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Let K/F be a quadratic extension of p-adic fields, and χ a character of F^*. A representation (π, V) of $\mathrm{GL}(n, K)$ is said to be χ-distinguished if there is a nonzero linear form L on V such that $L(\pi(h)v) = \chi \circ \det(h)L(v)$ for $h \in \mathrm{GL}(n, F)$ and $v \in V$. We classify here distinguished principal series representations of $\mathrm{GL}(n, K)$. Call $\eta_{K/F}$ the nontrivial character of F^* that is trivial on the norms of K^*, and σ the nontrivial element of the Galois group of K over F. A conjecture attributed to Jacquet asserts that admissible irreducible representations π of $\mathrm{GL}(n, K)$ are such that the smooth dual $\pi^\vee$ is isomorphic to $\pi \circ \sigma$ if and only if it is 1-distinguished or $\eta_{K/F}$-distinguished. Our classification gives a counterexample for $n \geq 3$.

1. Introduction

For K/F a quadratic extension of p-adic fields, let σ be the conjugation relative to this extension, and let $\eta_{K/F}$ be the character of F^* with kernel being norms of K^*.

Let π be a smooth irreducible representation of $\mathrm{GL}(n, K)$, let χ be a character of F^*, and let m be dimension of the space of linear forms on π's space that transform by χ under $\mathrm{GL}(n, F)$ with respect to the action $(L, g) \mapsto L \circ \pi(g)$. By [Flicker 1991, Proposition 11], m is known to be at most one. One says that π is χ-distinguished if $m = 1$; one says π is distinguished if it is 1-distinguished.

In this article, we give a description of distinguished principal series representations of $\mathrm{GL}(n, K)$.

The result, Theorem 3.4, is that the irreducible distinguished representations of the principal series of $\mathrm{GL}(n, K)$ are (up to isomorphism) those unitarily induced from a character $\chi = (\chi_1, \ldots, \chi_n)$ of the maximal torus of diagonal matrices such that there exists an $r \leq n/2$ for which $\chi_{i+1}^\sigma = {\chi_i}^{-1}$ for $i = 1, 3, \ldots, 2r-1$, and $\chi_i|_{F^*} = 1$ for $i > 2r$. For the quadratic extension $\mathbb{C}/\mathbb{R}$, it is known [Panichi 2001] that the analogous result is true for tempered representations.

For $n \geq 3$, this gives a counterexample (see Corollary 3.5) to a conjecture of Jacquet [Anandavardhanan 2005, Conjecture 1], which states that an irreducible

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representation π of $\mathrm{GL}(n, K)$ with central character trivial on F^* is isomorphic to $\check{\pi}^\sigma$ if and only if it is distinguished or $\eta_{K/F}$-distinguished (where $\eta_{K/F}$ is the character of order 2 of F^*, attached by local class field theory to the extension K/F). For discrete series representations, the conjecture is true, as proved in [Kable 2004].

Unitary irreducible distinguished principal series representations of $\mathrm{GL}(2, K)$ were described in [Hakim 1991]. The general case of distinguished irreducible principal series representations of $\mathrm{GL}(2, K)$ was treated in [Flicker and Hakim 1994]; we use this occasion to give a different proof. To do this, in Theorems 4.1 and 4.3, we extend a criterion of Hakim [1991, Theorem 4.1], which characterizes smooth unitary irreducible distinguished representations of $\mathrm{GL}(2, K)$ in terms of γ factors at 1/2, to all smooth irreducible distinguished representations of $\mathrm{GL}(2, K)$.

2. Preliminaries

Let ϕ be a group automorphism and x an element of the group. We sometimes write x^ϕ instead of $\phi(x)$ and write $x^{-\phi}$ for the inverse of x^ϕ. If $\phi = x \mapsto h^{-1}xh$ for h in the group, then we may write x^ϕ as x^h.

Let G be a locally compact totally disconnected group, and let H be a closed subgroup of G.

We denote by Δ_G the module of G given by the relation $d_G(gx) = \Delta_G(g)d_G(x)$, where d_G is the right Haar measure on G.

Let X be a locally closed subspace of G with $H.X \subset X$. If V is a complex vector space, we denote by $D(X, V)$ the space of smooth V-valued functions on X with compact support (if $V = \mathbb{C}$, we simply write $D(X)$).

