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EXPLICIT FORMULAS FOR BIHARMONIC SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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To the memory of Professor Neculai Papaghiuc

We classify all biharmonic Legendre curves in a Sasakian space form and obtain their explicit parametric equations in the $(2n+1)$-dimensional unit sphere endowed with the canonical and deformed Sasakian structures defined by Tanno. We also show that, under the flow-action of the characteristic vector field, a biharmonic integral submanifold becomes a biharmonic anti-invariant submanifold. Then, we obtain new examples of biharmonic submanifolds in the Euclidean sphere $\mathbb{S}^7$.

1. Introduction

Biharmonic maps between Riemannian manifolds $\phi : (M, g) \to (N, h)$ are the critical points of the *bienergy functional* $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2\, v_g$ and represent a natural generalization of the well-known *harmonic maps* [Eells and Sampson 1964], the critical points of the *energy functional* $E(\phi) = \frac{1}{2}\int_M |d\phi|^2\, v_g$. The Euler–Lagrange equation for the energy functional is $\tau(\phi) = 0$, where $\tau(\phi) = \operatorname{trace} \nabla d\phi$ is the tension field, and the corresponding Euler–Lagrange equation for the bienergy functional was derived by G. Y. Jiang [1986]:

$$\tau_2(\phi) = -\Delta\tau(\phi) - \operatorname{trace}\ R^N(d\phi, \tau(\phi))d\phi = 0.$$

Since any harmonic map is biharmonic, we are interested in nonharmonic biharmonic maps, which are called *proper-biharmonic*.

A special case of biharmonic maps is represented by the biharmonic Riemannian immersions, or biharmonic submanifolds, that is, submanifolds for which the inclusion map is biharmonic. We note that the biharmonic submanifolds in Euclidean spaces are the same as those defined by B.-Y. Chen [1996], that is, they are characterized by the equation $\Delta H = 0$, where H is the mean curvature vector field.

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There are several classification results for proper-biharmonic submanifolds in space forms [Balmuş et al. 2008; Caddeo et al. 2001a; Chen 1996; Dimitrić 1992; Montaldo and Oniciuc 2006], while in spaces of nonconstant sectional curvature only a few results were obtained [Arslan et al. 2007; Ichiyama et al. 2008; Inoguchi 2004; Sasahara 2005; Zhang 2007].

A different and active direction is the study of proper-biharmonic submanifolds in pseudo-Riemannian manifolds (for example, see [Arvanitoyeorgos et al. 2007] and [Chen 2008]).

Among proper-biharmonic submanifolds, particular attention has been paid to proper-biharmonic curves parametrized by arc length. R. Caddeo, S. Montaldo and P. Piu [Caddeo et al. 2001b] proved that the proper-biharmonic curves in the unit Euclidean 2-dimensional sphere $\mathbb{S}^2$ are circles of radius $1/\sqrt{2}$. Caddeo, Montaldo, and the second author [Caddeo et al. 2001a] also showed that the proper-biharmonic curves in $\mathbb{S}^3$ are the geodesics of the minimal (harmonic) Clifford torus $\mathbb{S}(1/\sqrt{2}) \times \mathbb{S}(1/\sqrt{2})$ with slope different from ± 1. The proper-biharmonic curves of $\mathbb{S}^3$ are helices. Further, the proper-biharmonic curves of $\mathbb{S}^n$, $n > 3$, are, up to a totally geodesic embedding of $\mathbb{S}^3$ in $\mathbb{S}^n$, those of $\mathbb{S}^3$ [Caddeo et al. 2002]. Classification results for proper-biharmonic curves in 3-dimensional spaces of nonconstant sectional curvature were obtained in [Caddeo et al. 2006; Cho et al. 2007; Fetcu and Oniciuc 2007; Inoguchi 2004], and it turn out that, in the studied cases, they are helices.

Biharmonic submanifolds in Euclidean spheres has proved to be an interesting subject. Since the odd-dimensional unit Euclidean spheres can be thought as a particular class of Sasakian space forms (which do not have, in general, constant sectional curvature), it seems that the next step would be the study of biharmonic submanifolds in Sasakian space forms.

In the present paper we classify all proper-biharmonic Legendre curves in Sasakian space forms of any dimension. Because of the complexity of the biharmonic equation, we must do case-by-case analysis, and the classification is given by Theorems 3.3, 3.6, 3.7 and 3.9. As a by-product we prove that in a 5-dimensional Sasakian space form, all proper-biharmonic curves are helices (Theorem 3.12). Then we consider the $(2n+1)$-dimensional unit sphere $\mathbb{S}^{2n+1}$ endowed with the canonical and deformed Sasakian structures defined by Tanno as a model for the Sasakian space forms, and obtain the explicit parametric equations of proper-biharmonic Legendre curves (Theorems 3.14, 3.17 and 3.18).

In Section 4 we prove that, by composing with the flow of the characteristic vector field of a Sasakian space form, we can render a proper-biharmonic integral submanifold onto a proper-biharmonic anti-invariant submanifold (Theorem 4.1). This result allows us to obtain all proper-biharmonic surfaces which are invariant under the flow-action of the characteristic vector field (Theorem 4.3) and to construct new examples of proper-biharmonic submanifolds (Section 5).

For a general account of biharmonic maps see [Montaldo and Oniciuc 2006] and *The bibliography of biharmonic maps* [BibBhM 2008].

Conventions. We work in the C^∞ category, which means manifolds, metrics, connections and maps are smooth. The Lie algebra of the vector fields on M is denoted by $C(TM)$.

2. Preliminaries

In this section we briefly recall basic things from the theory of Sasakian manifolds (for example, see [Blair 2002]) which we shall use throughout the paper.

A *contact metric structure* on an odd-dimensional manifold N^{2n+1} is given by (φ, ξ, η, g), where φ is a tensor field of type $(1, 1)$ on N, ξ is a vector field, η is an 1-form and g is a Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \; g(X, \varphi Y) = d\eta(X, Y), \quad \forall X, Y \in C(TN).$$

A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is called *Sasakian* if it is *normal*, meaning that

$$N_\varphi + 2d\eta \otimes \xi = 0,$$

where N_φ is the Nijenhuis tensor field of φ, given by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], \quad \forall X, Y \in C(TN);$$

or, equivalently, that

$$(\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad \forall X, Y \in C(TN).$$

We note that from the above formula it follows $\nabla_X \xi = -\varphi X$.

The *contact distribution* of a Sasakian manifold $(N, \varphi, \xi, \eta, g)$ is defined by $\{X \in TN : \eta(X) = 0\}$. We say that a submanifold M of N is an *integral submanifold* if $\eta(X) = 0$ for any vector X tangent to M; in particular, an integral curve is called a *Legendre* curve. The maximum dimension for an integral submanifold of N^{2n+1} is n. A submanifold M of N which is tangent to ξ is said to be *anti-invariant* if φ maps any vector tangent to M and normal to ξ to a vector normal to M.

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a 2-plane generated by X and φX, where X is an unit vector orthogonal to ξ, is called the *φ-sectional curvature* determined by X. If the φ-sectional curvature is a constant c, then $(N, \varphi, \xi, \eta, g)$ is called a *Sasakian space form* and it is denoted by $N(c)$.

The curvature tensor field of a Sasakian space form $N(c)$ is given by

$$
\begin{aligned}
&R(X,Y)Z\\
&\quad=\tfrac{1}{4}(c+3)\{g(Z,Y)X-g(Z,X)Y\}\\
&\qquad+\tfrac{1}{4}(c-1)\big\{\eta(Z)\eta(X)Y-\eta(Z)\eta(Y)X+g(Z,X)\eta(Y)\xi-g(Z,Y)\eta(X)\xi\\
&\qquad\qquad+g(Z,\varphi Y)\varphi X-g(Z,\varphi X)\varphi Y+2g(X,\varphi Y)\varphi Z\big\}.
\end{aligned}
$$

The classification of complete, simply connected Sasakian space forms $N(c)$ was given in [Tanno 1969]. When $c>-3$, $N(c)$ is isometric to the unit sphere $\mathbb{S}^{2n+1}$ endowed with the Sasakian structure defined by Tanno. This structure is given as follows (see [Tanno 1968]).

Let $\mathbb{S}^{2n+1}=\{z\in\mathbb{C}^{n+1}:|z|=1\}$ be the unit $(2n+1)$-dimensional sphere endowed with its standard metric field g_0. Consider the following structure tensor fields on $\mathbb{S}^{2n+1}$: $\xi_0=-\mathcal{J}z$ for each $z\in\mathbb{S}^{2n+1}$, where $\mathcal{J}$ is the usual complex structure on $\mathbb{C}^{n+1}$ defined by

$$\mathcal{J}z=(-y^1,\dots,-y^{n+1},x^1,\dots,x^{n+1}),$$

for $z=(x^1,\dots,x^{n+1},y^1,\dots,y^{n+1})$, and $\varphi_0=s\circ\mathcal{J}$, where $s:T_z\mathbb{C}^{n+1}\to T_z\mathbb{S}^{2n+1}$ denotes the orthogonal projection. Equipped with these tensors, $\mathbb{S}^{2n+1}$ becomes a Sasakian space form with the φ_0-sectional curvature equal to 1.

