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CONVEX EIGENFUNCTION OF A DRIFTING LAPLACIAN OPERATOR AND THE FUNDAMENTAL GAP

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We study the convexity of the first eigenfunction of the drifting Laplacian operator with zero Dirichlet boundary value provided a suitable assumption to the drifting term is added. We firstly generalize some results of N. Korevaar and S.-T. Yau to gain a Hessian estimate of the first eigenfunction. As an application, we use this Hessian estimate to get a lower bound of the difference of the first and second eigenvalues of the drifting Laplacian. At the end we also find a lower bound when the Hessian estimate does not hold.

1. Introduction

It is a significant problem in mathematical physics and differential geometry to study the eigenvalue estimates of self-adjoint operators in Hilbert spaces [Li and Yau 1986; Schoen and Yau 1994; Li and Wang 2005; Ma and Zhu 2007]. Given a smooth convex bounded domain $\Omega \subset \mathbb{R}^n$, we consider the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_h f + Vf = \lambda f, & \text{in } \Omega \\ f = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_h = \Delta - \nabla h \cdot \nabla$ and h, V are two given smooth functions on the closure of Ω. In the $h = 0$ case, Δ_0 is the standard Laplacian operator in $\mathbb{R}^n$ such that $\Delta u = u''$ when $n = 1$. See [Da Prato and Lunardi 2004] for interesting results with the drifting Laplacian operator. There are very few results on the eigenvalue estimates for the problem (1) — see [González and Negrin 1999] — and we only find some related interesting results in [Kawohl 1985; Ni 2004; Setti 1993].

Throughout this paper, we shall use the following basic properties of the operator $-\Delta_h + V$:

Property 1. The first and second eigenvalues λ_1 and λ_2 of the operator $-\Delta_h + V$ satisfy $0 < \lambda_1 < \lambda_2$.

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Property 2. The first and second eigenfunctions f_1 and f_2 are both smooth on $\overline{\Omega}$. Moreover, $f_1 > 0$.

Our overall plan is first to investigate the convexity of the first eigenfunction of problem (1), by enhancing some results of N. Korevaar [1983]. Then we use the convexity properties to extend results of S.-T. Yau [2003] (where $h = 0$) to the problem (1).

In the case when $h = 0$, one of these results is that for a convex domain Ω with a potential V, if the Hessian of V has a positive lower bound, then the first eigenfunction of the operator $-\Delta + V$ is Log concave. In our case when the drifting term is added, we will show that if the Hessian of

$$\psi := V - \tfrac{1}{2}\Delta h + \tfrac{1}{4}|\nabla h|^2$$

has a positive lower bound, then the first eigenfunction of the operator $-\Delta_h + V$ is Log concave compared with the drifting term h. To be precise:

Theorem 1. *Let Ω be a smooth convex bounded domain in $\mathbb{R}^n$. Suppose*

$$\mathrm{Hess}(\psi) - cI \geq 0$$

with some constant $c > 0$. Then we have

$$\mathrm{Hess}\left(\frac{h}{2} + \varphi\right) - \sqrt{\frac{c}{2}}\,I \geq 0,$$

where $\varphi = -\log f_1$.

Remark. When $V = 0$, the function $\psi = -\frac{1}{2}\Delta h + \frac{1}{4}|\nabla h|^2$ has a geometric meaning; see [Ma and Liu 2008].

After applying Theorem 1, we deduce the following corollary by using Theorem 1.1 in [Yau 2003].

Corollary 2. *Let Ω be a smooth convex bounded domain in $\mathbb{R}^n$. Suppose*

$$\mathrm{Hess}(\psi) - cI \geq 0$$

with some constant $c > 0$. Then

$$\lambda_2 - \lambda_1 \geq \frac{\theta^2(\beta)}{\mathrm{diam}(\Omega)^2} + \beta\sqrt{c}, \tag{2}$$

where $\theta(\beta) = \arcsin\left(1/\sqrt{1 + \beta/(\sqrt{2} - \beta)}\right)$ and $0 < \beta < \sqrt{2}$.

Even when ψ is not convex, we can find an estimate of the fundamental gap of $-\Delta_h + V$ by using the following gradient estimate for function $u = f_2/f_1$, where f_1 and f_2 are the first and second eigenfunctions of $-\Delta_h + V$. Actually, we follow the methods of S.-T. Yau [2003]. Since our results are more general than his results, we shall give complete proofs.

Theorem 3. *Let Ω be a smooth convex bounded domain in $\mathbb{R}^n$. Let $\kappa_i(x)$ $(1 \le i \le n)$ be the eigenvalues of* $\operatorname{Hess}(h/2+\varphi)$ *at x, and let $\lambda = \lambda_2 - \lambda_1$. For any $\varepsilon > 0$, let*

$$\alpha = 2\lambda(1+\varepsilon^{-1}) - 4 \min_{1\le i\le n} \inf_{x\in\Omega} \kappa_i .$$

Assume that

$$\min_{1\le i\le n} \inf_{x\in\Omega} \kappa_i(x) \le 0.$$

Then we have the following estimate for the gradient of $u = f_2/f_1$:

$$\frac{|\nabla u|}{c-u} \le \sqrt{\alpha}\big(\log c - \log (c-u)\big)^{1/2}, \tag{3}$$

where $c = (1+\varepsilon)\sup_{x\in\Omega} u$.

After using this gradient estimate, we can derive a lower bound for the difference of eigenvalues λ.

Corollary 4. *Let Ω be a smooth convex bounded domain in $\mathbb{R}^n$. Suppose*

$$\min_{1\le i\le n} \inf_{x} \kappa_i \ge -a, \qquad a \ge 0.$$

Then the fundamental gap of the operator $-\Delta_h + V$ satisfies

$$\lambda_2 - \lambda_1 \ge 2(\operatorname{diam}\Omega)^{-2} \exp\left(-a(\operatorname{diam}\Omega)^2 - 1\right). \tag{4}$$

We point out that the constant e^{-1} in [Yau 2003, (3.15)] is missing.

Remark. Because a convex domain can be approximated by strictly convex domains, we shall prove the results only for strictly convex domains. In the following we assume that Ω is a smooth strictly convex bounded domain in $\mathbb{R}^n$.

2. Preliminary results

By Property 2, f_1 is a positive function. Then $u = f_2/f_1$ is a well-defined smooth function in Ω. We firstly try to find the equation it satisfies. Recall that $\lambda = \lambda_2 - \lambda_1$.

Lemma 5. $\Delta_h u = -\lambda u - 2\nabla u \cdot \nabla \log f_1$.