Let ρ be a smooth representation of H in a complex vector space V_ρ. We denote by $D(H\backslash X, \rho, V_\rho)$ the space of smooth V_ρ-valued functions f on X, with compact support modulo H, such that $f(hx) = \rho(h)f(x)$ for $h \in H$ and $x \in X$ (if ρ is a character, we simply write $D(H\backslash X, \rho)$).

We denote by $\mathrm{ind}_H^G(\rho)$ the representation induced by right translations of G in $D(H\backslash G, (\Delta_G/\Delta_H)^{1/2}\rho, V_\rho)$.

Let F be a nonarchimedean local field of characteristic zero, and let K be a quadratic extension of F. We have $K = F(\delta)$ with δ^2 in F^*.

We denote by $|\cdot|_K$ the absolute value on K.

We denote by σ the nontrivial element of the Galois group $G(K/F)$ of K over F, and we use the same letter to designate its action on $M_n(K)$.

We denote by $N_{K/F}$ the norm of the extension K/F, and we write $\eta_{K/F}$ for the nontrivial character of F^* that is trivial on $N_{K/F}(K^*)$.

Whenever G is an algebraic group defined over F, we denote by $G(K)$ its K-points and by $G(F)$ its F-points.

We denote by G_n the group $\mathrm{GL}(n)$, by B_n its standard Borel subgroup, by U_n its unipotent radical, and by T_n the standard maximal split torus of diagonal matrices.

We denote by S the space of matrices M in $G_n(K)$ satisfying $MM^\sigma = 1$.

Everything in this section is more or less contained in [Flicker 1992], but we give detailed proofs here for convenience of the reader.

Proposition 2.1 [Serre 1968, Chapter 10, Proposition 3]. *The map $S_n : g \mapsto g^\sigma g^{-1}$ is a homeomorphism between $G_n(K)/G_n(F)$ and S.*

Proposition 2.2. *For its natural action on S, each orbit of $B_n(K)$ contains one and only one element of $\mathfrak{S}_n$ of order 2 or 1.*

Proof. We begin with the following:

Lemma 2.3. *Let w be an element of $\mathfrak{S}_n \subset G_n(K)$ of order at most 2.*

Let θ' be the involution of $T_n(K)$ given by $t \mapsto w^{-1}t^\sigma w$. Then any $t \in T_n(K)$ with $t\theta'(t) = 1$ is of the form $a/\theta'(a)$ for some $a \in T_n(K)$.

Proof of Lemma 2.3. There exists a $r \le n/2$ such that, up to conjugacy,

$$w = (1, 2)(3, 4) \cdots (2r - 1, 2r).$$

We write $t = \mathrm{diag}(z_1, z_1', \dots, z_r, z_r', z_{2r+1}, \dots, z_n)$. Hence for $i \le r$, we have $z_i\sigma(z_i') = 1$, and $z_j\sigma(z_j) = 1$ for $j \ge 2r + 1$.

Hilbert's Theorem 90 says each z_j for $j \ge 2r+1$ is of the form $u_{j-2r}/\sigma(u_{j-2r})$ for some $u_{j-2r} \in K^*$.

We then take $a = \mathrm{diag}(z_1, 1, \dots, z_r, 1, u_1, \dots, u_{n-2r})$. □

Lemma 2.4. *Let N be an algebraic connected unipotent group over K. Let θ be an involutive automorphism of $N(K)$. If $x \in N(K)$ satisfies $x\theta(x) = 1_N$, then there is an $a \in N$ such that $x = \theta(a^{-1})a$.*

Proof of Lemma 2.4. The group $N(K)$ has a composition series $1_N = N_0 \subset N_1 \subset \cdots \subset N_{n-1} \subset N_n = N(K)$ such that each quotient N_{i+1}/N_i is isomorphic to $(K, +)$ and such that each commutator subgroup $[N, N_{i+1}]$ is a subgroup of N_i.

Now we prove the lemma by induction on n. If $n = 1$, then $N(K)$ is isomorphic to $(K, +)$, one concludes taking $a = x/2$. For the induction step, suppose the lemma is true for every $N(K)$ of length n. Let $N(K)$ be of length $n + 1$.

