Pacific Journal of Mathematics

Volume 241 No. 1 May 2009

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by

E. F. Beckenbach (1906–1982) F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

Regular subscription rate for 2009: US$450 a year (10 issues). Special rate: US$225 a year to individual members of supporting institutions. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840 is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS
Vol. 241, No. 1, 2009

AN END-TO-END CONSTRUCTION FOR SINGLY PERIODIC MINIMAL SURFACES

LAURENT HAUSWIRTH, FILIPPO MORABITO
AND M. MAGDALENA RODRÍGUEZ

We construct families of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ with Scherk-type ends and arbitrary finite genus in the quotient. The construction follows by gluing small perturbations of pieces of already known minimal surfaces: Scherk minimal surfaces, Costa–Hoffman–Meeks surfaces and KMR examples.

1. Introduction

Besides the plane and the helicoid, the first singly periodic minimal surface in $\mathbb{R}^3$ was discovered by Scherk [1835]. This surface, known as *Scherk's second surface*, is a properly embedded minimal surface in $\mathbb{R}^3$ that is invariant by one translation T we can assume to be along the x_2 axis, and can be seen as the desingularization of two perpendicular planes P_1 and P_2 containing the x_2 axis. We assume P_1 and P_2 are symmetric with respect to the planes $\{x_1 = 0\}$ and $\{x_3 = 0\}$. By changing the angle between P_1 and P_2, we obtain a 1-parameter family of properly embedded singly periodic minimal surfaces, which we will refer to as *Scherk surfaces*. In the quotient $\mathbb{R}^3/T$ by its shortest period T, each Scherk surface has genus zero and four ends asymptotic to flat annuli contained in P_1/T and P_2/T. Such ends are called Scherk-type ends. From now on, T will denote a translation in the x_2 direction.

In 1982, C. Costa [1982; 1984] discovered a genus one minimal surface with three embedded ends: one top catenoidal end, one middle planar end and one bottom catenoidal end. D. Hoffman and W. H. Meeks [1985; 1989; 1990] proved the global embeddedness for this Costa example, and generalized it for higher genus. For each $k \geqslant 1$, the Costa–Hoffman–Meeks surface M_k is a properly embedded minimal surface of genus k and three ends: two catenoidal ends and one middle planar end.

MSC2000: 49Q05, 53A10.
Keywords: singly periodic minimal surfaces, gluing.
Rodriguez is partially supported by grants from Région Ile-de-France, Spanish MEC/FEDER Grant MTM2007-61775 and Regional J. Andalucía Grant P06-FQM-01642.

F. Martín and V. Ramos Batista [Martín and Ramos Batista 2006] have recently constructed a 1-parameter family of properly embedded singly periodic minimal surfaces that have genus one and six Scherk-type ends in the quotient $\mathbb{R}^3/T$. These are called *Scherk–Costa surfaces* and are based on the Costa surface. Roughly speaking, they remove each end of the Costa surface (asymptotic to a catenoid or a plane) and replace it by two Scherk-type ends. In this paper, we obtain surfaces in the same spirit as Martin and Ramos Batista's one, but with a completely different method. We construct properly embedded singly periodic minimal surfaces with genus $k \geqslant 1$ and six Scherk-type ends in the quotient $\mathbb{R}^3/T$ by gluing (in an analytic way) a compact piece of M_k to two halves of a Scherk surface at the top and bottom catenoidal ends, and one flat horizontal annulus P/T with a disk removed at the middle planar end.

Theorem 1.1. *Let T denote a translation in the x_2 direction. For each $k \geqslant 1$, there exists a 1-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ invariant by T whose quotient in $\mathbb{R}^3/T$ has genus k and six Scherk-type ends.*

V. Ramos Batista [2005] constructed a singly periodic Costa minimal surface with two catenoidal ends and two Scherk-type middle ends; this surface has genus one in the quotient $\mathbb{R}^3/T$. This example is not embedded outside a slab in $\mathbb{R}^3/T$ that contains the topology of the surface. We observe that the surface we obtain by gluing a compact piece of M_1 (Costa surface) at its middle planar end to a flat horizontal annulus with a disk removed has the same properties as Ramos Batista's.

In 1988, H. Karcher [1988; 1989] defined a family of properly embedded doubly periodic minimal surfaces, called *toroidal halfplane layers*, which have genus one and four horizontal Scherk-type ends in the quotient. In 1989, W. H. Meeks and H. Rosenberg [1989] developed a general theory for doubly periodic minimal surfaces having finite topology in the quotient, and used a minimax approach to obtain the existence of a family of properly embedded doubly periodic minimal surfaces, also with genus one and four horizontal Scherk-type ends in the quotient. Karcher's and Meeks and Rosenberg's surfaces have been generalized by M. M. Rodríguez [2007], who constructed a 3-parameter family $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ of surfaces, called *KMR examples* (sometimes they are also called toroidal halfplane layers). Such examples have been classified by J. Pérez, M. M. Rodríguez and M. Traizet [Pérez et al. 2005] as the only properly embedded doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient. Each $M_{\sigma,\alpha,\beta}$ is invariant by a horizontal translation T (by the period vector at its ends) and a nonhorizontal one $\widetilde{T}$. We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{R}^3/T$, which has genus zero, infinitely many horizontal Scherk-type ends, and two limit ends.

In [1992], F. S. Wei added a handle to a KMR example $M_{\sigma,0,0}$ in a periodic way, obtaining a properly embedded doubly periodic minimal surface invariant under reflection in three orthogonal planes, which has genus two and four horizontal Scherk-type ends in the quotient. Some years later, W. Rossman, E. C. Thayer and M. Wolgemuth [Rossman et al. 2000] added a different type of handle to a KMR example $M_{\sigma,0,0}$, also in a periodic way, obtaining a different minimal surface with the same properties as Wei's one. They also added two handles to a KMR example, getting doubly periodic examples of genus three in the quotient. L. Mazet and M. Traizet [2008] have added $N \geqslant 1$ handles to a KMR example $M_{\sigma,0,0}$, obtaining a genus N properly embedded minimal surface in $\mathbb{R}^3/T$ with an infinite number of horizontal Scherk-type ends and two limit ends. The idea of the construction is to connect N periods of the doubly periodic example of Wei with two halves KMR example. However they only control the asymptotic behavior in their construction. They have also constructed a properly embedded minimal surface in $\mathbb{R}^3/T$ with infinite genus, adding handles in a quasiperiodic way to a KMR example.

L. Hauswirth and F. Pacard [2007] have constructed higher genus Riemann minimal examples in $\mathbb{R}^3$, by gluing two halves of a Riemann minimal example with the intersection of a conveniently chosen Costa–Hoffman–Meeks surface M_k with a slab. We follow their ideas to generalize Mazet and Traizet's examples by constructing higher genus KMR examples: We construct two 1-parameter families of properly embedded singly periodic minimal examples whose quotient in $\mathbb{R}^3/T$ has arbitrary finite genus, infinitely many horizontal Scherk-type ends and two limit ends. More precisely, we glue a compact piece of a slightly deformed example M_k with tilted catenoidal ends, to two halves of a KMR example $M_{\sigma,\alpha,0}$ or $M_{\sigma,0,\beta}$ (see Figure 1) and a periodic horizontal flat annulus with a disk removed.

Theorem 1.2. *Let T denote a translation in the x_2 direction. For each $k \geqslant 1$, there exist two 1-parameter families $\mathcal{K}_1$ and $\mathcal{K}_2$ of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^3/T$ has genus k, infinitely many horizontal Scherk-type ends and two limit ends. The surfaces in $\mathcal{K}_1$ have a plane of symmetry orthogonal to the x_1 axis, and the surfaces in $\mathcal{K}_2$ have a plane of symmetry orthogonal to the x_2 axis.*

L. Mazet, M. Rodríguez and M. Traizet [2007] have constructed saddle towers with infinitely many ends: They are nonperiodic properly embedded minimal surfaces in $\mathbb{R}^3/T$ with infinitely many ends and one limit end. In this paper, we construct (nonperiodic) properly embedded minimal surfaces in $\mathbb{R}^3/T$ with arbitrary finite genus $k \geqslant 0$, infinitely many ends and one limit end. With this aim, we glue half of a Scherk example with half of a KMR example in the case $k = 0$; when $k \geqslant 1$, we glue a compact piece of M_k to half of a Scherk surface (at the top catenoidal end of M_k), a periodic horizontal flat annulus with a disk removed (at

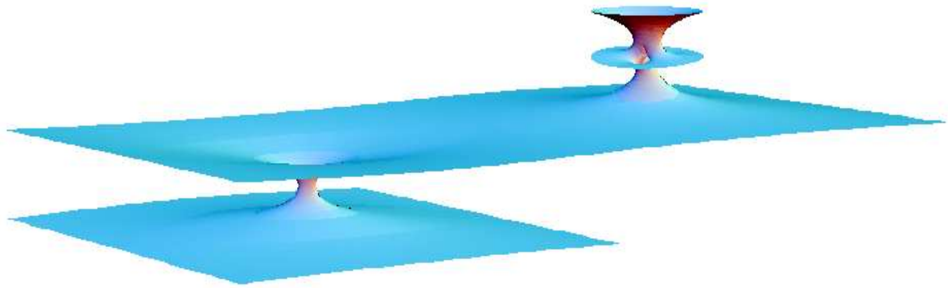

Figure 1. A sketch of half of a KMR example $M_{\sigma,0,0}$ glued to a compact piece of Costa surface.