Proof. By direct computation, we have

$$\begin{aligned}
&\Delta u \\
&= \frac{\Delta f_2}{f_1} - 2\frac{\nabla f_1\cdot\nabla f_2}{f_1^2} - \frac{f_2}{f_1^2}\Delta f_1 + 2\frac{f_2}{f_1^3}|\nabla f_1|^2 \\
&= \frac{1}{f_1^2}(-\lambda_2 f_1 f_2 + \lambda_1 f_1 f_2) + \frac{1}{f_1^2}(f_1\nabla h\cdot\nabla f_2 - f_2\nabla h\cdot\nabla f_1) - 2\frac{\nabla f_1\cdot\nabla f_2}{f_1^2} + 2f_2\frac{|\nabla f_1|^2}{f_1^3} \\
&= -\lambda u + \frac{\nabla h\cdot\nabla f_2}{f_1} - \frac{f_2}{f_1^2}\nabla h\cdot\nabla f_1 - 2\frac{\nabla f_1\cdot\nabla f_2}{f_1^2} + 2f_2\frac{|\nabla f_1|^2}{f_1^3}.
\end{aligned}$$

Now, taking into account the relations

$$\nabla u\cdot\nabla\log f_1=\frac{\nabla f_1\cdot\nabla f_2}{f_1^2}-f_2\frac{|\nabla f_1|^2}{f_1^3},\qquad \nabla h\cdot\nabla u=\frac{\nabla h\cdot\nabla f_2}{f_1}-\frac{f_2}{f_1^2}\nabla h\cdot\nabla f_1,$$

we obtain

$$\Delta u=-\lambda u+\nabla h\cdot\nabla u-2\nabla u\cdot\nabla\log f_1, \tag{5}$$

which proves the lemma. □

We now consider the smoothness of the function u up to the boundary. This is a standard matter, but for the sake of completeness we include it here.

Lemma 6. *Let $\Omega\subset\mathbb{R}^n$ be a smooth bounded domain. Then $u=f_2/f_1$ is smooth up to the boundary $\partial\Omega$. Moreover, it satisfies the Neumann condition on the boundary.*

Proof. For all $p\in\partial\Omega$, let us choose local coordinates $\{x_1,x_2,\dots,x_n\}$ on a sufficiently small neighborhood U such that $p\in U\cap\partial\Omega=U\cap\{x_1=0\}$.

Since

$$\begin{cases} f_1=0 & \text{on } \partial\Omega,\\ f_1>0 & \text{in } \Omega,\end{cases} \tag{6}$$

by the Hopf lemma we have $\partial f_1/\partial x_1\neq 0$ on $\partial\Omega$. Furthermore, f_1 is smooth up to the boundary, thus one can consider f_1 as a smooth function which is defined on U restricted to $U\cap\overline{\Omega}$. Using the Malgrange preparation theorem [Schoen and Yau 1994], we have locally

$$f_1=g_1\cdot x_1,\ \ x\in\overline{\Omega}\cap U,$$

where g_1 satisfies $g_1\neq 0$ and is smooth on $\overline{\Omega}\cap U$. Moreover, f_2 is identically zero on $\partial\Omega$. Applying the Malgrange preparation theorem again, we can write locally

$$f_2=g_2\cdot x_1,$$

where g_2 is also a smooth function on $\overline{\Omega}\cap U$. It is an immediate consequence that

$$u=\frac{f_2}{f_1}=\frac{g_2}{g_1}$$

must be smooth on $\overline{\Omega}\cap U$. Therefore, u is smooth up to the boundary $\partial\Omega$.

By using Equation (5), we have

$$2\nabla u\cdot\nabla\log f_1=-\Delta u-\lambda u+\nabla h\cdot\nabla u.$$

Since h is smooth up to the boundary, as we have assumed, Δu, $\nabla h \cdot \nabla u$ and u are all smooth up to the boundary and thus attain finite values on $\partial\Omega$. Therefore,

$$\nabla u \cdot \nabla \log f_1 = \frac{1}{f_1} u_1 (f_1)_1 + \frac{1}{f_1} \sum_{i=2}^{n} u_i (f_1)_i \tag{7}$$

achieves finite value on $\partial\Omega$ as well. Multiply both sides of Equation (7) by f_1. A simple computation shows

$$f_1(\nabla u \cdot \nabla \log f_1) - \sum_{i=2}^{n} u_i (f_1)_i = u_1 (f_1)_1 \tag{8}$$

From the fact that $f_1 = 0$ on $\partial\Omega$, we have $(f_1)_i = 0$ on $\partial\Omega$ for $i \in \{2, 3, \dots, n\}$. Thus we see that the left-hand side of (8) tends to 0 as x tends to $p \in \partial\Omega$. Therefore,

$$\lim_{x \to p} u_1 (f_1)_1 = 0.$$

Nevertheless, since $(f_1)_1 \neq 0$ on $\partial\Omega$, we get the important observation:

$$u_1(p) = 0, \ \ p \in \partial\Omega.$$

Thus we get $\partial u / \partial \nu = 0$ on $\partial\Omega$, where ν is the outward normal vector to $\partial\Omega$. That is to say u satisfies the Neumann condition on the boundary $\partial\Omega$. □

Let us compare (5) with (9) carefully. If $h/2 - \log f_1$ is strictly convex, then we can gain a lower bounded of $\lambda = \lambda_2 - \lambda_1$ by applying the following lemma, obtained by S.-T. Yau [2003].

Lemma 7. *Suppose the Ricci curvature of* Ω *is nonnegative and* $\partial\Omega$ *is convex. Let the function u be a solution of the problem*

$$\begin{cases} \Delta u = -(\lambda_2 - \lambda_1) u + 2W \cdot \nabla u, \\ \dfrac{\partial u}{\partial \nu} = 0, \end{cases} \tag{9}$$

where W is a vector field such that $W_{i,i} \geq \sqrt{c/2} > 0$. *Then*

$$\lambda_2 - \lambda_1 \geq \frac{\theta^2(\beta)}{(\operatorname{diam}\Omega)^2} + \beta\sqrt{c},$$

where β *is any number in* $(0, \sqrt{2})$ *and* $\theta(\beta) = \arcsin\left(1 + \dfrac{\beta}{\sqrt{2} - \beta}\right)^{-1/2}$.

Proof. This is Theorem 1.1 in [Yau 2003]. □

To find the condition under which $h/2 - \log f_1$ can be strictly convex, we will introduce the concavity function $\mathscr{C}$ and after that we will introduce two maximum principles for it.

Definition 8. Suppose u is defined on the closure of a bounded domain Ω. The function

$$\mathscr{C}(y_1, y_3, \mu) = u(y_2) - \mu u(y_3) - (1-\mu)u(y_1),$$

defined for $y_1, y_3 \in \overline{\Omega}$ such that $y_2 = \mu y_3 + (1-\mu)y_2 \in \overline{\Omega}$, $0 \le \mu \le 1$, is called the concavity function of u.

This function was introduced in [Korevaar 1983]. It is used to measure how much a function u fails to be convex. We can see that the function u is convex if and only if $\mathscr{C} \le 0$ for all y_1, y_2, y_3 as above.