By the induction hypothesis, one gets that there exists an element in $h \in N_1$ and an element u in $N(K)$ such that $x = \theta(u^{-1})uh$. Here h lies in the center of $N(K)$, because $[N(K), N_1] = 1_N$.

Because $x\theta(x) = 1$, we get $h\theta(h) = 1$. By the induction hypothesis again, we get $h = \theta(b^{-1})b$ for $b \in N_1$. We then take $a = ub$. □

We return to the proof of Proposition 2.2.

For w in $\mathfrak{S}_n$, denote by U_w the subgroup of U_n generated by the elementary subgroups U_α, with α positive and $w\alpha$ negative; denote by U_w' the subgroup of U_n

generated by the elementary subgroups U_α, with α positive and $w\alpha$ positive. Then $U_n = U'_w U_w$.

Let s be in S. According to Bruhat's decomposition, there is a w in $\mathfrak{S}_n$, an a in $T_n(K)$, an n_1 in $U_n(K)$, and an n_2^+ in U_w such that $s = n_1 a w n_2^+$; this decomposition is unique.

Then $s = s^{-\sigma} = (n_2^+)^{-\sigma} w^{-1} a^{-\sigma} n_1{}^{-\sigma}$. Thus we have $aw = (aw)^{-\sigma}$, that is, $w^2 = 1$ and $a^w = a^{-\sigma}$.

Now we write $n_1{}^{-\sigma} = u^- u^+$ with $u^- \in U'_w$ and $u^+ \in U_w$. Then, comparing s and $s^{-\sigma}$, we see u^+ must be equal to $n_2{}^+$. Hence $s = n_1 aw(u^-)^{-1} n_1{}^{-\sigma}$; thus we suppose $s = awn$, with n in U'_w.

From $s = s^{-\sigma}$, we have the relation $awn(aw)^{-1} = n^{-\sigma}$. Applying σ on each side, this becomes $(aw)^{-1} n^\sigma aw = n^{-1}$.

But $\theta : u \mapsto (aw)^{-1} u^\sigma aw$ is an involutive automorphism of U'_w; hence from Lemma 2.4, there is a u' in U'_w such that $n = \theta(u^{-1})u$. This gives $s = u^{-\sigma} awu$, so that we suppose $s = aw$. Again $wa^\sigma w = a^{-1}$, and applying Lemma 2.3 to $\theta' : x \mapsto wx^\sigma w$, we deduce that a is of the form $y\theta'(y^{-1})$, and $s = ywy^{-\sigma}$. □

Let $u \in M_2(K)$ equal $\left(\begin{smallmatrix} 1 & -\delta \\ 1 & \delta \end{smallmatrix}\right)$. We have $S_2(u) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; see Proposition 2.1.

We note for further use (in the proof of Proposition 3.3) that if we define the subgroup $\widetilde{T} := \{(\mathrm{diag}(z, z^\sigma) \in G_2(K) | z \in K^*\}$, then

$$u^{-1} \tilde{T} u = T = \left\{ \begin{pmatrix} x & \delta^2 y \\ y & x \end{pmatrix} \in G_2(F) \,\middle|\, x, y \in F \right\}.$$

For $r \leq n/2$, we denote by U_r the $n \times n$ matrix given by the block decomposition $\mathrm{diag}(u, \dots, u, I_{n-2r})$.

For w an element of $\mathfrak{S}_n$ naturally injected in $G_n(K)$, we write $U_r^w = w^{-1} U_r w$.

Corollary 2.5. *The elements U_r^w for $0 \leq r \leq n/2$, and $w \in \mathfrak{S}_n$ give a complete set of representatives of classes of $B_n(K) \backslash G_n(K) / G_n(F)$.*

Let $G_n = \coprod_{w \in \mathfrak{S}_n} B_n w B_n$ be the Bruhat decomposition of G_n. We call a double-class BwB a Bruhat cell.