Now, consider the deformed structure on $\mathbb{S}^{2n+1}$

$$\eta=a\eta_0,\quad \xi=\frac{1}{a}\xi_0,\quad \varphi=\varphi_0,\quad g=ag_0+a(a-1)\eta_0\otimes\eta_0,$$

where a is a positive constant. The structure (φ,ξ,η,g) is still a Sasakian structure and $(\mathbb{S}^{2n+1},\varphi,\xi,\eta,g)$ is a Sasakian space form with constant φ-sectional curvature $c=4/a-3$, $c>-3$.

We end this subsection recalling that a contact metric manifold (N,φ,ξ,η,g) is *regular* if for any point $p\in N$ there exists a cubic neighborhood such that any integral curve of ξ passes through it at most once; and it is *strictly regular* if all integral curves of ξ are homeomorphic to each other.

3. Biharmonic Legendre curves in Sasakian space forms

We shall work with Frenet curves of osculating order r, parametrized by arc-length, which we recall here (see [Baikoussis and Blair 1995]).

Definition 3.1. Let (N^m,g) be a Riemannian manifold and $\gamma:I\to N$ a curve parametrized by arc length, that is, $|\gamma'(s)|=1$. Then γ is called a *Frenet curve of osculating order* r, $1\le r\le m$, if there are orthonormal vector fields $E_1, E_2,\dots,E_r$

along γ such that

$$\begin{aligned} E_1 &= \gamma' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3 \\ &\;\vdots \\ \nabla_T E_r &= -\kappa_{r-1} E_{r-1}, \end{aligned}$$

where $\kappa_1, \dots, \kappa_{r-1}$ are positive functions on I.

Remark 3.2. A geodesic is a Frenet curve of osculating order 1, a *circle* is a Frenet curve of osculating order 2 with $\kappa_1 = \text{constant}$, and a *helix of order* r, $r \geq 3$, is a Frenet curve of osculating order r with $\kappa_1, \dots, \kappa_{r-1}$ constants. A helix of order 3 is called, simply, a *helix*.

Now let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r. Since

$$\nabla^3_T T = (-3\kappa_1\kappa_1')E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)E_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4,$$

$$R(T, \nabla_T T)T = -\frac{(c+3)\kappa_1}{4}E_2 - \frac{3(c-1)\kappa_1}{4} g(E_2, \varphi T)\varphi T,$$

we obtain the expression of the bitension vector field

$$\begin{aligned} \tau_2(\gamma) &= \nabla^3_T T - R(T, \nabla_T T)T \\ &= (-3\kappa_1\kappa_1')E_1 + \Big(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \frac{(c+3)\kappa_1}{4}\Big)E_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 \\ &\qquad + \kappa_1\kappa_2\kappa_3 E_4 + \frac{3(c-1)\kappa_1}{4} g(E_2, \varphi T)\varphi T. \end{aligned} \tag{3-1}$$

We shall solve the biharmonic equation $\tau_2(\gamma) = 0$. Because of the last term of $\tau_2(\gamma)$ we must do a case by case analysis.

Case I: $c = 1$. In this case, from (3-1), it follows that γ is proper-biharmonic if and only if

$$\kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant}, \quad \kappa_1^2 + \kappa_2^2 = 1, \quad \kappa_2\kappa_3 = 0.$$

One obtains:

Theorem 3.3. *Let $N^{2n+1}(1)$ be a Sasakian space form and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r. If $n \geq 2$, then γ is proper-biharmonic if and only if it is a circle with $\kappa_1 = 1$, or a helix with $\kappa_1^2 + \kappa_2^2 = 1$.*

Remark 3.4. If $n = 1$ and γ is a nongeodesic Legendre curve we have $\nabla_T T = \pm\kappa_1\varphi T$ and then $E_2 = \pm\varphi T$ and $\nabla_T E_2 = \pm\nabla_T \varphi T = \pm(\xi \mp \kappa_1 T) = -\kappa_1 T \pm \xi$. Therefore $\kappa_2 = 1$ and γ cannot be biharmonic.

Case II: $c \neq 1,\ E_2 \perp \varphi T$. From (3-1) we obtain that γ is proper-biharmonic if and only if

$$\kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant}, \quad \kappa_1^2 + \kappa_2^2 = (c+3)/4, \ \kappa_2\kappa_3 = 0.$$

Before stating the theorem we need the following lemma which imposes a restriction on the dimension of the manifold $N^{2n+1}(c)$.

Lemma 3.5. *Let γ be a Legendre Frenet curve of osculating order* 3 *such that $E_2 \perp \varphi T$. Then $\{T = E_1, E_2, E_3, \varphi T, \xi, \nabla_T \varphi T\}$ is linearly independent, in any point, and hence $n \geq 3$.*

Proof. Since γ is a Frenet curve of osculating order 3, we have

$$\begin{aligned} E_1 &= \gamma' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\ \nabla_T E_3 &= -\kappa_2 E_2. \end{aligned}$$

It is easy to see that, in an arbitrary point, the system

$$S_1 = \{T = E_1, E_2,\ E_3,\ \varphi T,\ \xi,\ \nabla_T \varphi T\}$$

has only nonzero vectors and

$$T \perp E_2, \quad T \perp E_3, \quad T \perp \varphi T, \quad T \perp \xi, \quad T \perp \nabla_T \varphi T.$$

Thus S_1 is linearly independent if and only if $S_2 = \{E_2, E_3, \varphi T, \xi, \nabla_T \varphi T\}$ is linearly independent. Further, since we have the relations

$$E_2 \perp \xi,\ E_2 \perp \nabla_T \varphi T,\ E_3 \perp \xi,\ E_3 \perp \nabla_T \varphi T,\ \varphi T \perp \xi,\ \varphi T \perp \nabla_T \varphi T,\ E_2 \perp E_3 \perp \varphi T,$$

it follows that S_2 is linearly independent if and only if $S_3 = \{\xi, \nabla_T \varphi T\}$ is linearly independent. But $\nabla_T \varphi T = \xi + \kappa_1 \varphi E_2$, $\kappa_1 \neq 0$, and therefore S_3 is linearly independent. □

Theorem 3.6. *Let $N^{2n+1}(c)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r such that $E_2 \perp \varphi T$.*

(1) *If $c \leq -3$ then γ is biharmonic if and only if it is a geodesic.*

(2) *If $c > -3$ then γ is proper-biharmonic if and only if either*

 (a) *$n \geq 2$ and γ is a circle with $\kappa_1^2 = (c+3)/4$, in which case the vectors $\{E_1, E_2, \varphi T, \xi\}$ are linearly independent, or*

 (b) *$n \geq 3$ and γ is a helix with $\kappa_1^2 + \kappa_2^2 = (c+3)/4$, in which case the vectors $\{E_1, E_2, E_3, \varphi T, \xi, \nabla_T \varphi T\}$ are linearly independent.*

Case III: $c \neq 1$, $E_2 \parallel \varphi T$. In this case, from (3-1), γ is proper-biharmonic if and only if

$$\kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant}, \quad \kappa_1^2 + \kappa_2^2 = c, \quad \kappa_2\kappa_3 = 0.$$

We can assume that $E_2 = \varphi T$. Then $\nabla_T T = \kappa_1 E_2 = \kappa_1 \varphi T$, $\nabla_T E_2 = \nabla_T \varphi T = \xi - \kappa_1 T$. That means $E_3 = \xi$ and $\kappa_2 = 1$. Hence $\nabla_T E_3 = \nabla_T \xi = -\varphi T = -E_2$.

Therefore:

Theorem 3.7. *Let $N^{2n+1}(c)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r such that $E_2 \parallel \varphi T$. Then $\{T, \varphi T, \xi\}$ is the Frenet frame field of γ and we have*:

(1) *If $c < 1$ then γ is biharmonic if and only if it is a geodesic.*

(2) *If $c > 1$ then γ is proper-biharmonic if and only if it is a helix with $\kappa_1^2 = c-1$ (and $\kappa_2 = 1$).*

Remark 3.8. If $n = 1$, for any Legendre curve $E_2 \parallel \varphi T$, and we reobtain Inoguchi's result in [2004].

Case IV: $c \neq 1$ and $g(E_2, \varphi T)$ is not constant 0, 1 or -1. Assume that γ is a proper-biharmonic Legendre Frenet curve of osculating order r such that $g(E_2, \varphi T)$ is not constant 0, 1 or -1. One can check that, in this case, $4 \leq r \leq 2n+1$, $n \geq 2$, and $\varphi T \in \operatorname{span}\{E_2, E_3, E_4\}$.