Notice that $\mathscr{C}$ is defined on a closed subset of $\overline{\Omega} \times \overline{\Omega} \times [0, 1]$. We slightly change our notation as follows.

Definition 9. We say that the triple (y_1, y_2, μ) is in the interior, provided each of y_1, y_2, y_3 is in Ω. It is on the boundary if at least one of y_1, y_2, y_3 is in $\partial\Omega$.

For a function $u \in C(\overline{\Omega})$, $\mathscr{C}$ defined on a closed subset of $\overline{\Omega} \times \overline{\Omega} \times [0, 1]$, is continuous on its domain. Hence $\mathscr{C}$ does attain its maximum value somewhere. The following lemma is a concavity maximum principle giving a sufficient condition for the positive maximum not to be attainable at interior points.

Lemma 10. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the elliptic equation*

$$\Delta u = b(x, u, \nabla u) \quad in\ \Omega,$$

where b satisfies $\partial b/\partial u \ge 0$, b jointly concave with respect to (x, u). Then if $\mathscr{C}$ is anywhere positive, it attains its positive maximum on the boundary (Definition 9).

Proof. This is a special case of Theorem 1.3 in [Korevaar 1983]. □

On the other hand, another concavity maximum principle gives a sufficient condition to that the positive maximum does be attained at the interior points.

Lemma 11. *Let Ω be smooth, strictly convex and bounded. Let u be such that its graph S_u has tangent planes π_x, for all $x \in \partial\Omega$. If each of these boundary planes lies beneath S_u (contacting it only at $(x, u(x))$), then $\mathscr{C}$ does not attain any positive maximum on the boundary* (Definition 9).

Proof. This is Lemma 2.1 in [Korevaar 1983]. □

A combination immediately yields that if a function u satisfies both Lemma 10 and Lemma 11, then u is convex (not strictly convex). One can get more results about the convexity of a function. (See [Korevaar 1983] for more information.)

3. Proofs of Theorem 1 and Corollary 2

In our particular situation (5), we have to show strict convexity for $h/2 - \log f_1$. Firstly we investigate the equation it satisfies. Recall that we use the notation $\varphi = -\log f_1$ and $\psi = V - \Delta h/2 + |\nabla h(x)|^2/4$.

Lemma 12. *We have the following equation for $h/2+\varphi$:*

$$\Delta\Big(\frac{h}{2}+\varphi\Big) = \Big|\nabla\Big(\frac{h}{2}+\varphi\Big)\Big|^2 - \psi + \lambda_1. \tag{10}$$

Proof. A direct calculation shows

$$\Delta\varphi = -\frac{\Delta f_1}{f_1} + \frac{|\nabla f_1|^2}{f_1^2} = \nabla h\cdot\nabla\varphi - V + \lambda_1 + |\nabla\varphi|^2. \tag{11}$$

Notice that

$$\Big|\nabla\Big(\frac{h}{2}+\varphi\Big)\Big|^2 = \frac{|\nabla h|^2}{4} + |\nabla\varphi|^2 + \nabla h\cdot\nabla\varphi$$

and thus

$$|\nabla\varphi|^2 + \nabla h\cdot\nabla\varphi = \Big|\nabla\Big(\frac{h}{2}+\varphi\Big)\Big|^2 - \frac{|\nabla h|^2}{4}. \tag{12}$$

Substituting (12) into (11), we conclude

$$\Delta\Big(\frac{h}{2}+\varphi\Big) = \frac{\Delta h}{2} - V + \lambda_1 + \Big|\nabla\Big(\frac{h}{2}+\varphi\Big)\Big|^2 - \frac{|\nabla h|^2}{4},$$

which implies the conclusion. □

Remark. Though we can try to apply Lemma 10 and Lemma 11 to the function $h/2+\varphi$, we can only get convexity (not strict convexity) of it. However, we need the strict convexity. Let

$$\Psi\Big(x, \nabla\Big(\frac{h}{2}+\varphi\Big)\Big) = \Big|\nabla\Big(\frac{h}{2}+\varphi\Big)\Big|^2 - \psi(x) + \lambda_1.$$

Equation (10) becomes

$$\Delta\Big(\frac{h}{2}+\varphi\Big) = \Psi\Big(x, \nabla\Big(\frac{h}{2}+\varphi\Big)\Big).$$

Compared with Lemma 10, $\Psi(x, \nabla(h/2+\varphi))$ does not depend on $h/2+\varphi$ itself. Luckily, in this case we can obtain strict convexity, provided $\Psi(x, \nabla(h/2+\varphi))$ is strictly convex with respect to x. We derive the following lemma to make this precise.

Lemma 13. *Let $\Omega \subset \mathbb{R}^n$ be a smooth strictly convex bounded domain. Let $u \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfy*

$$\Delta u = |\nabla u|^2 - \Phi(x) \quad \textit{for all } x \in \Omega, \tag{13}$$

where Φ *is a smooth function in* Ω*. Let* $\xi(x) = u(x) - \frac{1}{2}\sqrt{c/2}\sum_{i=1}^{n} x_i^2$*, where* c *is a nonnegative constant. Assume that*

(A1) *for all* $x \in \partial\Omega$*, the tangent plane* π_x *at* x *lies beneath the graph* S_ξ*, contacting it only at* $(x, \xi(x))$*, and*

(A2) *for all* $x \in \Omega$ *we have* $\operatorname{Hess}_x(\Phi) - cI \geq 0$*.*

Then

$$\operatorname{Hess}_x u - \sqrt{\frac{c}{2}} I \geq 0 \quad \textit{for all } x \in \Omega.$$

Proof. We can see that the conclusion equals to that the function ξ is convex. We will show this by applying Lemma 10 and Lemma 11 to function ξ.

By direct computation, we have

$$|\nabla u|^2 = |\nabla\xi|^2 + \frac{c}{2}\sum_{i=1}^{n} x_i^2 + 2\sqrt{\frac{c}{2}}\nabla\xi \cdot x \quad \text{and} \quad \Delta u = \Delta\xi + \sqrt{\frac{c}{2}}n.$$

From these two equations and (13), we obtain

$$\Delta\xi = |\nabla\xi|^2 + 2\sqrt{\frac{c}{2}}\nabla\xi \cdot x - \Big(\Phi(x) - \frac{c}{2}\sum_{i=1}^{n} x_i^2\Big) - \sqrt{\frac{c}{2}}n = B(x, \nabla\xi). \tag{14}$$

Since B does not depend on ξ itself, $\partial B/\partial\xi = 0$. All we have to check is $\operatorname{Hess}_x B \geq 0$. A direct computation shows that

$$\frac{\partial B}{\partial x_i} = 2\sqrt{\frac{c}{2}}\xi_i - \left(\frac{\partial\Phi}{\partial x_i} - cx_i\right) \quad \text{and} \quad \frac{\partial^2 B}{\partial x_j \partial x_i} = -\left(\frac{\partial^2\Phi}{\partial x_j \partial x_i} - c\delta_{ij}\right),$$

which implies $\operatorname{Hess}_x B = -(\operatorname{Hess}_x(\Phi) - cI)$. Using our assumption $\operatorname{Hess}_x(\Phi) - cI \geq 0$, we conclude that B is concave with respect to x.

