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We prove a uniform Sobolev inequality for Ricci flow that is independent of the number of surgeries. As an application, under fewer assumptions, we derive a noncollapsing result stronger than Perelman's κ-noncollapsing result with surgery. The proof is shorter and seems more accessible. The result also improves some earlier ones where the Sobolev inequality depended on the number of surgeries.

1. Introduction

A crucial step in Perelman's work on the Poincaré and the geometrization conjectures is the κ-noncollapsing result for Ricci flow with or without surgeries. The proof of this result in the surgery case requires a truly complicated calculation using new concepts such as reduced distance, admissible curve, barely admissible curve, gradient estimate of scalar curvature, and so on. This is thoroughly elucidated by Cao and Zhu [2006], Kleiner and Lott [2007] and Morgan and Tian [2007]. See also [Tao 2006] for a PDE point of view.

In this paper we prove a uniform Sobolev inequality for Ricci flow that is independent of the number of surgeries. It is well known that uniform Sobolev inequalities are essential in that they encode rich analytical and geometrical information about the manifold. These include noncollapsing and isoperimetric inequalities. As a consequence, we obtain a strong noncollapsing result, which includes Perelman's κ-noncollapsing result with surgery as a special case. Our result also requires fewer assumptions. For instance, we do not need the whole canonical neighborhood assumption for the manifold (see Remark 1.8 below). In the proof, we use only Perelman's W entropy and some analysis of the minimizer equation of the W entropy on hornlike manifolds. Hence it is shorter and seems more accessible.

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Keywords: strong noncollapsing, Sobolev imbedding, Ricci flow, surgery.

Let M be a compact Riemannian manifold with dimension $n \geq 3$ and metric g. Then a Sobolev inequality of the following form holds: there exist positive constants A and B such that, for all $v \in W^{1,2}(M, g)$,

$$\left(\int v^{2n/(n-2)}\, d\mu(g)\right)^{(n-2)/n} \leq A \int |\nabla v|^2\, d\mu(g) + B \int v^2\, d\mu(g). \tag{1-1}$$

This inequality was proved by Aubin [1976] for $A = K^2(n) + \epsilon$ with $\epsilon > 0$ and B depending on bounds on the injectivity radius and sectional curvatures. Here $K(n)$ is the best constant in the Sobolev imbedding for $\mathbb{R}^n$. Hebey [1996] showed that B can be chosen to depend only on ϵ, the injectivity radius, and the lower bound of the Ricci curvature. Hebey and Vaugon [1996] proved that one can even take $\epsilon = 0$. However, the constant B will also depend on the derivatives of the curvature tensor. Hence the controlling geometric quantities for B as stated above are not invariant under the Ricci flow in general. Theorem 1.6 states that a uniform Sobolev inequality of the type above holds under Ricci flow in finite time, even in the presence of an indefinite number of surgeries.

In order to state the theorem, we first introduce some notations. They are mainly taken from [Perelman 2002; 2003, Cao and Zhu 2006; Kleiner and Lott 2007; Morgan and Tian 2007].

We use $(M, g(t))$ to denote Hamilton's Ricci flow, satisfying $dg/dt = -2\,\mathrm{Ric}$. If a surgery occurs at time t, then $(M, g(t^-))$ denotes the preoperative manifold (the one just before the surgery) and $(M, g(t^+))$ denotes the postoperative manifold (the one just after). Denote by $B(x, t, r)$ the ball whose radius is r with respect to the metric $g(t)$ and whose center is at x. Denote the scalar curvature by $R = R(x, t)$. Put $R_0^- = \sup R^-(x, 0)$. Denote by $\mathcal{R}$ the full curvature tensor and by $d\mu(g(t))$ the volume element. The total volume of M under $g(t)$ is $\mathrm{vol}(M(g(t)))$.

In this paper we use the following definition of κ-noncollapsing by Perelman [2003], as elucidated in [Kleiner and Lott 2007, Definition 77.9].

Definition 1.1 (κ-noncollapsing). Let $(M, g(t))$ be a Ricci flow with surgery defined on $[a, b]$. Suppose that $x_0 \in M$, $t_0 \in [a, b]$ and $r > 0$ are such that $t_0 - r^2 \geq a$, $B(x_0, t_0, r) \subset M$ is a proper ball and the parabolic ball $P(x_0, t_0, r, -r^2)$ is unscathed. Then we say M is κ-collapsed at (x_0, t_0) at scale r if $|\mathcal{R}| \leq r^{-2}$ on $P(x_0, t_0, r, -r^2)$ and $\mathrm{vol}(B(x_0, t_0, r)) < \kappa r^3$; otherwise it is κ-noncollapsed.

Definition 1.2 (strong κ-noncollapsing). Let M be a Ricci flow with surgery defined on $[a, b]$. Suppose $x_0 \in M$, $t_0 \in [a, b]$ and $r > 0$ are such that $B(x_0, t_0, r) \subset M$ is a proper ball . Then M is strong κ-noncollapsed at (x_0, t_0) at scale r if $R \leq r^{-2}$ on $B(x_0, t_0, r)$ and $\mathrm{vol}(B(x_0, t_0, r)) \geq \kappa r^3$.

Strong κ-noncollapsing improves on κ-noncollapsing in two respects. First is that only information on the metric balls on one time level is needed. Thus it bypasses the complicated issue that a parabolic ball may be cut by a surgery. The other is that it places an upper bound on the scalar curvature instead of the full curvature tensor. When the Ricci flow is smooth, it is already known by Perelman that it is strong noncollapsed. However, this is not the case when surgeries are present.

Definition 1.3 (normalized manifold). A compact Riemannian manifold is normalized if $|\mathcal{R}| \leq 1$ everywhere and if the volume of every unit ball is at least half that of the Euclidean unit ball.

Definition 1.4 (ϵ-neck, ϵ-horn, double ϵ-horn, and ϵ-tube). An ϵ-neck (of radius r) is an open set that has a metric and, after scaling the metric by the factor r^{-2}, is ϵ-close in the $C^{\epsilon^{-1}}$ topology to the standard neck $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$. Here and later $C^{\epsilon^{-1}}$ means $C^{[\epsilon^{-1}]+1}$.

Let I be an open interval in R^1. An ϵ-horn (of radius r) is $S^2 \times I$ with a metric and the properties that each point is contained in some ϵ-neck, one end is contained in an ϵ-neck of radius r, and the scalar curvature tends to infinity at the other end.

An ϵ-tube is $S^2 \times I$ with a metric and the properties that each point is contained in some ϵ-neck and the scalar curvature stays bounded on both ends.

A double ϵ-horn is $S^2 \times I$ with a metric and the properties that each point is contained in some ϵ-neck and the scalar curvature tends to infinity at both ends.

Definition 1.5. A standard capped infinite cylinder is $\mathbb{R}^3$ equipped with a rotationally symmetric metric with nonnegative sectional curvature and positive scalar curvature such that, outside a compact set, it is a semiinfinite standard round cylinder $S^2 \times (-\infty, 0)$.

A few more basic facts concerning Ricci flow with surgery, such as (r, δ) surgery and δ-neck, are given in the appendix. For detailed information and related terminology, see [Cao and Zhu 2006; Kleiner and Lott 2007; Morgan and Tian 2007].

Here is our main result:

Theorem 1.6. *Given real numbers $T_1 < T_2$, let $(M, g(t))$ be a 3-dimensional Ricci flow with normalized initial condition defined on the time interval containing $[T_1, T_2]$. Suppose the following conditions are met.*

(a) *There are finitely many (r, δ) surgeries in $[T_1, T_2]$, occurring in ϵ-horns of radii r. Here $r \leq r_0$ and $\epsilon \leq \epsilon_0$, with r_0 and ϵ_0 being fixed sufficiently small positive numbers less than 1. The surgery radii are $h \leq \delta^2 r$, that is, the surgeries occur in δ-necks of radius $h \leq \delta^2 r$. Here $0 < \delta \leq \delta_0$, where $\delta_0 = \delta_0(r, \epsilon_0) > 0$ is sufficiently small. Outside of the ϵ-horns, the Ricci flow is smooth.*

(b) *For a constant $c > 0$ and any point x in all the above ϵ-horns, there is a region U satisfying $B(x, c\epsilon^{-1}R^{-1/2}(x)) \subset U \subset B(x, 2c\epsilon^{-1}R^{-1/2}(x))$ such that, after scaling by a factor $R(x)$, it is ϵ-close in the $C^{\epsilon^{-1}}$ topology to $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$.*

Also, for any x in the modified part of the ϵ-horn immediately after a surgery, the ball $B(x, \epsilon^{-1}R^{-1/2}(x))$, is, after scaling by a factor $R(x)$, ϵ-close in the $C^{\epsilon^{-1}}$ topology to the corresponding ball of the standard capped infinite cylinder.