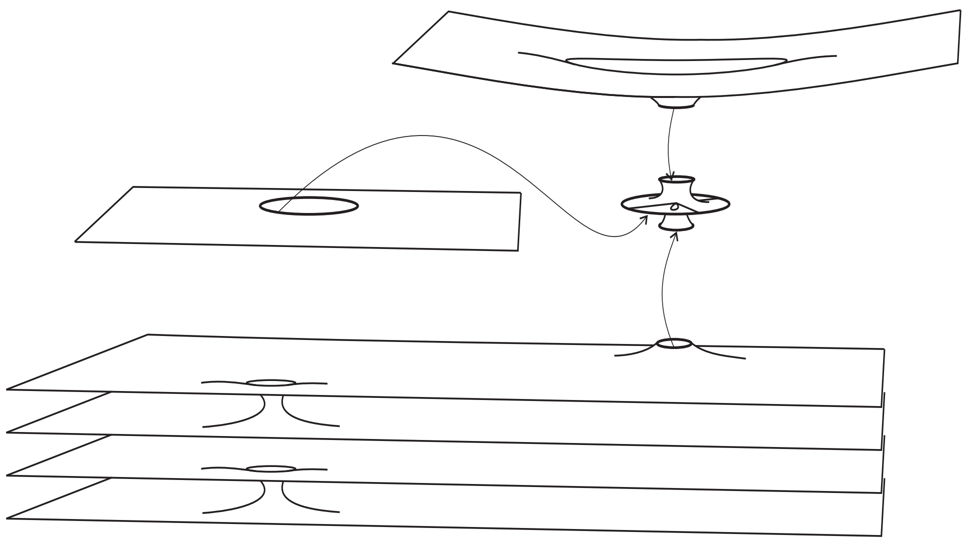

Figure 2. A sketch of a surface in the family of Theorem 1.3.

the middle planar end) and half of a KMR example (at the bottom catenoidal end); see Figure 2.

Theorem 1.3. *Let T denote a translation in the x_2 direction. For each $k \geqslant 0$, there exists a 1-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^3/T$ has genus k, infinitely many horizontal Scherk-type ends and one limit end.*

The family of KMR examples is a three parameter family that contains two subfamilies whose surfaces have a plane of symmetry. In the construction of examples satisfying Theorems 1.2 and 1.3, we need to have at least one plane of symmetry in order to control the kernel of the Jacobi operator on each glued piece.

F. Morabito [2008a] has recently proved there is a bounded Jacobi field that does not come from isometries of $\mathbb{R}^3$ on M_k with tilted ends. For this reason, we are not able to produce a 3-parameter family of KMR examples with higher genus in Theorem 1.2.

The paper is organized as follows. In Section 2 we briefly describe the Costa–Hoffman–Meeks examples M_k and obtain, for each genus k, a 1-parameter family of surfaces $M_k(\xi)$ by bending the catenoidal ends of $M_k = M_k(0)$ while keeping a plane of symmetry. This is used to prescribe the flux of the deformed surface M_k, which has to be the same as the corresponding KMR example we want to glue (to prove Theorem 1.2). To simplify the construction of examples satisfying Theorems 1.1 and 1.3, we consider a "not bent" example M_k. In Section 3 we perturb $M_k(\xi)$ using the implicit function theorem. We get an infinite dimensional family of minimal surfaces that have three boundaries.

In Section 4, we apply an implicit function theorem to solve the Dirichlet problem for the minimal graph equation on a horizontal flat periodic annulus with a disk B removed, prescribing the boundary data on ∂B and the asymptotic direction of the Scherk-type ends. We construct the flat annulus with a disk removed that we will glue to the example M_k at its middle planar end. Varying the asymptotic direction of the ends and the flux of the surface, we obtain the pieces of Scherk surface that we will glue at the top and bottom catenoidal ends of M_k (proving Theorem 1.1) and to half of a KMR example (to prove Theorem 1.3).

In Section 5, we study the KMR examples $M_{\sigma,\alpha,\beta}$ and describe a conformal parameterization of these examples on a cylinder. We also obtain an expansion of pieces of the KMR examples as the flux of $M_{\sigma,\alpha,\beta}$ becomes horizontal (that is, near the catenoidal limit). Section 6 is devoted to the study of the mapping properties of the Jacobi operator about such $M_{\sigma,\alpha,\beta}$ near the catenoidal limit. And we apply in Section 7 the implicit function theorem to perturb half of a KMR example $M_{\sigma,\alpha,0}$ (respectively $M_{\sigma,0,\beta}$), obtaining a family of minimal surfaces asymptotic to half of a $M_{\sigma,\alpha,0}$ (respectively $M_{\sigma,0,\beta}$) and whose boundary is a Jordan curve. We prescribe the boundary data of such a surface. Sections 5, 6, 7 are of independent interest: They are devoted to the global analysis on KMR examples.

Finally, we do the end-to-end construction in Section 8: We explain how the boundary data of the corresponding minimal surfaces constructed in Sections 3, 4 and 7 can be chosen so that their union forms smooth minimal surfaces satisfying Theorems 1.1, 1.2 and 1.3.

2. A Costa–Hoffman–Meeks type surface with bent catenoidal ends

In this section we recall the result shown in [Hauswirth and Pacard 2007] about the existence of a family of minimal surfaces $M_k(\xi)$ close to the Costa–Hoffman–Meeks surface $M_k(0)=M_k$ of genus $k \geqslant 1$, with one planar end and two catenoidal ends slightly bent by an angle ξ.

2.1. ***Costa–Hoffman–Meeks surfaces.*** We briefly present here the family of the surfaces M_k studied in [Costa 1982; 1984; Hoffman and Meeks 1985; 1989; 1990]. For each natural $k \geqslant 1$, M_k is a properly embedded minimal surface of genus k and three ends. After suitable rotations and translations, we can assume its ends are horizontal (in particular, they can be ordered by heights). The surface M_k enjoys the following properties:

(1) M_k has one middle planar end E_m asymptotic to the $\{x_3=0\}$ plane, and two catenoidal ends: one top E_t and one bottom E_b asymptotic, respectively, to the upper and lower end of a catenoid having as axis of revolution the x_3 axis.

(2) M_k intersects the $\{x_3=0\}$ plane in $k+1$ straight lines, which intersect at equal angles $\pi/(k+1)$ at the origin. The intersection of M_k with any one of the remaining horizontal planes is a single Jordan curve. Thus the intersection of M_k with the upper half-space $\{x_3>0\}$ (respectively the lower half-space $\{x_3<0\}$) is topologically an open annulus.

(3) The isometry group of M_k is generated by rotations by π about the $k+1$ lines contained in the surface at height zero, together with reflections in planes that bisect those lines. Assume one such plane of symmetry is the $\{x_2=0\}$ plane. In particular, M_k is invariant by the rotation by $2\pi/(k+1)$ about the x_3 axis and by the composition of a rotation by $\pi/(k+1)$ about the x_3 axis with a reflection across the $\{x_3=0\}$ plane.

Now we give describe locally the surfaces M_k near its ends, and we introduce coordinates that we will use.

The planar end. See [Hauswirth and Pacard 2007]. The planar end E_m of M_k can be parameterized by

$$X_m(x)=\Big(\frac{x}{|x|^2},u_m(x)\Big)\in\mathbb{R}^3 \quad \text{for } x\in \bar{B}^*_{\rho_0}(0),$$

where $\bar{B}^*_{\rho_0}(0)$ is the punctured closed disk in $\mathbb{R}^2$ of radius $\rho_0>0$ small centered at the origin, and $u_m=\mathcal{O}_{C^{2,\alpha}_b}(|x|^{k+1})$ is a solution of

$$|x|^4\operatorname{div}\Big(\frac{\nabla u}{(1+|x|^4|\nabla u|^2)^{1/2}}\Big)=0. \tag{1}$$

Moreover, u_m can be extended continuously to the puncture, using Weierstrass representation (in fact, it can be extended as a $C^{2,\alpha}$ function). Here $\mathcal{O}_{C_b^{n,\alpha}}(g)$ denotes a function that, together with its partial derivatives of order no greater than $n+\alpha$, is bounded by a constant times g. In the sequel, where necessary, we will consider on $B_{\rho_0}(0)$ also the polar coordinates $(\rho, \theta) \in [0, \rho_0] \times \mathbb{S}^1$.

If we linearize in $u = 0$ the nonlinear Equation (1), we obtain the expression of an operator that is the Jacobi operator about the plane; that is, $\mathscr{L}_{\mathbb{R}^2} = |x|^4 \Delta_0$. To be more precise, the linearization of (1) gives

$$L_u v = |x|^4 \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4 \nabla u \frac{\nabla u \cdot \nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}}\right). \tag{2}$$

Equation (1) means that the surface Σ_u parameterized by $x \mapsto (x/|x|^2, u(x))$ is minimal. We will express the mean curvature H_{u+v} of Σ_{u+v} in terms of the mean curvature H_u of Σ_u.

