrow c$ and $c \rightarrow b$, then the contact metric structure remains the same. Hence the condition for a 3-dimensional contact metric manifold to be semi-K contact is equivalent to $a - \lambda + 1 = b = 0$.

Remark 2.3. If $M^3 = U_0$ (as in [Gouli-Andreou and Xenos 1998b]), Lemma 2.1 is expressed in a similar form, where $\lambda = 0$, e is a unit vector field belonging to

the contact distribution, the Equation (2-11) is identically zero, and the functions A, B, D, H, I, K and Z satisfy

$$A = B = Z = H = K = 0, \quad I = D = 1, \quad C = \frac{1}{2}r - 2.$$

Proposition 2.4. *For a 3-dimensional contact metric manifold, we have*

$$(2-21) \quad Q\phi = \phi Q \quad \text{if and only if} \quad \xi \cdot \lambda = 2b\lambda - (\phi e \cdot \lambda) = 2c\lambda - (e \cdot \lambda) = a\lambda = 0.$$

Proof. By (2-4), (2-12), (2-16) and (2-20), the relations (2-18) yield

$$\begin{aligned} (Q\phi - \phi Q)e &= 2Ze + 4a\lambda\phi e + B\xi, \\ (Q\phi - \phi Q)\phi e &= 4a\lambda e - 2Z\phi e - A\xi, \\ (Q\phi - \phi Q)\xi &= Be - A\phi e. \end{aligned}$$

Proposition 2.4 follows immediately. \square

3. Pseudosymmetric contact metric 3-manifolds

Let (M, η, g, ϕ, ξ) be a contact metric 3-manifold. In the case $M = U_0$, that is, when (ξ, η, ϕ, g) is a Sasakian structure, M is a pseudosymmetric space of constant type [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ -basis as in Lemma 2.1.

Lemma 3.1. *A contact metric 3-manifold (M, η, g, ϕ, ξ) is pseudosymmetric if and only if*

$$\begin{aligned} B(\xi \cdot \lambda) + (-2a\lambda - \lambda^2 + 1)A &= LA, \\ A(\xi \cdot \lambda) + (2a\lambda - \lambda^2 + 1)B &= LB, \\ (\xi \cdot \lambda)(\frac{1}{2}r + 2\lambda^2 - 2) + AB &= L(\xi \cdot \lambda), \end{aligned}$$

and

$$\begin{aligned} A^2 - |(\xi \cdot \lambda)|^2 + (2a\lambda - \lambda^2 + 1)(-2a\lambda - 3\lambda^2 + 3 - \frac{1}{2}r) &= L(-2a\lambda - 3\lambda^2 + 3 - \frac{1}{2}r), \\ B^2 - |(\xi \cdot \lambda)|^2 + (-2a\lambda - \lambda^2 + 1)(2a\lambda - 3\lambda^2 + 3 - \frac{1}{2}r) &= L(2a\lambda - 3\lambda^2 + 3 - \frac{1}{2}r), \end{aligned}$$

where L is the function in the pseudosymmetry definition (2-1).

Proof. Setting $X_1 = e$, $X_2 = \phi e$ and $X_3 = \xi$ in Equation (2-1), we obtain

$$(R(X, Y) \cdot R)(e, \phi e, \xi) = L(((X \wedge Y) \cdot R)(e, \phi e, \xi)).$$

First we set $X = e$ and $Y = \phi e$. By virtue of (2-2), (2-3), (2-14) and (2-15) we obtain

$$(B(\xi \cdot \lambda) + (-2a\lambda - \lambda^2 + 1)A)e + (A(\xi \cdot \lambda) + (2a\lambda - \lambda^2 + 1)B)\phi e = L(Ae + B\phi e),$$

from which the first two equations of the lemma follow at once.

Similarly, setting $X = \phi e$ and $Y = \xi$ we obtain

$$(A^2 - |(\xi \cdot \lambda)|^2 + (2a\lambda - \lambda^2 + 1)(-2a\lambda - 3\lambda^2 + 2 - \frac{1}{2}r))e + ((\xi \cdot \lambda)(\frac{1}{2}r + 2\lambda^2 - 2) + AB)\phi e = L((-2a\lambda - 3\lambda^2 + 2 - \frac{1}{2}r)e + (\xi \cdot \lambda)\phi e),$$

from which we get the next two equations.

Finally, setting $X = e$ and $Y = \xi$ we have

$$(B^2 - |(\xi \cdot \lambda)|^2 + (-2a\lambda - \lambda^2 + 1)(2a\lambda - 3\lambda^2 + 2 - \frac{1}{2}r))\phi e + ((\xi \cdot \lambda)(\frac{1}{2}r + 2\lambda^2 - 2) + AB)e = L((2a\lambda - 3\lambda^2 + 2 - \frac{1}{2}r)\phi e + (\xi \cdot \lambda)e),$$

from which we obtain the last equation. □

Using Equations (2-16) and (2-20), the five equations take the more convenient form

$$(3-1) \quad \begin{aligned} ZB + IA &= LA, \\ ZA + DB &= LB, & A^2 - Z^2 + D(I - C) &= L(I - C), \\ ZC + AB &= LZ, & B^2 - Z^2 + I(D - C) &= L(D - C). \end{aligned}$$

Remark 3.2. If $L = 0$, the manifold is semisymmetric and (3-1) is in accordance with [Calvaruso and Perrone 2002, Equations (3.1)–(3.5)].