(c) *For $A_1 > 0$ and $n = 3$, the Sobolev imbedding*

$$\Big(\int v^{2n/(n-2)}\,d\mu(g(T_1))\Big)^{(n-2)/n} \le A_1 \int (4|\nabla v|^2 + Rv^2 + v^2)\,d\mu(g(T_1))$$

holds for all $v \in W^{1,2}(M, g(T_1))$.

Then for all $t \in (T_1, T_2]$, the Sobolev imbedding

$$\Big(\int v^{2n/(n-2)}\,d\mu(g(t))\Big)^{(n-2)/n} \le A_2 \int (4|\nabla v|^2 + Rv^2 + v^2)\,d\mu(g(t))$$

holds for all $v \in W^{1,2}(M, g(t))$. Here

$$A_2 = C\big(A_1, \sup R^-(x,0), T_2, T_1, \sup_{t\in[T_1,T_2]} \mathrm{vol}(M(g(t)))\big),$$

is independent of the number of surgeries or r.

Moreover, the Ricci flow is strong κ-noncollapsed in the whole interval $[T_1, T_2]$ under scale 1, where κ depends only on A_2.

Remark 1.7. By [Hebey 1996], at any given time, a Sobolev imbedding always holds with constants depending on lower bound of Ricci curvature and injectivity radius. So one can replace assumption (c) by the assumptions that $(M, g(T_1))$ is κ-noncollapsed and that the canonical neighborhood assumption (with a fixed radius $r_0 > 0$ and $\epsilon_0 > 0$) holds at time T_1. It is easy to see that these together imply the Sobolev imbedding at T_1.

We assume as usual that, at a surgery, we throw away all compact components with positive sectional curvature, and also capped horns, double horns and all compact components lying in the region where $R > (\delta r)^{-2}$. In the extra assumption that the Ricci flow is smooth outside of the ϵ-horns, we have excluded these deleted items. By keeping track of the constants in the proof, one can see that A_2 is bounded from above by $C \max\{1, T_2 - T_1, \sup_{t\in[T_1,T_2]} \mathrm{vol}(M(g(t))\}$. It is known that $\mathrm{vol}(M(g(t)) \le C(1 + t^{3/2})$. If one can choose the initial scalar curvature to be nonnegative everywhere, then A_2 can be chosen as a constant independent of the lifespan of the Ricci flow. This is due to the facts that the volume does not increase with time and that the Sobolev constant is uniformly bounded in this case; see [Zhang 2007c, Remark 1.2(2)].

Remark 1.8. With the exception of using the monotonicity of Perelman's W entropy, the proof of Theorem 1.6 uses only long-established results. Under (r, δ) surgery, assumption (b) is clearly implied by, but much weaker than, the *canonical neighborhood assumption* on the whole manifold M, which was used in all other papers so far. In particular, there is no need for the gradient estimate on the scalar curvature, which is difficult to prove by itself.

However, in proving long time existence of Ricci flow with surgery, one must show that the canonical neighborhood assumption holds, using a delicate contradiction argument; see [Perelman 2003; Cao and Zhu 2006; Kleiner and Lott 2007; Morgan and Tian 2007]. In this argument, one supposes the canonical neighborhood assumption at a fixed scale first breaks down at a certain time. Then it can be shown that this same assumption holds simultaneously at a larger scale. Using this, one can prove the noncollapsing property, which in turn will lead to a contradiction through a blow-up argument. During this process, the gradient estimate for the scalar curvature is still required. Also, this proof of noncollapsing with surgeries via Theorem 1.6 seems to work only in the case of the Poincaré conjecture. For the full geometrization conjecture, so far one must use Perelman's argument via reduced distance. We hope to address this problem in the future. The motivation is that a Sobolev imbedding implies more information than just the noncollapsing property.

Remark 1.9. Zhang [2007a] showed that under Ricci flow with a finite number of surgeries in finite time, a uniform Sobolev imbedding holds. Recently, the preprint [Ye 2007] stated without proof a similar result depending on the number of surgeries.

Remark 1.10. The strong noncollapsing result clearly implies Hamilton's little loop conjecture with surgeries. That is, if the curvature tensor in a small geodesic ball is bounded, then the injectivity radius is bounded from below. The conjecture was proved by Perelman only in the case without surgery. In the case with surgery, using among other things the method of reduced distance, Perelman proved the lower bound of the injectivity radius under the more restrictive assumption that the curvature tensor is bounded in a parabolic cube.

Let us finish the introduction by outlining the proof. Recall Perelman's W entropy and its monotonicity, which are in fact the monotonicity of the best constants of the log Sobolev inequality with certain parameters. If a Ricci flow is smooth over a finite time interval, then these best constants do not decrease as the parameters change. If a Ricci flow undergoes a (r, δ) surgery with δ sufficiently small, then the best constant only decreases by at most a constant times the change in volume. This proves the essential monotonicity of the W entropy under surgeries; see (2-21) below. This is achieved by a weighted estimate of Agmon type for the minimizing equation of the W entropy. The method is motivated by those at the ends of [Perelman 2003] and [Kleiner and Lott 2007], which studied the change of eigenvalues of the linear operator $4\Delta - R$. Since our case is nonlinear and contains an extra parameter, more analysis and estimates are needed. In the end we prove, in finite time, the best constant of the log Sobolev inequality (see [Gross 1975]) with certain parameters is uniformly bounded from below by a negative constant, regardless of the number of surgeries. This uniform log Sobolev inequality is then converted by known methods to the desired uniform Sobolev inequality, which in turn yields the strong noncollapsing property. The estimate of the change of the best constant of the log Sobolev inequality under one surgery seems to be of interest independent of the study of Ricci flow.

This paper was first posted as [Zhang 2007b] (in December), and the announcement appeared in [Zhang 2008] the following May.

2. Proof of Theorem 1.6.

The theorem will require three lemmas. Much of the analysis is focused on the ϵ-horn where a surgery takes place. So we will fix some notations and basic facts concerning the ϵ-horn and the surgery cap. Also we will use c with or without index to denote a generic positive constant.

Recall that a (r, δ) surgery occurs deep inside an ϵ-horn of radius r. The horn is cut open at the place where the radius is $h \le \delta^2 r$. Then a cap is attached and a smooth metric is constructed by interpolating between the metric on the horn and the metric on the cap. The manifold just after surgery is denoted by M^+, and the ϵ-horn thus surgically modified is called a capped ϵ-horn with radius r.

Let D be a capped ϵ-horn. By assumption, a region N around the boundary ∂D equipped with the scaled metric $cr^{-2}g$ is ϵ-close, in the $C^{\epsilon^{-1}}$ topology, to the standard round neck $S^2 \times (-\epsilon, \epsilon)$. Here c is a generic positive constant such that cr^{-2} equals the scalar curvature at a point on ∂D. For this reason we will often take $c = 1$.

Let Π be the diffeomorphism, from the definition of ϵ-closeness, that maps from the standard round neck to N. Denote by z a number in $(-\epsilon^{-1}, \epsilon^{-1})$. For $\theta \in S^2$, (θ, z) is a parametrization of N via the diffeomorphism Π. In this way, we can identify the metric on N with its pullback by Π on the round neck. We normalize the parameters so that the capped ϵ-horn lies in the region where $z \ge 0$.

Next we define

$$Y(D) = \inf\left\{ \frac{\int 4|\nabla v|^2 + Rv^2}{\left(\int v^{2n/(n-2)}\right)^{(n-2)/n}} \;\middle|\; v \in C_0^\infty(D \cup N),\ v > 0 \right\}. \tag{2-1}$$

Proposition 2.1. *For sufficiently small* $\epsilon > 0$, *there exist positive constants* C_1 *and* C_2 *such that* $C_1 \le Y(D) \le C_2$.