Lemma 2.1. *There exists a function Q_u satisfying $Q_u(0,0) = 0$ and $\nabla Q_u(0,0) = 0$ such that*

$$2H_{u+v} = 2H_u + L_u v + |x|^4 Q_u(|x|^2 \nabla v, |x|^2 \nabla^2 v).$$

Proof. Define $f(t) = 1/\sqrt{1+|x|^4|\nabla(u+tv)|^2}$ and apply Taylor expansion. □

Since u satisfies (1), $H_u = 0$. Then, if we put

$$\mathcal{Q}_u(\,\cdot\,) := \sqrt{1+|x|^4|\nabla u|^2}\, Q_u(|x|^2 \nabla\, \cdot\,, |x|^2 \nabla^2\, \cdot\,)$$

to simplify the notation, the minimal surface equation satisfied by the function v defined on the graph of the function u is

$$|x|^4 \big(\Delta_0 v + \sqrt{1+|x|^4|\nabla u|^2}\, \bar{L}_u v + \mathcal{Q}_u(v)\big) = 0, \tag{3}$$

where $\bar{L}_u$ is a second order linear operator whose coefficients are in $\mathcal{O}_{C^{2,\alpha}}(|x|^{k+1})$.

The catenoidal ends. We will denote by X_c the parameterization of the standard catenoid C whose axis of revolution is the x_3 axis. Its expression is

$$X_c(s, \theta) := (\cosh s \cos\theta, \cosh s \sin\theta, s) \in \mathbb{R}^3,$$

where $(s, \theta) \in \mathbb{R} \times \mathbb{S}^1$. The unit normal vector field about C is

$$n_c(s, \theta) := \frac{1}{\cosh s}(\cos\theta, \sin\theta, -\sinh s) \quad \text{for } (s, \theta) \in \mathbb{R} \times \mathbb{S}^1.$$

Up to a dilation, we can assume that the two ends E_t and E_b of M_k are asymptotic to some translated copy in the vertical direction of the catenoid parameterized by X_c.

Therefore, E_t and E_b can be parameterized, respectively, by

$$X_t := X_c + w_t n_c + \sigma_t e_3 \quad \text{in } (s_0, \infty) \times \mathbb{S}^1,$$
$$X_b := X_c - w_b n_c - \sigma_b e_3 \quad \text{in } (-\infty, -s_0) \times \mathbb{S}^1,$$

where $\sigma_t, \sigma_b \in \mathbb{R}$, and w_t (respectively w_b) is a function defined in $(s_0, \infty) \times \mathbb{S}^1$ (respectively $(-\infty, -s_0) \times \mathbb{S}^1$) that tends exponentially fast to 0 as s goes to $+\infty$ (respectively $-\infty$), reflecting that the ends are asymptotic to a catenoidal end.

We recall that the surface parameterized by $X := X_c + w\, n_c$ is minimal if and only if the function w satisfies the minimal surface equation, which for normal graphs over a catenoid has the form

$$\mathbb{L}_C w + \frac{1}{\cosh^2 s}\Big(Q_2\Big(\frac{w}{\cosh s}\Big) + \cosh s\, Q_3\Big(\frac{w}{\cosh s}\Big)\Big) = 0, \tag{4}$$

where $\mathbb{L}_C$ is the Jacobi operator about the catenoid, that is,

$$\mathbb{L}_C w = \frac{1}{\cosh^2 s}\Big(\partial^2_{ss} w + \partial^2_{\theta\theta} w + \frac{2w}{\cosh^2 s}\Big),$$

and Q_2 and Q_3 are nonlinear second order differential operators that are bounded in $\mathscr{C}^k(\mathbb{R} \times \mathbb{S}^1)$ for every k and satisfy $Q_2(0) = Q_3(0) = 0$, $\nabla Q_2(0) = \nabla Q_3(0) = 0$, and $\nabla^2 Q_3(0) = 0$ together with

$$\begin{aligned}\|Q_j(v_2) - Q_j(v_1)\|_{\mathscr{C}^{0,\alpha}([s,s+1]\times\mathbb{S}^1)} \\ \leqslant c\Big(\sup_{i=1,2} \|v_i\|_{\mathscr{C}^{2,\alpha}([s,s+1]\times\mathbb{S}^1)}\Big)^{j-1} \|v_2 - v_1\|_{\mathscr{C}^{2,\alpha}([s,s+1]\times\mathbb{S}^1)}\end{aligned} \tag{5}$$

for all $s \in \mathbb{R}$ and all v_1, v_2 such that $\|v_i\|_{\mathscr{C}^{2,\alpha}([s,s+1]\times\mathbb{S}^1)} \leqslant 1$. The constant $c > 0$ does not depend on s.

The family of Costa–Hoffman–Meeks surfaces with bent catenoidal ends. We denote by R_ξ the rotation by ξ about the x_2 axis oriented by e_2. The following result may be proved using an elaborate version of the implicit function theorem and by following [Jleli 2004] and [Kusner et al. 1996].

Theorem 2.2 [Hauswirth and Pacard 2007]. *There exists $\xi_0 > 0$ and a smooth 1-parameter family of minimal surfaces $\{M_k(\xi) \mid \xi \in (-\xi_0, \xi_0)\}$ with the properties that $M_k(0) = M_k$ and each $M_k(\xi)$ is invariant by reflection across the $\{x_2 = 0\}$ plane, has one horizontal planar end E_m and has two catenoidal ends $E_t(\xi)$ and $E_b(\xi)$ asymptotic respectively, up to a translation, to the upper and lower end of the catenoid $R_\xi C$ (that is, the standard catenoid whose axis of revolution is directed by $R_\xi e_3$). Moreover, $E_t(\xi)$ and $E_b(\xi)$ can be parameterized respectively* *by*

$$X_{t,\xi} = R_\xi(X_c + w_{t,\xi}\, n_c) + \sigma_{t,\xi}\, e_3 + \varsigma_{t,\xi}\, e_1, \tag{6}$$

$$X_{b,\xi} = R_\xi(X_c - w_{b,\xi}\, n_c) - \sigma_{b,\xi}\, e_3 - \varsigma_{b,\xi}\, e_1, \tag{7}$$

where the functions $w_{t,\xi}, w_{b,\xi}$ *and the numbers* $\sigma_{t,\xi}, \varsigma_{t,\xi}, \sigma_{b,\xi}, \varsigma_{b,\xi} \in \mathbb{R}$ *depend smoothly on* ξ *and satisfy*

$$|\sigma_{t,\xi} - \sigma_t| + |\sigma_{b,\xi} - \sigma_b| + |\varsigma_{t,\xi}| + |\varsigma_{b,\xi}| \\ + \|w_{t,\xi} - w_t\|_{C^{2,\alpha}_{-2}([s_0,+\infty)\times\mathbb{S}^1)} + \|w_{b,\xi} - w_b\|_{C^{2,\alpha}_{-2}((-\infty,-s_0]\times\mathbb{S}^1)} \leqslant c|\xi|,$$

where

$$\|w\|_{\mathscr{C}^{\ell,\alpha}_\delta([s_0,+\infty)\times\mathbb{S}^1)} = \sup_{s\geqslant s_0}\big(e^{-\delta s}\, \|w\|_{\mathscr{C}^{\ell,\alpha}([s,s+1]\times\mathbb{S}^1)}\big),$$
$$\|w\|_{\mathscr{C}^{\ell,\alpha}_\delta((-\infty,-s_0])\times\mathbb{S}^1)} = \sup_{s\leqslant -s_0}\big(e^{\delta s}\, \|w\|_{\mathscr{C}^{\ell,\alpha}([s-1,s]\times\mathbb{S}^1)}\big).$$

For all $s > s_0$ and $\rho < \rho_0$, we define

$$M_k(\xi, s, \rho) := \\ M_k(\xi) - \big(X_{t,\xi}([s,+\infty)\times\mathbb{S}^1) \cup X_m(B_\rho(0)) \cup X_{b,\xi}((-\infty,-s]\times\mathbb{S}^1)\big). \tag{8}$$

The parameterizations of the three ends of $M_k(\xi)$ induce a decomposition of $M_k(\xi)$ into slightly overlapping components: a compact piece $M_k(\xi, s_0+1, \rho_0/2)$ and three noncompact pieces

$$X_{t,\xi}((s_0,+\infty)\times\mathbb{S}^1), \quad X_{b,\xi}((-\infty,-s_0)\times\mathbb{S}^1), \quad X_m(\bar{B}_{\rho_0}(0)).$$

We define the weighted space of functions on $M_k(\xi)$.

Definition 2.3. Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, we define $\mathscr{C}^{\ell,\alpha}_\delta(M_k(\xi))$ as the space of functions in $\mathscr{C}^{\ell,\alpha}_{\mathrm{loc}}(M_k(\xi))$ invariant by reflections across the $\{x_2 = 0\}$ plane (that is, $w(p) = w(\bar{p})$ for all $p = (p_1, p_2, p_3) \in M_k(\xi)$, where $\bar{p} := (p_1, -p_2, p_3)$) and for which the following norm is finite:

$$\|w\|_{\mathscr{C}^{\ell,\alpha}_\delta(M_k(\xi))} := \|w\|_{\mathscr{C}^{\ell,\alpha}(M_k(\xi,s_0+1,\rho_0/2))} + \|w\circ X_m\|_{\mathscr{C}^{\ell,\alpha}(B_{\rho_0}(0))} \\ + \|w\circ X_{t,\xi}\|_{\mathscr{C}^{\ell,\alpha}_\delta([s_0,+\infty)\times\mathbb{S}^1)} + \|w\circ X_{b,\xi}\|_{\mathscr{C}^{\ell,\alpha}_\delta((-\infty,-s_0]\times\mathbb{S}^1)}.$$

We remark that there is no weight on the planar end E_m of $M_k(\xi)$. In fact, we can compactify this end and consider a weighted space of functions defined on a two-ended surface. In the next section we will consider normal perturbations of $M_k(\xi)$ by functions $u \in \mathscr{C}^{2,\alpha}_\delta(M_k(\xi))$, and the planar end E_m will be just vertically translated.