Proposition 3.3. *Let M^3 be a 3-dimensional contact metric manifold satisfying $Q\phi = \phi Q$. Then M^3 is a pseudosymmetric space of constant type.*

Proof. In [2005], Cho and Inoguchi have proved that contact metric 3-manifolds that satisfy $Q\phi = \phi Q$ are pseudosymmetric. We can improve their result by proving that these manifolds are also pseudosymmetric of constant type. We know from [Blair et al. 1990] that for these manifolds the Ricci operator has the form $QX = \alpha X + \beta\eta(X)\xi$ or equivalently the Ricci tensor is given by the equation

$$(3-2) \quad S = \alpha g + \beta\eta \otimes \eta,$$

where $\alpha = (r - \text{Tr} l)/2$, $\beta = (3 \text{Tr} l - r)/2$ and the ϕ -sectional curvature and $\text{Tr} l$ are both constant functions. Also, from [Koufogiorgos 1995] we have that the ϕ -sectional curvature is given by the equation $r/2 - \text{Tr} l$, and hence in contact metric 3-manifolds with $Q\phi = \phi Q$, the function $r = \text{Tr} Q$ is also constant; obviously the functions α and β in (3-2) are constant as well. The manifold is quasi-Einstein and hence pseudosymmetric, and because β is constant it is pseudosymmetric of constant type, that is, L is constant; see also [Cho and Inoguchi 2005]. □

Remark 3.4. If the manifold M^3 is Sasakian, we know from Cho and Inoguchi [2005] that M^3 is a pseudosymmetric space of constant type. Also, by using Remark 2.3, the system (3-1) is reduced to the equation $(C - 1)(L - 1) = 0$. Hence

a Sasakian 3-manifold satisfying the condition $R(X, Y) \cdot R = L((X \wedge Y) \cdot R)$ with $L \neq 1$ is a space of constant scalar curvature $r = 6$, where L is some constant function on M^3 .

4. Pseudosymmetric contact metric 3-manifolds with $\nabla_\xi \tau = 0$

Theorem 4.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold satisfying $\nabla_\xi \tau = 0$. Then M^3 is of constant type, and it is either Sasakian, flat, or locally isometric to either $SU(2)$ or $SL(2, R)$, where these two Lie groups are equipped with a left invariant metric.*

Proof. We consider the open subsets

$$(4-1) \quad \begin{aligned} U_0 &= \{p \in M : \lambda = 0 \text{ in a neighborhood of } p\}, \\ U &= \{p \in M : \lambda \neq 0 \text{ in a neighborhood of } p\} \end{aligned}$$

of M . Suppose $M = U_0$, that is, (ξ, η, ϕ, g) is a Sasakian structure. In [2005], Cho and Inoguchi proved that M is a pseudosymmetric space of constant type.

If U is not empty, let $\{e, \phi e, \xi\}$ be a ϕ -basis. In contact metric 3-manifolds, the assumption $\nabla_\xi \tau = 0$ is equivalent to $a = Z = 0$; see [Gouli-Andreou and Xenos 1998a]. Hence (3-1) becomes

$$\begin{aligned} AB &= 0 \\ B(D - L) &= 0, & A^2 + D(D - C) &= L(D - C), \\ A(D - L) &= 0, & B^2 &= A^2, \end{aligned}$$

or equivalently

$$(4-2) \quad \begin{aligned} Z &= A = B = a = 0, \\ (D - L)(D - C) &= 0, \end{aligned}$$

where the functions A, B, C, D, I and Z are given by (2-12) and (2-16). Using Proposition 2.4 we obtain $Q\phi = \phi Q$, and hence the manifold is pseudosymmetric of constant type (by [Cho and Inoguchi 2005] and Proposition 3.3). The equation $(D - L)(D - C) = 0$ does not contribute any further information to our problem, and hence the proof is completed by [Blair et al. 1990, Theorem 3.3] and [Blair and Chen 1992, Main Theorem]. \square

5. Pseudosymmetric contact metric 3-manifolds of constant type with $\nabla_\xi \tau = 2a\tau\phi$

Theorem 5.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold of constant type satisfying the condition $\nabla_\xi \tau = 2a\tau\phi$, where a is a smooth function. Then the functions a and $\text{Tr } l$ are constant on M^3 , and M^3 is either*

- Sasakian;
- flat;
- locally isometric to either $SU(2)$ or $SL(2, R)$, where both Lie groups are equipped with a left invariant metric;
- semi-K contact of constant type $L = -4$, with $\text{Tr} l = 0$ and with constant scalar curvature $r = -4$;
- semi-K contact of constant type $L = a^2$, with $\text{Tr} l = -2a(2 + a)$;
- of constant scalar curvature $r = 4a$ and of constant type $L = 2a$, with $\text{Tr} l = 0$;
- of constant scalar curvature $r = 2a(4 - a)$ and of constant type $L = a^2$, with $\text{Tr} l = 2a(2 - a)$;
- of constant scalar curvature $r = 2a(4 - a)$ and of constant type $L = \text{Tr} l - a^2$, with $\text{Tr} l = 2a(2 - a)$.