Proof. Since R is positive in $D \cup N$, $Y(D)$ is bounded from above and below by constant multiples of the Yamabe constant

$$Y_0(D) = \inf\left\{ \int \frac{4\frac{(n-1)}{(n-2)}|\nabla v|^2 + Rv^2}{(\int v^{2n/(n-2)})^{(n-2)/n}} \,\middle|\, v \in C_0^\infty(D \cup N),\ v > 0 \right\}.$$

So it suffices to prove that the Yamabe constant is bounded between two positive constants.

Let $g = g(x)$ be the metric on $D \cup N$. Then $Y_0(D)$ is the same under the metric $g_1(x) = R(x)g(x)$.

Consider the manifold $(D \cup N, g_1)$. By assumption and the (r, δ) surgery procedure, there is a fixed $r_0 > 0$ such that for any $x \in D \cup N$, the ball $B(x, r_0)$ is ϵ-close under $g_2(y) = R(x)g(y)$ for $y \in B(x, r_0)$ (in the $C^{\epsilon^{-1}}$ topology) to a part of the standard capped infinite cylinder. Therefore, for y in the same geodesic ball, the scaled scalar curvature $R^{-1}(x)R(y)$ is ϵ-close, in the $C^{\epsilon^{-1}-2}$ norm, to a positive function. This positive function is the scalar curvature in the standard capped cylinder, which is both uniformly bounded away from 0 and bounded from above. Actually, $R(y) = R(x)(h(y) + \xi(y, \epsilon))$, where $h(y) = 1$ when y is away from the surgery cap and $h(y)$ is the scalar curvature of the surgery cap. The $C^{\epsilon^{-1}-2}$ norm of ξ is less than $c \cdot \epsilon$.

Hence, for y away from the surgery cap and under the metric $g_1(y) = R(y)g(y)$, the same geodesic ball is ϵ-close (in $C^{\epsilon^{-1}-2}$ topology) to a part of the standard capped infinite cylinder. For y in the surgery cap, the curvatures in the metric $g_1(y) = R(y)g(y)$ are uniformly bounded since $h = h(y)$ has bounded C^2 norm.

Since ϵ is sufficiently small, we know that the injectivity radius of $(D \cup N, g_1)$ is bounded from below by a positive constant and that its Ricci curvature is bounded from below. Actually, it is easy to see that these hold for a much larger domain containing $(D \cup N, g_1)$. By [Hebey 1996, Proposition 6], we can find a positive constant C such that

$$\left(\int v^{2n/(n-2)}\, d\mu(g_1)\right)^{(n-2)/n} \le C \int (|\nabla_1 v|^2 + v^2)\, d\mu(g).$$

Recall the scalar curvature of $(D \cup N, g_1)$ is bounded between two positive constants outside of the surgery cap. Inside the surgery cap, the scalar curvature is bounded from below by an absolute negative constant. Therefore for a constant still named C,

$$\left(\int v^{2n/(n-2)}\, d\mu(g_1)\right)^{(n-2)/n} \le C \int \left(4\,\tfrac{n-1}{n-2}\, |\nabla_1 v|^2 + R_1 v^2 + v^2\alpha^2\right) d\mu(g_1)$$

for all $v \in C_0^\infty(D \cup N)$. Here α is a nonnegative, smooth function supported in a neighborhood of the surgery cap; it is bounded from above by an absolute constant. Also, ∇_1 and R_1 are the gradient and scalar curvature under g_1, respectively. Note that R_1 may not be positive inside the surgery cap.

Now we scale back to the metric $g = R^{-1}(y) g_1(y)$. By conformal invariance of all but the last term, it is easy to check that, after renaming $R^{(n-2)/4} v$ by v,

$$\Big(\int v^{2n/(n-2)}\, d\mu(g)\Big)^{(n-2)/n} \le C \int \big(4\tfrac{n-1}{n-2}|\nabla v|^2 + Rv^2 + v^2(x)R(x)\alpha^2(x)\big)\, d\mu(g)$$

for all $v \in C_0^\infty(D \cup N)$. Now the scalar curvature is positive everywhere.

Hence we see that $Y_0(D)$ is bounded from below by a positive constant when ϵ is sufficiently small. It is also bounded from above by the Yamabe constant of S^n. Since $Y_0(D)$ and $Y(D)$ are comparable, we have that $0 < c_1 \le Y(D) \le c_2$ for sufficiently small ϵ. □

Lemma 2.2. *Let (M^+, g) be a manifold just after (r, δ) surgery. Let $D \subset M^+$ be a capped ϵ-horn of radius r, where ϵ is a sufficiently small positive number.*

Suppose u with $\|u\|_{L^2(M^+)} = 1$ is a positive solution to the equation

$$\sigma^2(4\Delta u - Ru) + 2u \ln u + \Lambda u + n(\ln \sigma)u = 0. \tag{2-2}$$

Here $\sigma > 0$ and $\Lambda \le 0$.

Then there exists a positive constant C depending only on $Y(D)$ and n, but not on the smallness of ϵ, such that $\sup_D u^2 \le C \max(r^{-n}, \sigma^{-n})$.

Proof. Under the scaling

$$g_1 = \sigma^{-2} g, \quad R_1 = \sigma^2 R, \quad u_1 = \sigma^{n/2} u,$$

we see that u_1 satisfies $4\Delta_1 u_1 - R_1 u_1 + 2u_1 \ln u_1 + \Lambda u_1 = 0$. Since the result in the lemma is independent of this scaling, we need only prove it for $\sigma = 1$.

So let u be a positive solution in M^+ of $4\Delta u - Ru + 2u \ln u + \Lambda u = 0$ with unit L^2 norm. Given any $p \ge 1$, it is easy to see that

$$-4\Delta u^p + pRu^p \le 2pu^p \ln u. \tag{2-3}$$

We select a smooth cutoff function ϕ that is 1 in D and 0 outside of $D \cup N$. Writing $w = u^p$ and using $w\phi^2$ as a test function in (2-3), we find

$$4\int \nabla(w\phi^2)\nabla w + p\int R(w\phi)^2 \le 2p \int (w\phi)^2 \ln u.$$

Since the scalar curvature R is positive in the support of ϕ, and $p \ge 1$, this shows

$$4\int \nabla(w\phi^2)\nabla w + \int R(w\phi)^2 \le p\int (w\phi)^2 \ln u^2.$$

Using integration by parts, we have

$$4\int|\nabla(w\phi)|^2+\int R(w\phi)^2 \le 4\int|\nabla\phi|^2w^2+p\int(w\phi)^2\ln u^2. \tag{2-4}$$

We need to dominate the last term in (2-4) by the left side of (2-4). For one positive number a to be chosen later, it is clear that $\ln u^2 \le u^{2a}+c(a)$. Hence for any fixed $q>n/2$, the Hölder inequality implies

$$\begin{aligned} p\int(w\phi)^2\ln u^2 &\le p\int(w\phi)^2u^{2a}+pc(a)\int(w\phi)^2 \\ &\le p\Big(\int u^{2aq}\Big)^{1/q}\,\Big(\int(w\phi)^{2q/(q-1)}\Big)^{(q-1)/q}+pc(a)\int(w\phi)^2. \end{aligned}$$

We take $a=1/q$, so that $2aq=2$. Since the L^2 norm of u is 1 by assumption, the above implies

$$p\int(w\phi)^2\ln u^2 \le p\Big(\int(w\phi)^{2q/(q-1)}\Big)^{(q-1)/q}+pc(a)\int(w\phi)^2.$$

By the interpolation inequality (see for example [Han and Lin 1997, page 84]), we have, for any $b>0$, that

$$\Big(\int(w\phi)^{2q/(q-1)}\Big)^{(q-1)/q} \le b\Big(\int(w\phi)^{2n/(n-2)}\Big)^{(n-2)/n} + c(n,q)b^{-n/(2q-n)}\int(w\phi)^2.$$

Therefore

$$p\int(w\phi)^2\ln u^2 \le pb\Big(\int(w\phi)^{2n/(n-2)}\Big)^{(n-2)/n} + c(n,q)pb^{-n/(2q-n)}\int(w\phi)^2+pc(a)\int(w\phi)^2. \tag{2-5}$$

By the definition of $Y(D)$ in (2-1), we see that (2-4) gives

$$Y(D)\Big(\int(w\phi)^{2n/(n-2)}\Big)^{(n-2)/n} \le 4\int|\nabla\phi|^2w^2+p\int(w\phi)^2\ln u^2.$$