Lemma 2.6. *One can order the Bruhat cells $C_1, C_2, \dots, C_{n!}$ so that for every $1 \leq i \leq n!$, the cell C_i is closed in $G_n - \coprod_{k=1}^{i-1} C_i$.*

Proof. Choose $C_1 = B_n$. It is closed in G_n. Now let w_2 be an element of $\mathfrak{S}_n - \mathrm{Id}$ with minimal length. Then from [Springer 1998, 8.5.5], the Bruhat cell Bw_2B is closed in $G_n - B_n$ in the Zariski topology; hence in the p-adic topology, we may take this cell to be C_2. We conclude by repeating this process. □

Corollary 2.7. *One can order the classes $A_1, \dots, A_t$ of $B_n(K) \backslash G_n(K) / G_n(F)$ so that A_i is closed in $G_n(K) - \coprod_{k=1}^{i-1} A_i$.*

Proof. From the proof of Proposition 2.2, we know that if C is a Bruhat cell of G_n, then $S_n \cap C$ is either empty or it corresponds through the homeomorphism S_n to a class A of $B_n(K)\backslash G_n(K)/G_n(F)$. The conclusion then follows from the previous lemma. □

Corollary 2.8. *Each A_i is locally closed in $G_n(K)$ in the Zariski topology.*

Lemma 2.9. *Let G, H, X, and (ρ, V_ρ) be as in the beginning of the section. The map*

$$\Phi : D(X) \otimes V_\rho \to D(H\backslash X, \rho, V_\rho), \quad f \otimes v \mapsto \Big(x \mapsto \int_H f(hx)\rho(h^{-1})v dh \Big)$$

is surjective.

Proof. Let $v \in V_\rho$. Let U be an open subset of G intersecting X and small enough for $h \mapsto \rho(h)v$ to be trivial on $H \cap UU^{-1}$. Let f' be the function with support in $H(X \cap U)$ defined by $hx \mapsto \rho(h)v$. Such functions generate $D(H\backslash X, \rho, V_\rho)$ as a vector space.

Now let f be the function of $D(X, V_\rho)$ defined by $x \mapsto 1_{U\cap X}(x)v$. Then $\Phi(f)$ is a multiple of f'.

But for x in $U \cap X$, $\Phi(f)(x) = \int_H \rho(h^{-1}) f(hx) dh$ because $h \mapsto \rho(h)v$ is trivial on $H \cap UU^{-1}$. Also $h \mapsto f(hx)$ is a positive function that multiplies v, and $f(x) = V$. Thus $F(f)(x)$ is v multiplied by a strictly positive scalar. □

Corollary 2.10. *Let Y be an H-stable closed subset of X. Then the restriction map from $D(H\backslash X, \rho, V_\rho)$ to $D(H\backslash Y, \rho, V_\rho)$ is surjective.*

Proof. This is a consequence of the known surjectivity of the restriction map from $D(X)$ to $D(Y)$, which implies the surjectivity of the restriction from $D(X, V_\rho)$ to $D(Y, V_\rho)$, and of the commutativity of the diagram

$$\begin{array}{ccc} D(X) & \longrightarrow & D(Y) \\ \Phi \downarrow & & \downarrow \Phi \\ D(H\backslash X, \rho) & \longrightarrow & D(H\backslash Y, \rho). \end{array}$$

□

3. Distinguished principal series

If π is a smooth representation of $G_n(K)$ on the space V_π and χ is a character of F^*, we say that π is χ-distinguished if there exists on V_π a nonzero linear form L such that $L(\pi(g)v) = \chi(\det(g))L(v)$ whenever g is in $G_n(F)$ and v belongs to V_π. If χ is trivial, we simply say that π is distinguished.

We first recall the following:

Theorem 3.1 [Flicker 1991, Proposition 12]. *Let π be a smooth irreducible distinguished representation of $G_n(K)$. Then $\pi^\sigma \simeq \check{\pi}$.*

Let $\chi_1, \dots, \chi_n$ be n characters of K^*, with none of their quotients equal to $|\cdot|_K$. We denote by χ the character of $b \in B_n(K)$ defined by $\chi(b) = \chi_1(b_1) \cdots \chi_n(b_n)$, where the b_i are the diagonal entries of b.

We denote by $\pi(\chi)$ the representation of $G_n(K)$ by right translation on the space of functions $D(B_n(K)\backslash G_n(K), \Delta_{B_n}^{-1/2}\chi)$. This representation is smooth and irreducible; we call it the principal series attached to χ. For a smooth representation π of $G_n(K)$, we denote by $\check{\pi}$ its smooth contragredient.