In view of Lemma 10, we know that if the concavity function $\mathscr{C}$ of ξ is anywhere positive, it attains its positive maximum on the boundary (Definition 9). On the other hand, Lemma 11 tells us that $\mathscr{C}$ does not attain any positive maximum on the boundary (Definition 9). So the concavity function $\mathscr{C}$ of ξ is nonpositive, which implies that ξ is convex. □

Remark. Noticing that $h/2 - \log f_1$ has no definition on $\partial\Omega$, we only can use Lemma 13 on a subset of Ω. Fortunately, if we can show that $h/2 - \log f_1$ is uniformly and strictly convex on any subset of Ω, then it is strictly convex on Ω. In order to show this we have to find a positive constant b such that

$$\frac{h}{2} - \log f_1 - b\sum_{i=1}^{n} x_i$$

satisfies assumption (A1) in Lemma 13 near the boundary $\partial\Omega$. More generally, we will show it holds for a wide class of smooth transformations:

Theorem 14. *Let Ω be a smooth bounded strictly convex domain in $\mathbb{R}^n$. Let $u \in C^2(\overline{\Omega})$ satisfy*

$$u = 0 \quad \text{on } \partial\Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad Du \cdot \nu > 0 \quad \text{on } \partial\Omega, \tag{15}$$

where ν is the interior normal to $\partial\Omega$. Let a transformation function F be

$$F(x,t) = g(x) + f(t), \quad x \in \Omega,\ t \in \mathbb{R}^+.$$

Assume $g \in C^2(\overline{\Omega})$ and assume $f(t) \in C^2(\mathbb{R}^+)$ satisfies

$$f' < 0, \quad \lim_{t\to 0^+} f' = -\infty, \quad f'' > 0, \quad \lim_{t\to 0^+} \frac{f'}{f''} = 0, \quad \lim_{t\to 0^+} \frac{f}{f'} = 0. \tag{16}$$

Then, for $\delta > 0$ small enough, the function $w(x) = F(x, u(x))$ is such that π_x lies beneath S_w (contacting only at $(x, w(x))$), for all $x \in \partial\Omega_\delta$, where

$$\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) > \delta\}.$$

Remark. This theorem is a generalization of a result in [Korevaar 1983], which deals with the case of a homogeneous transformation function F. However, in studying the convexity of the first eigenfunction of problem (1), we have to deal with nonhomogeneous F.

Proof. The conclusion equals to that if δ is small enough, then

$$A_x^\delta := \{y \in \Omega_\delta \mid S_w(y) \text{ lies beneath } \pi_x(y) \text{ or } S_w(y) = \pi_x(y)\}$$

is an empty set, for all $x \in \partial\Omega_\delta$. We will prove this by the following two facts. Fact 1 says when x is near to $\partial\Omega$, A_x^δ is also near $\partial\Omega$. While Fact 2 tells us that we do find a narrow strip between $\partial\Omega$ and A_x^δ, no matter how small δ is. Obviously, these two facts are totally incompatible, unless A_x^δ is empty.

Fact 1. *Given $\varepsilon > 0$, the exists $\delta_0 > 0$ such that $A_x^\delta \cap \Omega_\varepsilon = \varnothing$ for all $0 < \delta < \delta_0$ and all $x \in \partial\Omega_\delta$.*

Proof. We show this by comparing the height of graph S_w with the height of the tangent plane π_x directly.

Let $y = (y_1, y_2, \dots, y_n) \in \Omega$ and let $x = (x_1, x_2, \dots, x_n) \in \partial\Omega_\delta$. Then the coordinate of the graph of function $w(y) = F(y, u(y)) = g(y) + f(u(y))$ is

$$S_w(y) = \big(y_1, y_2, \dots, y_n, S_w^{n+1}(y)\big),$$

where $S_w^{n+1}(y) = g(y) + f(u(y))$. The coordinate of the tangent plane at x is

$$\pi_x(y) = \big(y_1, y_2, \dots, y_n, \pi_x^{n+1}(y)\big).$$

One of the normal directions of π_x is

$$\mu = \big(D_x F(x, u(x)), -1\big) = \big(D_x g + f'(u(x)) D_x u(x), -1\big).$$

From the definition of a normal vector, we know

$$0=\big(y-x,\pi_x^{n+1}(y)-S_w^{n+1}(x)\big)\cdot\mu,$$

which implies

$$\pi_x^{n+1}(y)=(y-x)\cdot\big(D_xg+f'(u(x))D_xu(x)\big)+S_w^{n+1}(x).$$

Hence,

$$\begin{aligned}
&S_w^{n+1}(y)-\pi_x^{n+1}(y)\\
&\quad=S_w^{n+1}(y)-(y-x)\cdot\big(D_xg+f'(u(x))D_xu(x)\big)-S_w^{n+1}(x)\\
&\quad=g(y)+f(u(y))-g(x)-f(u(x))-(y-x)\cdot D_xg-f'(u(x))(y-x)\cdot D_xu(x)\\
&\quad=f'(u(x))\left(\frac{Q(x,y)}{f'(u(x))}-\frac{f(u(x))}{f'(u(x))}-(y-x)\cdot D_xu(x)\right),
\end{aligned}$$

where

$$Q(x,y):=g(y)+f(u(y))-g(x)-(y-x)\cdot D_xg.$$

Notice that $Q(x,y)$ is bounded on $\Omega\times\Omega_\varepsilon$, since $g\in C^1(\overline{\Omega})$, $f\in C^2(\mathbb{R}^+)$ and Ω is bounded by assumption. That is to say, we can choose a positive constant $C_1>0$ such that

$$|Q(x,y)|<C_1\quad\text{for all }(x,y)\in\Omega\times\Omega_\varepsilon. \tag{17}$$

Extending the normal vector field ν smoothly in a neighborhood of $\partial\Omega$, we can talk about normal directions in the entire neighborhood. Since $\partial\Omega$ is a level set of u by (15), $Du(x)$ is a positive multiple of the interior normal $\nu(x)$, for $x\in\partial\Omega$. So when δ is small enough, $Du(x)$ is close to $\nu(x)$ for $x\in\partial\Omega_\delta$. Hence, we can choose $\delta_1>0$ small enough and a positive constant C_2 such that

$$(y-x)\cdot Du(x)>C_2>0\quad\text{for all }y\in\Omega_\varepsilon\text{ and }x\in\Omega\backslash\Omega_{\delta_1}. \tag{18}$$

We have used the strict convexity of Ω and the compactness of $\partial\Omega$ to gain estimate (18).