The Jacobi operator. The Jacobi operator about $M_k(\xi)$ is

$$\mathbb{L}_{M_k(\xi)} := \Delta_{M_k(\xi)} + |A_{M_k(\xi)}|^2,$$

where $|A_{M_k(\xi)}|$ is the norm of the second fundamental form on $M_k(\xi)$.

In the parameterization of the ends of $M_k(\xi)$ introduced above, the volume form $d\mathrm{vol}_{M_k(\xi)}$ can be written as $\gamma_t\, ds d\theta$ (respectively $\gamma_b\, ds d\theta$, $\gamma_m dx_1 dx_2$) near $E_t(\xi)$ (respectively $E_b(\xi)$, E_m). We define globally on $M_k(\xi)$ a smooth function

$$\gamma : M_k(\xi) \to [0, +\infty)$$

that equals 1 on $M_k(\xi, s_0 - 1, 2\rho_0)$ and equals γ_t (respectively γ_b, γ_m) on the end $E_t(\xi)$ (respectively $E_b(\xi)$, E_m). Observe that

$$\begin{aligned}
(\gamma \circ X_{t,\xi})(s,\theta) &\sim \cosh^2 s \quad \text{on } (s_0, +\infty) \times \mathbb{S}^1, \\
(\gamma \circ X_{b,\xi})(s,\theta) &\sim \cosh^2 s \quad \text{on } (-\infty, -s_0) \times \mathbb{S}^1, \\
(\gamma \circ X_m)(x) &\sim |x|^{-4} \qquad \text{on } B_{\rho_0}.
\end{aligned}$$

Given the defined spaces above, one can check that

$$\mathcal{L}_{\xi,\delta} : \mathscr{C}^{2,\alpha}_\delta(M_k(\xi)) \to \mathscr{C}^{0,\alpha}_\delta(M_k(\xi)), \quad w \mapsto \gamma\, \mathbb{L}_{M_k(\xi)}(w)$$

is a bounded linear operator. The subscript δ is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that the function γ is here to counterbalance the effect of the conformal factor $1/\sqrt{|g_{M_k(\xi)}|}$ in the expression of the Laplacian in the coordinates we use to parameterize the ends of the surface $M_k(\xi)$. This is precisely what is needed to have the operator defined from the space $\mathscr{C}^{2,\alpha}_\delta(M_k(\xi))$ into the target space $\mathscr{C}^{0,\alpha}_\delta(M_k(\xi))$.

To better grasp what is going on, let us linearize the nonlinear Equation (4) at $w = 0$. We get the expression of the Jacobi operator about the standard catenoid

$$\mathbb{L}_C := \frac{1}{\cosh^2 s}\Big(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\Big).$$

The operator $\cosh^2 s \mathbb{L}_C$ maps the space $(\cosh s)^\delta \mathscr{C}^{2,\alpha}((s_0, +\infty) \times \mathbb{S}^1)$ into the space $(\cosh s)^\delta \mathscr{C}^{0,\alpha}((s_0, +\infty) \times \mathbb{S}^1)$.

Similarly, if we linearize the nonlinear Equation (1) at $u = 0$, we obtain (see (2) with $u = 0$) the expression of the Jacobi operator about the plane $\mathbb{L}_{\mathbb{R}^2} := |x|^4 \Delta_0$. Again, the operator $|x|^{-4}\mathbb{L}_{\mathbb{R}^2} = \Delta_0$ clearly maps the space $\mathscr{C}^{2,\alpha}(\bar{B}_{\rho_0})$ into the space $\mathscr{C}^{0,\alpha}(\bar{B}_{\rho_0})$. Now, the function γ plays for the ends of the surface $M_k(\xi)$ the role the function $\cosh^2 s$ plays for the ends of the standard catenoid and the role the function $|x|^{-4}$ plays for the plane. Since the Jacobi operator about $M_k(\xi)$ is asymptotic to $\mathbb{L}_{\mathbb{R}^2}$ at E_m and is asymptotic to $\mathbb{L}_C$ at $E_t(\xi)$ and $E_b(\xi)$, we conclude that the operator $\mathcal{L}_{\xi,\delta}$ maps $\mathscr{C}^{2,\alpha}_\delta(M_k(\xi))$ into $\mathscr{C}^{0,\alpha}_\delta(M_k(\xi))$.

Definition 2.4 [Hauswirth and Pacard 2007]. A surface $M_k(\xi)$ is said to be nondegenerate if $\mathscr{L}_{\xi,\delta}$ is injective for all $\delta < -1$.

It is useful to observe that a duality argument in the weighted Lebesgue spaces implies $\mathscr{L}_{\xi,\delta}$ is injective if and only if $\mathscr{L}_{\xi,-\delta}$ is surjective, provided $\delta \notin \mathbb{Z}$. For details, see [Jleli 2004; Melrose 1993].

The nondegeneracy of $M_k(\xi)$ follows from the study of the kernel of $\mathscr{L}_{\xi,\delta}$.

The Jacobi fields. It is known that a smooth 1-parameter group of isometries containing the identity generates a Jacobi field, that is, a solution of $\mathbb{L}_{M_k(\xi)} u = 0$. The solutions that are invariant under reflection across the $\{x_2 = 0\}$ plane are generated by dilations, vertical translations and horizontal translations along the x_1 axis (see [Hauswirth and Pacard 2007]):

- The vertical translations generated by the Killing vector field $\Xi(p) = e_3$ give rise to the Jacobi field $\Phi^{0,+}(p) := n(p) \cdot e_3$.
- The vector field $\Xi(p) = p$ associated to the 1-parameter group of dilations generates the Jacobi field $\Phi^{0,-}(p) := n(p) \cdot p$.
- The Killing vector field $\Xi(p) = e_1$ that generates the group of translations along the x_1 axis is associated to a Jacobi field $\Phi^{1,+}(p) := n(p) \cdot e_1$.
- Finally, we denote by $\Phi^{1,-}(p) := n(p) \cdot (e_2 \times p)$ the Jacobi field associated to the Killing vector field $\Xi(p) = e_2 \times p$ that generates the group of rotations about the x_2 axis.

There are other Jacobi fields we do not take into account because they are not invariant by reflection across the $\{x_2 = 0\}$ plane.

With these notations, we define the deficiency space

$$\mathscr{D} := \operatorname{Span}\{\chi_t \Phi^{j,\pm}, \chi_b \Phi^{j,\pm} : j = 0, 1\}$$

where χ_t is a cutoff function that equals 1 on $X_{t,\xi}((s_0 + 1, +\infty) \times \mathbb{S}^1)$, equals 0 on $M_k(\xi) - X_{t,\xi}((s_0, +\infty) \times \mathbb{S}^1)$, is invariant under reflection across the $\{x_2 = 0\}$ plane, and satisfies $\chi_b(\,\cdot\,) := \chi_t(-\,\cdot\,)$. Clearly

$$\widetilde{\mathscr{L}}_{\xi,\delta} : \mathscr{C}^{2,\alpha}_\delta(M_k(\xi)) \oplus \mathscr{D} \to \mathscr{C}^{0,\alpha}_\delta(M_k(\xi)), \quad w \mapsto \gamma\, \mathbb{L}_{M_k(\xi)}(w)$$

is a bounded linear operator for $\delta < 0$.

A result of S. Nayatani [1992; 1993], which the second author extended in [Morabito 2008b], states that any bounded Jacobi field invariant by reflection across the $\{x_2 = 0\}$ plane is a linear combination of $\Phi^{0,+}$ and $\Phi^{1,+}$.

From that we get the following result about the operator $\mathscr{L}_{\xi,\delta}$.

Proposition 2.5. *We fix $\delta \in (1, 2)$. Then (reducing ξ_0 if this is necessary) the operator $\mathscr{L}_{\xi,\delta}$ is surjective and has a kernel of dimension* 4. *Moreover, there exists*

$G_{\xi,\delta}$, a right inverse for $\mathcal{L}_{\xi,\delta}$ that depends smoothly on ξ and in particular whose norm is bounded uniformly as $|\xi| < \xi_0$.

This fact together with an adaptation to our setting of the linear decomposition lemma proved in [Kusner et al. 1996] for constant mean curvature surfaces (see also [Jleli 2004] for minimal hypersurfaces), allows us to prove the following result.

Proposition 2.6. *We fix $\delta \in (-2, -1)$. Then (reducing ξ_0 if this is necessary) the operator $\widetilde{\mathcal{L}}_{\xi,\delta}$ for $|\xi| < \xi_0$ is surjective and has a kernel of dimension* 4.

3. Infinite dimensional family of minimal surfaces close to $M_k(\xi)$

In this section we consider a truncature of $M_k(\xi)$. First we write portions of the ends of $M_k(\xi)$ as vertical graphs over the $\{x_3 = 0\}$ plane.

We set $r_\varepsilon = 1/(2\sqrt{\varepsilon})$.

