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Let X be a smooth projective variety of dimension n and G a simple linear algebraic group of exceptional type acting regularly and nontrivially on X. Then it is known that n has a lower bound r_G which only depends on the Dynkin type of G. In this article we give a classification of X with an action of G in the case where $n = r_G + 1$.

1. Introduction

Let X be a smooth projective variety of dimension n and r_G the minimum of the dimension of a homogeneous variety of a simple linear algebraic group G, that is, the minimum codimension of a maximal parabolic subgroup of G. M. Andreatta [2001] proved that if $r_G < n$, the only regular action of G on X is trivial, and if $r_G = n$, then X is homogeneous. He also gave a classification of smooth projective varieties on which a simple linear algebraic group of classical type acts regularly and nontrivially in the case where $n = r_G + 1$. Our main purpose of this article is to prove the following:

Theorem 1.1. *Let X be a smooth projective variety of dimension n and G a simple, simply connected and connected linear algebraic group of exceptional type acting regularly and nontrivially on X. Assume that $n = r_G + 1$. Then X is one of the following; the action of G is unique for each case:*

(i) $\mathbb{P}^6$,

(ii) $\mathbb{Q}^6$,

(iii) $E_6(\omega_1)$,

(iv) $G_2(\omega_1 + \omega_2)$,

(v) $Y \times Z$, *where Y is* $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ *or* $G_2(\omega_2)$, *and Z is a smooth projective curve*,

(vi) $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(m))$, *where Y is as in* (v) *and $m > 0$.*

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Note that G-orbits on X are very simple (for example a projective space and a quadric) in the case where G is classical type, but they are not in our case. So we need other arguments than Andreatta's in several points.

Throughout this paper we work over the complex number field $\mathbb{C}$.

2. Preliminaries

We denote a simple linear algebraic group of Dynkin type G simply by G and for a dominant integral weight ω of G, the minimal closed orbit of G in $\mathbb{P}(V_\omega)$ by $G(\omega)$, where V_ω is the irreducible representation space of G with highest weight ω. For example, $E_6(\omega_1)$ is the minimal closed orbit of an algebraic group of type E_6 in $\mathbb{P}(V_{\omega_1})$, where ω_1 is the first fundamental dominant weight in the standard notation of Bourbaki [1968]. Then we call $G(\omega)$ a *rational homogeneous variety*.

Lemma 2.1 [Andreatta 2001, Lemmas 1.4, 1.5]. *Let X be a smooth projective variety on which a connected linear algebraic group G acts regularly and nontrivially. Then X has an extremal contraction $\phi : X \to Z$ which is G-equivariant, and G acts regularly on Z.*

Definition 2.2 [Andreatta 2001, Definition 1.8]. Let G be a simple linear algebraic group. We define r_G to be the minimal codimension of parabolic subgroups of G.

Example 2.3 [Andreatta 2001, Example 1.0.1]. If G is an exceptional linear algebraic group, we have $r_{E_6} = 16$, $r_{E_7} = 27$, $r_{E_8} = 57$, $r_{F_4} = 15$ and $r_{G_2} = 5$.

Proposition 2.4 [Andreatta 2001, Proposition 2.1]. *Suppose that a connected reductive linear algebraic group G acts effectively on a complete normal variety Z. Then the following are equivalent*:

(1) *There exists a fixed point z such that its projectivized tangent cone, that is the variety $P_z = \operatorname{Proj}\big(\bigoplus_k m_z^k/m_z^{k+1}\big)$, is a G-homogeneous variety.*

(2) *Z is a projective quasihomogeneous cone over a homogeneous variety with respect to G.*

Proposition 2.5 [Andreatta 2001, Lemma 2.2 and Proposition 3.1]. *Let X be a smooth projective variety of dimension n and G a simple, simply connected, connected linear algebraic group acting regularly and nontrivially on X. Then*

(1) $n \geq r_G$;

(2) *if moreover $n = r_G$, then X is homogeneous*;