Proof. We consider again the open subsets of (4-1). If $M = U_0$, then M is Sasakian and hence it is a pseudosymmetric space of constant type [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ -basis. Using (2-10) and (2-11), we can prove that the assumption $\nabla_{\xi} \tau = 2a\tau\phi$ is equivalent to $Z = \xi \cdot \lambda = 0$, and hence the system (3-1) becomes

$$\begin{aligned} AB &= 0, \\ DB &= LB, & A^2 + D(I - C) &= L(I - C), \\ IA &= LA, & B^2 + I(D - C) &= L(D - C), \end{aligned}$$

or using (2-16), it is

$$\begin{aligned} 0 &= AB, \\ 0 &= B(2a\lambda - \lambda^2 + 1 - L), \\ 0 &= A(-2a\lambda - \lambda^2 + 1 - L), \\ 0 &= A^2 + (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ &\quad \times (2a\lambda - \lambda^2 + 1 - L), \\ 0 &= B^2 + (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ &\quad \times (-2a\lambda - \lambda^2 + 1 - L), \\ 0 &= Z = \xi \cdot \lambda \end{aligned}$$

To study this system we consider these open subsets of U :

$$\begin{aligned} U' &= \{p \in U : A = 2b\lambda - (\phi e \cdot \lambda) = 0 \text{ in a neighborhood of } p\}, \\ U_3 &= \{p \in U : A \neq 0 \text{ in a neighborhood of } p\}, \end{aligned}$$

where $U' \cup U_3$ is open and dense in the closure of U . In U' , we have

$$\begin{aligned} 0 &= (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L), \\ 0 &= B(2a\lambda - \lambda^2 + 1 - L), \\ 0 &= B^2 + (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ &\quad \times (-2a\lambda - \lambda^2 + 1 - L), \\ 0 &= Z = \xi \cdot \lambda. \end{aligned}$$

We consider these open subsets of U' :

$$\begin{aligned} U_1 &= \{p \in U' : B = 2c\lambda - (e \cdot \lambda) = 0 \text{ in a neighborhood of } p\}, \\ U_2 &= \{p \in U' : B \neq 0 \text{ in a neighborhood of } p\}, \end{aligned}$$

where $U_1 \cup U_2$ is open and dense in the closure of U' . Because $B \neq 0$ in U_2 , we have $2a\lambda - \lambda^2 + 1 - L = 0$ there. Hence U_1 can also be described as the set of $p \in U$ satisfying

$$\begin{aligned} 0 &= 2b\lambda - (\phi e \cdot \lambda), \quad 0 = 2c\lambda - (e \cdot \lambda) = 0, \quad 0 = \xi \cdot \lambda, \\ 0 &= (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(-2a\lambda - \lambda^2 + 1 - L), \\ 0 &= (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L), \end{aligned}$$

in a neighborhood of p , whereas U_2 is the set of $p \in U$ satisfying

$$\begin{aligned} 0 &= 2b\lambda - (\phi e \cdot \lambda), \quad 0 = 2a\lambda - \lambda^2 + 1 - L, \quad 0 = \xi \cdot \lambda, \\ 0 &= (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ &\quad \times (-2a\lambda - \lambda^2 + 1 - L) + B^2 \end{aligned}$$

in a neighborhood of p . In U_3 we have $A \neq 0$ (or equivalently $B = 2c\lambda - (e \cdot \lambda) = 0$) and the system becomes

$$\begin{aligned} 0 &= -2a\lambda - \lambda^2 + 1 - L, \\ 0 &= A^2 + (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ &\quad \times (2a\lambda - \lambda^2 + 1 - L), \\ 0 &= Z = \xi \cdot \lambda. \end{aligned}$$

The set U_3 is also described as the set of $p \in U$ for which there is a neighborhood satisfying

$$\begin{aligned} 0 &= 2c\lambda - (e \cdot \lambda), \quad 0 = -2a\lambda - \lambda^2 + 1 - L, \quad 0 = \xi \cdot \lambda, \\ 0 &= A^2 + (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ &\quad \times (2a\lambda - \lambda^2 + 1 - L). \end{aligned}$$

We shall study the initial system at each U_i for $i = 1, 2, 3$.

In U_1 , we have

$$(5-1) \quad (\phi e \cdot \lambda) = 2b\lambda,$$

$$(5-2) \quad (e \cdot \lambda) = 2c\lambda,$$

$$(5-3) \quad 0 = \xi \cdot \lambda,$$

$$(5-4) \quad 0 = (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ \times (-2a\lambda - \lambda^2 + 1 - L),$$

$$(5-5) \quad 0 = (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ \times (2a\lambda - \lambda^2 + 1 - L).$$

Differentiating the equations (5-1) and (5-2) with respect to e and ϕe , respectively, and subtracting we get $[e, \phi e]\lambda = 2b(e \cdot \lambda) + 2\lambda(e \cdot b) - 2c(\phi e \cdot \lambda) - 2\lambda(\phi e \cdot c)$ or, because of (2-13), (5-1), (5-2) and (5-3),

$$(5-6) \quad e \cdot b = \phi e \cdot c.$$

Differentiating the equations (5-1), (5-3) with respect to ξ and ϕe , respectively, and subtracting, we get $[\xi, \phi e]\lambda = 2\lambda(\xi \cdot b)$ or, because of (2-13), (2-17) and (5-2),

$$(5-7) \quad \xi \cdot b = c(\lambda - a - 1),$$

$$(5-8) \quad e \cdot a = 2c\lambda.$$

Differentiating the equations (5-2) and (5-3) with respect to ξ and e , respectively, and subtracting, we get $[\xi, e]\lambda = 2\lambda(\xi \cdot c)$ or, because of (2-13), (2-17) and (5-1),

$$(5-9) \quad \xi \cdot c = b(\lambda + a + 1),$$

$$(5-10) \quad \phi e \cdot a = -2b\lambda.$$

In order to study the system of (5-4) and (5-5) we consider these open subsets of U_1 :

$$V = \{p \in U_1 : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \\ \text{in a neighborhood of } p\},$$