Now, we denote $f(s) = g(E_2, \varphi T)$ and differentiating it we obtain

$$\begin{aligned} f'(s) &= g(\nabla_T E_2, \varphi T) + g(E_2, \nabla_T \varphi T) = g(\nabla_T E_2, \varphi T) + g(E_2, \xi + \kappa_1 \varphi E_2) \\ &= g(\nabla_T E_2, \varphi T) = g(-\kappa_1 T + \kappa_2 E_3, \varphi T) \\ &= \kappa_2 g(E_3, \varphi T). \end{aligned}$$

Since $\varphi T = g(\varphi T, E_2)E_2 + g(\varphi T, E_3)E_3 + g(\varphi T, E_4)E_4$, the curve γ is proper-biharmonic if and only if

$$\begin{aligned} &\kappa_1 = \text{constant} > 0, && \kappa_1^2 + \kappa_2^2 = \tfrac{1}{4}(c+3) + \tfrac{3}{4}(c-1)f^2, \\ &\kappa_2' = -\tfrac{3}{4}(c-1)fg(\varphi T, E_3), && \kappa_2\kappa_3 = -\tfrac{3}{4}(c-1)fg(\varphi T, E_4). \end{aligned}$$

Using the expression of $f'(s)$ we see that the third equation of this system is equivalent to

$$\kappa_2^2 = -\tfrac{3}{4}(c-1)f^2 + \omega_0,$$

where $\omega_0 = \text{constant}$. Substituting in the second equation, it follows that

$$\kappa_1^2 = \frac{c+3}{4} - \omega_0 + \frac{3(c-1)}{2} f^2,$$

which implies $f = \text{constant}$. Thus $\kappa_2 = \text{constant} > 0$, $g(E_3, \varphi T) = 0$ and then $\varphi T = f E_2 + g(\varphi T, E_4)E_4$. It follows that there exists a unique constant $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ such that $f = \cos\alpha_0$ and $g(\varphi T, E_4) = \sin\alpha_0$.

We can state:

Theorem 3.9. *Let $N^{2n+1}(c)$ be a Sasakian space form with $c \neq 1$, $n \geq 2$, and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r such that $g(E_2, \varphi T)$ is not constant* 0, 1 *or* -1.

(1) *If $c \leq -3$ then γ is biharmonic if and only if it is a geodesic.*
(2) *If $c > -3$ then γ is proper-biharmonic if and only if $\varphi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$, $\kappa_1, \kappa_2, \kappa_3 = \text{constant} > 0$,*
$$\kappa_1^2 + \kappa_2^2 = \tfrac{1}{4}(c+3) + \tfrac{3}{4}(c-1)\cos^2\alpha_0 \quad \textit{and} \quad \kappa_2\kappa_3 = -\tfrac{3}{8}(c-1)\sin 2\alpha_0,$$
where $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ is a constant such that $c+3+3(c-1)\cos^2\alpha_0 > 0$ and $3(c-1)\sin 2\alpha_0 < 0$.

Remark 3.10. In this case we may obtain biharmonic curves which are not helices.

Proposition 3.11. *Assume that $c > -3$, $c \neq 1$, and $n = 2$. Let γ be a proper-biharmonic Legendre Frenet curve of osculating order r, such that $g(E_2, \varphi T)$ is not constant* 0, -1 *or* 1. *Then γ is a helix of order* 4 *or* 5.

Proof. We know that $r \in \{4, 5\}$. If $r = 4$, then the result is obvious from Theorem 3.9.

Assume now $r = 5$. Since $\varphi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$, and $\xi \perp \varphi T$, $\xi \perp E_2$, we get $\xi \perp E_4$, and then, along γ, $\xi \in \text{span}\{E_3, E_5\}$.

From the Frenet equations of γ it follows that
$$g(\nabla_T E_3, \xi) = g(-\kappa_2 E_2 + \kappa_3 E_4, \xi) = 0,$$
$$g(\nabla_T E_5, \xi) = g(-\kappa_4 E_4, \xi) = 0.$$
Then, since $\nabla g = 0$, we obtain $(g(E_3, \xi))' = 0$ and $(g(E_5, \xi))' = 0$, that is, $a = g(E_3, \xi) = \text{constant}$ and $b = g(E_5, \xi) = \text{constant}$.

Now, we have
$$g(\nabla_T E_4, \xi) = -\kappa_3 g(E_3, \xi) + \kappa_4 g(E_5, \xi) = -\kappa_3 a + \kappa_4 b$$
and, since $g(\nabla_T E_4, \xi) = g(E_4, \varphi T) = \sin\alpha_0$, we get
$$\sin\alpha_0 = -\kappa_3 a + \kappa_4 b \tag{3-2}$$
which implies that $b = 0$ or $\kappa_4 = \text{constant}$.

Case $b = 0$. Since $\xi \in \text{span}\{E_3, E_5\}$, we have $E_3 = \pm\xi$ and therefore
$$\nabla_T E_3 = \mp\varphi T = \mp\cos\alpha_0 E_2 \mp \sin\alpha_0 E_4.$$

From the third Frenet equation, $\kappa_2 = \pm\cos\alpha_0$, $\kappa_3 = \mp\sin\alpha_0$, and then, from Theorem 3.9, $\kappa_2\kappa_3 = -\frac{1}{2}\sin 2\alpha_0 = -(3(c-1)/8)\sin 2\alpha_0$. Thus, we have $c = \frac{7}{3}$ and, again using Theorem 3.9, $\kappa_1 = 2/\sqrt{3}$.

We shall prove now $\kappa_4 = \kappa_1$, so γ is a helix of order 5. From the last Frenet equation, we obtain

$$g(\nabla_T E_5, \varphi T) = -\kappa_4 g(E_4, \varphi T) = -\kappa_4 \sin\alpha_0. \tag{3-3}$$

Since $g(E_5, \varphi T) = 0$ we have $g(\nabla_T E_5, \varphi T) + g(E_5, \nabla_T \varphi T) = 0$. We can check that $g(E_5, \nabla_T \varphi T) = \kappa_1 g(E_5, \varphi E_2)$, therefore, using (3-3), we get

$$\kappa_1 g(E_5, \varphi E_2) = \kappa_4 \sin\alpha_0. \tag{3-4}$$

Next, from the fourth Frenet equation and (3-4),

$$g(\nabla_T E_4, \varphi E_2) = \kappa_4 g(E_5, \varphi E_2) = \frac{\kappa_4^2}{\kappa_1}\sin\alpha_0. \tag{3-5}$$

Since $\varphi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$ it results that $g(E_4, \varphi E_2) = 0$. It follows that

$$\begin{aligned} g(\nabla_T E_4, \varphi E_2) &= -g(E_4, \nabla_T \varphi E_2) \\ &= -g(E_4, \varphi\nabla_T E_2) = \kappa_1 g(E_4, \varphi T) = \kappa_1 \sin\alpha_0. \end{aligned} \tag{3-6}$$

From (3-5) and (3-6) we obtain $\kappa_4 = \kappa_1 = 2/\sqrt{3}$.

Case $b \neq 0$. Of course, due to (3-2) $\kappa_4 =$ constant and so γ is a helix. Moreover, we can obtain an additional relation between the curvatures.

Indeed, since $\xi \in \operatorname{span}\{E_3, E_5\}$ it follows $a^2 + b^2 = 1$. On the other hand

$$g(\nabla_T E_2, \xi) = g(E_2, \varphi T) = \cos\alpha_0 = g(-\kappa_1 T + \kappa_2 E_3, \xi) = \kappa_2 a$$

and as $-\kappa_3 a + \kappa_4 b = \sin\alpha_0$, replacing in $a^2 + b^2 = 1$ we get

$$(\kappa_2 \sin\alpha_0 + \kappa_3 \cos\alpha_0)^2 + \kappa_4^2(\cos\alpha_0)^2 = \kappa_2^2\kappa_4^2. \qquad \square$$

From Theorems 3.3, 3.6 and 3.7 and Proposition 3.11 we conclude:

Theorem 3.12. *Let γ be a proper-biharmonic Legendre curve in $N^5(c)$. Then $c > -3$ and γ is a helix of order r with $2 \leq r \leq 5$.*

Remark 3.13. In [Fetcu 2008a], a preliminary version of the full classification of the proper-biharmonic Legendre curves in Sasakian space forms was obtained.

In the following, we shall choose the unit $(2n+1)$-dimensional sphere $\mathbb{S}^{2n+1}$ with its canonical and deformed Sasakian structures as a model for the complete, simply connected Sasakian space form with constant φ-sectional curvature $c > -3$, and we shall find the explicit equations of biharmonic Legendre curves obtained in the first three cases, viewed as curves in $\mathbb{R}^{2n+2}$.