Lemma 3.1 [Hauswirth and Pacard 2007]. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $|\xi| \leqslant \varepsilon$, an annular part of the ends $E_t(\xi)$, $E_b(\xi)$ and E_m of $M_k(\xi)$ can be written, respectively, as vertical graphs over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ for the functions*

$$\begin{aligned} U_t(r,\theta) &= \sigma_{t,\xi} + \ln(2r) - \xi r\cos\theta + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon), \\ U_b(r,\theta) &= -\sigma_{b,\xi} - \ln(2r) - \xi r\cos\theta + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon), \\ U_m(r,\theta) &= \mathcal{O}_{\mathcal{C}_b^\infty}(r^{-(k+1)}). \end{aligned}$$

Here (r,θ) are the polar coordinates in the $\{x_3 = 0\}$ plane. The functions $\mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon)$ are defined in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ and are bounded in the $\mathcal{C}_b^\infty$ topology by a constant (independent on ε) multiplied by ε, where the partial derivatives are computed with respect to the vector fields $r\partial_r$ and ∂_θ.

In particular, a portion of the two catenoidal ends $E_t(\varepsilon/2)$ and $E_b(\varepsilon/2)$ of $M_k(\varepsilon/2)$ are graphs over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2} \subset \{x_3 = 0\}$ for functions U_t and U_b. We set $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$, $\rho_\varepsilon = 2\varepsilon^{1/2}$ and

$$M_k^T(\varepsilon/2) = \\ M_k(\varepsilon/2) - \big(X_{t,\varepsilon/2}((s_\varepsilon, +\infty)\times\mathbb{S}^1) \cup X_{b,\varepsilon/2}((-\infty, -s_\varepsilon)\times\mathbb{S}^1) \cup X_m(B_{\rho_\varepsilon}(0))\big).$$

We prove, following [Hauswirth and Pacard 2007, Section 6], the existence of a family of surfaces close to $M_k^T(\xi)$. In a first step, we modify the parameterization of the ends $E_t(\varepsilon/2)$, $E_b(\varepsilon/2)$, E_m, for appropriates values of s, so that, when $r \in [3r_\varepsilon/4, 3r_\varepsilon/2]$, the curves given by

$$\begin{aligned} \theta &\to (r\cos\theta, r\sin\theta, U_t(r,\theta)), \\ \theta &\to (r\cos\theta, r\sin\theta, U_b(r,\theta)), \\ \theta &\to (r\cos\theta, r\sin\theta, U_m(r,\theta)) \end{aligned}$$

correspond respectively to the curves $\{s = \ln(2r)\}$, $\{s = -\ln(2r)\}$, $\{\rho = 1/r\}$.

The second step is the modification of the unit normal vector field on $M_k(\varepsilon/2)$ to produce a transverse unit vector field $\tilde{n}_{\varepsilon/2}$ that coincides with the normal vector field $n_{\varepsilon/2}$ on $M_k(\varepsilon/2)$, is equal to e_3 on the graph over $B_{3r_\varepsilon/2} - B_{3r_\varepsilon/4}$ of the functions U_t and U_b, and interpolates smoothly between the different definitions of $\tilde{n}_{\varepsilon/2}$ in different subsets of $M_k^T(\varepsilon/2)$.

Finally we observe that close to $E_t(\varepsilon/2)$, we can give the estimate

$$\left|\cosh^2 s \left(\mathbb{L}_{M_k(\varepsilon/2)} v - \cosh^{-2} s (\partial^2_{ss} v + \partial^2_{\theta\theta} v)\right)\right| \leqslant c |\cosh^{-2} s\, v|. \tag{9}$$

This follows easily from (4) together with the fact that $w_{t,\xi}$ (see (6)) decays at least like $\cosh^{-2} s$ on $E_t(\varepsilon/2)$. Similar considerations hold close to the bottom end $E_b(\varepsilon/2)$. Near the middle planar end E_m, we have the estimate

$$\left||x|^{-4} \left(\mathbb{L}_{M_k(\varepsilon/2)} v - |x|^4 \Delta_0 v\right)\right| \leqslant c \left||x|^{2k+3} \nabla v\right|. \tag{10}$$

This follows easily from (2) and the fact that u_m decays at least like $|x|^{k+1}$ on E_m.

The graph of a function u, using the vector field $\tilde{n}_{\varepsilon/2}$, will be a minimal surface if and only if u is a solution of a second order nonlinear elliptic equation of the form

$$\mathbb{L}_{M_k^T(\varepsilon/2)} u = \tilde{L}_{\varepsilon/2} u + Q_\varepsilon(u),$$

where $\mathbb{L}_{M_k^T(\varepsilon/2)}$ is the Jacobi operator about $M_k^T(\varepsilon/2)$, Q_ε is a nonlinear second order differential operator, and $\tilde{L}_{\varepsilon/2}$ is a linear operator that takes into account the change of the normal vector field (only for the top and bottom ends) and the change of the parameterization. This operator is supported in neighborhoods of $\{\pm s_\varepsilon\} \times \mathbb{S}^1$, where its coefficients are uniformly bounded by a constant times ε^2, and a neighborhood of $\{\rho_\varepsilon\} \times \mathbb{S}^1$, where its coefficients are uniformly bounded by a constant times ε^3.

Now, we consider three even functions $\varphi_t, \varphi_b, \varphi_m \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ such that φ_t and φ_b are L^2-orthogonal to 1 and $\theta \mapsto \cos\theta$, while φ_m is L^2-orthogonal to 1. Assume that they satisfy

$$\|\varphi_t\|_{\mathscr{C}^{2,\alpha}} + \|\varphi_b\|_{\mathscr{C}^{2,\alpha}} + \|\varphi_m\|_{\mathscr{C}^{2,\alpha}} \leqslant \kappa\varepsilon. \tag{11}$$

We set $\Phi := (\varphi_t, \varphi_b, \varphi_m)$ and we define w_Φ to be the function equal to

(1) $\chi_+(s) H_{\varphi_t}(s_\varepsilon - s, \cdot)$ on the image of $X_{t,\varepsilon/2}$, where χ_+ is a cutoff function that equals 0 for $s \leqslant s_0 + 1$ and equals 1 for $s \in [s_0 + 2, s_\varepsilon]$;

(2) $\chi_-(s) H_{\varphi_b}(s + s_\varepsilon, \cdot)$ on the image of $X_{b,\varepsilon/2}$, where χ_- is a cutoff function that equals 0 for $s \geqslant -s_0 - 1$ and equals 1 for $s \in [-s_\varepsilon, -s_0 - 2]$;

(3) $\chi_m(\rho) \tilde{H}_{\rho_\varepsilon, \varphi_m}(\cdot, \cdot)$ on the image of X_m, where χ_m is a cutoff function that equals 0 for $\rho \geqslant \rho_0$ and equals 1 for $\rho \in [\rho_\varepsilon, \rho_0/2]$;

(4) 0 on the remaining part of the surface $M_k^T(\varepsilon/2)$,

where $\tilde{H}$ and H are, respectively, harmonic extensions of the operators introduced in Propositions A.2 and A.4.

We would like to prove that, under appropriate hypotheses, the graph over $M_k^T(\varepsilon/2)$ of the function $u = w_\Phi + v$ is a minimal surface. This is equivalent to solving the equation

$$\mathbb{L}_{M_k^T(\varepsilon/2)}(w_\Phi + v) = \tilde{L}_{\varepsilon/2}(w_\Phi + v) + Q_\varepsilon(w_\Phi + v).$$

The solution of this equation is obtained thanks to the fixed point problem

$$v = T(\Phi, v) := G_{\varepsilon/2,\delta} \circ \mathscr{E}_\varepsilon\big(\gamma\left(\tilde{L}_{\varepsilon/2}(w_\Phi + v) - \mathbb{L}_{M_k^T(\varepsilon/2)}\, w_\Phi + Q_\varepsilon(w_\Phi + v)\right)\big), \tag{12}$$

where $\delta \in (1, 2)$, the operator $G_{\varepsilon/2,\delta}$ is the right inverse provided in Proposition 2.5, and $\mathscr{E}_\varepsilon$ is a linear extension operator

$$\mathscr{E}_\varepsilon : \mathscr{C}_\delta^{0,\alpha}(M_k^T(\varepsilon/2)) \to \mathscr{C}_\delta^{0,\alpha}(M_k(\varepsilon/2)).$$