$$V' = \{p \in U_1 : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0 \\ \text{in a neighborhood of } p\},$$

where $V \cup V'$ is open and dense in the closure of U_1 . In the set V , the Equation (5-5) also holds; hence we consider these open subsets of V :

$$V_1 = \{p \in V : \quad -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ \quad 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \\ \text{in a neighborhood of } p\}$$

and

$$V_2 = \left\{ p \in V : \begin{aligned} 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) &= 0, \\ 2a\lambda - \lambda^2 + 1 - L &= 0 \end{aligned} \quad \text{in a neighborhood of } p \right\},$$

where $V_1 \cup V_2$ is open and dense in the closure of V . Similarly for V' , where $-2a\lambda - \lambda^2 + 1 - L = 0$, we consider the open subsets

$$\begin{aligned} V_3 &= \left\{ p \in V' : \begin{aligned} -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) &= 0, \\ -2a\lambda - \lambda^2 + 1 - L &= 0 \end{aligned} \quad \text{in a neighborhood of } p \right\}, \\ V_4 &= \left\{ p \in V' : \begin{aligned} -2a\lambda - \lambda^2 + 1 - L &= 0, \\ 2a\lambda - \lambda^2 + 1 - L &= 0 \end{aligned} \quad \text{in a neighborhood of } p \right\}, \end{aligned}$$

where $V_3 \cup V_4$ is open and dense in the closure of V' and the set $\bigcup V_i$ is open and dense in the closure of U_1 . We shall prove that the functions λ and a are constant at every V_i for $i = 1, 2, 3, 4$

Now

$$\text{in } V_1, \quad \begin{cases} -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0. \end{cases}$$

Subtracting these two equations, we deduce that $a = 0$ in $V_1 \subset U$. Hence from (5-8) and (5-10) we have $c = b = 0$, and from (5-1) and (5-2) we have $\phi e \cdot \lambda = e \cdot \lambda = 0$. These, together with (5-3), give $\lambda = \text{constant}$ in V_1 . Moreover, if we put $a = b = c = 0$ in one of the equations of the set V_1 , we finally get $\lambda^2 = 1$ and the structure is flat.

Next,

$$(5-11) \quad \text{in } V_2 \quad \begin{cases} 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ 2a\lambda - \lambda^2 + 1 - L = 0, \end{cases}$$

$$(5-12) \quad \text{in } V_3 \quad \begin{cases} -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ -2a\lambda - \lambda^2 + 1 - L = 0. \end{cases}$$

In V_2 we differentiate the equation $2a\lambda - \lambda^2 + 1 - L = 0$ with respect to ξ , ϕe and e , and because of (5-3), (5-8), (5-10) we obtain respectively

$$(5-13) \quad \xi \cdot a = 0,$$

$$(5-14) \quad b(a - 2\lambda) = 0,$$

$$(5-15) \quad ac = 0,$$

while in V_3 from $-2a\lambda - \lambda^2 + 1 - L = 0$ we obtain Equation (5-13), and

$$(5-16) \quad ba = 0, \quad c(a + 2\lambda) = 0.$$

Differentiating the relations (5-8) and (5-13) with respect to ξ and e , respectively, and subtracting, we get $[\xi, e]a = 2\lambda(\xi \cdot c)$ or, because of (2-13), (5-9) and (5-10),

$$(5-17) \quad b(\lambda + a + 1) = 0.$$

Similarly, differentiating (5-10) and (5-13) with respect to ξ and ϕe , respectively, and subtracting, we have $[\xi, \phi e]a = -2\lambda(\xi \cdot b)$ or, because of (2-13), (5-7) and (5-8),

$$(5-18) \quad c(\lambda - a - 1) = 0.$$

To study the system of (5-17) and (5-18), we consider these open subsets of V_2 :

$$G = \{p \in V_2 : b = 0 \text{ in a neighborhood of } p\},$$

$$G' = \{p \in V_2 : b \neq 0 \text{ in a neighborhood of } p\},$$

where $G \cup G'$ is open and dense in the closure of V_2 .

In G we have $c(\lambda - a - 1) = 0$, hence we consider these open subsets of G :

$$G_1 = \{p \in G : c = 0 \text{ in a neighborhood of } p\},$$

$$G_2 = \{p \in G : c \neq 0 \text{ in a neighborhood of } p\},$$

where $G_1 \cup G_2$ is open and dense in the closure of G . These sets are described more specifically as

$$G_1 = \{p \in G \subset V_2 : b = c = 0 \text{ in a neighborhood of } p\},$$

$$G_2 = \{p \in G \subset V_2 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\}.$$

The set G' (where $b \neq 0$ or equivalently $\lambda + a + 1 = 0$) is decomposed similarly as

$$G_3 = \{p \in G' : c = 0 \text{ in a neighborhood of } p\},$$

$$G_4 = \{p \in G' : c \neq 0 \text{ in a neighborhood of } p\},$$

where $G_3 \cup G_4$ is open and dense in the closure of G' . These can also be written

$$G_3 = \{p \in G' \subset V_2 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$

$$G_4 = \{p \in G' \subset V_2 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\}.$$

We have $V_2 \subset U$ where $\lambda \neq 0$; hence $G_4 = \emptyset$.