Theorem 3.14. *Let $\gamma : I \to (\mathbb{S}^{2n+1}, \varphi_0, \xi_0, \eta_0, g_0)$, $n \geq 2$, be a proper-biharmonic Legendre curve parametrized by arc length. Then the equation of γ in the Euclidean space $\mathbb{E}^{2n+2} = (\mathbb{R}^{2n+2}, \langle\,,\rangle)$ is either*

$$\gamma(s) = \frac{1}{\sqrt{2}} \cos(\sqrt{2}s)e_1 + \frac{1}{\sqrt{2}} \sin(\sqrt{2}s)e_2 + \frac{1}{\sqrt{2}} e_3$$

where $\{e_i, \mathcal{J}e_j\}_{i,j=1}^3$ are orthogonal constant unit vectors, or

$$\gamma(s) = \frac{1}{\sqrt{2}} \cos(As)e_1 + \frac{1}{\sqrt{2}} \sin(As)e_2 + \frac{1}{\sqrt{2}} \cos(Bs)e_3 + \frac{1}{\sqrt{2}} \sin(Bs)e_4,$$

where

$$A = \sqrt{1+\kappa_1}, \quad B = \sqrt{1-\kappa_1}, \quad \kappa_1 \in (0, 1), \tag{3-7}$$

and $\{e_i\}_{i=1}^4$ are orthogonal constant unit vectors, satisfying

$$\langle e_1, \mathcal{J}e_3\rangle = \langle e_1, \mathcal{J}e_4\rangle = \langle e_2, \mathcal{J}e_3\rangle = \langle e_2, \mathcal{J}e_4\rangle = 0, \quad A\langle e_1, \mathcal{J}e_2\rangle + B\langle e_3, \mathcal{J}e_4\rangle = 0.$$

Proof. Let us denote by $\dot{\nabla}$ and by $\widetilde{\nabla}$ the Levi-Civita connections on $(\mathbb{S}^{2n+1}, g_0)$ and $(\mathbb{R}^{2n+2}, \langle\,,\rangle)$, respectively.

First, assume that γ is the biharmonic circle, that is, $\kappa_1 = 1$. From the Gauss and Frenet equations we get

$$\widetilde{\nabla}_T T = \dot{\nabla}_T T - \langle T, T\rangle \gamma = \kappa_1 E_2 - \gamma,$$
$$\widetilde{\nabla}_T \widetilde{\nabla}_T T = (-\kappa_1^2 - 1)T = -2T,$$

which implies

$$\gamma''' + 2\gamma' = 0.$$

The general solution of this equation is

$$\gamma(s) = \cos(\sqrt{2}s)c_1 + \sin(\sqrt{2}s)c_2 + c_3,$$

where the c_i are constant vectors in $\mathbb{E}^{2n+2}$.

Now, as γ satisfies

$$\langle \gamma, \gamma\rangle = 1,\ \langle \gamma', \gamma'\rangle = 1,\ \langle \gamma, \gamma'\rangle = 0,\ \langle \gamma', \gamma''\rangle = 0,\ \langle \gamma'', \gamma''\rangle = 2,\ \langle \gamma, \gamma''\rangle = -1,$$

and since in $s = 0$ we have $\gamma = c_1 + c_3$, $\gamma' = \sqrt{2}c_2$, $\gamma'' = -2c_1$, we obtain

$$c_{11} + 2c_{13} + c_{33} = 1,\ c_{22} = \tfrac{1}{2},\ c_{12} + c_{23} = 0,\ c_{12} = 0,\ c_{11} = \tfrac{1}{2}, c_{11} + c_{13} = \tfrac{1}{2},$$

where c_{ij} denotes $\langle c_i, c_j\rangle$. The above relations imply that $\{c_i\}$ are orthogonal vectors in $\mathbb{E}^{2n+2}$ with $|c_1| = |c_2| = |c_3| = 1/\sqrt{2}$.

Finally, using that γ is a Legendre curve one obtains easily that $\langle c_i, \mathcal{J}c_j\rangle = 0$ for any $i, j = 1, 2, 3$. If we denote $e_i = \sqrt{2}c_i$ we obtain the first part of the Theorem.

Suppose now γ is the biharmonic helix, that is, $\kappa_1^2+\kappa_2^2=1$, $\kappa_1\in(0,1)$. From the Gauss and Frenet equations we get

$$\begin{aligned}
\widetilde{\nabla}_T T &= \dot{\nabla}_T T-\langle T,T\rangle\gamma=\kappa_1 E_2-\gamma,\\
\widetilde{\nabla}_T\widetilde{\nabla}_T T &= \kappa_1\widetilde{\nabla}_T E_2-T=\kappa_1(-\kappa_1 T+\kappa_2 E_3)-T=-(\kappa_1^2+1)T+\kappa_1\kappa_2 E_3,\\
\widetilde{\nabla}_T\widetilde{\nabla}_T\widetilde{\nabla}_T T &= -(\kappa_1^2+1)\widetilde{\nabla}_T T+\kappa_1\kappa_2\widetilde{\nabla}_T E_3\\
&= -(\kappa_1^2+1)\widetilde{\nabla}_T T-\kappa_1\kappa_2^2 E_2=-2\gamma''-\kappa_2^2\gamma.
\end{aligned}$$

Hence

$$\gamma^{iv}+2\gamma''+\kappa_2^2\gamma=0,$$

whose general solution is

$$\gamma(s)=\cos(As)c_1+\sin(As)c_2+\cos(Bs)c_3+\sin(Bs)c_4,$$

where A, B are given by (3-7) and $\{c_i\}$ are constant vectors in $\mathbb{E}^{2n+2}$.

Since γ satisfies

$$\begin{array}{llll}
\langle\gamma,\gamma\rangle=1, & \langle\gamma,\gamma'\rangle=0, & \langle\gamma,\gamma''\rangle=-1 & \langle\gamma,\gamma'''\rangle=0,\\
\langle\gamma',\gamma'\rangle=1, & \langle\gamma',\gamma''\rangle=0, & \langle\gamma',\gamma'''\rangle=-(1+\kappa_1^2), & \\
\langle\gamma'',\gamma''\rangle=1+\kappa_1^2, & \langle\gamma'',\gamma'''\rangle=0, & \langle\gamma''',\gamma'''\rangle=3\kappa_1^2+1, &
\end{array}$$

and since in $s=0$ we have $\gamma=c_1+c_3$, $\gamma'=Ac_2+Bc_4$, $\gamma''=-A^2c_1-B^2c_3$, $\gamma'''=-A^3c_2-B^3c_4$, we obtain

$$c_{11}+2c_{13}+c_{33}=1, \tag{3-8}$$

$$A^2c_{22}+2ABc_{24}+B^2c_{44}=1, \tag{3-9}$$

$$Ac_{12}+Ac_{23}+Bc_{14}+Bc_{34}=0, \tag{3-10}$$

$$A^3c_{12}+AB^2c_{23}+A^2Bc_{14}+B^3c_{34}=0, \tag{3-11}$$

$$A^4c_{11}+2A^2B^2c_{13}+B^4c_{33}=1+\kappa_1^2, \tag{3-12}$$

$$A^2c_{11}+(A^2+B^2)c_{13}+B^2c_{33}=1, \tag{3-13}$$

$$A^4c_{22}+(AB^3+A^3B)c_{24}+B^4c_{44}=1+\kappa_1^2, \tag{3-14}$$

$$A^5c_{12}+A^3B^2c_{23}+A^2B^3c_{14}+B^5c_{34}=0, \tag{3-15}$$

$$A^3c_{12}+A^3c_{23}+B^3c_{14}+B^3c_{34}=0, \tag{3-16}$$

$$A^6c_{22}+2A^3B^3c_{24}+B^6c_{44}=3\kappa_1^2+1, \tag{3-17}$$

where $c_{ij}=\langle c_i,c_j\rangle$. Since the determinant of the system given by (3-10), (3-11), (3-15) and (3-16) is $-A^2B^2(A^2-B^2)^4\neq 0$ it follows that

$$c_{12}=c_{23}=c_{14}=c_{34}=0.$$

The equations (3-8), (3-12) and (3-13) give

$$c_{11} = \tfrac{1}{2}, \quad c_{13} = 0, \quad c_{33} = \tfrac{1}{2},$$

and, from (3-9), (3-14) and (3-17) follows that

$$c_{22} = \tfrac{1}{2}, \quad c_{24} = 0, \quad c_{44} = \tfrac{1}{2}.$$

Therefore, we obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{E}^{2n+2}$ with $|c_1| = |c_2| = |c_3| = |c_4| = 1/\sqrt{2}$.

Finally, since γ is a Legendre curve one obtains the second part of the theorem. $\square$

Remark 3.15. Vectors $\{e_i\}$ satisfying the conditions in the theorem can be easily found.

Remark 3.16. If γ is a proper-biharmonic Legendre circle, then $E_2 \perp \varphi T$. If γ is a proper-biharmonic Legendre helix, then $g_0(E_2, \varphi T) = \sqrt{1+\kappa_1}\langle e_1, \mathcal{I} e_2\rangle$ and we have two cases: either $g_0(E_2, \varphi T) = 0$ and then $\{e_i, \mathcal{I} e_j\}_{i,j=1}^4$ is an orthonormal system in $\mathbb{E}^{2n+2}$, so $n \geq 3$, or $g_0(E_2, \varphi T) \neq 0$ and in this case $g_0(E_2, \varphi T) \in (-1, 1) \setminus \{0\}$.

Next we shall use the deformed Sasakian structure (φ, ξ, η, g) on $\mathbb{S}^{2n+1}$.