Here $\mathscr{C}_\delta^{0,\alpha}(M_k^T(\varepsilon/2))$ denotes the space of functions of $\mathscr{C}_\delta^{0,\alpha}(M_k(\varepsilon/2))$ restricted to $M_k^T(\varepsilon/2)$, and $\mathscr{E}_\varepsilon$ is defined so that $\mathscr{E}_\varepsilon v$ equals v in $M_k^T(\varepsilon/2)$, vanishes in the image of $[s_\varepsilon + 1, +\infty) \times \mathbb{S}^1$ by $X_{t,\varepsilon/2}$, in the image of $(-\infty, -s_\varepsilon - 1) \times \mathbb{S}^1$ by $X_{b,\varepsilon/2}$ and in the image of $B_{\rho_\varepsilon/2}$ by X_m, and is an interpolation of these values in the remaining part of $M_k(\varepsilon/2)$:

$$(\mathscr{E}_\varepsilon v) \circ X_{t,\varepsilon/2}(s, \theta) = (1 + s_\varepsilon - s)(v \circ X_{t,\varepsilon/2}(s_\varepsilon, \theta)) \quad \text{for } (s, \theta) \in [s_\varepsilon, s_\varepsilon + 1] \times \mathbb{S}^1,$$

$$(\mathscr{E}_\varepsilon v) \circ X_{b,\varepsilon/2}(s, \theta) = (1 + s_\varepsilon + s)(v \circ X_{b,\varepsilon/2}(s_\varepsilon, \theta)) \quad \text{for } (s, \theta) \in [-s_\varepsilon - 1, -s_\varepsilon] \times \mathbb{S}^1,$$

$$(\mathscr{E}_\varepsilon v) \circ X_m(\rho, \theta) = \Big(\frac{2\rho}{\rho_\varepsilon} - 1\Big)(v \circ X_m(\rho_\varepsilon, \theta)) \quad \text{for } (\rho, \theta) \in [\rho_\varepsilon/2, \rho_\varepsilon] \times \mathbb{S}^1.$$

Remark 3.2. As consequence of the properties of $\mathscr{E}_\varepsilon$, if $\operatorname{supp} v \cap \big(B_{\rho_\varepsilon} - B_{\rho_\varepsilon/2}\big) \neq \varnothing$ then

$$\|(\mathscr{E}_\varepsilon v) \circ X_m\|_{\mathscr{C}^{0,\alpha}(\bar{B}_{\rho_0})} \leqslant c\rho_\varepsilon^{-\alpha} \|v \circ X_m\|_{\mathscr{C}^{0,\alpha}(B_{\rho_0} - B_{\rho_\varepsilon})}.$$

This explosion of the norm does not occur near the catenoidal type ends:

$$\|(\mathscr{E}_\varepsilon v) \circ X_{t,\varepsilon/2}\|_{\mathscr{C}^{0,\alpha}([s_0,+\infty)\times\mathbb{S}^1)} \leqslant c\|v \circ X_{t,\varepsilon/2}\|_{\mathscr{C}^{0,\alpha}([s_0,s_\varepsilon]\times\mathbb{S}^1)}.$$

A similar inequality holds for the bottom end.

In the sequel we will assume $\alpha > 0$ and close to zero.

The existence of a solution $v \in \mathscr{C}_\delta^{2,\alpha}(M_k^T(\varepsilon/2))$ for Equation (12) is a consequence of the following result, which proves that $T(\Phi, \cdot)$ is a contracting mapping.

Proposition 3.3. *Choose* $\delta \in (1,2)$, $\alpha \in (0,1/2)$, $\Phi = (\varphi_t, \varphi_b, \varphi_m) \in [\mathscr{C}^{2,\alpha}(\mathbb{S}^1)]^3$ *satisfying* (11) *and enjoying the properties described above. There exist constants* $c_\kappa > 0$ *and* $\varepsilon_\kappa > 0$ *such that*

$$\|T(\Phi, 0)\|_{\mathscr{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))} \leqslant c_\kappa \varepsilon^{3/2}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant \tfrac{1}{2}\|v_2 - v_1\|_{\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))},$$

$$\|T(\Phi_2, v) - T(\Phi_1, v)\|_{\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant c\varepsilon\|\Phi_2 - \Phi_1\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}, \tag{13}$$

where

$$\|\Phi_2 - \Phi_1\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} = \|\varphi_{t,2} - \varphi_{t,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} + \|\varphi_{b,2} - \varphi_{b,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} + \|\varphi_{m,2} - \varphi_{m,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)},$$

for all $v, v_1, v_2 \in \mathscr{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))$ *such that* $\|v\|_{\mathscr{C}^{2,\alpha}_\delta} \leqslant 2c_\kappa \varepsilon^{3/2}$ *and for all boundary data* $\Phi_1, \Phi_2 \in [\mathscr{C}^{2,\alpha}(\mathbb{S}^1)]^3$ *enjoying the same properties as* Φ.

Proof. We recall that the Jacobi operator associated to $M_k(\varepsilon/2)$ is asymptotic to the operator of the catenoid near the catenoidal ends, and it is asymptotic to the Laplacian near of the planar end. The function w_Φ is identically zero far from the ends where the explicit expression of $\mathbb{L}_{M_k(\varepsilon/2)}$ is not known: This is the reason of our particular choice in the definition of w_Φ. Then from the definition of w_Φ, thanks to Proposition 2.5 and to (9) and (10), we obtain the estimate

$$\begin{aligned}\|\mathscr{E}_\varepsilon(\gamma \mathbb{L}_{M_k^T(\varepsilon/2)} w_\Phi)\|_{\mathscr{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} &\leqslant c\|\cosh^{-2} s(w_\Phi \circ X_{t,\varepsilon/2})\|_{\mathscr{C}^{0,\alpha}_\delta([s_0+1, s_\varepsilon]\times\mathbb{S}^1)}\\ &\quad + c\|\cosh^{-2} s(w_\Phi \circ X_{b,\varepsilon/2})\|_{\mathscr{C}^{0,\alpha}_\delta([-s_\varepsilon, -s_0-1]\times\mathbb{S}^1)}\\ &\quad + c\varepsilon^{-\alpha/2}\|\rho^{2k+3}\nabla(w_\Phi \circ X_m)\|_{\mathscr{C}^{0,\alpha}([\rho_\varepsilon, \rho_0]\times\mathbb{S}^1)} \leqslant c_\kappa \varepsilon^{3/2}.\end{aligned}$$

Using the properties of $\tilde{L}_{\varepsilon/2}$, we obtain

$$\begin{aligned}\|\mathscr{E}_\varepsilon(\gamma \tilde{L}_{\varepsilon/2} w_\Phi)\|_{\mathscr{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} &\leqslant c\varepsilon\|w_\Phi \circ X_{t,\varepsilon/2}\|_{\mathscr{C}^{0,\alpha}_\delta([s_0+1, s_\varepsilon]\times\mathbb{S}^1)}\\ &\quad + c\varepsilon\|w_\Phi \circ X_{b,\varepsilon/2}\|_{\mathscr{C}^{0,\alpha}_\delta([-s_\varepsilon, -s_0-1]\times\mathbb{S}^1)}\\ &\quad + c\varepsilon^{1-\alpha/2}\|w_\Phi \circ X_m\|_{\mathscr{C}^{0,\alpha}([\rho_\varepsilon, \rho_0]\times\mathbb{S}^1)} \leqslant c_\kappa \varepsilon^{3/2}.\end{aligned}$$

As for the last term, we recall that the operator Q_ε has two different expressions if we consider the catenoidal type end and the planar end (see (4) and (3)). We leave it to the reader to check that

$$\|\mathscr{E}_\varepsilon(\gamma Q_\varepsilon(w_\Phi))\|_{\mathscr{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant c_\kappa \varepsilon^{3/2}. \qquad \square$$

Theorem 3.4. *Let*

$$B := \{w \in \mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2)) \mid \|w\|_{\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant 2c_\kappa \varepsilon^{3/2}\} \quad \textit{and} \quad \Phi \in [\mathscr{C}^{2,\alpha}(\mathbb{S}^1)]^3$$

be as above. Then the nonlinear mapping $T(\Phi, \cdot)$ defined above has a unique fixed point v in B.

Proof. The previous proposition shows that, if ε is chosen small enough, the nonlinear mapping $T(\Phi, \cdot)$ is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^{3/2}$ in $\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))$ into itself. This value follows from the estimate of the norm of $T(\Phi, 0)$. Consequently by the Schauder fixed point theorem, $T(\Phi, \cdot)$ has a unique fixed point w in this ball. □

This argument provides a minimal surface $M_k^T(\varepsilon/2, \Phi)$ that is close to $M_k^T(\varepsilon/2)$ and has three boundaries. This surface is, close to its upper and lower boundary, a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$ whose parameterizations are respectively given by

$$U_t(r,\theta) = \quad \sigma_{t,\varepsilon/2} + \ln(2r) - \tfrac{1}{2}\varepsilon r\cos\theta + H_{\varphi_t}(s_\varepsilon - \ln(2r), \theta) + V_t(r,\theta), \tag{14}$$

$$U_b(r,\theta) = -\sigma_{b,\varepsilon/2} - \ln(2r) - \tfrac{1}{2}\varepsilon r\cos\theta + H_{\varphi_b}(s_\varepsilon - \ln(2r), \theta) + V_b(r,\theta), \tag{15}$$

where $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$. The boundaries of the surface correspond to $r_\varepsilon = \frac{1}{2}\varepsilon^{-1/2}$. Near the middle boundary the surface is a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$, Its parameterization is

$$U_m(r,\theta) = \tilde{H}_{\rho_\varepsilon,\varphi_m}(1/r, \theta) + V_m(r,\theta), \tag{16}$$

where $\rho_\varepsilon = 2\varepsilon^{1/2}$. All the functions V_i for $i = t, b, m$ depend nonlinearly on ε, φ.