Substituting (2-5) in the right side of this, we get

$$Y(D)\Big(\int w^{2n/(n-2)}\Big)^{(n-2)/n} \le 4\int|\nabla\phi|^2w^2+pb\Big(\int(w\phi)^{2n/(n-2)}\Big)^{(n-2)/n} + c(n,q)pb^{-n/(2q-n)}\int(w\phi)^2+pc(a)\int(w\phi)^2.$$

Take b so that $pb=Y(D)/2$. Clearly there is a positive constant $c=c(Y(D),n,q)$ and an $\alpha=\alpha(n,q)$ such that

$$\Big(\int(w\phi)^{2n/(n-2)}\Big)^{(n-2)/n} \le c(p+1)^\alpha\int(|\nabla\phi|^2+1)w^2. \tag{2-6}$$

From here one can use Moser's iteration to prove the desired bound. Let z be the longitudinal parameter for D described before the lemma. For z_2 and z_1 such that $-1 \le z_2 < z_1 < 0$, we construct a smooth function ξ of z such that $\xi(z) = 1$ when $z \ge z_1$, $\xi(z) = 0$ when $z < z_2$, and $\xi(z) \in (0, 1)$ for all other z. Set the test function $\phi = \xi(z) = \xi(z(x))$. Then it is clear that

$$|\nabla\phi| \le \frac{c}{r(z_1 - z_2)}. \tag{2-7}$$

Write $D_i = \{x \in M^+ \mid z(x) > z_i\}$ for $i = 1, 2$. By (2-6) and (2-7),

$$\Big(\int_{D_1} w^{2n/(n-2)}\Big)^{(n-2)/n} \le c \max\Big\{\frac{1}{((z_1 - z_2)r)^2}, 1\Big\}(p+1)^\alpha \int_{D_2} w^2. \tag{2-8}$$

Recall that $w = u^p$. We iterate (2-8) with $p = (n/(n-2))^i$ for $i = 0, 1, 2, \dots,$ while choosing

$$z_1 = -(1/2 + 1/2^{i+2}) \quad \text{and} \quad z_2 = -(1/2 + 1/2^{i+1}).$$

Following Moser, we get $\sup_D u^2 \le C \max(r^{-n}, 1) \int u^2$. □

Remark. One can avoid using Proposition 2.1 by working directly on each ϵ-neck and the surgery cap as above. Then one can show $u^2(x) \le C \max\{R^{n/2}(x), \sigma^{-n}\}$. This weaker bound suffices for proving the main result, as will be made clear in the proof below.

The next lemma is a nonlinear version of the result in [Perelman 2003] and [Kleiner and Lott 2007, Lemma 92.10]. The estimate has its origin in the weighted Agmon-type estimate of eigenfunctions of the Laplacian.

Lemma 2.3. *Let (M, g) be any compact manifold without boundary. Suppose u is a positive solution to the inequality*

$$4\Delta u - Ru + 2u \ln u + \Lambda u \ge 0 \quad \textit{with } \Lambda \le 0. \tag{2-9}$$

Given a nonnegative function $\phi \in C^\infty(M)$ with $\phi \le 1$, suppose there is a smooth function f that, when $R \ge 0$ in the support of ϕ, satisfies

$$4|\nabla f|^2 \le R - 2\ln^+ u + |\Lambda|/2 \quad \textit{in the support of } \phi.$$

Then

$$\tfrac{1}{2}|\Lambda| \|e^f \phi u\|_2 \le 8 \sup_{x \in \operatorname{supp} \nabla\phi} \Big(e^f (R - 2\ln^+ u + |\Lambda|/2)^{1/2} + \|e^f \nabla\phi\|_\infty\Big) \|u\|_2.$$

Proof. The main point of the lemma is that the right side depends only on information in the support of $\nabla\phi$.

Using integration by parts,

$$\int e^f\phi u\left(-4\Delta+R-2\ln u-\Lambda-4|\nabla f|^2\right)(e^f\phi u)$$
$$=4\int|\nabla(e^f\phi u)|^2+\int(e^f\phi u)^2(R-2\ln u-\Lambda-4|\nabla f|^2).$$

By assumption, $|\Lambda|/2\le R-2\ln u-\Lambda-4|\nabla f|^2$. Hence

$$\begin{aligned}\frac{|\Lambda|}{2}\int(e^f\phi u)^2&\le\int e^f\phi u\bigl(-4\Delta+R-2\ln u-\Lambda-4|\nabla f|^2\bigr)(e^f\phi u)\\&=\int(e^f\phi)^2u\,(-4\Delta u+Ru-2u\ln u-\Lambda u)\\&\quad-\int e^f\phi u\left[8\nabla(e^f\phi)\nabla u+4\Delta(e^f\phi)u\right]-4\int(e^f\phi u)^2|\nabla f|^2\\&\le-\int e^f\phi u\left[8\nabla(e^f\phi)\nabla u+4\Delta(e^f\phi)u\right]-4\int(e^f\phi u)^2|\nabla f|^2.\end{aligned}$$

The equality above is by a straightforward calculation, and the last step follows from (2-9). Performing integration by parts on the term containing Δ, we deduce

$$\begin{aligned}&\frac{|\Lambda|}{2}\int(e^f\phi u)^2\\&\quad\le-8\int e^f\phi u\nabla(e^f\phi)\nabla u+\int4\nabla(e^f\phi)\nabla(e^f\phi u^2)-4\int(e^f\phi u)^2|\nabla f|^2\\&\quad\le4\int|\nabla(e^f\phi)|^2u^2-4\int(e^f\phi u)^2|\nabla f|^2\\&\quad\le4\int\bigl(|(e^f\phi)|^2|\nabla f|^2+2e^{2f}(\nabla f\nabla\phi)\phi+e^{2f}|\nabla\phi|^2\bigr)u^2-4\int(e^f\phi u)^2|\nabla f|^2\\&\quad=8\int e^{2f}(\nabla f\nabla\phi)\phi u^2+4\int e^{2f}|\nabla\phi|^2u^2.\end{aligned}$$

(In the last step, the first and the last terms canceled.) Note that the integrations on the right side only take place in the support of $\nabla\phi$. Thus this shows, by the assumption on $|\nabla f|^2$, that

$$\begin{aligned}\frac{|\Lambda|}{2}\int(e^f\phi u)^2&\le4\int_{\operatorname{supp}\nabla\phi}e^{2f}|\nabla f|^2\phi^2u^2+8\int e^{2f}|\nabla\phi|^2u^2\\&\le\int_{\operatorname{supp}\nabla\phi}e^{2f}(R-2\ln^+u+|\Lambda|/2)\phi^2u^2+8\int e^{2f}|\nabla\phi|^2u^2.\end{aligned}$$

The lemma follows by pulling out the supremum of the non-u^2 terms. □

Lemma 2.4. *Let (M, g) be any compact manifold without boundary and X be a domain in M. Define*

$$\lambda_X = \inf\Big\{\textstyle\int (4|\nabla v|^2 + Rv^2 - v^2 \ln v^2) \mid v \in C_0^\infty(X),\ \|v\|_2 = 1\Big\},$$
$$\lambda_M = \inf\Big\{\textstyle\int (4|\nabla v|^2 + Rv^2 - v^2 \ln v^2) \mid v \in C^\infty(M),\ \|v\|_2 = 1\Big\}.$$

Let $u \in C_0^\infty(X)$ be the (positive) minimizer for λ_M. For any smooth cutoff function $\eta \in C_0^\infty(X)$ with $0 \le \eta \le 1$, we have

$$\lambda_X \le \lambda_M + 4\frac{\int u^2|\nabla\eta|^2}{\int (u\eta)^2} - \frac{\int (u\eta)^2 \ln \eta^2}{\int (u\eta)^2}.$$

Proof. Since $\eta u/\|\eta u\|_2 \in C_0^\infty(X)$, with an L^2 norm of 1, we have by definition

$$\lambda_X \le \int \bigg(4\frac{|\nabla(\eta u)|^2}{\|\eta u\|_2^2} + R\frac{(\eta u)^2}{\|\eta u\|_2^2} - \frac{(\eta u)^2}{\|\eta u\|_2^2} \ln \frac{(\eta u)^2}{\|\eta u\|_2^2}\bigg).$$