Theorem 3.17. *Let $\gamma : I \to (\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$, $n \geq 2$, $a > 0$, $a \neq 1$ (so $c = 4/a - 3 > -3$ and $c \neq 1$), be a proper-biharmonic Legendre curve parametrized by arc length such that $E_2 \perp \varphi T$. Then the equation of γ in the Euclidean space $\mathbb{E}^{2n+2}$ is either*

$$\gamma(s) = \frac{1}{\sqrt{2}} \cos\Big(\sqrt{\frac{2}{a}}s\Big) e_1 + \frac{1}{\sqrt{2}} \sin\Big(\sqrt{\frac{2}{a}}s\Big) e_2 + \frac{1}{\sqrt{2}} e_3,$$

for $n \geq 2$, where $\{e_i, \mathcal{I} e_j\}_{i,j=1}^3$ are orthogonal constant unit vectors, or

$$\gamma(s) = \frac{1}{\sqrt{2}} \cos(As) e_1 + \frac{1}{\sqrt{2}} \sin(As) e_2 + \frac{1}{\sqrt{2}} \cos(Bs) e_3 + \frac{1}{\sqrt{2}} \sin(Bs) e_4,$$

for $n \geq 3$, where

$$A = \sqrt{\frac{1+\kappa_1\sqrt{a}}{a}}, \quad B = \sqrt{\frac{1-\kappa_1\sqrt{a}}{a}}, \quad \kappa_1 \in \Big(0, \frac{1}{a}\Big), \tag{3-18}$$

and $\{e_i, \mathcal{I} e_j\}_{i,j=1}^3$ are orthogonal constant unit vectors.

Proof. Again let us denote by ∇, $\dot{\nabla}$ and by $\widetilde{\nabla}$ the Levi-Civita connections on $(\mathbb{S}^{2n+1}, g)$, $(\mathbb{S}^{2n+1}, g_0)$ and $(\mathbb{R}^{2n+2}, \langle\, , \rangle)$, respectively. From the definition of the Levi-Civita connection, as $g_0(X, \varphi_0 Y) = d\eta_0(X, Y)$ and $g(X, \varphi Y) = d\eta(X, Y)$, we obtain $g(\nabla_X Y, Z) = a g_0(\dot{\nabla}_X Y, Z)$, for any vector field Z and for any X, Y which satisfy $X \perp \xi$, $Y \perp \xi$ and $X \perp \varphi Y$. Further, it is easy to check that we have

$$\nabla_X Y = \dot{\nabla}_X Y, \quad \forall X, Y \in C(T\mathbb{S}^{2n+1}) \text{ with } X \perp \xi,\ Y \perp \xi,\ X \perp \varphi Y. \tag{3-19}$$

First we consider the case when γ is the biharmonic circle, that is, $\kappa_1^2 = (c+3)/4$. Let $T = \gamma'$ be the unit tangent vector field (with respect to the metric g) along γ. Using (3-19) we obtain $\dot{\nabla}_T T = \nabla_T T$ and $\dot{\nabla}_T E_2 = \nabla_T E_2$.

From the Gauss and Frenet equations we get

$$\widetilde{\nabla}_T T = \dot{\nabla}_T T - \langle T, T\rangle \gamma = \kappa_1 E_2 - \frac{1}{a}\gamma \quad \text{and} \quad \widetilde{\nabla}_T \widetilde{\nabla}_T T = \Big(-\kappa_1^2 - \frac{1}{a}\Big) T = -\frac{2}{a} T.$$

Hence

$$a\gamma''' + 2\gamma' = 0,$$

whose general solution is

$$\gamma(s) = \cos\Big(\sqrt{\frac{2}{a}}s\Big) c_1 + \sin\Big(\sqrt{\frac{2}{a}}s\Big) c_2 + c_3,$$

where the c_i are constant vectors in $\mathbb{E}^{2n+2}$.

Since γ satisfies

$$\langle \gamma, \gamma\rangle = 1,\ \langle \gamma', \gamma'\rangle = \frac{1}{a},\ \langle \gamma, \gamma'\rangle = 0,\ \langle \gamma', \gamma''\rangle = 0,\ \langle \gamma'', \gamma''\rangle = \frac{2}{a^2},\ \langle \gamma, \gamma''\rangle = -\frac{1}{a},$$

and in $s = 0$ we have $\gamma = c_1 + c_3$, $\gamma' = (\sqrt{2/a})c_2$, $\gamma'' = -(2/a)c_1$, one obtains

$$c_{11} + 2c_{13} + c_{33} = 1,\ c_{22} = \tfrac{1}{2},\ c_{12} + c_{23} = 0,\ c_{12} = 0,\ c_{11} = \tfrac{1}{2}, c_{11} + c_{13} = \tfrac{1}{2},$$

where $c_{ij} = \langle c_i, c_j\rangle$. Consequently, we obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{E}^{2n+2}$ with $|c_1| = |c_2| = |c_3| = 1/\sqrt{2}$.

Finally, using the facts that γ is a Legendre curve and $g(\nabla_{\gamma'}\gamma', \varphi\gamma') = 0$ one obtains easily that $\langle c_i, \mathcal{J} c_j\rangle = 0$ for any $i, j = 1, 2, 3$.

Now we assume that γ is a biharmonic helix, that is, $\kappa_1^2 + \kappa_2^2 = (c+3)/4$, $\kappa_1^2 \in (0, (c+3)/4)$. First, using (3-19), we obtain $\dot{\nabla}_T T = \nabla_T T$, $\dot{\nabla}_T E_2 = \nabla_T E_2$ and $\dot{\nabla}_T E_3 = \nabla_T E_3$.

From the Gauss and Frenet equations we get

$$\begin{aligned}
\widetilde{\nabla}_T T &= \dot{\nabla}_T T - \langle T, T\rangle\gamma = \kappa_1 E_2 - \frac{1}{a}\gamma,\\
\widetilde{\nabla}_T \widetilde{\nabla}_T T &= \kappa_1 \widetilde{\nabla}_T E_2 - \frac{1}{a} T = \kappa_1(-\kappa_1 T + \kappa_2 E_3) - \frac{1}{a} T = -\Big(\kappa_1^2 + \frac{1}{a}\Big) T + \kappa_1\kappa_2 E_3,\\
\widetilde{\nabla}_T \widetilde{\nabla}_T \widetilde{\nabla}_T T &= -\Big(\kappa_1^2 + \frac{1}{a}\Big)\widetilde{\nabla}_T T + \kappa_1\kappa_2 \widetilde{\nabla}_T E_3 = -\Big(\kappa_1^2 + \frac{1}{a}\Big)\widetilde{\nabla}_T T - \kappa_1\kappa_2^2 E_2\\
&= -\frac{2}{a}\gamma'' - \frac{1}{a}\kappa_2^2\gamma.
\end{aligned}$$

Therefore
$$a\gamma^{iv}+2\gamma''+\kappa_2^2\gamma=0,$$
whose general solution is
$$\gamma(s)=\cos(As)c_1+\sin(As)c_2+\cos(Bs)c_3+\sin(Bs)c_4,$$
where A, B are given by (3-18) and $\{c_i\}$ are constant vectors in $\mathbb{E}^{2n+2}$.

The curve γ satisfies
$$\begin{array}{llll}
\langle\gamma,\gamma\rangle=1, & \langle\gamma,\gamma'\rangle=0, & \langle\gamma,\gamma''\rangle=-\dfrac{1}{a}, & \langle\gamma,\gamma'''\rangle=0,\\
\langle\gamma',\gamma'\rangle=\dfrac{1}{a}, & \langle\gamma',\gamma''\rangle=0, & \langle\gamma',\gamma'''\rangle=-\dfrac{1+a\kappa_1^2}{a^2}, & \\
\langle\gamma'',\gamma''\rangle=\dfrac{1+a\kappa_1^2}{a^2}, & \langle\gamma'',\gamma'''\rangle=0, & \langle\gamma''',\gamma'''\rangle=\dfrac{3a\kappa_1^2+1}{a^3}, &
\end{array}$$
and in $s=0$ we have
$$\gamma=c_1+c_3,\quad \gamma'=Ac_2+Bc_4,\quad \gamma''=-A^2c_1-B^2c_3,\quad \gamma'''=-A^3c_2-B^3c_4.$$
It follows that
$$c_{11}+2c_{13}+c_{33}=1, \tag{3-20}$$
$$A^2c_{22}+2ABc_{24}+B^2c_{44}=\frac{1}{a}, \tag{3-21}$$
$$Ac_{12}+Ac_{23}+Bc_{14}+Bc_{34}=0, \tag{3-22}$$
$$A^3c_{12}+AB^2c_{23}+A^2Bc_{14}+B^3c_{34}=0, \tag{3-23}$$
$$A^4c_{11}+2A^2B^2c_{13}+B^4c_{33}=\frac{1+a\kappa_1^2}{a^2}, \tag{3-24}$$
$$A^2c_{11}+(A^2+B^2)c_{13}+B^2c_{33}=\frac{1}{a}, \tag{3-25}$$
$$A^4c_{22}+(AB^3+A^3B)c_{24}+B^4c_{44}=\frac{1+a\kappa_1^2}{a^2}, \tag{3-26}$$
$$A^5c_{12}+A^3B^2c_{23}+A^2B^3c_{14}+B^5c_{34}=0, \tag{3-27}$$
$$A^3c_{12}+A^3c_{23}+B^3c_{14}+B^3c_{34}=0, \tag{3-28}$$
$$A^6c_{22}+2A^3B^3c_{24}+B^6c_{44}=\frac{3a\kappa_1^2+1}{a^3}, \tag{3-29}$$
where $c_{ij}=\langle c_i,c_j\rangle$.