In G_1 , $b = c = 0$. From (5-1), (5-2), (5-3), (5-8), (5-10) and (5-13), we can conclude that λ and a are constant in G_1 , and from the first of (5-11), we have $(\lambda - 1)(a - \lambda - 1) = 0$. We consider these open subsets of G_1 :

$$K_1 = \{p \in G_1 : \lambda = 1 \text{ in a neighborhood of } p\},$$

$$K_2 = \{p \in G_1 : \lambda \neq 1 \text{ in a neighborhood of } p\}.$$

In K_1 , we get $\text{Tr} l = 0$, $L = 2a$, and, from (2-19), $r = 4a$. In K_2 , we have $a - \lambda - 1 = 0$, $\text{Tr} l = 2a(2 - a)$, $L = a^2$, and $r = 2a(4 - a)$.

In G_2 , $b = \lambda - a - 1 = 0$. Using this to eliminate a from the second equation of (5-11), we obtain $\lambda^2 - 2\lambda + 1 - L = 0$. We suppose that there is a point p in G_2 at which $e \cdot \lambda \neq 0$. Then there is a neighborhood S of p in which $e \cdot \lambda \neq 0$. We differentiate $\lambda^2 - 2\lambda + 1 - L = 0$ with respect to e twice and obtain $e \cdot \lambda = 0$ in S , a contradiction. Hence $e \cdot \lambda = 0$ (and $c = 0$), and from (5-1) and (5-3), we conclude that λ is constant in G_2 and similarly $a = \lambda - 1$. In particular, from (5-11), we obtain $\lambda = 1$ and $a = 0$; hence the structure is flat.

We have proved that λ and a are constant on G_1 and G_2 . The set $G_1 \cup G_2$ is open and dense in the closure of G . Hence λ and a are constant everywhere in G .

In G_3 , $c = \lambda + a + 1 = 0$. Using this to eliminate a from the second equation of (5-11), we obtain $-3\lambda^2 - 2\lambda + 1 - L = 0$. If we assume that there is a point p in G_3 at which $\phi e \cdot \lambda \neq 0$, then there is a neighborhood S of p in which $\phi e \cdot \lambda \neq 0$. We differentiate $-3\lambda^2 - 2\lambda + 1 - L = 0$ with respect to ϕe twice and obtain $\phi e \cdot \lambda = 0$ in S , a contradiction. Thus $\phi e \cdot \lambda = 0$ everywhere in G_3 , which gives $b = 0$. We note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$, so λ is constant in G_3 , and obviously $a = -\lambda - 1$. Moreover, if we put $b = c = 0$ and $a = -\lambda - 1$ in the system (5-11), we get $\lambda^2 = 1$ (hence $\text{Tr} l = 0$), $a = 0$ or -2 , and $L = -2a(a + 1)$. If $a = 0$, we obtain a flat structure, while if $a = -2$, we have a semi-K contact structure with constant scalar curvature $r = 4a = -8$.

The functions λ and a are constant in G and G' . The set $G \cup G'$ is open and dense in the closure of V_2 ; hence λ and a are constant in V_2 , and Equations (5-14) and (5-15) are satisfied because $b = c = 0$.

We similarly consider these open subsets of V_3 :

$$G'_1 = \{p \in V_3 : b = c = 0 \text{ in a neighborhood of } p\},$$

$$G'_2 = \{p \in V_3 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$

$$G'_3 = \{p \in V_3 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$

$$G'_4 = \{p \in V_3 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\}.$$

The set $\bigcup G'_i$ is open and dense subset of V_3 and $G'_4 = \emptyset$.

In G'_1 , $b = c = 0$. From (5-1), (5-2), (5-3), (5-8), (5-10) and (5-13) we can conclude that λ and a are constant in G'_1 . From the first of (5-12) we have $(\lambda + 1)(a + \lambda - 1) = 0$. We consider these open subsets of G'_1 :

$$K'_1 = \{p \in G'_1 : \lambda = -1 \text{ in a neighborhood of } p\},$$

$$K'_2 = \{p \in G'_1 : \lambda \neq -1 \text{ in a neighborhood of } p\}.$$

In K'_1 , we get $\text{Tr} l = 0$, $L = 2a$, and, from (2-19), $r = 4a$. In K'_2 we have $a + \lambda - 1 = 0$, $\text{Tr} l = 2a(2 - a)$, $L = a^2$, and $r = 2a(4 - a)$.

In G'_2 , $b = \lambda - a - 1 = 0$. Using this to eliminate a from the second equation of (5-12), we obtain $-3\lambda^2 + 2\lambda + 1 - L = 0$. If we assume $e \cdot \lambda \neq 0$, we may differentiate

this equation twice with respect to e , obtaining $e \cdot \lambda = 0$, a contradiction. Hence $e \cdot \lambda = 0$ (and $c = 0$), and from (5-1) and (5-3) we can conclude that λ is constant in G'_2 and similarly $a = \lambda - 1$. In particular, from the system (5-12), we have $\lambda^2 = 1$ (hence $\text{Tr} l = 0$), $a = 0$ or -2 , and $L = -2a(a + 1)$. If $a = 0$ we obtain a flat structure, and if $a = -2$ we have a semi-K contact structure with constant scalar curvature $r = 4a = -8$.

In G'_3 , $c = \lambda + a + 1 = 0$. Using this to eliminate a from the second equation of V_3 , we get $\lambda^2 + 2\lambda + 1 - L = 0$. If we assume $\phi e \cdot \lambda \neq 0$, we may differentiate this equation with twice respect to ϕe , obtaining $\phi e \cdot \lambda = 0$, a contradiction. Thus $\phi e \cdot \lambda = 0$ everywhere in G'_3 , which gives $b = 0$. We note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$. Thus λ is constant in G'_3 , and obviously $a = -\lambda - 1$. Moreover, if we put $b = c = 0$ and $a = -\lambda - 1$ in the system (5-12), we get $\lambda = -1$ and $a = 0$; hence the structure is flat.