(3) *if G is exceptional and $n = r_G + 1$, X has no fixed points.*

Lemma 2.6 [Andreatta 2001, Lemma 4.2]. *Let X and Y be smooth projective varieties on which a simple exceptional linear algebraic group G acts regularly and nontrivially. Assume that $r_G = \dim X - 1 = \dim Y - 1$. If X and Y each have a dense open orbit which is G-isomorphic, then we have a G-isomorphism $X \cong Y$.*

Proposition 2.7 [Watanabe 2008]. *Let X be a smooth projective variety and A a rational homogeneous variety $G(\omega)$, where G is exceptional. If A is an ample divisor on X, (X, A) is isomorphic to* $(\mathbb{P}^6, \mathbb{Q}^5)$, $(\mathbb{Q}^6, \mathbb{Q}^5)$ *or* $(E_6(\omega_1), F_4(\omega_4))$.

Remark that a 5-dimensional smooth quadric $\mathbb{Q}^5$ is G_2-homogeneous.

3. Proof of Theorem 1.1

By Lemma 2.1 we have a G-equivariant extremal contraction of a ray $\phi : X \to Z$.

Assume that $\rho(X) \geq 2$.

Case 1. *ϕ is birational.* Let ϕ be birational and E the exceptional locus of ϕ. Since r_G is equal to $n-1$ and X has no fixed points, ϕ is a divisorial contraction and E is contracted to a point z. Furthermore E is isomorphic to $E_6(\omega_1)(= E_6(\omega_5))$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)(= Q^5)$ and $G_2(\omega_2)$. The conormal bundle of the exceptional divisor is ${N_{E/X}}^* \cong \mathbb{O}(k)$ with $1 \leq k \leq i(E) - 1$, where $i(E)$ is the Fano index of E.

Applying Proposition 2.4, we see that X is a completion of an open orbit G/K (see [Ahiezer 1977]). Here K is the kernel of the character map $\rho : P \to \mathbb{C}^*$ associated to the homogeneous line bundle ${N_{E/X}}^* \cong \mathbb{O}(k)$, where P is the parabolic subgroup which satisfies $E \cong G/P$.

On the other hand, $X_k = \mathbb{P}({N_{E/X}}^* \oplus \mathbb{O})$ is also a completion of an open orbit G/K. By Lemma 2.6, X is isomorphic to $X_k = \mathbb{P}({N_{E/X}}^* \oplus \mathbb{O})$.

Case 2. *ϕ is a fibering type.* Let ϕ be a contraction of fibering type.

First we assume that the induced action of G on Z is trivial. In this case, any fiber of ϕ is isomorphic to $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$ and $\dim Z = 1$. Since rational homogeneous varieties are locally rigid, there is no ϕ which has both $F_4(\omega_1)$ and $F_4(\omega_4)$ (respectively $G_2(\omega_1)$ and $G_2(\omega_2)$) as fibers. So all fibers of ϕ are isomorphic to each other. Then we have $X = E_6(\omega_1) \times Z$, $E_7(\omega_1) \times Z$, $E_8(\omega_1) \times Z$, $F_4(\omega_1) \times Z$, $F_4(\omega_4) \times Z$, $G_2(\omega_1) \times Z$ or $G_2(\omega_2) \times Z$. This follows from [Mabuchi 1979, Theorem 1.2.1].

Second we assume that the induced action of G on Z is not trivial. Then Z is isomorphic to $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$. It follows that all fibers have dimension one. Moreover, all fibers of ϕ are isomorphic to each other. So ϕ is a conic bundle which fibers are isomorphic to $\mathbb{P}^1$. Since the Brauer group of Z is trivial, X is $\mathbb{P}(\mathscr{E})$ with $\mathscr{E}$ a rank 2 vector bundle on Z.