The solution of the system given by (3-22), (3-23), (3-27) and (3-28) is
$$c_{12}=c_{23}=c_{14}=c_{34}=0.$$

From equations (3-20), (3-24) and (3-25) we get

$$c_{11} = \tfrac{1}{2}, \quad c_{13} = 0, \quad c_{33} = \tfrac{1}{2},$$

and, from (3-21), (3-26), (3-29),

$$c_{22} = \tfrac{1}{2}, \quad c_{24} = 0, \quad c_{44} = \tfrac{1}{2}.$$

We obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{E}^{2n+2}$ with $|c_1| = |c_2| = |c_3| = |c_4| = 1/\sqrt{2}$.

Finally, since γ is a Legendre curve and $g(\nabla_{\gamma'}\gamma', \varphi\gamma') = 0$, one obtains the conclusion. □

In the third case, just like for $\mathbb{S}^3$ (see [Fetcu and Oniciuc 2007]), we obtain:

Theorem 3.18. *Let $\gamma : I \to (\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$, $0 < a < 1$ (so $c > 1$), be a proper-biharmonic Legendre curve parametrized by arc length such that $E_2 \parallel \varphi T$. Then the equation of γ in the Euclidean space $\mathbb{E}^{2n+2}$ is*

$$\begin{aligned}\gamma(s) &= \sqrt{\frac{B}{A+B}}\cos(As)e_1 - \sqrt{\frac{B}{A+B}}\sin(As)\mathcal{J}e_1 \\ &\qquad + \sqrt{\frac{A}{A+B}}\cos(Bs)e_3 + \sqrt{\frac{A}{A+B}}\sin(Bs)\mathcal{J}e_3 \\ &= \sqrt{\frac{B}{A+B}}\exp(-\mathrm{i}As)e_1 + \sqrt{\frac{A}{A+B}}\exp(\mathrm{i}Bs)e_3,\end{aligned}$$

where e_1, e_3 are constant unit orthogonal vectors in $\mathbb{E}^{2n+2}$ with e_3 orthogonal to $\mathcal{J}e_1$, and

$$A = \sqrt{\frac{3-2a-2\sqrt{(a-1)(a-2)}}{a}}, \quad B = \sqrt{\frac{3-2a+2\sqrt{(a-1)(a-2)}}{a}}. \tag{3-30}$$

Remark 3.19. For the fourth case the ODE satisfied by proper-biharmonic Legendre curves in the $(2n+1)$-sphere may be also obtained but the computations are rather complicated.

4. Biharmonic submanifolds in Sasakian space forms

A method to obtain biharmonic submanifolds in a Sasakian space form is provided by the following Theorem.

Theorem 4.1. *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian space form with constant φ-sectional curvature c and let $\mathbf{i} : M \to N$ be an r-dimensional integral submanifold of N, $1 \le r \le n$. Consider*

$$F : \widetilde{M} = I \times M \to N, \quad F(t, p) = \phi_t(p) = \phi_p(t),$$

where $I = \mathbb{S}^1$ or $I = \mathbb{R}$ and $\{\phi_t\}_{t\in I}$ is the flow of the vector field ξ. Then the map $F : (\widetilde{M}, \tilde{g} = dt^2 + \mathbf{i}^*g) \to N$ is a Riemannian immersion, and it is proper-biharmonic if and only if M is a proper-biharmonic submanifold of N.

Proof. From the definition of the flow of ξ we have

$$dF(t,p)\Big(\frac{\partial}{\partial t}\Big) = \frac{d}{ds}\Big|_{s=t}\{\phi_p(s)\} = \dot{\phi}_p(t) = \xi(\phi_p(t)) = \xi(F(t,p)),$$

that is, $\partial/\partial t$ is F-correlated to ξ and

$$\Big|dF(t,p)\Big(\frac{\partial}{\partial t}\Big)\Big| = |\xi(F(t,p))| = 1 = \Big|\frac{\partial}{\partial t}\Big|.$$

The vector $X_p \in T_pM$ can be identified to $(0, X_p) \in T_{(t,p)}(I \times M)$ and we have

$$dF_{(t,p)}(X_p) = (dF)_{(t,p)}(\dot{\gamma}(0)) = \frac{d}{ds}\Big|_{s=0}\{\phi_t(\gamma(s))\} = (d\phi_t)_p(X_p).$$

Since ϕ_t is an isometry $|dF_{(t,p)}(X_p)| = |(d\phi_t)_p(X_p)| = |X_p|$. Moreover,

$$\begin{aligned} g\Big(dF_{(t,p)}\Big(\frac{\partial}{\partial t}\Big), dF_{(t,p)}(X_p)\Big) &= g\big(\xi(\phi_p(t)), (d\phi_t)_p(X_p)\big) \\ &= g\big((d\phi_t)_p(\xi_p), (d\phi_t)_p(X_p)) = g(\xi_p, X_p\big) = 0 \\ &= \tilde{g}\Big(\frac{\partial}{\partial t}, X_p\Big), \end{aligned}$$

and therefore $F : (I \times M, \tilde{g}) \to N$ is a Riemannian immersion.

Let $F^{-1}(TN)$ be the pullback bundle over $\widetilde{M}$ and ∇^F the pullback connection determined by the Levi-Civita connection on N. We shall prove that

$$\tau(F)_{(t,p)} = (d\phi_t)_p(\tau(\mathbf{i})) \ \text{ and } \ \tau_2(F)_{(t,p)} = (d\phi_t)_p(\tau_2(\mathbf{i})),$$

so, from the point of view of harmonicity and biharmonicity, $\widetilde{M}$ and M have the same behaviour.

We start with two remarks. First, let $\sigma \in C(F^{-1}(TN))$ be a section in $F^{-1}(TN)$ defined by $\sigma_{(t,p)} = (d\phi_t)_p(Z_p)$, where Z is a vector field along M, that is, $Z_p \in T_pN$, $\forall p \in M$. One can easily check that

$$(\nabla^F_X\sigma)_{(t,p)} = (d\phi_t)_p(\nabla^N_X Z), \quad \forall X \in C(TM). \tag{4-1}$$

Then, if $\sigma \in C(F^{-1}(TN))$, it follows that $\varphi\sigma$ given by $(\varphi\sigma)_{(t,p)} = \varphi_{\phi_p(t)}(\sigma_{(t,p)})$ is a section in $F^{-1}(TN)$ and

$$\nabla^F_{\partial/\partial t}\varphi\sigma = \varphi\nabla^F_{\partial/\partial t}\sigma. \tag{4-2}$$

Now, we consider $\{X_1, \dots, X_r\}$ a local orthonormal frame field on U, where U is an open subset of M. The tension field of F is given by

$$\tau(F)=\nabla^F_{\partial/\partial t}dF\Big(\frac{\partial}{\partial t}\Big)-dF\Big(\nabla^{\widetilde{M}}_{\partial/\partial t}\frac{\partial}{\partial t}\Big)+\sum_{a=1}^{r}\big\{\nabla^F_{X_a}dF(X_a)-dF(\nabla^{\widetilde{M}}_{X_a}X_a)\big\}. \tag{4-3}$$

Since

$$\nabla^F_{\partial/\partial t}dF\Big(\frac{\partial}{\partial t}\Big)=\nabla^N_{\xi}\xi=0, \qquad (\nabla^F_{X_a}dF(X_a))_{(t,p)}=(d\phi_t)_p(\nabla^N_{X_a}X_a),$$
$$\nabla^{\widetilde{M}}_{\partial/\partial t}\frac{\partial}{\partial t}=\nabla^I_{\partial/\partial t}\frac{\partial}{\partial t}=0, \qquad dF_{(t,p)}(\nabla^{\widetilde{M}}_{X_a}X_a)=(d\phi_t)_p(\nabla^M_{X_a}X_a),$$

substituting in (4-3) we get

$$\tau(F)_{(t,p)}=(d\phi_t)_p(\tau(\mathbf{i})).$$

To obtain $\tau_2(F)_{(t,p)}=(d\phi_t)_p(\tau_2(\mathbf{i}))$, we prove first that $\nabla^F_{\partial/\partial t}\tau(F)=-\varphi(\tau(F))$.