From (17) and the assumptions $\lim_{t\to0^+}f'=-\infty$ and $\lim_{t\to0^+}f/f'=0$ in (16), we can choose a positive $\delta_2<\delta_1$ such that

$$\left|\frac{f(u(x))}{f'(u(x))}\right|<\tfrac14C_2\ \text{ and }\ \left|\frac{Q(x,y)}{f'(u(x))}\right|<\tfrac14C_2\quad\text{for all }y\in\Omega_\varepsilon\text{ and }x\in\Omega\backslash\Omega_{\delta_2}. \tag{19}$$

From (18) (19) and the assumption $f'<0$, we have

$$S_w^{n+1}(y)-\pi_x^{n+1}(y)>-\frac{C_1}{2}f'(u(x))>0\quad\text{for all }y\in\Omega_\varepsilon\ \text{ for all }x\in\Omega\backslash\Omega_{\delta_2},$$

which implies $A_x^\delta\cap\Omega_\varepsilon=\varnothing$, for all $x\in\partial\Omega_\delta$, $0<\delta<\delta_2$. ☐

We now show that w is convex in a boundary strip about $\partial\Omega$.

Fact 2. *There exists* $\varepsilon > 0$ *such that* $\operatorname{Hess}(w(x)) > 0$ *for all* $x \in \Omega\backslash\Omega_\varepsilon$.

Proof. To show this, we study the terms comprising

$$\operatorname{Hess}(w) = \operatorname{Hess}(g) + f''(u)(D_x u)(D_x u)^t + f'(u)\operatorname{Hess}(u).$$

As in the proof of Fact 1, we extend the normal vector field $\nu(x)$ smoothly into a strip about $\partial\Omega$ and then we can continue to talk about tangential directions ($\nu(x)\cdot\eta = 0$) and nontangential ones.

Let $\eta(x) = (\eta_1(x), \eta_2(x), \dots, \eta_n(x))$ be a vector at point x. The conclusion equals to $\eta(x)\operatorname{Hess}(w(x))\eta^t(x) > 0$, for all $\eta(x) \neq 0$, for all $x \in \Omega\backslash\Omega_\varepsilon$. Actually, we only have to show this for a set of orthonormal basis. When ε is sufficiently small, we can choose a set of smooth vector field $\{e_1(x), e_2(x), \dots, e_n(x)\}$, such that $\{e_1(x), e_2(x), \dots, e_n(x)\}$ is an orthonormal basis at $x \in \Omega\backslash\Omega_\varepsilon$, $e_1(x)$ is close to $\nu(x)$ and each $e_i(x)$ $(i \neq 1)$ is close to some tangential direction. Moreover, since the boundary $\partial\Omega$ is compact and $Du(x)$ is a positive multiple of the interior normal ν when $x \in \partial\Omega$, we can assume that for any $\frac{1}{2} > a > 0$ there exists $\varepsilon_1 > 0$ such that

$$\begin{aligned} |e_i(x)\cdot Du(x)| &< a \qquad \text{for all } x \in \Omega\backslash\Omega_{\varepsilon_1} \text{ and } i \neq 1,\\ e_1(x)\cdot Du(x) &> 1-a \quad \text{for all } x \in \Omega\backslash\Omega_{\varepsilon_1}. \end{aligned} \tag{20}$$

For $\eta = e_1$, which is close to the normal direction, we have

$$\begin{aligned} \eta\operatorname{Hess}(w)\eta^t &= \eta\operatorname{Hess}(g)\eta^t + f''(u)\eta(D_x u)(D_x u)^t\eta^t + f'(u)\eta\operatorname{Hess}(u)\eta^t\\ &= f''(u)\big(P(x) + \eta(D_x u)(D_x u)^t\eta^t\big), \end{aligned} \tag{21}$$

where

$$P(x) := \frac{\eta\operatorname{Hess}(g)\eta^t}{f''(u)} + \frac{f'(u)}{f''(u)}\eta\operatorname{Hess}(u)\eta^t.$$

From the assumptions $f'' > 0$, $\lim_{t\to 0^+} f' = -\infty$ and $\lim_{t\to 0^+} f'/f'' = 0$ in (16), we have

$$\lim_{t\to 0^+} f''(t) = +\infty. \tag{22}$$

By the continuity of u_{ij} and g on $\overline{\Omega}$, combined with (22) and the assumption that $\lim_{t\to 0^+} f'/f'' = 0$, there exists a positive $\varepsilon_2 < \varepsilon_1$ such that

$$|P(x)| < \tfrac{1}{2}(1-a)^2 \quad \text{for all } x \in \Omega\backslash\Omega_{\varepsilon_2}.$$

Therefore, using (20) and assumption that $f'' > 0$, we have

$$\eta\operatorname{Hess}(w)\eta^t > f''(u)(-\tfrac{1}{2}(1-a)^2 + (1-a)^2) > 0 \quad \text{for all } x \in \Omega\backslash\Omega_{\varepsilon_2}.$$

As to $\eta = e_i$ $(i \neq 1)$, which is close to the tangential direction,

$$\begin{aligned}(23)\quad \eta\,\mathrm{Hess}(w)\eta^t &= \eta\,\mathrm{Hess}(g)\eta^t + f''(u)\eta(D_xu)(D_xu)^t\eta^t + f'(u)\eta\,\mathrm{Hess}(u)\eta^t\\ &\geq \eta\,\mathrm{Hess}(g)\eta^t + f'(u)\eta\,\mathrm{Hess}(u)\eta^t.\end{aligned}$$

We have used the positivity of f'' and positive semidefiniteness of the matrix $(D_xu)(D_xu)^t$ to gain (23).

If $x \in \partial\Omega$, the matrix $\mathrm{Hess}\,u(x)$ is negative definite in all tangential directions, that is, there exists a positive constant $k > 0$ such that $\eta\,\mathrm{Hess}(u)\eta < -k|\eta|^2 = -k$ for any tangential direction η. From the compactness of $\partial\Omega$ and the assumption $u \in C^2(\overline{\Omega})$, there exists a positive $\varepsilon_3 < \varepsilon_2$ such that

$$(24)\qquad \eta(x)\,\mathrm{Hess}\,u(x)\eta^t(x) < -k, \quad \text{for all } x \in \Omega\backslash\Omega\varepsilon_3.$$

From the continuity of g_{ij} on $\overline{\Omega}$ and the assumption that $\lim_{t\to0^+} f' = -\infty$, we can choose a positive $\varepsilon_4 < \varepsilon_3$ such that

$$(25)\qquad \frac{\eta\,\mathrm{Hess}(g)\eta^t}{-f'(u)} > -\frac{1}{2}k > \eta(x)\,\mathrm{Hess}\,u(x)\eta^t(x), \text{ for all } x \in \Omega\backslash\Omega\varepsilon_4.$$