This implies

$$\lambda_X \|\eta u\|_2^2 \le \int \big(4|\nabla(\eta u)|^2 + R(\eta u)^2 - (\eta u)^2 \ln(\eta u)^2\big) + \|\eta u\|_2^2 \ln\|\eta u\|_2^2. \tag{2-10}$$

On the other hand, u is a smooth positive solution (see [Rothaus 1981]) of the equation $4\Delta u - Ru + 2u \ln u + \lambda_M u = 0$. Using $\eta^2 u$ as a test function for the equation, we deduce

$$\lambda_M \int (\eta u)^2 = -4\int (\Delta u)\eta^2 u + \int R(\eta u)^2 - 2\int (\eta u)^2 \ln u.$$

By direct calculation, $-4\int (\Delta u)\eta^2 u = 4\int |\nabla(\eta u)|^2 - 4\int u^2|\nabla\eta|^2$. Hence

$$\lambda_M \int (\eta u)^2 = 4\int |\nabla(\eta u)|^2 - 4\int u^2|\nabla\eta|^2 + \int R(\eta u)^2 - 2\int (\eta u)^2 \ln u.$$

Comparing this with (2-10) and noting that $\|\eta u\|_2 < 1$, we obtain

$$\lambda_X \|\eta u\|_2^2 \le \lambda_M \|\eta u\|_2^2 + 4\int |\nabla\eta|^2 u^2 - \int (\eta u)^2 \ln \eta^2. \qquad \square$$

Proof of Theorem 1.6. At a given time t in a Ricci flow $(M, g(t))$ and for $\sigma > 0$, let us define

$$\lambda_{\sigma^2}(g(t)) = \inf\bigg\{\int (\sigma^2(4|\nabla v|^2 + Rv^2) - v^2 \ln v^2)\, d\mu(g(t)) - n \ln \sigma \ \bigg|\ v \in C^\infty(M),\ \|v\|_2 = 1\bigg\}. \tag{2-11}$$

Sometimes we refer to λ_{σ^2} as the best log Sobolev constant with parameter σ. If t happens to be a surgery time, then $\lambda_{\sigma^2}(g(t^+))$ is the best log Sobolev constant with parameter σ for the manifold just after surgery, and

$$\lambda_{\sigma^2}(g(t^-)) \equiv \lim_{s\to t^-} \lambda_{\sigma^2}(g(s)).$$

We will see in Step 2 below that such a limit exists.

The main aim is to find a uniform lower bound for $\lambda_{\sigma^2}(g(t))$ for $t \in [T_1, T_2]$ and $\sigma \in (0, 1]$. So without loss of generality, we may assume it is negative.

The rest of the proof is divided into five steps.

Step 1. We estimate the change of $\lambda_{\sigma^2}(t)$, the best constant of the log Sobolev inequality after one (r, δ) surgery.

It will be clear that the proof below is independent of the number of cutoffs occurring at one surgery time T. Therefore we may assume that there is just one ϵ-horn and one cutoff at T.

Let $(M, g(T^+))$ be the manifold right after the surgery, and let $\Lambda \equiv \lambda_{\sigma^2}(g(T^+))$ be the best constant for this postsurgical manifold, defined in (2-11).

By [Rothaus 1981], there is a smooth positive function u that reaches the infimum in (2-11) and u solves

$$\sigma^2(4\Delta u - Ru) + 2u \ln u + \Lambda u + n(\ln\sigma)u = 0.$$

After taking the scaling

$$g_1 = \sigma^{-2} g(T^+), \quad R_1 = \sigma^2 R, \quad d_1 = \sigma^{-1} d, \quad u_1 = \sigma^{n/2} u,$$

we see that u_1 satisfies

$$4\Delta_1 u_1 - R_1 u_1 + 2u_1 \ln u_1 + \Lambda u_1 = 0 \tag{2-12}$$

and

$$\Lambda = \inf\left\{ \int ((4|\nabla_{g_1} v|^2 + R_1 v^2 - v^2 \ln v^2)\, d\mu(g_1) \;\middle|\; v \in C^\infty(M^+),\ \|v\|_2 = 1 \right\}.$$

Denote by U the $\sigma^{-1}h$ neighborhood of the surgery cap C under g_1, that is,

$$U = \left\{ x \in (M, g_1(T^+)) \middle| d_1(x, C) < \sigma^{-1} h \right\} = \left\{ x \in M^+ \middle| d(x, C) < h \right\}.$$

Note that $U - C$ is the part of the ϵ-tube that is unaffected by the surgery. Therefore $U - C$ is ϵ-close to a portion of the standard round neck under the scaled metric $\sigma^2 h^{-2} g_1$. Actually, it is even $\delta(< \epsilon)$-close, since it is part of the strong δ-neck. But we do not need this fact. Following the description at the beginning of the section, there is a longitudinal parametrization z of $U - C$ that maps $U - C$ to $(-1, 0) \subset (-\epsilon^{-1}, \epsilon^{-1})$. Let $\zeta : [-1, 0] \to [0, 1]$ be a smooth decreasing function such that $\zeta(-1) = 1$ and $\zeta(0) = 0$. Then $\eta \equiv \zeta(z(x))$ maps $U - C$ to $(0, 1)$. We then extend η to be a cutoff function on the whole manifold by setting $\eta = 1$ in $M^+ - U$ and $\eta = 0$ in C.

Define

$$\Lambda_X = \inf\Big\{ \int ((4|\nabla_{g_1} v|^2 + R_1 v^2 - v^2 \ln v^2)\, d\mu(g_1) \;\Big|\; v \in C_0^\infty(M^+ - C),\ \|v\|_2 = 1 \Big\}.$$

Then it is clear that $\lambda_{\sigma^2}(g(T^-)) \le \Lambda_X$. By Lemma 2.4,

$$\Lambda_X \le \Lambda + 4\,\frac{\int u_1^2 |\nabla_{g_1}\eta|^2\, d\mu(g_1)}{\int (u_1\eta)^2\, d\mu(g_1)} - \frac{\int (u_1\eta)^2 \ln \eta^2\, d\mu(g_1)}{\int (u_1\eta)^2\, d\mu(g_1)}.$$

Observe that the supports of $\nabla_{g_1}\eta$ and $\eta \ln \eta$ are in $U - C$. Moreover

$$|\nabla_{g_1}\eta| \le c\sigma/h \quad \text{and} \quad -\eta^2 \ln \eta^2 \le c.$$

Therefore the above shows that

$$\text{(2-13)}\quad \lambda_{\sigma^2}(g(T^-)) \le \Lambda_X \le \Lambda + \frac{4c\sigma^2}{h^2}\,\frac{\int_U u_1^2\, d\mu(g_1)}{1 - \int_U u_1^2\, d\mu(g_1)} + c\,\frac{\int_U u_1^2\, d\mu(g_1)}{1 - \int_U u_1^2\, d\mu(g_1)}.$$

Recall that $\Lambda = \lambda_{\sigma^2}(g(T^+))$. So, to bound it from below, we need to show that $\int_U u_1^2\, d\mu(g_1)$ is small. This is where we will use Lemmas 2.2 and 2.3.

Under the metric $g_1 = \sigma^{-2} g$, the capped ϵ-horn D of radius r under $g(T^+)$ is just a capped ϵ-horn of radius $r_1 = \sigma^{-1} r$. Using the longitudinal parametrization z of D described at the beginning of this section, we can construct a cutoff function $\phi = \phi(z(x))$ for $x \in M^+$, with the following properties:

(i) The set $\{x \in M \mid z(x) = 0\}$ is the boundary of D.

(ii) If $z \le 0$, then $\phi(z) = 0$, and if $z \ge 1$, then $\phi(z) = 1$.

(iii) $0 \le \phi \le 1$ and $|\nabla_{g_1}\phi| \le c/r_1$.

(iv) ϕ is 0 outside of D and is 1 to the right of the set $\{x \in M^+ \mid z(x) = 1\}$.