Since $[\partial/\partial t, X_a]=0$, $a=1,\dots,r$, it follows that

$$\nabla^F_{\partial/\partial t}dF(X_a)=\nabla^F_{X_a}dF\Big(\frac{\partial}{\partial t}\Big).$$

But

$$\Big(\nabla^F_{X_a}dF\Big(\frac{\partial}{\partial t}\Big)\Big)_{(t,p)}=\nabla^N_{dF_{(t,p)}X_a}\xi=\nabla^N_{(d\phi_t)_pX_a}\xi=-\varphi((d\phi_t)_p(X_a))$$
$$=-(d\phi_t)_p(\varphi X_a),$$

so

$$\big(\nabla^F_{\partial/\partial t}dF(X_a)\big)_{(t,p)}=-(d\phi_t)_p(\varphi X_a). \tag{4-4}$$

We note that

$$R^F\Big(\frac{\partial}{\partial t},X_a\Big)dF(X_a)=\nabla^F_{\partial/\partial t}\nabla^F_{X_a}dF(X_a)-\nabla^F_{X_a}\nabla^F_{\partial/\partial t}dF(X_a)$$

and, on the other hand, as N is a Sasakian space form,

$$\Big(R^F\Big(\frac{\partial}{\partial t},X_a\Big)dF(X_a)\Big)_{(t,p)}=R^N_{\phi_t(p)}(\xi,(d\phi_t)_p(X_a))(d\phi_t)_p(X_a)=\xi.$$

Therefore

$$\nabla^F_{\partial/\partial t}\nabla^F_{X_a}dF(X_a)-\nabla^F_{X_a}\nabla^F_{\partial/\partial t}dF(X_a)=\xi. \tag{4-5}$$

Using (4-1) and (4-4), $\nabla^F_{X_a}\nabla^F_{\partial/\partial t}dF(X_a)$ can be written as

$$\big(\nabla^F_{X_a}\nabla^F_{\partial/\partial t}dF(X_a)\big)_{(t,p)}=-(d\phi_t)_p(\nabla^N_{X_a}\varphi X_a)=-(d\phi_t)_p(\xi+\varphi\nabla^N_{X_a}X_a). \tag{4-6}$$

Moreover, from (4-4),

$$(4\text{-}7)\quad \big(\nabla^F_{\partial/\partial t} dF(\nabla^{\widetilde{M}}_{X_a} X_a)\big)_{(t,p)} = \big(\nabla^F_{\partial/\partial t} dF(\nabla^M_{X_a} X_a)\big)_{(t,p)} = -(d\phi_t)_p(\varphi\nabla^M_{X_a} X_a).$$

Substituting (4-6) in (4-5) and using (4-7), we obtain

$$\begin{aligned}\xi &= \nabla^F_{\partial/\partial t}\nabla^F_{X_a} dF(X_a) - \nabla^F_{\partial/\partial t} dF(\nabla^{\widetilde{M}}_{X_a} X_a) + \nabla^F_{\partial/\partial t} dF(\nabla^{\widetilde{M}}_{X_a} X_a) - \nabla^F_{X_a}\nabla^F_{\partial/\partial t} dF(X_a)\\ &= \nabla^F_{\partial/\partial t}\nabla dF(X_a, X_a) - (d\phi_t)_p(\varphi\nabla^M_{X_a} X_a) + (d\phi_t)_p(\xi + \varphi\nabla^N_{X_a} X_a)\\ &= \nabla^F_{\partial/\partial t}\nabla dF(X_a, X_a) + \varphi(d\phi_t)_p(\nabla^N_{X_a} X_a - \nabla^M_{X_a} X_a) + \xi,\end{aligned}$$

so

$$(4\text{-}8)\qquad \big(\nabla^F_{\partial/\partial t}\nabla dF(X_a, X_a)\big)_{(t,p)} = -\varphi(d\phi_t)_p(\nabla d\mathbf{i}(X_a, X_a)).$$

Since $\nabla dF(\partial/\partial t, \partial/\partial t) = 0$, summing up in (4-8) we obtain

$$(4\text{-}9)\qquad \nabla^F_{\partial/\partial t}\tau(F) = -\varphi(\tau(F)).$$

From (4-2) and (4-9) we have

$$(4\text{-}10)\quad \nabla^F_{\partial/\partial t}\nabla^F_{\partial/\partial t}\tau(F) = -\nabla^F_{\partial/\partial t}\varphi(\tau(F)) = -\varphi\nabla^F_{\partial/\partial t}\tau(F) = \varphi^2\tau(F) = -\tau(F),$$

and from (4-1)

$$(4\text{-}11)\qquad \big(\nabla^F_{X_a}\nabla^F_{X_a}\tau(F)\big)_{(t,p)} = (d\phi_t)_p\big(\nabla^N_{X_a}\nabla^N_{X_a}\tau(\mathbf{i})\big),$$

$$(4\text{-}12)\qquad \big(\nabla^F_{\nabla^{\widetilde{M}}_{X_a} X_a}\tau(F)\big)_{(t,p)} = (d\phi_t)_p\big(\nabla^N_{\nabla^M_{X_a} X_a}\tau(\mathbf{i})\big).$$

From (4-10), (4-11) and (4-12) we obtain

$$\begin{aligned}(4\text{-}13)\quad -(\Delta^F\tau(F))_{(t,p)} &= \nabla^F_{\partial/\partial t}\nabla^F_{\partial/\partial t}\tau(F) + \sum_{a=1}^{r}\big\{\nabla^F_{X_a}\nabla^F_{X_a}\tau(F) - \nabla^F_{\nabla^{\widetilde{M}}_{X_a} X_a}\tau(F)\big\}\\ &= -\tau(F)_{(t,p)} - (d\phi_t)_p(\Delta^{\mathbf{i}}\tau(\mathbf{i})).\end{aligned}$$

Using the form of the curvature tensor field R^N, after a straightforward computation, we get

$$(4\text{-}14)\qquad \operatorname{trace} R^F(dF, \tau(F))dF = -\tau(F) + (d\phi_t)_p(\operatorname{trace} R^N_p(d\mathbf{i}, \tau(\mathbf{i}))d\mathbf{i}).$$

Finally, from (4-13) and (4-14) we conclude

$$\tau_2(F)_{(t,p)} = (d\phi_t)_p(\tau_2(\mathbf{i})). \qquad \square$$

Remark 4.2. The previous result was expected for the following reason. Assume that $(N^{2n+1}, \varphi, \xi, \eta, g)$ is a compact strictly regular Sasakian manifold and let $G : M \to N$ be an arbitrary smooth map from a compact Riemannian manifold M. If F is biharmonic, then the map G is biharmonic, where $F : \widetilde{M} = \mathbb{S}^1 \times M \to N$, $F(t, p) = \phi_t(G(p))$.

Indeed, an arbitrary variation $\{G_s\}_s$ of G induces a variation $\{F_s\}_s$ of F. We can check that $\tau_{(p,t)}(F_s) = (d\phi_t)_{G_s(p)}(\tau_p(G_s))$ and, from the biharmonicity of F and the Fubini Theorem, we get

$$0 = \frac{d}{ds}\Big|_{s=0}\{E_2(F_s)\} = \frac{1}{2}\frac{d}{ds}\Big|_{s=0}\int_{\widetilde{M}} |\tau(F_s)|^2\, v_{\tilde{g}} = \frac{1}{2}2\pi\,\frac{d}{ds}\Big|_{s=0}\int_M |\tau(G_s)|^2\, v_g$$
$$= 2\pi\,\frac{d}{ds}\Big|_{s=0}\{E_2(G_s)\}.$$

Since $\frac{d}{ds}\big|_{s=0}\{E_2(G_s)\} = 0$ for any variation $\{G_s\}_s$ of G, it follows that G is biharmonic. In particular, if M is a submanifold of N and G is the inclusion map $\mathbf{i}$, then we have the direct implication of the Theorem.

Theorem 4.3. *Let M^2 be a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then M is proper-biharmonic if and only if locally, it is given by $x(t,s) = \phi_t(\gamma(s))$, where γ is a proper-biharmonic Legendre curve.*

Proof. A surface M of N^{2n+1} invariant under the flow-action of the characteristic vector field ξ, that is, $\phi_t(p) \in M$, for any t and any $p \in M$, can be written, locally, $x(t,s) = \phi_t(\gamma(s))$, where γ is a Legendre curve in N. Then, from Theorem 4.1, such a surface is proper-biharmonic if and only if γ is proper-biharmonic. □

Corollary 4.4. *Let M^2 be a surface of $\mathbb{S}^{2n+1}$ endowed with its canonical Sasakian structure which is invariant under the flow-action of the characteristic vector field ξ. Then M is proper-biharmonic if and only if locally, it is given by $x(t,s) = \phi_t(\gamma(s))$, where γ is a proper-biharmonic Legendre curve given by Theorem 3.14.*

Next, consider the unit $(2n+1)$-dimensional sphere $\mathbb{S}^{2n+1}$ endowed with its canonical or deformed Sasakian structure. The flow of ξ is $\phi_t(z) = \exp(-\mathrm{i}\frac{t}{a})z$, and from Theorems 3.17, 3.18 and 4.1 we obtain explicit examples of proper-biharmonic surfaces in $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$, $a > 0$, of constant mean curvature.

Moreover, we reobtain a result of [Arslan et al. 2007].