As in case of V_2 , we have that the functions λ and a are constant in V_3 , and the equations (5-16) are satisfied because $b = c = 0$.

In V_4 , $2a\lambda - \lambda^2 + 1 - L = 0$ and $-2a\lambda - \lambda^2 + 1 - L = 0$. Working as in the set V_1 , we find that $a = 0$ (hence Theorem 4.1 applies), $b = c = 0$, and λ is constant in V_4 , that is, $\lambda^2 = 1 - L \geq 0$. Also, from (2-19), $r = \text{Tr} l = 2L$.

The functions λ and a are constant in each V_i for $i = 1, 2, 3, 4$, and the set $\bigcup V_i$ is open and dense in the closure of U_1 . Hence λ and a are constant in U_1 .

In U_2 ,

$$(5-1) \quad \phi e \cdot \lambda = 2b\lambda,$$

$$(5-3) \quad 0 = \xi \cdot \lambda,$$

$$(5-19) \quad 0 = 2a\lambda - \lambda^2 + 1 - L,$$

$$(5-20) \quad 0 = B^2 + (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \times (-2a\lambda - \lambda^2 + 1 - L).$$

We differentiate (5-19) with respect to ξ , ϕe , and e and because of (5-1) and (5-3) we obtain respectively

$$(5-21) \quad 0 = \xi \cdot a,$$

$$(5-22) \quad \phi e \cdot a = 2b\lambda - 2ab,$$

$$(5-23) \quad 0 = (a - \lambda)(e \cdot \lambda) + \lambda(e \cdot a) = 0.$$

We differentiate Equations (5-1) and (5-3) with respect to ξ and ϕe respectively and subtract. Then, because of (5-3), (2-13) and (2-17), we obtain

$$(5-24) \quad (3\lambda - a - 1)(e \cdot \lambda) - 2\lambda(e \cdot a) = 2\lambda c(\lambda - a - 1)$$

We consider the system of (5-23) and (5-24) with unknown functions the derivatives of $e \cdot \lambda$ and $e \cdot a$. We consider these open subsets of U_2 :

$$F_1 = \{p \in U_2 \subset U : -a - \lambda + 1 \neq 0 \text{ in a neighborhood of } p\},$$

$$F_2 = \{p \in U_2 \subset U : -a - \lambda + 1 = 0 \text{ in a neighborhood of } p\}.$$

In the neighborhood F_1 solving the system of (5-23) and (5-24), we obtain

$$(5-25) \quad (-a - \lambda + 1)(e \cdot \lambda) = -2\lambda c(\lambda - a - 1).$$

From (2-17), (5-1) and (5-22), we obtain

$$(5-26) \quad \xi \cdot c = -b(3\lambda - 3a - 1).$$

We differentiate (5-3), (5-25) with respect to e and ξ , respectively, and subtract. Then using (5-3), (5-21), we obtain $(-a - \lambda + 1)[e, \xi]\lambda = 2\lambda(\lambda - a - 1)(\xi \cdot c)$ or, by (2-13), (5-1), (5-26), and $\lambda \neq 0$ (as $F_1 \subset U$), we finally obtain

$$(5-27) \quad b(a^2 + \lambda^2 - a\lambda + a - \lambda) = 0$$

We work in the open subset F_1 and suppose that there is a point p in F_1 at which $b \neq 0$ (or equivalently, by (5-1), $\phi e \cdot \lambda \neq 0$). The function b is smooth, and because of its continuity there is an open neighborhood S of p such that $S \subset F_1$ and $b \neq 0$ everywhere in S . Hence from (5-27) we have in S

$$(5-28) \quad a^2 + \lambda^2 - a\lambda + a - \lambda = 0.$$

We differentiate this equation with respect to ϕe , use (5-1) and (5-22), and with the assumption $b \neq 0$ obtain $2a^2 - 2a\lambda - \lambda^2 + a = 0$. This last equation together with (5-28) gives $-3\lambda^2 - a + 2\lambda = 0$. We differentiate this equation with respect to ϕe . Then using (5-1) and (5-22), we get $2b(-6\lambda^2 + a + \lambda) = 0$ or equivalently $-6\lambda^2 + a + \lambda = 0$. The equations $-3\lambda^2 - a + 2\lambda = 0$ and $-6\lambda^2 + a + \lambda = 0$ give $-9\lambda^2 + 3\lambda = 0$. We differentiate this equation twice with respect to ϕe and obtain $\phi e \cdot \lambda = 0$ or equivalently $b = 0$ everywhere in S , a contradiction. Hence, from (5-27) we deduce that $b = 0$ everywhere in F_1 .