Lemma 3.5. *The functions $V_i(\varepsilon, \varphi_i)$ for $i = t, b$ satisfy*

$$\begin{aligned} &\|V_i(\varepsilon,\varphi_i)(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \leqslant c\varepsilon, \\ &\|V_i(\varepsilon,\varphi_{i,2})(r_\varepsilon\,\cdot\,,\cdot) - V_i(\varepsilon,\varphi_{i,1})(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \\ &\qquad\qquad \leqslant c\varepsilon^{1-\delta/2}\|\varphi_{i,2} - \varphi_{i,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}. \end{aligned} \tag{17}$$

The function $V_m(\varepsilon, \varphi)$ satisfies

$$\begin{aligned} &\|V_m(\varepsilon,\varphi)(\rho_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \leqslant c\varepsilon, \\ &\|V_m(\varepsilon,\varphi_{m,2})(\rho_\varepsilon\,\cdot\,,\cdot) - V_m(\varepsilon,\varphi_{m,1})(\rho_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \\ &\qquad\qquad \leqslant c\varepsilon\|\varphi_{m,2} - \varphi_{m,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}. \end{aligned} \tag{18}$$

Proof. The first estimate follows from

$$\begin{aligned} &\|V_i(\varepsilon,\varphi_2)(\,\cdot\,,\cdot) - V_i(\varepsilon,\varphi_1)(\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_{2r_\varepsilon} - B_{r_\varepsilon/2})} \\ &\qquad\qquad \leqslant ce^{\delta s_\varepsilon}\|(T(\Phi_2, V_i) - T(\Phi_1, V_i)) \circ X_{i,\varepsilon/2}\|_{\mathscr{C}^{2,\alpha}_\delta(\Omega_i \times \mathbb{S}^1)}, \end{aligned}$$

for $i = t, b$, with $\Omega_t = [s_0, s_\varepsilon]$ and $\Omega_b = [-s_\varepsilon, -s_0]$. The second one follows from

$$\|V_m(\varepsilon, \varphi_2)(\cdot,\cdot) - V_m(\varepsilon, \varphi_1)(\cdot,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_{2\rho_\varepsilon} - B_{\rho_\varepsilon/2})} \leqslant c\|(T(\Phi_2, V_m) - T(\Phi_1, V_m)) \circ X_m\|_{\mathscr{C}^{2,\alpha}([\rho_\varepsilon,\rho_0]\times\mathbb{S}^1)}$$

and the estimate (13) of Proposition 3.3. □

4. An infinite family of Scherk-type minimal surfaces close to a horizontal periodic flat annulus

This section has two purposes. The first is to find an infinite family of minimal surfaces close to a horizontal periodic flat annulus Σ with a disk D_s removed. The surfaces of this family have two horizontal Scherk-type ends E_1 and E_2 and will be glued on the middle planar end of a Costa–Hoffman–Meeks surface M_k. We will prescribe the boundary data φ on ∂D_s. Assume the period T of Σ points in the x_2 direction. Then the asymptotic direction of E_1 and E_2 is along x_1 axis.

The second and more general purpose of this section is to show the existence of an infinite family of minimal graphs over $\Sigma - D_s$, whose ends have slightly modified asymptotic directions. When the asymptotic directions are not horizontal, these surfaces are close to half of a Scherk surface, seen as a graph over $\Sigma - D_s$ (see Figure 2). A piece of such a surface will be glued to the catenoidal ends of the surface M_k and to an end of a KMR example $M_{\sigma,0,0}$ introduced in Section 5. We will prescribe the boundary data on ∂D_s. Since we need to prescribe the flux along ∂D_s, we will modify the asymptotic direction of the ends, and we will choose $|T|$ large.

4.1. ***Scherk-type ends.*** Conformally parameterize the annulus $\Sigma \subset \mathbb{R}^3/T$ on $\mathbb{C}^*$, with the notation $(x_1, x_2, x_3) = (x_1 + ix_2, x_3)$, by the mapping

$$A(w) = \Big(-\frac{|T|}{2\pi}\ln(w), 0\Big) \quad \text{for } w \in \mathbb{C}^*.$$

The horizontal Scherk-type end E_1 described above can be written as the graph of a function $h_1 \in \mathscr{C}^{2,\alpha}(B_r^*(0))$, where $B_r^*(0)$ is the punctured disk $B_r(0) - \{0\}$ of radius $r \in (0, 1)$ centered at the origin. The function $h_1(w)$ is bounded and extends to the puncture; see [Hauswirth and Traizet 2002]. The end E_1 can be parameterized by

$$X_1(w) = A(w) + h_1(w)e_3 = \Big(-\frac{|T|}{2\pi}\ln(w), h_1(w)\Big) \in \mathbb{R}^3/T \quad \text{for } w \in B_r^*(0)$$

in the orthonormal frame $\mathscr{F} = (e_1, e_2, e_3)$. The end has asymptotic direction e_1.

The horizontal Scherk-type end E_2 can be parameterized in $\mathbb{C} - B_{r^{-1}}(0)$ similarly. Via an inversion, we can parameterize E_2 by

$$X_2(w) = \Big(-\frac{|T|}{2\pi}\ln(w), h_2(w)\Big) \in \mathbb{R}^3/T \quad \text{for } w \in B_r^*(0)$$

in the frame $\mathscr{F}^- = (-e_1, -e_2, e_3)$, where $h_2 \in \mathscr{C}^{2,\alpha}(B_r^*(0))$ is a bounded function that can be extended to the puncture. Now the end has asymptotic direction $-e_1$.

Let us now parameterize a general Scherk-type end, not necessarily horizontal. Let R_θ denote a rotation in $\mathbb{R}^3/T$ by θ about the x_2 axis (oriented by e_2). We can parameterize a not necessarily horizontal Scherk-type end $\widetilde{E}_1$ with asymptotic direction $\cos\theta_1 e_1 + \sin\theta_1 e_3$ and limit normal vector $R_{\theta_1}(e_3)$, with $\theta_1 \in [0, \pi/2)$, by

$$\widetilde{X}_1(z) = \Big(-\frac{|T|}{2\pi}\ln(z), \widetilde{h}_1(z)\Big) \quad \text{for } z \in B_r^*(0)$$

in the frame $\mathscr{F}(\theta_1) = R_{\theta_1}\mathscr{F}$, where $\widetilde{h}_1 \in \mathscr{C}^{2,\alpha}(B_r^*(0))$ is a bounded function that can be extended to the origin.

Finally, a Scherk-type end $\widetilde{E}_2$ with asymptotic direction $-\cos\theta_2 e_1 + \sin\theta_2 e_3$ and limit normal vector $R_{-\theta_2}(e_3)$, with $\theta_2 \in [0, \pi/2)$, can be parameterized by

$$\widetilde{X}_2(z) = \Big(-\frac{|T|}{2\pi}\ln(z), \widetilde{h}_2(z)\Big) \quad \text{for } z \in B_r^*(0)$$

in the frame $\mathscr{F}^-(\theta_2) = R_{-\theta_2}\mathscr{F}^-$, where $\widetilde{h}_2 \in \mathscr{C}^{2,\alpha}(B_r^*(0))$ is a bounded function that can be extended to the origin.

4.2. ***Construction of the infinite families.*** Given an $r \in (0, 1)$ and a $\Theta = (\theta_1, \theta_2)$ in $[0, \theta_0]^2$, with $\theta_0 > 0$ small, we denote by $A_\Theta : \mathbb{C}^* \to \mathbb{R}^3/T$ the immersion obtained as the smooth interpolation of

$$\begin{array}{ll} (R_{\theta_1} \circ A)(z) & \text{if } |z| < r/2, \\ A(z) & \text{if } r < |z| < r^{-1}, \\ (R_{-\theta_2} \circ A)(z) & \text{if } |z| > 2r^{-1}. \end{array}$$

Let N_Θ be the vector field obtained as the smooth interpolation of $R_{\theta_1}(e_3)$ on $\{|z| < r/2\}$, of e_3 on $\{r < |z| < r^{-1}\}$ and of $R_{-\theta_2}(e_3)$ on $\{|z| > 2r^{-1}\}$. For any $h \in C^{2,\alpha}(\overline{\mathbb{C}})$, we define the immersion

$$X_{\Theta,h}(z) = A_\Theta(z) + h(z)N_\Theta(z) \quad \text{for } z \in \mathbb{C}^*.$$

The immersion $X_{\Theta,h}$ has two Scherk-type ends E_1 and E_2 with asymptotic directions $\cos\theta_1 e_1 + \sin\theta_1 e_3$ and $-\cos\theta_2 e_1 + \sin\theta_2 e_3$, respectively.

At the end E_1 (respectively E_2), $X_{\Theta,h}(z) = A(z) + h_1(z)e_3$ in the orthogonal frame $\mathscr{F}(\theta_1)$ (respectively $X_{\Theta,h}(z) = A(z^{-1}) + h_2(z)e_3$ in the frame $\mathscr{F}^-(\theta_2)$), with $z \in B_r^*(0)$, where $h_1(z) = h(z)$ and $h_2(z) = h(z^{-1})$. L. Hauswirth and M. Traizet [2002] proved that, in terms of the z coordinate, the mean curvature of $X_{\Theta,h}$ at E_i is

$$H = \frac{2\pi^2|z|^2}{|T|^2}\,\mathrm{div}_0(P^{-1/2}\nabla_0 h_i),$$

where $P = 1 + (4\pi^2|z|^2/|T|^2)\|\nabla_0 h_i\|_0^2$ and the subscript 0 means that the corresponding object is computed with respect to the flat metric of the z plane. We denote by λ the smooth function without zeros defined by $\lambda(z) = |T|^2/(4\pi^2|z|^2)$ for $z \in B_r^*(0)$. Then at E_i we have

$$2\lambda H = P^{-1/2}\Delta_0 h_i - \tfrac{1}{2}P^{-3/2}\langle\nabla_0 P, \nabla_0 h_i\rangle_0.$$

So the mean curvature at the end E_i vanishes if h_i satisfies the equation

$$\Delta_0 h - \frac{1}{2P}\langle\nabla_0 P, \nabla_0 h\rangle_0 = 0. \tag{19}$$

Definition 4.1. Given $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ we define $C^{k,\alpha}(\overline{\mathbb{C}})$ as the space of functions $u \in C^{k,\alpha}_{\text{loc}}(\overline{\mathbb{C}})$ such that

$$\|u\|_{C^{k,\alpha}(\overline{\mathbb{C}})} := [u]_{k,\alpha,\overline{\mathbb{C}}} < +\infty,$$

where $[u]_{k,\alpha,\overline{\mathbb{C}}}$ denotes the usual $C^{k,\alpha}$ Hölder norm on $\overline{\mathbb{C}}$.