Note that the support of $\nabla\phi$ is in the set where z is between 0 and 1. Applying Lemma 2.2 on u_1, which satisfies (2-12), we know that

$$u_1(x) \le c \max\{1/r_1^{n/2}, 1\} \quad \text{for } \in D.$$

Hence, for a negative number Λ_0 with $|\Lambda_0|$ sufficiently large,

$$\text{(2-14)}\quad \begin{aligned} &R_1(x) - 2\ln^+ u_1(x) + \tfrac12|\Lambda_0| \le c r_1^{-2} + \tfrac12|\Lambda_0| \qquad \text{for } x \in \operatorname{supp} \nabla_{g_1}\phi,\\ &R_1(x) - 2\ln^+ u_1(x) + \tfrac12|\Lambda_0| \ge \tfrac12 R_1(x) + c r_1^{-2}\\ &\qquad\qquad\qquad\qquad - c_1 \ln^+ \max\{1/r_1, 1\} + \tfrac12|\Lambda_0|\\ &\qquad\qquad\qquad\qquad \ge \tfrac12 R_1(x) + \tfrac14|\Lambda_0| \quad \text{for } x \in D. \end{aligned}$$

We stress that Λ_0 is independent of the size of $r_1 = \sigma/r$, which could be large or small due to the scaling factor σ.

Recall that we desire a uniform lower bound for Λ. If $\Lambda = \lambda_{\sigma^2}(g(T^+)) \geq \Lambda_0$, then we are in good shape. So we assume throughout that $\Lambda \leq \Lambda_0$. Then, by (2-12), we have

$$4\Delta_1 u_1 - R_1 u_1 + 2u_1 \ln u_1 + \Lambda_0 u_1 \geq 0. \tag{2-15}$$

Motivated by [Kleiner and Lott 2007, Lemma 92.10], we choose a function $f = f(x)$ as the distance between x and the set $z^{-1}(0)$ under the metric

$$\tfrac{1}{4}(R_1(x) - 2\ln^+ u_1(x) + \tfrac{1}{2}|\Lambda_0|)g_1(x) \quad \text{for } x \in D.$$

By the first inequality in (2-14),

$$4|\nabla_{g_1} f|^2 \leq \begin{cases} c r_1^{-2} + \frac{1}{2}|\Lambda_0| & \text{in the support of } \nabla_{g_1}\phi, \\ R_1(x) - 2\ln^+ u_1(x) + \frac{1}{2}|\Lambda_0| & \text{in } D. \end{cases} \tag{2-16}$$

Note that the “in D” case of (2-16) is positive by the second inequality in (2-14).

Inequalities (2-16) and (2-15) allow us to use Lemma 2.3 (with Λ replaced by Λ_0) to conclude

$$\tfrac{1}{2}|\Lambda_0| \|e^f \phi u_1\|_2 \leq 8\|u_1\|_2 \Big(\sup_{x \in \operatorname{supp} \nabla_{g_1}\phi} e^f (R_1 - 2\ln^+ u_1 + \tfrac{1}{2}|\Lambda_0|)^{1/2} + \|e^f \nabla_{g_1}\phi\|_\infty \Big).$$

Here the underlying metric is g_1. By the first inequality of (2-14), this shows

$$\tfrac{1}{2}|\Lambda_0| \|e^f \phi u_1\|_2 \leq c\|u_1\|_2 \sup_{x \in \operatorname{supp} \nabla_{g_1}\phi} \big(e^f ((1/r_1^2 + |\Lambda_0|))^{1/2} \big). \tag{2-17}$$

From (2-17), we will derive a bound for $\|u_1\|_{L^2(U)}$ that holds for all finite σ. Here and later, $\|u_1\|_{L^2(U)}$ stands for integration under the metric g_1.

First, we note from (2-17)

$$\tfrac{1}{2}|\Lambda_0| \inf_U e^f \|u_1\|_{L^2(U)} \leq c\|u_1\|_2 \sup_{x \in \operatorname{supp} \nabla_{g_1}\phi} \big(e^f ((\sigma^2/r^2 + |\Lambda_0|))^{1/2} \big). \tag{2-18}$$

Let us remember that U lies deep inside the capped ϵ-horn D. Going from ∂D (that is, from $z^{-1}(0)$) to U, one must traverse a number of disjoint ϵ-necks. The ratio of scalar curvatures between the two ends of an ϵ-neck is bounded by $e^{c_2\epsilon}$ for some fixed $c_2 > 0$. The ratio of the scalar curvatures between ∂U and ∂D is $c_3 r^2 h^{-2}$, which is independent of the scaling factor σ. Therefore one must traverse at least

$$K \equiv \frac{1}{c_2\epsilon} \ln(c_3 r^2 h^{-2})$$

ϵ-necks to reach U. Note that K is independent of σ.

Let G_i be one of the ϵ-necks. Under the metric g, the distance between its two ends is comparable to $2\epsilon^{-1} R^{-1/2}(x_i)$, where x_i is a point in G_i. So, under the metric $\frac{1}{4}(R_1(x) - 2\ln^+ u_1(x) + \frac{1}{2}|\Lambda_0|)g_1(x)$, the distance between the two ends is bounded from below by

$$c_4 \inf_{x\in G_i} \left(\tfrac{1}{4}(R_1(x) - 2\ln^+ u_1(x) + \tfrac{1}{2}|\Lambda_0|)\right)^{1/2} R_1^{-1/2}(x_i)\epsilon^{-1} \ge c_5\epsilon^{-1}.$$

Here the last inequality comes from the second item in (2-14). This means that the function f increases by at least $c_5\epsilon^{-1}$ when traversing one ϵ-neck.

Next we observe that $\inf_{G_2} f \ge \sup_{\mathrm{supp}\,\nabla_{g_1}\phi} f$, since the support of $\nabla_{g_1}\phi$ is contained in the first ϵ-neck G_1. Therefore

$$\inf_U f \ge c_5\epsilon^{-1}(K-2) + \inf_{G_2} f \ge c_5\epsilon^{-1}(K-2) + \sup_{\mathrm{supp}\,\nabla_{g_1}\phi} f.$$

Substituting this into (2-18), we find

$$\|u_1\|_{L^2(U)} \le 2c|\Lambda_0^{-1}|e^{-c_5\epsilon^{-1}(K-2)}\|u_1\|_2(\sigma^2/r^2 + |\Lambda_0|)^{1/2}.$$

Therefore, by the formula for K in the above,

$$\|u_1\|_{L^2(U)} \le c_6|\Lambda_0^{-1}|(r^{-2}h^2)^{c_7\epsilon^{-2}}\|u_1\|_2(\sigma^2/r^2 + |\Lambda_0|)^{1/2}.$$

Since $r \le 1$ by assumption, we know that

$$\|u_1\|_{L^2(U)} \le c_8 C(\Lambda_0)(\sigma+1)r^{-1}\|u\|_2(r^{-2}h^2)^{c_7\epsilon^{-2}}.$$

Since $h \le \delta^2 r \le 1$, it is easy to see that we can choose δ as a suitable power of r so that

$$\|u\|_{L^2(U,d\mu(g))} = \|u_1\|_{L^2(U)} \le c_9(\sigma+1)h^5\|u\|_2$$

if ϵ is made sufficiently small.

Substituting this into (2-13), we see that

$$\lambda_{\sigma^2}(g(T^-)) \le \Lambda + c_{10}(\sigma+1)^3h^3\frac{1}{1-c_9(\sigma+1)h^5}.$$

Hence, given any $\sigma_0 > 0$, we have, for all $\sigma \in (0,\sigma_0)$, either

$$\begin{aligned} &\lambda_{\sigma^2}(g(T^+)) \ge \Lambda_0 \\ \text{or}\quad &\lambda_{\sigma^2}(g(T^-)) \le \Lambda + c_{11}(\sigma+1)^3h^3 = \lambda_{\sigma^2}(g(T^+)) + c_{11}(\sigma+1)^3h^3, \end{aligned}$$

provided that $h \le (2(\sigma_0+1)c_9)^{-1/5}$. This shows, for all $\sigma \in (0,\sigma_0]$, either $\lambda_{\sigma^2}(g(T^+)) \ge \Lambda_0$ or

$$\lambda_{\sigma^2}(g(T^-)) \le \lambda_{\sigma^2}(g(T^+)) + c_{12}|\mathrm{vol}(M(T^-)) - \mathrm{vol}(M(T^+))|. \tag{2-19}$$

Here $\mathrm{vol}(M(T^-))$ and $\mathrm{vol}(M(T^+))$ are the volumes of the preoperative and postoperative manifolds at T, respectively.

Step 2. We estimate the change of the best constant in the log Sobolev inequality in a given time interval without surgery.