Equation (5-1) and $b = 0$ give $\phi e \cdot \lambda = 0$, which together with (5-3) gives $[\xi, \phi e]\lambda = 0$, or because of (2-13), $(\lambda - a - 1)(e \cdot \lambda) = 0$. Let's suppose that there is a point $q \in F_1$ at which $e \cdot \lambda \neq 0$. Then, there is a neighborhood Y of q in which $e \cdot \lambda \neq 0$. In Y we then have $\lambda - a - 1 = 0$, and hence from (5-25) we have $(-a - \lambda + 1)(e \cdot \lambda) = 0$, which in $Y \subset F_1$ gives $e \cdot \lambda = 0$, a contradiction. Hence $e \cdot \lambda = 0$ everywhere in F_1 . Then $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$ implies that λ is constant in F_1 . From (5-21), (5-22) and (5-23) we obtain that a is constant in F_1 and from

(5-25) we have $c(\lambda - a - 1) = 0$. We consider these two open subsets of F_1 :

$$J_1 = \{p \in F_1 : c = 0 \text{ in a neighborhood of } p\},$$

$$J_2 = \{p \in F_1 : c \neq 0 \text{ in a neighborhood of } p\}.$$

In J_1 , Equation (5-20) yields $a(\lambda - 1)(a - \lambda - 1) = 0$, where a and λ are constant and $b = 0$. We consider these two open subsets of J_1 :

$$H_1 = \{p \in J_1 : a = 0 \text{ in a neighborhood of } p\},$$

$$H_2 = \{p \in J_1 : a \neq 0 \text{ in a neighborhood of } p\}.$$

In H_1 , we have $\nabla_\xi \tau = 0$ (hence Theorem 4.1 applies) and from (2-19) $r = 2L$. In H_2 we have $(\lambda - 1)(a - \lambda - 1) = 0$, and hence we consider these two open subsets of H_2 :

$$H_3 = \{p \in J_1 : \lambda = 1 \text{ in a neighborhood of } p\},$$

$$H_4 = \{p \in J_1 : \lambda \neq 1 \text{ in a neighborhood of } p\}$$

In H_3 , we have $\text{Tr} l = 0$, $L = 2a$ by (5-19), and $r = 4a$ by (2-19). In H_4 we obtain $a - \lambda - 1 = 0$, $\text{Tr} l = 2a(2 - a)$, $L = a^2$, and $r = 2a(4 - a)$.

In J_2 , we have $b = a - \lambda + 1 = 0$ (a semi-K contact structure with constant a and λ); hence $\text{Tr} l = -2a(2 + a)$, $L = a^2$ by (5-19), and $c^2 + a(e \cdot c) + 4a^2 = 0$ by (5-20). The set $J_1 \cup J_2$ is open and dense inside the closure of F_1 ; hence we conclude that $b = 0$ and that a and λ are constant in F_1 .

In the open set F_2 we have $-a - \lambda + 1 = 0$, which together with (5-19) gives $-3\lambda^2 + 2\lambda + 1 - L = 0$. If we assume $e \cdot \lambda \neq 0$, we may differentiate this equation twice with respect to e and obtain $e \cdot \lambda = 0$, a contradiction. Hence $e \cdot \lambda = 0$. Similarly we can deduce that $\phi e \cdot \lambda = 0$ (so (5-1) implies $b = 0$), and hence λ is constant in F_2 . Obviously $a = -\lambda + 1$ is constant in F_2 . The system of (5-23) and (5-24) gives $ca = 0$. We consider these two open subsets of F_2 :

$$Q_1 = \{p \in F_2 : c = 0 \text{ in a neighborhood of } p\},$$

$$Q_2 = \{p \in F_2 : c \neq 0 \text{ in a neighborhood of } p\}.$$

In Q_1 , (5-19) implies $L = \text{Tr} l - a^2$, where $\text{Tr} l = 2a(2 - a)$ and $r = 2a(4 - a)$. In Q_2 , we have a 3- τ manifold structure with $\text{Tr} l = L = 0$.

We have proved that λ and a are constant in F_1 and F_2 . Since $F_1 \cup F_2$ is open and dense inside the closure of U_2 , we conclude that λ and a are constant in U_2 .

In U_3 ,

$$(5-2) \quad 2c\lambda = (e \cdot \lambda),$$

$$(5-3) \quad 0 = \xi \cdot \lambda,$$

$$(5-29) \quad 0 = -2a\lambda - \lambda^2 + 1 - L,$$

$$(5-30) \quad 0 = A^2 + (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \\ \times (2a\lambda - \lambda^2 + 1 - L).$$

We differentiate the relation (5-29) with respect to ξ , e and ϕe , and because of (5-2) and (5-3) we obtain respectively (5-21),

$$(5-31) \quad e \cdot a = -2ac - 2c\lambda,$$

$$(5-32) \quad 0 = (a + \lambda)(\phi e \cdot \lambda) + \lambda(\phi e \cdot a).$$

We differentiate (5-2) and (5-3) with respect to ξ and e respectively and subtract. Then by (5-21), we obtain $[\xi, e]\lambda = 2\lambda(\xi \cdot c)$, or because of (2-13) and (2-17),

$$(5-33) \quad (3\lambda + a + 1)(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot a) = 2\lambda b(\lambda + a + 1).$$

Meanwhile from (2-17), (5-2) and (5-31), we obtain

$$(5-34) \quad \xi \cdot b = -c(3a + 3\lambda + 1).$$

Now consider the system of (5-32) and (5-33) and these two open subsets of U_3 :

$$F'_1 = \{p \in U_3 \subset U : a - \lambda - 1 \neq 0 \text{ in a neighborhood of } p\},$$

$$F'_2 = \{p \in U_3 \subset U : a - \lambda - 1 = 0 \text{ in a neighborhood of } p\}.$$

In the open set F'_1 of p in which $a - \lambda - 1 \neq 0$, we may solve the system of (5-32) and (5-33) to obtain

$$(5-35) \quad (a - \lambda - 1)(\phi e \cdot \lambda) = -2\lambda b(\lambda + a + 1).$$