Lemma 3.2 [Flicker 1992, Proposition 26]. *Let $\overline{m} = (m_1, \dots, m_l)$ be a partition of a positive integer m, let $P_{\overline{m}}$ be the corresponding standard parabolic subgroup, and for each $1 \le i \le l$, let π_i be a smooth distinguished representation of $G_{m_i}(K)$. Then*

$$\pi_1 \times \cdots \times \pi_l = \operatorname{ind}_{P_{\overline{m}}(K)}^{G_m(K)}(\Delta_{P_{\overline{m}}(K)}^{-1/2}(\pi_1 \otimes \cdots \otimes \pi_l))$$

is distinguished.

We now come to the principal result:

Proposition 3.3. *Let $\chi = (\chi_1, \dots, \chi_n)$ be a character of $B_n(K)$ with none of the characters χ_i/χ_j equal to $|\cdot|_K$. Suppose that the principal series representation $\pi(\chi)$ is distinguished. Then there exists a reordering of the χ_i and an $r \le n/2$ satisfying $\chi_{i+1}^{\sigma} = {\chi_i}^{-1}$ for $i = 1, 3, \dots, 2r-1$ and $\chi_i|_{F^*} = 1$ for $i > 2r$.*

Proof. We write $B = B_n(K)$ and $G = G_n(K)$. From Corollaries 2.7 and 2.10, we have the following exact sequence of smooth $G_n(F)$-modules:

$$D(B\backslash G - A_1, \Delta_B^{-1/2}\chi) \hookrightarrow D(B\backslash G, \Delta_B^{-1/2}\chi) \to D(B\backslash A_1, \Delta_B^{-1/2}\chi).$$

Hence there is a nonzero distinguished linear form either on $D(B\backslash A_1, \Delta_B^{-1/2}\chi)$, or on $D(B\backslash G - A_1, \Delta_B^{-1/2}\chi)$.

In the second case, we have the exact sequence

$$D(B\backslash G - A_1 \sqcup A_2, \Delta_B^{-1/2}\chi) \hookrightarrow D(B\backslash G - A_1, \Delta_B^{-1/2}\chi) \to D(B\backslash A_2, \Delta_B^{-1/2}\chi).$$

Repeating the process, we deduce the existence of a nonzero distinguished linear form on one of the spaces $D(B\backslash A_i, \Delta_B^{-1/2}\chi)$.

By Corollary 2.5, we may choose w in S_n and $r \le n/2$ with $A_i = BU_r^w G_n(F)$. The application $f \mapsto (x \mapsto f(U_r^w x))$ gives an isomorphism of $G_n(F)$-modules between

$$D(B\backslash A_i, \Delta_B^{-1/2}\chi) \quad \text{and} \quad D(U_r^{-w} B U_r^w \cap G_n(F)\backslash G_n(F), \Delta'\chi'),$$

where $\Delta'(x) = \Delta_B^{-1/2}(U_r^w x U_r^{-w})$ and $\chi'(x) = \chi(U_r^w x U_r^{-w})$.

Now there exists a nonzero $G_n(F)$-invariant linear form on

$$D(U_r^{-w} B U_r^w \cap G_n(F)\backslash G_n(F), \Delta'\chi')$$

if and only if $\Delta'\chi'$ is equal to the inverse of the module of $U_r^{-w} B U_r^w \cap G_n(F)$; see [Bushnell and Henniart 2006, Chapter 1, Proposition 3.4]. From this we deduce that χ' is positive on $U_r^{-w} B U_r^w \cap G_n(F)$ or equivalently χ is positive on $B \cap U_r^w G_n(F) U_r^{-w}$.

Let $\overline{T}_r$ be the F-torus of matrices of the form

$$\operatorname{diag}(z_1, z_1^\sigma, \dots, z_r, z_r^\sigma, x_1, \dots, x_t),$$

where $2r + t = n$, $z_i \in K^*$, and $x_i \in F^*$. Then $\overline{T}_r^w \subset B \cap U_r^w G_n(F) U_r^{-w}$, so that χ must be positive on $\overline{T}_r^w$.

If χ is unitary, then χ is trivial on $\overline{T}_r^w$, and $\pi(\chi)$ is of the desired form.