Combining (23) (24) and (25), we have for all $x \in \Omega\backslash\Omega\varepsilon_4$

$$\eta\,\mathrm{Hess}(w)\eta^t \geq -f'(u)\left(\frac{\eta\,\mathrm{Hess}(g)\eta^t}{-f'(u)} - \eta\,\mathrm{Hess}(u)\eta^t\right) > -\tfrac{1}{2}kf'(u) > 0.$$

In conclusion, if $\varepsilon < \varepsilon_4$, then $\eta^t(x)\,\mathrm{Hess}(w)(x)\eta(x) > 0$ for all $x \in \Omega\backslash\Omega\varepsilon_4$ and for all $\eta(x) \neq 0$, which implies Fact 2. □

Theorem 14 now follows from Fact 1 and Fact 2 together: Pick $\varepsilon > 0$ such that $\mathrm{Hess}(w)(x) > 0$ for $x \in \Omega\backslash\Omega_\varepsilon$. For this $\varepsilon > 0$, pick δ_0 so that for $0 < \delta < \delta_0$ and $x \in \partial\Omega_\delta$, we have $A_x \cap \Omega_\varepsilon = \varnothing$. Because $\mathrm{Hess}(w)(x) > 0$ in $\Omega\backslash\Omega_\varepsilon$, we also have $A_x \cap (\Omega_\delta\backslash\Omega_\varepsilon) = \varnothing$. Hence for $0 < \delta < \delta_0$, $A_x = \varnothing$, which implies for small enough δ, tangent planes π_x lies beneath S_w for all $x \in \partial\Omega_\delta$. □

Proof of Theorem 1. Recall that in Lemma 12 we have shown

$$\Delta\left(\frac{h}{2}+\varphi\right) = \left|\nabla\left(\frac{h}{2}+\varphi\right)\right|^2 - \psi(x) + \lambda_1,$$

where $\varphi = -\log f_1$ and $\psi = V - \frac{1}{2}\Delta h + \frac{1}{4}|\nabla h|^2$.

First we will show for small enough $\delta > 0$, $\xi = \frac{1}{2}h + \varphi - \frac{1}{2}\sqrt{c/2}\sum_{i=1}^n x_i$ satisfies assumption (A1) in Lemma 13: for all $x \in \partial\Omega$, π_x lies beneath S_ξ, contacting it only at $(x, \xi(x))$.

Choosing the transformation function $F(x,t) = g(x) + f(t)$, where

$$g(x) = \frac{h(x)}{2} - \frac{1}{2}\sqrt{\frac{c}{2}}\sum_{i=1}^n x_i \quad \text{and} \quad f(t) = -\log t,$$

we can write

$$\xi = \frac{h}{2} + \varphi - \frac{1}{2}\sqrt{\frac{c}{2}}\sum_{i=1}^{n} x_i = F(x, f_1(x)).$$

Thus, using Theorem 14 we see that π_x lies beneath S_ξ for all $x \in \partial\Omega_\delta$ with $\delta > 0$ small enough.

Let $\Phi = \psi - \lambda_1$. Since $\mathrm{Hess}_x\, \psi - cI \geq 0$ for all $x \in \Omega$, we have $\mathrm{Hess}_x\, \Phi = \mathrm{Hess}_x\, \psi \geq cI$ for all $x \in \Omega$. Therefore, for $\delta > 0$ small enough, $h/2 + \varphi$ satisfies Lemma 13 in the domain Ω_δ. Since Ω is strictly convex, we can still assume Ω_δ is strictly convex. By using Lemma 13 on Ω_δ, we get

$$\mathrm{Hess}_x \left(\frac{h}{2} + \varphi\right) - \sqrt{\frac{c}{2}} I \geq 0 \quad \text{in } \Omega_\delta. \tag{26}$$

Since δ can be any sufficiently small positive constant, (26) is also valid in Ω. □

Proof of Corollary 2. Recall from Equation (5) that

$$\Delta u = -\lambda u + 2\nabla u \cdot \nabla\left(\frac{h}{2} - \log f_1\right).$$

We already know that $h/2 + \varphi$ is strictly convex and that u satisfies the Neumann boundary condition $\partial u/\partial\nu = 0$ (Lemma 6). Combining Lemma 7 and Theorem 1, we obtain the estimate (2). □

4. Proofs of Theorem 3 and Corollary 4

Equation (5) will satisfies the hypothesis of Lemma 7 if

$$\mathrm{Hess}_x \left(\frac{h}{2} + \varphi\right) - \sqrt{\frac{c}{2}} I \geq 0,$$

otherwise we can still obtain the following estimate.

Lemma 15. *Let* $\Omega \subset \mathbb{R}^n$ *be a smooth and bounded domain. Let* $\tau_i(x)$ $(i = 1, \dots, n)$ *be the eigenvalues of* $\mathrm{Hess}_x\, \varphi$ *at the point* x *and let* $\kappa_i(x)$ $(i = 1, \dots, n)$ *be the eigenvalues of* $\mathrm{Hess}_x(h/2 + \varphi)$ *at* x. *Then*

$$\min_{1 \leq i \leq n} \inf_{x \in \Omega} \tau_i(x) > -\infty;$$

equivalently, there exists a constant $a \geq 0$ *such that*

$$\min_{1 \leq i \leq n} \inf_{x \in \Omega} \tau_i(x) \geq -a.$$

Since h *is smooth, the same holds for* $\min_{1 \leq i \leq n} \inf_{x \in \Omega} \kappa_i(x)$.

Proof. The conclusion is equivalent to the existence of a constant $a \geq 0$ such that $\mathrm{Hess}\, \varphi(x) + aI \geq 0$ for all $x \in \Omega$. We find the constant by computing the Hessian of φ directly. Since φ is smooth in Ω, we only need to study what happens when x is near to the boundary.

For any $p \in \partial\Omega$, we choose the same local coordinates $\{x_1, x_2, \dots, x_n\}$ and the neighborhood U as in Lemma 6. Similar as in there we can write locally $f_1 = x_1 \cdot g$. Recall that g is a smooth function and $g \neq 0$ in $\overline{\Omega} \cap U$.