Proposition 4.5 [Arslan et al. 2007]. *Let $F : \widetilde{M}^3 \to (\mathbb{S}^5, \varphi_0, \xi_0, \eta_0, g_0) \subset \mathbb{R}^6$ be a proper-biharmonic anti-invariant immersion. Then*

$$F(t,u,v) = \frac{\exp(-\mathrm{i}t)}{\sqrt{2}}\big(\exp(\mathrm{i}u), \mathrm{i}\exp(-\mathrm{i}u)\sin(\sqrt{2}v), \mathrm{i}\exp(-\mathrm{i}u)\cos(\sqrt{2}v)\big).$$

Proof. It was proved in [Sasahara 2005] that the proper-biharmonic integral surface of $(\mathbb{S}^5, \varphi_0, \xi_0, \eta_0, g_0)$ is given by

$$f(u,v) = \frac{1}{\sqrt{2}}\big(\exp(\mathrm{i}u), \mathrm{i}\exp(-\mathrm{i}u)\sin\sqrt{2}v, \mathrm{i}\exp(-\mathrm{i}u)\cos\sqrt{2}v\big).$$

Now, composing with the flow of ξ_0 we obtain the result. □

5. Biharmonic submanifolds of $(\mathbb{S}^7, g_0)$

First we shall recall the definition of a Sasakian 3-structure. If a manifold N admits three Sasakian structures $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, satisfying

$$\varphi_c = -\varphi_a\varphi_b + \eta_b \otimes \xi_a = \varphi_b\varphi_a - \eta_a \otimes \xi_b,$$
$$\xi_c = -\varphi_a\xi_b = \varphi_b\xi_a, \quad \eta_c = -\eta_a \circ \varphi_b = \eta_b \circ \varphi_a,$$

for an even permutation (a, b, c) of $(1, 2, 3)$, then the manifold is said to have a *Sasakian* 3-*structure* [Blair 2002]. The dimension of such a manifold is of the form $4n+3$. The maximum dimension of a submanifold of a 3-Sasakian manifold N^{4n+3} which is an integral submanifold with respect to all three Sasakian structures is n.

We consider now the Euclidean space $\mathbb{E}^8$ with three complex structures,

$$\mathcal{I} = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \\ 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{K} = -\mathcal{I}\mathcal{J},$$

where I_n denotes the $n \times n$ identity matrix. We define three vector fields on $\mathbb{S}^7$ by

$$\xi_1 = -\mathcal{I}z, \quad \xi_2 = -\mathcal{J}z, \quad \xi_3 = -\mathcal{K}z, \quad z \in \mathbb{S}^7,$$

and consider their dual 1-forms $\eta_1 = \eta_0$, η_2, η_3. Let φ_a defined by

$$\varphi_1 = \varphi_0 = s \circ \mathcal{I}, \quad \varphi_2 = s \circ \mathcal{J}, \quad \varphi_3 = s \circ \mathcal{K}.$$

Then $(\varphi_a, \xi_a, \eta_a, g_0)$, $a = 1, 2, 3$, determine a Sasakian 3-structure on $\mathbb{S}^7$ (see [Baikoussis and Blair 1995]).

In the following, we shall indicate a method to construct proper-biharmonic submanifolds in $(\mathbb{S}^7, g_0)$. We consider $\gamma = \gamma(s)$ a proper-biharmonic curve in $(\mathbb{S}^7, g_0)$, parametrized by arc-length, which is a Legendre curve for two of the three contact structures (it was proved in [Fetcu 2008b] that there is no proper-biharmonic curve which is Legendre with respect to all three contact structures on $\mathbb{S}^7$). For example, assume that γ is a Legendre curve for η_1 and η_2. Composing with the flow of ξ_1 (or ξ_2) we obtain a biharmonic surface which is Legendre with respect to η_2 (or η_1). Then, composing with the flow of ξ_2 (or ξ_1) we get a biharmonic 3-dimensional submanifold of $(\mathbb{S}^7, g_0)$.

Using this method, from Theorems 3.14 and 4.1, we obtain 4 classes of proper-biharmonic surfaces in $(\mathbb{S}^7, g_0)$ and 4 classes of proper-biharmonic 3-dimensional submanifolds of $(\mathbb{S}^7, g_0)$, all of constant mean curvature.

For example, from Theorems 3.14 and 4.1, composing first with the flow of ξ_1 and then with that of ξ_2, we get the explicit parametric equations of proper-biharmonic 3-dimensional submanifolds of $(\mathbb{S}^7, g_0)$.

Proposition 5.1. *Let M be a 3-dimensional submanifold in $\mathbb{S}^7$ such that its position vector field in $\mathbb{E}^8$ is either*

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{x}_1(u,t,s)\\ &= \tfrac{1}{\sqrt{2}}\big(\cos(u)\cos(\sqrt{2}s)\cos(t)e_1+\cos(u)\sin(\sqrt{2}s)\cos(t)e_2\\ &\qquad +\cos(u)\cos(t)e_3-\cos(u)\cos(\sqrt{2}s)\sin(t)\mathcal{I}e_1\\ &\qquad -\cos(u)\sin(\sqrt{2}s)\sin(t)\mathcal{I}e_2-\cos(u)\sin(t)\mathcal{I}e_3\\ &\qquad -\sin(u)\cos(\sqrt{2}s)\cos(t)\mathcal{J}e_1-\sin(u)\sin(\sqrt{2}s)\cos(t)\mathcal{J}e_2\\ &\qquad -\sin(u)\cos(t)\mathcal{J}e_3-\sin(u)\cos(\sqrt{2}s)\sin(t)\mathcal{K}e_1\\ &\qquad -\sin(u)\sin(\sqrt{2}s)\sin(t)\mathcal{K}e_2-\sin(u)\sin(t)\mathcal{K}e_3\big),\end{aligned}$$

where $\{e_i,\mathcal{I}e_j\}_{i,j=1}^3$ and $\{e_i,\mathcal{J}e_j\}_{i,j=1}^3$ are systems of constant orthonormal vectors in $\mathbb{E}^8$, or

$$\begin{aligned}\mathbf{x}_2 &= \mathbf{x}_2(u,t,s)\\ &= \tfrac{1}{\sqrt{2}}\big(\cos(u)\cos(As)\cos(t)e_1+\cos(u)\sin(As)\cos(t)e_2\\ &\qquad +\cos(u)\cos(Bs)\cos(t)e_3+\cos(u)\sin(Bs)\cos(t)e_4\\ &\qquad -\cos(u)\cos(As)\sin(t)\mathcal{I}e_1-\cos(u)\sin(As)\sin(t)\mathcal{I}e_2\\ &\qquad -\cos(u)\cos(Bs)\sin(t)\mathcal{I}e_3-\cos(u)\sin(Bs)\sin(t)\mathcal{I}e_4\\ &\qquad -\sin(u)\cos(As)\cos(t)\mathcal{J}e_1-\sin(u)\sin(As)\cos(t)\mathcal{J}e_2\\ &\qquad -\sin(u)\cos(Bs)\cos(t)\mathcal{J}e_3-\sin(u)\sin(Bs)\cos(t)\mathcal{J}e_4\\ &\qquad -\sin(u)\cos(As)\sin(t)\mathcal{K}e_1-\sin(u)\sin(As)\sin(t)\mathcal{K}e_2\\ &\qquad -\sin(u)\cos(Bs)\sin(t)\mathcal{K}e_3-\sin(u)\sin(Bs)\sin(t)\mathcal{K}e_4\big),\end{aligned}$$

where

$$A=\sqrt{1+\kappa_1},\quad B=\sqrt{1-\kappa_1},\quad \kappa_1=\text{constant}\in(0,1),$$

and the e_i, $i=1,2,3,4$ are constant orthonormal vectors in $\mathbb{E}^8$ such that

$$\begin{gathered}\langle e_1,\mathcal{I}e_3\rangle=\langle e_1,\mathcal{I}e_4\rangle=\langle e_2,\mathcal{I}e_3\rangle=\langle e_2,\mathcal{I}e_4\rangle=0,\\ \langle e_1,\mathcal{J}e_3\rangle=\langle e_1,\mathcal{J}e_4\rangle=\langle e_2,\mathcal{J}e_3\rangle=\langle e_2,\mathcal{J}e_4\rangle=0,\\ A\langle e_1,\mathcal{I}e_2\rangle+B\langle e_3,\mathcal{I}e_4\rangle=A\langle e_1,\mathcal{J}e_2\rangle+B\langle e_3,\mathcal{J}e_4\rangle=0.\end{gathered}$$

Then M is a proper-biharmonic submanifold of $(\mathbb{S}^7,g_0)$.

Proof. As the flows of ξ_1 and ξ_2 are given by

$$\phi_t^1(z)=(\cos t)z-(\sin t)\mathcal{I}z,\qquad \phi_t^2(z)=(\cos t)z-(\sin t)\mathcal{J}z,$$

the Proposition follows by a straightforward computation. □

Remark 5.2. Note that there exist vectors $\{e_i\}$ which satisfy the hypotheses of the above Proposition. For example the first three, respectively four vectors, from the canonical basis of $\mathbb{E}^8$ satisfy the required properties.

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