For the general case, we deduce from Theorem 3.1, that there exist three integers $p \geq 0$, $q \geq 0$, and $s \geq 0$ such that up to reordering the χ_i are as follows: For $1 \leq i \leq p$, we have $\chi_{2i} = \chi_{2i-1}^{-\sigma}$. For $1 \leq k \leq q$, we have $\chi_{2p+k}|_{F^*} = 1$, and these χ_{2p+k} are distinct (meaning $\chi_{2p+k} \neq \chi_{2p+k'}^{-\sigma}$ for $k \neq k'$). For $1 \leq j \leq s$, we have $\chi_{2p+q+j}|_{F^*} = \eta_{K/F}$, and these χ_{2p+q+j} are distinct.

We write $\mu_k = \chi_{2p+k}$ for $q \geq k \geq 1$ and $\nu'_k = \chi_{2p+q+k'}$ for $s \geq k' \geq 1$.

We will show that if such a character χ is positive on a conjugate of $\overline{T}_r$ by an element of S_n, then $s = 0$.

Suppose ν_1 appears. Then either ν_1 is positive on F^*, which is not possible, or it is paired with another χ_i, and (ν_1, χ_i) is positive on elements (z, z^σ) for z in K^*.

Suppose $\chi_i = \nu_j$ for some $j \neq 1$. Then (ν_1, χ_i) is unitary, so it must be trivial on pairs (z, z^σ), which implies $\nu_1 = \nu_j{}^{-\sigma} = \nu_j$, which is absurd.

The character χ_i cannot be of the form μ_j, because this would imply $\nu_1|_{F^*} = 1$.

Suppose finally that $i \leq 2p$. In this case $\nu_1{}^{-\sigma} = \nu_1$ must be the unitary part of χ_i because of the positivity of (ν_1, χ_i) on the pairs (z, z^σ).

But $\chi_i{}^{-\sigma}$ also appears and is not trivial on F^*. Hence it must be paired with another character χ_j with $j \leq 2p$ and $j \neq i$ such that $(\chi_i^{-\sigma}, \chi_j)$ is positive on the elements (z, z^σ) for z in K^*. This implies that χ_j has unitary part $\nu_1^{-\sigma} = \nu_1$. The character χ_j cannot be a μ_k because of its unitary part.

If it is a χ_k with $k \leq 2p$, we consider $\chi_k^{-\sigma}$ again.

By repeating the process long enough, we can suppose that χ_j is of the form ν_k with $k \neq 1$. Taking unitary parts, we see that $\nu_k = \nu_1^{-\sigma} = \nu_1$, which is in contradiction with the fact that the ν_i are all different. We conclude that $s = 0$. □

Theorem 3.4. *Let $\chi = (\chi_1, \dots, \chi_n)$ be a character of $T_n(K)$. Then the principal series representation $\pi(\chi)$ is distinguished if and only if there exists an $r \leq n/2$ such that $\chi_{i+1}^\sigma = \chi_i^{-1}$ for $i = 1, 3, \dots, 2r - 1$ and $\chi_i|_{F^*} = 1$ for $i > 2r$.*

Proof. There is one implication left.

Suppose χ is of the desired form. Then $\pi(\chi)$ is parabolically (unitarily) induced from representations of the type $\pi(\chi_i, \chi_i^{-\sigma})$ of $G_2(K)$ and from distinguished characters of K^*.

Hence by Lemma 3.2 we need only show that the representations $\pi(\chi_i, \chi_i^{-\sigma})$ are distinguished. But this is just Corollary 4.2 of the next section. □

This gives a counterexample to a conjecture of Jacquet [Anandavardhanan 2005, Conjecture 1], which asserts that if an irreducible admissible representation π of $G_n(K)$ is such that $\check{\pi}$ is isomorphic to π^σ, then it is distinguished if n is odd, and is distinguished or $\eta_{K/F}$-distinguished if n is even.