We differentiate (5-3) and (5-35) with respect to ϕe and ξ respectively and subtract. Then by (5-3) and (5-21) we obtain $(a - \lambda - 1)[\phi e, \xi]\lambda = 2\lambda(\lambda + a + 1)(\xi \cdot b)$ or, because of (2-13), $(a - \lambda - 1)(a - \lambda + 1)(e \cdot \lambda) = 2\lambda(\lambda + a + 1)(\xi \cdot b)$. Then, using (5-2), (5-34) and $\lambda \neq 0$ (in $F'_1 \subset U$), we get

$$c(a^2 + \lambda^2 + a\lambda + a + \lambda) = 0.$$

As in the case of Equation (5-27) we can deduce in F'_1 that $c = 0$. Equation (5-2), because $c = 0$, gives $e \cdot \lambda = 0$. This together with (5-3) gives $[e, \xi]\lambda = 0$ or, because of (2-13), $(a + \lambda + 1)(\phi e \cdot \lambda) = 0$. Suppose that there is a point $q \in F'_1$ at which $\phi e \cdot \lambda \neq 0$. Then there is a neighborhood S of q in which $\phi e \cdot \lambda \neq 0$, and hence $a + \lambda + 1 = 0$. Because $a + \lambda + 1 = 0$, Equation (5-35) gives $(a - \lambda - 1)(\phi e \cdot \lambda) = 0$.

Working in $S \subset F'_1$, where $a - \lambda - 1 \neq 0$, we can conclude that $\phi e \cdot \lambda = 0$, a contradiction. Hence, $\phi e \cdot \lambda = 0$ everywhere in F'_1 . In the neighborhood F'_1 , we have $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$ or equivalently λ is constant in F'_1 . From (5-21), (5-31) and (5-33), we obtain that a is constant in F'_1 , and from (5-32) we have $b(\lambda + a + 1) = 0$. We consider these two open subsets of F'_1 :

$$\begin{aligned} J'_1 &= \{p \in F'_1 : b = 0 \text{ in a neighborhood of } p\}, \\ J'_2 &= \{p \in F'_1 : b \neq 0 \text{ in a neighborhood of } p\}. \end{aligned}$$

In J'_1 , we obtain from (5-30) that $a(\lambda + 1)(a + \lambda - 1) = 0$, where a and λ are constant and $c = 0$. We consider these two open subsets of J'_1 :

$$\begin{aligned} H'_1 &= \{p \in J'_1 : a = 0 \text{ in a neighborhood of } p\}, \\ H'_2 &= \{p \in J'_1 : a \neq 0 \text{ in a neighborhood of } p\}. \end{aligned}$$

In H'_1 , we obtain $\nabla_\xi \tau = 0$ (hence Theorem 4.1 applies). and $r = 2L$. In H'_2 , we have $(\lambda + 1)(a + \lambda - 1) = 0$. We consider these two open subsets of H'_2 :

$$\begin{aligned} H'_3 &= \{p \in J_1 : \lambda = -1 \text{ in a neighborhood of } p\}, \\ H'_4 &= \{p \in J_1 : \lambda \neq -1 \text{ in a neighborhood of } p\}. \end{aligned}$$

In H'_3 , we have $\text{Tr} l = 0$, $L = 2a$ by (5-29), and $r = 4a$ by (2-19). In H'_4 , we obtain $a + \lambda - 1 = 0$, $\text{Tr} l = 2a(2 - a)$, $L = a^2$ and $r = 2a(4 - a)$.

In J'_2 we have $c = a + \lambda + 1 = 0$ (a semi-K contact structure with constant a and λ) and hence $\text{Tr} l = -2a(2 + a)$. Then $L = a^2$ from (5-29), and from (5-30) we obtain $b^2 + a(\phi e \cdot b) + 4a^2 = 0$ with a a constant. The set $J'_1 \cup J'_2$ is open and dense inside the closure of F'_1 ; hence we can conclude that $c = 0$ and that a and λ are constant in F'_1 .

In the open set F'_2 we have $a - \lambda - 1 = 0$, which together with (5-29) gives $-3\lambda^2 - 2\lambda + 1 - L = 0$. If we assume $\phi e \cdot \lambda \neq 0$, we may differentiate this equation twice with respect to e , obtaining $\phi e \cdot \lambda = 0$, a contradiction. Hence $\phi e \cdot \lambda = 0$. Similarly we find that $e \cdot \lambda = 0$, and hence λ is constant in F'_2 . Obviously $a = \lambda + 1$ is constant in F'_2 . The system of (5-32) and (5-33) gives $ba = 0$. We consider these two open subsets of F'_2 :

$$\begin{aligned} Q'_1 &= \{p \in F'_2 : b = 0 \text{ in a neighborhood of } p\}, \\ Q'_2 &= \{p \in F'_2 : b \neq 0 \text{ in a neighborhood of } p\}. \end{aligned}$$

In Q'_1 , (5-29) implies $L = \text{Tr} l - a^2$, where $\text{Tr} l = 2a(2 - a)$ and $r = 2a(4 - a)$. In Q'_2 , we have a 3- τ manifold structure with $\text{Tr} l = L = 0$.

We have proved that λ and a are constant in F'_1 and F'_2 , while $F'_1 \cup F'_2$ is open and dense inside the closure of U_3 . Hence we conclude that λ and a are constant in U_3 .

Finally because λ and a are constant in each U_i for $i = 1, 2, 3$, and because the set $\bigcup U_i$ is open and dense inside of the closure of U , we conclude that λ and a are constant in U . Then by (2-9), $\text{Tr}l = 2(1 - \lambda^2)$ is also constant in U and obviously on M^3 . \square

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