Suppose the Ricci flow is smooth from time t_1 to t_2. Let $t \in (t_1, t_2)$ and $\sigma > 0$. Recall that, for $(M, g(t))$, Perelman's W entropy with parameter τ is

$$W(g, f, \tau) = \int_M (\tau(R + |\nabla f|^2) + f - n)\tilde{u}\ d\mu(g(t)), \text{ where } \tilde{u} = e^{-f}/(4\pi\tau)^{n/2}.$$

We are using $\tilde{u}$ in this step to distinguish from u in the last step.

We define $\tau = \tau(t) = \sigma^2 + t_2 - t$ so that $\tau_1 = \epsilon^2 + t_2 - t_1$ and $\tau_2 = \sigma^2$ (by taking $t = t_1$ and $t = t_2$ respectively). Let $\tilde{u}_2$ be a minimizer of the entropy $W(g(t), f, \tau_2)$ over all $\tilde{u}$ such that $\int \tilde{u}\, d\mu(g(t_2)) = 1$.

We solve the conjugate heat equation with the final value chosen as $\tilde{u}_2$ at $t = t_2$. Let $\tilde{u}_1$ be the value of the solution of the conjugate heat equation at $t = t_1$. As usual, we define functions f_i for $i = 1, 2$ by the relation $\tilde{u}_i = e^{-f_i}/(4\pi\tau_i)^{n/2}$ for $i = 1, 2$. Then, by the monotonicity of the W entropy [Perelman 2002],

$$\begin{aligned}\inf_{\int \tilde{u}_0\, d\mu(g(t_1))=1} W(g(t_1), f_0, \tau_1) &\le W(g(t_1), f_1, \tau_1) \le W(g(t_2), f_2, \tau_2)\\ &= \inf_{\int \tilde{u}\, d\mu(g(t_2))=1} W(g(t_2), f, \tau_2).\end{aligned}$$

Here f_0 and f are given by the formulas

$$\tilde{u}_0 = e^{-f_0}/(4\pi\tau_1)^{n/2} \quad \text{and} \quad \tilde{u} = e^{-f}/(4\pi\tau_2)^{n/2}.$$

Using these notations we can rewrite the above as

$$\begin{aligned}&\inf_{\|\tilde{u}\|_1=1} \int_M \big(\sigma^2(R + |\nabla \ln \tilde{u}|^2) - \ln \tilde{u} - \ln(4\pi\sigma^2)^{n/2}\big)\, \tilde{u}\ d\mu(g(t_2))\\ &\quad\ge \inf_{\|\tilde{u}_0\|_1=1} \int_M \big((\sigma^2 + t_2 - t_1)(R + |\nabla \ln \tilde{u}_0|^2)\\ &\qquad\qquad - \ln \tilde{u}_0 - \ln(4\pi(\sigma^2 + t_2 - t_1))^{n/2}\big)\tilde{u}_0\ d\mu(g(t_1)).\end{aligned}$$

Write $v = \sqrt{\tilde{u}}$ and $v_0 = \sqrt{\tilde{u}_0}$, we convert this inequality to

$$\begin{aligned}&\inf_{\|v\|_2=1} \int_M \big(\sigma^2(Rv^2 + 4|\nabla v|^2) - v^2 \ln v^2\big)\ d\mu(g(t_2)) - \ln(4\pi\sigma^2)^{n/2}\\ &\quad\ge \inf_{\|v_0\|_2=1} \int_M \big(4(\sigma^2 + t_2 - t_1)(\tfrac{1}{4} R v_0^2 + |\nabla v_0|^2) - v_0^2 \ln v_0^2\big)\ d\mu(g(t_1))\\ &\qquad\qquad - \ln(4\pi(\sigma^2 + t_2 - t_1))^{n/2}.\end{aligned}$$

That is,

$$\lambda_{\sigma^2}(g(t_2)) \ge \lambda_{\sigma^2 + t_2 - t_1}(g(t_1)). \tag{2-20}$$

Step 3. We estimate the change of the best constant in the log Sobolev inequality in the time interval $[T_1, T_2]$, with surgeries.

Suppose $T_1 \le t_1 < t_2 < \cdots < t_k \le T_2$, where t_i for $i = 1, 2, \dots, k$ are all the surgery times from T_1 to T_2. Here, without loss of generality, we may assume that T_1 and T_2 are not surgery times. Otherwise we can just directly apply Step 1 two more times at T_1 and T_2. We also fix a $\sigma_0 = T_2 - T_1 + 1$, where σ_0 is the upper bound for the parameter σ in Step 1's inequality (2-19).

For any $\sigma \in (0, 1]$, by (2-20), we have

$$\lambda_{\sigma^2}(g(T_2)) \ge \lambda_{\sigma^2+T_2-t_k}(g(t_k^+)).$$

By (2-19), either

$$\begin{aligned} &\lambda_{\sigma^2+T_2-t_k}(g(t_k^+)) \ge \Lambda_0 \\ \text{or}\quad &\lambda_{\sigma^2+T_2-t_k}(g(t_k^+)) \ge \lambda_{\sigma^2+T_2-t_k}(g(t_k^-)) - c_{12}\,|\mathrm{vol}(M(t_k^-) - \mathrm{vol}(M(t_k^+))|. \end{aligned}$$

In the first case, we have $\lambda_{\sigma^2}(g(T_2)) \ge \Lambda_0$, so a uniform lower bound is already found.

In the second case,

$$\lambda_{\sigma^2}(g(T_2)) \ge \lambda_{\sigma^2+T_2-t_k}(g(t_k^-)) - c_{12}\,|\mathrm{vol}(M(t_k^-) - \mathrm{vol}(M(t_k^+))|.$$

From here we start with $\lambda_{\sigma^2+T_2-t_k}(g(t_k^-))$ and repeat the process above. We have, from (2-20), with σ^2 in (2-20) replaced by $\sigma^2 + T_2 - t_k$,

$$\lambda_{\sigma^2+T_2-t_k}(g(t_k^-)) \ge \lambda_{\sigma^2+T_2-t_{k-1}}(g(t_{k-1}^+)).$$

Continue like this, until T_2, we have either

$$\begin{aligned} &\lambda_{\sigma^2}(g(T_2)) \ge \lambda_{\sigma^2+T_2-T_1}(g(T_1)) - c_{12}\textstyle\sum_{i=1}^k |\mathrm{vol}(M(t_i^-) - \mathrm{vol}(M(t_i^+))| \\ &\text{or } \lambda_{\sigma^2}(g(T_2)) \ge \Lambda_0 - c_{12}\textstyle\sum_{i=1}^k |\mathrm{vol}(M(t_i^-) - \mathrm{vol}(M(t_i^+))|. \end{aligned} \tag{2-21}$$

Note that the process above can be carried out since all the subscripted parameters of λ are bounded from above by σ_0.

It is known that

$$\sum_{i=1}^k |\mathrm{vol}(M(t_i^-) - \mathrm{vol}(M(t_i^+))| \le \sup_{t\in[T_1,T_2]} \mathrm{vol}(M(t)).$$

Hence, either

$$\begin{aligned} &\lambda_{\sigma^2}(g(T_2)) \ge \lambda_{\sigma^2+T_2-T_1}(g(T_1)) - c_{12} \sup_{t\in[T_1,T_2]} \mathrm{vol}(M(t)) \\ \text{or}\quad &\lambda_{\sigma^2}(g(T_2)) \ge \Lambda_0 - c_{12} \sup_{t\in[T_1,T_2]} \mathrm{vol}(M(t)). \end{aligned} \tag{2-22}$$

In either case, the lower bound is independent of the number of surgeries.