Corollary 3.5. *For $n \geq 3$, there exists a smooth irreducible representation π of $G_n(K)$, with central character trivial on F^*, that is neither distinguished nor $\eta_{K/F}$-distinguished but whose smooth contragredient $\check{\pi}$ is isomorphic to π^σ.*

Proof. Take $\chi_1, \dots, \chi_n$, all different, such that $\chi_1|_{F^*} = \chi_2|_{F^*} = \eta_{K/F}$, and $\chi_j|_{F^*} = 1$ for $3 \leq j \leq n$. Because each χ_i has trivial restriction to $N_{K/F}(K^*)$, it is equal to $\chi_i^{-\sigma}$; hence $\check{\pi}$ is isomorphic to π^σ. Another consequence is that if k and l are two distinct integers between 1 and n, then $\chi_k \neq \chi_l^{-\sigma}$, because we assumed the χ_i are all different.

Then it follows from Theorem 3.4 that $\pi = \pi(\chi_1, \dots, \chi_n)$ is neither distinguished nor $\eta_{K/F}$-distinguished, but clearly the central character of π is trivial on F^*, and $\check{\pi}$ is isomorphic to π^σ. □

4. Distinguishability and gamma factors for GL(2)

In this section we generalize to smooth infinite dimensional irreducible representations of $G_2(K)$ a criterion of Hakim [1991, Theorem 4.1], which characterizes smooth unitary irreducible distinguished representations of $G_2(K)$. In the proof of that theorem, Hakim deals with unitary representations so that the integrals of Kirillov functions on F^* with respect to a Haar measure of F^* converge. We bypass the convergence problems using [Jacquet and Langlands 1970, Proposition 2.9 of Chapter 1].

We denote $M(K)$ by the mirabolic subgroup of $G_2(K)$ of matrices of the form $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$ with a in K^* and x in K, and by $M(F)$ its intersection with $G_2(F)$. We let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let π be a smooth infinite dimensional irreducible representation of $G_2(K)$. It is known that π is generic (see for example [Zelevinsky 1980]). Let $K(\pi, \psi)$ be its Kirillov model corresponding to ψ [Jacquet and Langlands 1970, Theorem 2.13]. This model contains the subspace $D(K^*)$ of functions with compact support on the group K^*. If ϕ belongs to $K(\pi, \psi)$ and x belongs to K, then $\phi - \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\phi$ belongs to $D(K^*)$ [Jacquet and Langlands 1970, Chapter 1, Proposition 2.9]. From this follows that $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$.

We now recall a consequence of the functional equation at $1/2$ for Kirillov representations [Bump 1997, Section 4.7].

For all ϕ in $K(\pi, \psi)$ and χ a character of K^*, we have

$$\int_{K^*} \pi(w)\phi(x)(c_\pi \chi)^{-1}(x)d^*x = \gamma(\pi \otimes \chi, \psi) \int_{K^*} \phi(x)\chi(x)d^*x \tag{1}$$

whenever both sides converge absolutely, where d^*x is a Haar measure on K^* and c_π is the central character of π.

Theorem 4.1. *Let π be a smooth irreducible representation of $G_2(K)$ of infinite dimension with central character trivial on F^*, and let ψ be a nontrivial character of K trivial on F. If $\gamma(\pi \otimes \chi, \psi) = 1$ for every character χ of K^* trivial on F^*, then π is distinguished.*

Proof. In fact, using a Fourier inversion in the functional equation (1) and the change of variable $x \mapsto x^{-1}$, we deduce that

$$c_\pi(x) \int_{F^*} \pi(w)\phi(tx^{-1})d^*t = \int_{F^*} \phi(tx)d^*t \quad \text{for all } \phi \in D(K^*) \cap \pi(w)D(K^*),$$

where d^*t is a Haar measure on F^*. For $x = 1$, this gives

$$\int_{F^*} \pi(w)\phi(t)d^*t = \int_{F^*} \phi(t)d^*t.$$

Now we define on $K(\pi, \psi)$ a linear form λ by

$$\lambda(\phi_1 + \pi(w)\phi_2) = \int_{F^*} \phi_1(t)d^*t + \int_{F^*} \phi_2(t)d^*t \quad \text{for } \phi_1, \phi_2 \in D(K^*).$$

This is well defined by because of the previous equality and the fact that $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$.

It is clear that λ is w-invariant. Since the central character of π is trivial on F^*, λ is also F^*-invariant. Because $\mathrm{GL}_2(F)$ is generated by $M(F)$, its center, and w, it remains to show that λ is $M(F)$-invariant.