Then locally we have

$$\varphi_i = -\frac{(f_1)_i}{f_1} = -\frac{(x_1 g)_i}{x_1 g}.$$

When $i = 1$, we have

$$\varphi_{11} = -\frac{(f_1)_{11}}{f_1} + \frac{(f_1)_1^2}{f_1^2},$$

from which we can see that

$$f_1^2 \varphi_{11} = -(f_1)_{11} f_1 + (f_1)_1^2. \tag{27}$$

Since f_1 is smooth up to the boundary and $f_1 = 0$ on $\partial\Omega$. The Hopf lemma shows that $\partial f_1/\partial x_1 \neq 0$ on $\partial\Omega$. So the right-hand side of (27) tends to a finite positive number as $x \to p \in \partial\Omega$. Therefore

$$\lim_{x \to p} \varphi_{11} = +\infty. \tag{28}$$

For $2 \leq i \leq n$, we have

$$\varphi_i = -\frac{g_i}{g}.$$

For $1 \leq j \leq n$, we have

$$\varphi_{ij} = -\frac{g_{ij}}{g} + \frac{g_i g_j}{g^2}, \tag{29}$$

which tends to finite value as $x \to p \in \partial\Omega$. In conclusion, $\varphi_{11} \to +\infty$ as $x \to p$ and φ_{ij} ($i \neq 1$ or $j \neq 1$) tend to finite numbers as $x \to p$. So for any small neighborhood V of p, we can choose a sufficiently large a such that

$$\operatorname{Hess} \varphi(x) + aI \geq 0 \quad \text{for all } x \in V.$$

Since Ω is a bounded domain and φ is smooth in Ω, there exists an uniform number a such that

$$\operatorname{Hess} \varphi(x) + aI \geq 0 \quad \text{for all } x \in \Omega.$$

Thus, we obtain the conclusion. □

In view of Lemma 15, we will assume

$$\min_{1 \leq i \leq n} \inf_x \kappa_i \geq -a,$$

where a is a nonnegative constant.

Proof of Theorem 3. Following [Yau 2003], we consider the function

$$F(x)=\frac{|\nabla u(x)|^2}{(c-u(x))^2}+\alpha\log(c-u(x)),$$

for $c>\sup_x u$ and $\alpha>0$ as selected below. Actually, we try to find those constants α and c such that $|\nabla u|=0$ at the maximum points of F.

By some computations, we have

$$F_i=2\sum_{j=1}^{n}u_ju_{ji}(c-u)^{-2}+2|\nabla u|^2(c-u)^{-3}u_i-\alpha(c-u)^{-1}u_i,\tag{30}$$

$$\begin{aligned}\Delta F=2|D^2u|^2(c-u)^{-2}+2(\nabla u\cdot\nabla\Delta u)(c-u)^{-2}+6(c-u)^{-4}|\nabla u|^4\\ +2(c-u)^{-3}(\Delta u)|\nabla u|^2+8\sum_{i,j=1}^{n}u_ju_{ji}u_i(c-u)^{-3}\\ -\alpha|\nabla u|^2(c-u)^{-2}-\alpha(c-u)^{-1}\Delta u.\end{aligned}\tag{31}$$

Case 1. Suppose F attains its maximum on $\partial\Omega$ at a point x_0. We can choose an orthonormal frame $\{l_1,l_2,\dots,l_n\}$ around x_0 such that l_n is perpendicular to $\partial\Omega$ and pointing outward. We also use the notation $\partial/\partial x_n$ to denote the restriction of l_n on $\partial\Omega$.

A computation shows that, at the maximum point $x_0\in\partial\Omega$,

$$\begin{aligned}0\le\frac{\partial F}{\partial x_n}(x_0)&=2\sum_{j=1}^{n-1}u_ju_{jn}(c-u)^{-2}+2|\nabla u|^2(c-u)^{-3}u_n-\alpha(c-u)^{-1}u_n\\&=2\sum_{j=1}^{n-1}u_ju_{jn}(c-u)^{-2}.\end{aligned}$$

We have used that $(\partial u/\partial x_n)(x)=0$ for $x\in\partial\Omega$ (see Lemma 6). From the definition of the second fundamental form of a hypersurface in $\mathbb{R}^n$, we have

$$u_{jn}=-\sum_{k=1}^{n-1}h_{jk}u_k\quad\text{for all }1\le j\le n-1,$$

where h_{jk} is the second fundamental form of $\partial\Omega$. Therefore we obtain

$$0\le\frac{\partial F}{\partial x_n}=-2\sum_{j,k=1}^{n-1}u_jh_{jk}u_k(c-u)^{-2}\le 0.$$

We have used the positivity of h_{jk}, arising from the assumption that $\partial\Omega$ is strictly convex. Therefore, $|\nabla u|=0$ at x_0.

Thus for all $x \in \overline{\Omega}$, we have

$$F(x) \le F(x_0) = \alpha \log (c - u(x_0)) \le \alpha \log c. \tag{32}$$

Case 2. Suppose that F attains its maximum in an interior point x_0 of Ω and that $\nabla u(x_0) = 0$. In this case, we still can get (32).

Case 3. Suppose that F attains its maximum in an interior point x_0 of Ω and that $\nabla u(x_0) \ne 0$.

In this case, we can choose a coordinate so that

$$u_1(x_0) \ne 0, \quad u_i(x_0) = 0,\ 2 \le i \le n. \tag{33}$$

Using (33) we can rewrite (30) as

$$F_i(x_0) = 2u_1u_{1i}(c-u)^{-2} + 2u_1^2(c-u)^{-3}u_i - \alpha(c-u)^{-1}u_i.$$

Since $F_1(x_0) = 0$, we get

$$u_{11}(c-u)^{-1} + u_1^2(c-u)^{-2} = \frac{\alpha}{2}, \tag{34}$$

from which we can see that

$$u_{11} = \Big(\frac{\alpha}{2} - u_1^2(c-u)^{-2}\Big)(c-u).$$

Thus, we have

$$2|D^2u|^2(c-u)^{-2} \ge 2u_{11}(c-u)^{-2} = \frac{\alpha^2}{2} - 2\alpha u_1^2(c-u)^{-2} + 2u_1^4(c-u)^{-4}.$$

We can estimate the second term in the right-hand side of Equation (31) as follows:

$$\begin{aligned}
2\nabla u \cdot \nabla(\Delta u)(c-u)^{-2} &= 2\nabla u \cdot \nabla\Big(-\lambda u + 2\nabla\Big(\frac{h}{2}+\varphi\Big)\cdot\nabla u\Big)(c-u)^{-2}\\
&= -2\lambda|\nabla u|^2(c-u)^{-2} + 4u_i\Big(\frac{h}{2}+\varphi\Big)_{ji}u_j(c-u)^{-2}\\
&\qquad + 4u_i\Big(\frac{h}{2}+\varphi\Big)_i u_{ij}(c-u)^{-2}\\
&\ge -2\lambda|\nabla u|^2(c-u)^{-2} + 4u_i\Big(\frac{h}{2}+\varphi\Big)_i u_{ij}(c-u)^{-2}\\
&\qquad + 4|\nabla u|^2 \min_i \inf_x \kappa_i(x)(c-u)^{-2}.
\end{aligned} \tag{35}$$