If the first of (2-22) holds, then we must find a lower bound for $\lambda_{\sigma^2+T_2-T_1}(g(T_1))$ that is independent of σ. Remember that it is assumed that $(M, g(T_1))$ satisfies a Sobolev inequality with constant A_1. It is well known that this implies a log Sobolev inequality. Indeed, from

$$\Big(\int v^{2n/(n-2)}\,d\mu(g(T_1))\Big)^{(n-2)/n} \le A_1\int (4|\nabla v|^2 + Rv^2 + v^2)\,d\mu(g(T_1)),$$

we may use the Hölder inequality and the Jensen inequality for ln to obtain that those $v \in W^{1,2}(M, g(T_1))$ satisfying $\|v\|_2 = 1$ also satisfy

$$\int v^2 \ln v^2\,d\mu(g(T_1)) \le \tfrac{1}{2}n \ln\Big(A_1\int (4|\nabla v|^2 + Rv^2)\,d\mu(g(T_1)) + A_1\Big). \tag{2-23}$$

Recall the elementary inequality that $\ln z \le qz - \ln q - 1$ for all $z, q > 0$. By (2-23), this shows

$$\int v^2 \ln v^2\,d\mu(g(T_1)) \le \tfrac{1}{2}nq\Big(A_1\int (4|\nabla v|^2 + Rv^2)\,d\mu(g(T_1)) + A_1\Big) - \tfrac{1}{2}n\ln q - \tfrac{1}{2}n.$$

Take q such that $\frac{1}{2}nqA_1 = \sigma^2 + T_2 - T_1$. Since $\sigma \le 1$, this shows, for some $B = B(A_1, T_1, T_2, n) = c\,(T_2 - T_1) + c > 0$, that

$$\begin{aligned}\lambda_{\sigma^2+T_2-T_1}&(g(T_1))\\ &\equiv \inf_{\|v\|_2=1}\int \Big((\sigma^2 + T_2 - T_1)(4|\nabla v|^2 + Rv^2) - v^2\ln v^2\Big)\,d\mu(g(T_1))\\ &\qquad\qquad -\tfrac{1}{2}n\ln(\sigma^2 + T_2 - T_1) \ge -B.\end{aligned}$$

Therefore we can conclude from (2-22) that

$$\lambda_{\sigma^2}(g(T_2)) \ge \min\{-B, \Lambda_0\} - c_{12}\sup_{t\in[T_1,T_2]}\operatorname{vol}(M(t)) \equiv A_2 \quad \text{for all } \sigma \in (0, 1].$$

By definition (2-11), this is nothing but a (restricted) log Sobolev inequality for $(M, g(T_2))$. That is,

$$\int v^2\ln v^2\,d\mu(g(T_2)) \le \sigma^2\int (4|\nabla v|^2 + Rv^2)\,d\mu(g(T_2)) - \tfrac{1}{2}n\ln\sigma^2 - A_2, \tag{2-24}$$

where $\sigma \in (0, 1]$.

Step 4. The log Sobolev inequality (2-24) implies a certain heat kernel estimate.

Let $p(x, y, t)$ be the heat kernel of $\Delta - \frac{1}{4}R$ in $(M, g(T_2))$ (with respect to the fixed metric $g(T_2)$). Then (2-24) implies, for $t \in (0, 1]$, that

$$p(x, y, t) \le \frac{1}{(4\pi t)^{n/2}}\exp(4(T_2 + 1) + \tfrac{1}{2}n|\ln|A_2|| + c + R_0^-) \equiv \frac{\Lambda}{t^{n/2}}, \tag{2-25}$$

where $R_0 = \sup R^-(x, 0)$ again. This follows from a generalization of Davies's argument [1989], as in [Zhang 2007c, Step 3, pages 12–15]. We omit the details.

Step 5. The heat kernel estimate (2-25) implies the Sobolev inequality perturbed with scalar curvature R and with the strong noncollapsing property.

This is more or less standard. By adapting the standard method in heat kernel estimates of [Davies 1989], as demonstrated in [Zhang 2007c, Step 4, page 15], it is known that (2-25) implies the desired Sobolev imbedding for $g(T_2)$. That is, for all $v \in W^{1,2}(M, g(T_2))$, there is a $B_2 > 0$ such that

$$\Big(\int v^{2n/(n-2)}\, d\mu(g(T_2))\Big)^{(n-2)/n} \le B_2 \int (4|\nabla v|^2 + Rv^2 + v^2)\, d\mu(g(T_2)).$$

This is the desired Sobolev inequality.

The strong noncollapsing result follows from the work of Carron [1996], as given in [Zhang 2007a]. Please see Lemma A.2. □

Appendix

We collect some basic facts concerning Ricci flow with surgery. For details, see [Perelman 2002; 2003; Cao and Zhu 2006; Kleiner and Lott 2007; Morgan and Tian 2007].

Definition A.1 ((r, δ) surgery). A surgery occurs at a δ-neck, called N, of radius h such that $(N, h^{-2}g)$ is δ-close in the $C^{[\delta^{-1}]}$ topology to the standard round neck $S^2 \times (-\delta^{-1}, \delta^{-1})$ of scalar curvature 1. Let Π be the diffeomorphism, from the definition of δ-closeness, that maps the standard round neck to N. Denote by z a number in $(-\delta^{-1}, \delta^{-1})$. For $\theta \in S^2$, (θ, z) is a parametrization of N via the diffeomorphism Π. In this way, we can identify the metric on N with its pullback by Π on the round neck.

In the notations of [Cao and Zhu 2006, page 424] (based on [Hamilton 1997]), the metric $\tilde g = \tilde g(T_2)$ just after the surgery is given by

$$\tilde g = \begin{cases} \bar g & \text{if } z \le 0, \\ e^{-2f}\bar g & \text{if } z \in [0, 2], \\ \phi e^{-2f}\bar g + (1-\phi)e^{-2f}h^2 g_0 & \text{if } z \in [2, 3], \\ e^{-2f}h^2 g_0 & \text{if } z \in [3, 4]. \end{cases}$$

Here $\bar g$ is the nonsingular part of $\lim_{t\to T_2^-} g(t)$, while g_0 is the standard metric on the round neck, and $f = f(z)$ is a smooth function given by (see [Cao and Zhu 2006, page 424]) $f(z) = 0$ if $z \le 0$, $f(z) = ce^{-P/z}$ if $z \in (0, 3]$, $f''(z) > 0$ if $z \in [3, 3.9]$, and $f(z) = -\frac{1}{2}\ln(16 - z^2)$ if $z \in [3.9, 4]$. Here a small $c > 0$ and a large $P > 0$ are chosen so that the Hamilton–Ivey pinching condition remains valid. The function ϕ is a smooth bump with $\phi = 1$ for $z \le 2$ and $\phi = 0$ for $z \ge 3$.

The next result relates the Sobolev imbedding to local noncollapsing of volume of geodesic balls. We follow the idea in [Carron 1996].

Lemma A.2 [Zhang 2007a, Lemma A.2]. *Let (M, g) be a Riemannian manifold. Suppose $x_0 \in M$ and $r \in (0, 1]$. Let $B(x_0, r)$ be a proper geodesic ball, that is, $M - B(x_0, r)$ is nonempty. Suppose the scalar curvature R satisfies $|R(x)| \le 1/r^2$ in $B(x_0, r)$ and the following Sobolev imbedding holds: For all $v \in W_0^{1,2}(B(x_0, r))$, and a constant $A \ge 1$,*

$$\Big(\int v^{2n/(n-2)}\, d\mu(g)\Big)^{(n-2)/n} \le A\int (|\nabla v|^2 + \tfrac{1}{4}Rv^2)\, d\mu(g) + A\int v^2\, d\mu(g).$$

Then $|B(x_0, r)| \ge 2^{-(n+5)n/2}A^{-n/2}r^n$.

Proof. Since $R \le 1/r^2$, $r \le 1$ and $A \ge 1$ by assumption, the Sobolev imbedding can be simplified to

$$\Big(\int v^{2n/(n-2)}\, d\mu(g)\Big)^{(n-2)/n} \le A\int |\nabla v|^2\, d\mu(g) + \frac{2A}{r^2}\int v^2\, d\mu(g).$$

Under the scaled metric $g_1 = g/r^2$, we have, for all $v \in W_0^{1,2}(B(x_0, 1, g_1))$,

$$\Big(\int v^{2n/(n-2)}\, d\mu(g_1)\Big)^{(n-2)/n} \le A\int |\nabla v|^2\, d\mu(g_1) + 2A\int v^2\, d\mu(g_1).$$

Now, by [Carron 1996] (see [Hebey 1999, page 33, line 4]),

$$|B(x_0, 1, g_1)|_{g_1} \ge \min\Big\{\frac{1}{2\sqrt{2A}}, \frac{1}{2^{(n+4)/2}\sqrt{2A}}\Big\}^n.$$

Therefore $|B(x_0, r, g)|_g \ge 2^{-(n+5)n/2}A^{-n/2}r^n$. □

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