Since ψ is trivial on F, we have $\lambda(\pi(m)\phi) = \lambda(\phi)$ if $\phi \in D(K^*)$ and $m \in M(F)$.

Now if $\phi = \pi(w)\phi_2 \in \pi(w)D(K^*)$ and if a belongs to F^*, then

$$\pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \pi(w)\phi_2 = \pi(w)\pi \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \phi_2 = \pi(w)\pi \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \phi_2$$

because the central character of π is trivial on F^*, and $\lambda\big(\pi \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \phi\big) = \lambda(\phi)$.

If $x \in F$, then $\pi \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \phi - \phi$ is a function in $D(K^*)$, which vanishes on F^*. Hence $\lambda\pi\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \phi - \phi\big) = 0$.

Eventually λ is $M(F)$-invariant and hence $G_2(F)$-invariant; it is clear that its restriction to $D(K^*)$ is nonzero. □

Corollary 4.2. *Let μ be a character of K^*. Then $\pi(\mu, \mu^{-\sigma})$ is distinguished.*

Proof. First note that the central character $\mu\mu^{-\sigma}$ of $\pi(\mu, \mu^{-\sigma})$ is trivial on F^*.

Now let χ be a character of K^*/F^*. Then

$$\gamma(\pi(\mu, \mu^{-\sigma}) \otimes \chi, \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-\sigma}\chi, \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-1}\chi^{\sigma}, \psi^{\sigma}),$$

and since $\psi|_F = 1$ and $\chi|_{F^*} = 1$, we have $\psi^{\sigma} = \psi^{-1}$ and $\chi^{\sigma} = \chi^{-1}$. Thus $\gamma(\pi(\chi, \chi^{-\sigma}), \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-1}\chi^{-1}, \psi^{-1}) = 1$. The conclusion then follows from Theorem 4.1. □

Using [Aizenbud and Gourevitch 2007, Theorem 1.2], Theorem 4.1's converse is also true:

Theorem 4.3. *Let π be a smooth irreducible representation of infinite dimension of $G_2(K)$ with central character trivial on F^*, and let ψ be a nontrivial character of K/F. Then π is distinguished if and only if $\gamma(\pi \otimes \chi, \psi) = 1$ for every character χ of K^* that is trivial on F^*.*

Proof. It suffices to show that if π is a smooth irreducible distinguished representation of infinite dimension of $G_2(K)$ and ψ is a nontrivial character of K/F, then $\gamma(\pi, \psi) = 1$. Suppose λ is a nonzero $G_2(F)$-invariant linear form on $K(\pi, \psi)$. The proof of the corollary to [Hakim 1991, Proposition 3.3] shows that $\lambda(\phi)$ is equal to $\int_{F^*} \phi(t)d^*t$ for ϕ in $D(K^*)$. Hence for any function ϕ in $D(K^*) \cap \pi(w)D(K^*)$, we must have $\int_{F^*} \phi(t)d^*t = \int_{F^*} \pi(w)\phi(t)d^*t$.

From this we deduce that for any function ϕ in $D(K^*) \cap \pi(w)D(K^*)$, we have

$$\begin{aligned}
\int_{K^*} \pi(w)\phi(x)c_\pi^{-1}(x)\,d^*x &= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi(w)\phi(ta)\,d^*t\,da \\
&= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \pi(w)\phi(t)\,d^*t\,da \\
&= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi(w)c_\pi(a)\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \phi(t)\,d^*t\,da \\
&= \int_{K^*/F^*} \int_{F^*} \pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \phi(t)\,d^*t\,da \\
&= \int_{K^*/F^*} \int_{F^*} \phi(ta^{-1})\,d^*t\,da \\
&= \int_{K^*/F^*} \int_{F^*} \phi(ta)\,d^*t\,da = \int_{K^*} \phi(x)\,d^*x.
\end{aligned}$$

This implies that either $\gamma(\pi, \psi)$ is equal to one or $\int_{K^*} \phi(x)d^*x$ is equal to zero on $D(K^*) \cap \pi(w)D(K^*)$. The latter cannot be the case, because we could then define two independent K^*-invariant linear forms on $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$ given by $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*} \phi_1(x)d^*x$ and $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*} \phi_2(x)d^*x$, which contradicts [Aizenbud and Gourevitch 2007, Theorem 1.2]. □

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