The assumption that $n = r_G + 1$ implies that the dimension of any orbit of G in $\mathbb{P}(\mathscr{E})$ is at least $n-1$. If $\mathbb{P}(\mathscr{E})$ is G-homogeneous, then $\mathbb{P}(\mathscr{E})$ has another natural fibration structure $\mathbb{P}(\mathscr{E}) \to Z'$, where Z' is a G-homogeneous variety whose Picard number is 1 [Baston and Eastwood 1989, 2.4]. Since $\dim Z + 1 = \dim X > \dim Z'$, (Z, Z') (or (Z', Z)) is $(E_6(\omega_1), E_6(\omega_5))$, $(F_4(\omega_1), F_4(\omega_4))$ or $(G_2(\omega_1), G_2(\omega_2))$ [Snow 1989, 9.3]. However, if (Z, Z') is $(E_6(\omega_1), E_6(\omega_5))$ or $(F_4(\omega_1), F_4(\omega_4))$, the fiber of $\mathbb{P}(\mathscr{E}) \to Z$ is not $\mathbb{P}^1$. Hence (Z, Z') is $(G_2(\omega_1), G_2(\omega_2))$ and we have $\mathbb{P}(\mathscr{E}) \cong G_2(\omega_1 + \omega_2)$.

If $\mathbb{P}(\mathscr{E})$ is not G-homogeneous, we have the G-orbit decomposition $\mathbb{P}(\mathscr{E}) = \left(\bigsqcup_{i \in I} Gx_i\right)$ or $\mathbb{P}(\mathscr{E}) = Gx \sqcup \left(\bigsqcup_{i \in I} Gx_i\right)$, where $x, x_i \in \mathbb{P}(\mathscr{E})$. Here, Gx is a G-orbit of dimension n and Gx_i is a rational homogeneous variety of dimension $n-1$ whose Picard number is 1. Since $\dim Gx_i = \dim Z$, $\phi_{Gx_i} : Gx_i \to Z$ is a finite morphism. If the ramification divisor R of ϕ_{Gx_i} is not empty, G acts on R. But this contradicts homogeneity of Gx_i. So ϕ_{Gx_i} is étale. Hence we see that $\phi_{Gx_i} : Gx_i \to Z$ is isomorphic, because a Fano variety is simply connected. So Gx_i is a section of ϕ. Since any G-homogeneous vector bundle has no a transitive action of G, we have $\sharp I \neq 1$. So $\mathbb{P}(\mathscr{E})$ has two sections which do not intersect each other. Hence $\mathscr{E}$ is decomposable. The uniqueness of action can be proved as above.

Assume that $\rho(X) = 1$. By using the list of parabolic subgroups of codimension n corresponding to one node of the Dynkin diagram, we see that X is not G-homogeneous. So X has a closed orbit H which is isomorphic to $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$. The condition $\rho(X) = 1$ implies X is a Fano variety. Furthermore, Pic(X) $\cong \mathbb{Z}$. Hence H is an ample divisor of X. By Proposition 2.7, we see that (X, H) is $(\mathbb{P}^6, \mathbb{Q}^5)$, $(\mathbb{Q}^6, \mathbb{Q}^5)$ or $(E_6(\omega_1), F_4(\omega_4))$.

These X satisfy the assumption of the Theorem. In fact, we see that $F_4 \subset E_6$, $G_2 \subset \mathrm{SO}(7) \subset \mathrm{SO}(8)$. Here $\mathrm{SO}(k)$ means the special orthogonal group.

At last, we shall prove the uniqueness of action. We only deal with the case where X is $E_6(\omega_1)$. We can prove other cases as the same.

Let V_{27} be the irreducible representation space of E_6 with highest weight ω_1. Then E_6 acts on V_{27}. If G whose Dynkin type is F_4 acts on $E_6(\omega_1)$, we obtain a 27-dimensional representation $G \to \mathrm{GL}(V_{27})$. By the Weyl dimension theorem and our assumption, it is easy to see that V_{27} is a direct sum of a 26-dimensional irreducible representation space V_{26} and a 1-dimensional irreducible representation space V_1. Furthermore, we see that irreducible representations $G \to \mathrm{GL}(V_{26})$ and $G \to \mathrm{GL}(V_1)$ are unique. This implies that the action of G on $E_6(\omega_1)$ is unique.

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