By computation, we obtain

$$\begin{aligned}
&2|\nabla u|^2(\Delta u)(c-u)^{-3}\\
&\quad= 2|\nabla u|^2\Big(-\lambda u + 2\nabla\Big(\frac{h}{2}+\varphi\Big)\cdot\nabla u\Big)(c-u)^{-3}\\
&\quad= -2\lambda u|\nabla u|^2(c-u)^{-3} + 4|\nabla u|^2\Big(\nabla\Big(\frac{h}{2}+\varphi\Big)\cdot\nabla u\Big)(c-u)^{-3}.
\end{aligned} \tag{36}$$

At the maximum point x_0, we have

$$\begin{aligned}(37)\quad 0 &= \nabla F\cdot\nabla\Big(\frac{h}{2}+\varphi\Big)\\ &= 2u_ju_{ji}\Big(\frac{h}{2}+\varphi\Big)_i(c-u)^{-2}+2|\nabla u|^2(c-u)^{-3}\nabla\Big(\frac{h}{2}+\varphi\Big)\cdot\nabla u\\ &\qquad -\alpha(c-u)^{-1}\nabla\Big(\frac{h}{2}+\varphi\Big)\cdot\nabla u.\end{aligned}$$

We substitute (35), (36) and (37) into (31) and obtain

$$\begin{aligned}\Delta F(x_0) \ge \frac{\alpha^2}{2} &- 2\alpha u_1^2(c-u)^{-2}+2u_1^4(c-u)^{-4}\\ &- 2\lambda|\nabla u|^2(c-u)^{-2}+4|\nabla u|^2\min_i\inf_x\kappa_i(c-u)^{-2}\\ &+ 6u_1^4(c-u)^{-4}-2\lambda u|\nabla u|^2(c-u)^{-3}\\ &+ 8u_ju_{ji}u_i(c-u)^{-3}-\alpha|\nabla u|^2(c-u)^{-2}+\alpha(c-u)^{-1}\lambda u.\end{aligned}$$

By using (34), we can compute that

$$\begin{aligned}8|\nabla u|^4(c-u)^{-4}+8u_ju_{ji}u_i(c-u)^{-3} &= 8u_1^4(c-u)^{-4}+8u_1u_{11}u_1(c-u)^{-3}\\ &= 8(c-u)^{-2}u_1^2\big((c-u)^{-2}u_1^2+u_{11}u_1^{-1}\big)\\ &= 4\alpha(c-u)^{-2}|\nabla u|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}0 &\ge \Delta F(x_0)\\ &\ge \tfrac{1}{2}\alpha^2+\alpha(c-u)^{-2}|\nabla u|^2-2\lambda|\nabla u|^2(c-u)^{-2}\\ &\qquad -2\lambda u|\nabla u|^2(c-u)^{-3}+\alpha\lambda(c-u)^{-1}u+4|\nabla u|^2\min_i\inf_x\kappa_i(x)(c-u)^{-2}\\ &\ge \tfrac{1}{2}\alpha^2+(c-u)^{-2}|\nabla u|^2\big(\alpha-2\lambda-2\lambda u(c-u)^{-1}+4\min_i\inf_x\kappa_i(x)\big)\\ &\ge \tfrac{1}{2}\alpha^2+(c-u)^{-2}|\nabla u|^2\big(\alpha-2\lambda-2\lambda\sup_x u(c-\sup_x u)^{-1}+4\min_i\inf_x\kappa_i(x)\big).\end{aligned}$$

Choosing $c=(1+\varepsilon)\sup_x u$ and $\alpha=2\lambda(1+\varepsilon^{-1})-4\min_i\inf_{x\in\Omega}\kappa_i(x)$, we get $\Delta F(x_0)>0$, which is a contradiction. Therefore, $\nabla u(x_0)=0$, which means (32) is valid in this case as well.

Our argument above shows that (32) is valid in all cases. A simple computation shows (3). □

At last we shall derive our lower bound

$$2(\operatorname{diam}\Omega)^{-2}\exp\big(-a(\operatorname{diam}\Omega)^2-1\big)\le\lambda_2-\lambda_1.$$

Proof of Corollary 4. From (3) we have, for all $\varepsilon>0$,

$$(38)\qquad \left|\nabla\sqrt{\log\frac{c}{c-u}}\right|\le\tfrac{1}{2}\sqrt{\alpha},$$

where $c=(1+\varepsilon)\sup_x u$ and $\alpha=2\lambda(1+\varepsilon^{-1})-4\min_i\inf_{x\in\Omega}(h/2+\varphi)_{ii}$.

Let q_1, q_2 be two points of $\overline{\Omega}$ such that $u(q_1) = \sup_x u$, $u(q_2) = 0$ and γ is the line segment joining them. Since Ω is convex by assumption, γ lies in Ω. By integrating both sides of inequality (38) along γ from q_1 to q_2, we have

$$\int_{\sup_x u}^{0} \left| \frac{d\big(\log(c/(c-u))\big)^{1/2}}{du}\, du \right| \le \int_{q_1}^{q_2} \tfrac{1}{2}\sqrt{\alpha}\, ds \le \tfrac{1}{2}\sqrt{\alpha}\,(\operatorname{diam}\Omega).$$

By elementary calculus, we have

$$\Big(\log \frac{c}{c-\sup_x u}\Big)^{1/2} \le \tfrac{1}{2}\sqrt{\alpha}\,(\operatorname{diam}\Omega),$$

which implies

$$\alpha \ge 4\,(\operatorname{diam}\Omega)^{-2} \log\,(1+1/\varepsilon). \tag{39}$$

Putting $\alpha = 2\lambda(1+\varepsilon^{-1}) - 4\min_i \inf_{x\in\Omega} \kappa_i(x)$ into (39), and defining $\varepsilon' = 1+1/\varepsilon$, we obtain

$$\begin{aligned}\lambda_2 - \lambda_1 &\ge \varepsilon'^{-1}\big(2(\operatorname{diam}\Omega)^{-2}\log\varepsilon' + 2\min_i \inf_x \kappa_i(x)\big)\\ &= 2(\operatorname{diam}\Omega)^{-2}\varepsilon'^{-1}\log\big(\varepsilon' \exp((\operatorname{diam}\Omega)^2 \min_i \inf_x \kappa_i(x))\big).\end{aligned}$$

Since ε can be any positive number and the right-hand side of the preceding equation is at most $2(\operatorname{diam}\Omega)^{-2}\exp(\min_i \inf_x \kappa_i(x)(\operatorname{diam}\Omega)^2 - 1)$, we obtain

$$\lambda_2 - \lambda_1 \ge 2(\operatorname{diam}\Omega)^{-2}\exp\big(\min_i \inf_x \kappa_i(x)(\operatorname{diam}\Omega)^2 - 1\big).$$

Therefore, if $\min_i \inf_x \kappa_i(x) \ge -a$, then

$$\lambda_2 - \lambda_1 \ge 2(\operatorname{diam}\Omega)^{-2}\exp\,(-a(\operatorname{diam}\Omega)^2 - 1). \qquad \square$$

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