Let B_s be a disk in $\mathbb{C}^*$ such that

$$D_s = A(B_s) \subset \Sigma = \{z \in \mathbb{C} \mid -|T| < 2y \leqslant |T|\}$$

is a geodesic disk centered at the origin of $\mathbb{R}^3/T$. Denote by $C^{k,\alpha}(\overline{\mathbb{C}} - B_s)$ the space of functions in $C^{k,\alpha}(\overline{\mathbb{C}})$ restricted to $\overline{\mathbb{C}} - B_s$. We denote by $H(\Theta, h)$ the mean curvature of $X_{\Theta,h}$, and $\overline{H}(\Theta, h) = \lambda H(\Theta, h)$, where λ is the smooth function defined in a neighborhood of each puncture by $\lambda(z) = |T|^2/(4\pi^2|z|^2)$. [Hauswirth and Traizet 2002, Lemma 4.1] shows that

$$\overline{H} : \mathbb{R}^2 \times \mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_s) \to \mathscr{C}^{0,\alpha}(\overline{\mathbb{C}} - B_s)$$

is an analytical operator. Denote by $\mathscr{L}_\Theta$ the Jacobi operator about A_Θ. We set $\overline{\mathscr{L}}_\Theta = \lambda\mathscr{L}_\Theta$.

Remark 4.2. The operators H and $\mathscr{L}_\Theta$ are the mean curvature operator and the Jacobi operator with respect to the metric $|dz|^2$ of $\overline{\mathbb{C}}$. Defining operators $\overline{H} = \lambda H$ and $\overline{\mathscr{L}}_\Theta = \lambda\mathscr{L}_\Theta$ means considering a different metric on $\overline{\mathbb{C}}$. Actually, $\overline{H}$ and $\overline{\mathscr{L}}_\Theta$ are the mean curvature operator and Jacobi operator with respect to the metric $g_\lambda = |dz|^2/\lambda$. From the definition of λ, it follows that the volume of $\overline{\mathbb{C}}$ with respect this metric is finite.

The Jacobi operator $\overline{\mathscr{L}}_\Theta$ is a second order linear elliptic operator satisfying $|\overline{\mathscr{L}}_\Theta u - \Delta u| \leqslant c(|\theta_1| + |\theta_2|)|u|$, and the coefficients of $F_\Theta = \Delta - \overline{\mathscr{L}}_\Theta$ have compact support.

Now we fix $s_0 > 0$. Given $\varepsilon > 0$ and $|T| \in [4/\sqrt{\varepsilon}, +\infty)$ large enough, we choose $s \in (0, s_0)$ so that $D_s = A(B_s)$ is the geodesic disk of radius $1/2\sqrt{\varepsilon}$ centered at the origin.

Proposition 4.3. *There exists $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and every $|T| \in (\eta_0, +\infty)$, there exists an operator*

$$G_{\varepsilon,|T|} : \mathscr{C}^{0,\alpha}(\overline{\mathbb{C}} - B_s) \to \mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_s)$$

such that, given $f \in \mathscr{C}^{0,\alpha}(\overline{\mathbb{C}} - B_s)$, $w = G_{\varepsilon,|T|}(f)$ satisfies

$$\begin{cases} \Delta w = f & \text{on } \overline{\mathbb{C}} - B_s, \\ w \in \operatorname{Span}\{1\} & \text{on } \partial B_s, \end{cases}$$

and $\|w\|_{\mathscr{C}^{2,\alpha}} \le c\|f\|_{\mathscr{C}^{0,\alpha}}$ for some constant $c > 0$ that does not depend on ε or $|T|$.

Proof. Let be u a solution of $\Delta u = f$ on $\overline{\mathbb{C}} - B_s$ with $u = 0$ on ∂B_s. We recall that the metric in use on $\overline{\mathbb{C}}$ is given by $g_\lambda = |dz|^2/\lambda$. With respect to this metric

$$R := \operatorname{vol}(\overline{\mathbb{C}} - B_s) < +\infty \quad \text{and} \quad \int_{\overline{\mathbb{C}} - B_s} u\, d\mathrm{vol}_{g_\lambda} < \infty.$$

We set $w = u - (1/R) \int_{\overline{\mathbb{C}} - B_s} u\, d\mathrm{vol}_{g_\lambda}$. The function w is well defined and satisfies $\int_{\overline{\mathbb{C}} - B_s} w\, d\mathrm{vol}_{g_\lambda} = 0$; also $w \in \operatorname{Span}\{1\}$ on ∂B_s. If the theorem is false, there is a sequence of functions f_n, of solutions w_n, and of real numbers s_n such that

$$\sup_{\overline{\mathbb{C}} - B_{s_n}} |f_n| = 1 \quad \text{and} \quad A_n := \sup_{\overline{\mathbb{C}} - B_{s_n}} |w_n| \to +\infty \quad \text{as } n \to +\infty,$$

where $s_n \in [0, s_0]$. Now we set $\tilde{w}_n := w_n/A_n$. Elliptic estimates imply that s_n and $\tilde{w}_n$ converge up to a subsequence, respectively, to $s_\infty \in [0, s_0]$ and to $\tilde{w}_\infty$ on $\overline{\mathbb{C}} - B_{s_\infty}$. This function satisfies $\Delta \tilde{w}_\infty = 0$. Then $\tilde{w}_\infty$ is constant on $\overline{\mathbb{C}} - B_{s_\infty}$ and $\int_{\overline{\mathbb{C}} - B_{s_\infty}} \tilde{w}_\infty\, d\mathrm{vol}_{g_\lambda} = 0$, which contradicts that $\sup|\tilde{w}_\infty| = 1$. ☐

Now we fix $|T| \geqslant 4/\sqrt{\varepsilon}$, $\Theta \in (0, \varepsilon)^2$, $s_\varepsilon = 1/(2\sqrt{\varepsilon})$, and let $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ be even (or odd) L^2-orthogonal to 1, with $\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon$ for some $\kappa > 0$. Let w_φ be the unique bounded harmonic extension of φ. We would like to solve the minimal surface equation $H(\Theta, v + w_\varphi) = 0$ with fixed boundary data φ, prescribed asymptotic direction Θ and period $|T|$. Then we have to solve the equation

$$\Delta v = F_\Theta(v + w_\varphi) + Q_\Theta(v + w_\varphi),$$

with Q_Θ a quadratic term such that $|Q_\Theta(v_1) - Q_\Theta(v_2)| \leqslant c|v_1 - v_2|^2$. The resolution of the previous equation is obtained by showing the existence of a fixed point

$$v = S(\Theta, \varphi, v) := G_{\varepsilon,|T|}(F_\Theta(v + w_\varphi) + Q_\Theta(v + w_\varphi)).$$

Proposition 4.4. *Let $\varphi \in \mathbb{S}^1$ satisfy $\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon$ and enjoy the properties described above. There exist $c_\kappa > 0$ and $\varepsilon_\kappa > 0$ such that*

$$\|S(\Theta, \varphi, 0)\|_{\mathscr{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^2 \quad \textit{for all } |T| \geqslant 4/\sqrt{\varepsilon}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\|S(\Theta, \varphi, v_1) - S(\Theta, \varphi, v_2)\|_{\mathscr{C}^{2,\alpha}} \leqslant \tfrac{1}{2}\|v_2 - v_1\|_{\mathscr{C}^{2,\alpha}},$$
$$\|S(\Theta, \varphi_1, v) - S(\Theta, \varphi_2, v)\|_{\mathscr{C}^{2,\alpha}} \leqslant c\varepsilon\|\varphi_2 - \varphi_1\|_{\mathscr{C}^{2,\alpha}}$$

for all $v, v_1, v_2 \in \mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_{s_\epsilon})$ *whose* $\mathscr{C}^{2,\alpha}$ *norm is bounded by* $2c_\kappa\varepsilon^2$, *for all boundary data* $\varphi_1, \varphi_2 \in \mathbb{S}^1$ *with the same properties as* φ *and for all* $\Theta = (\theta_1, \theta_2)$ *such that* $|\theta_1| + |\theta_2| \leqslant \varepsilon$.

Proof. Using Proposition 4.3, the inequality $|\overline{\mathscr{L}}u - \Delta u| \leqslant c(|\theta_1| + |\theta_2|)|u|$, and the quadratic behavior of Q_Θ, we derive the stated estimate. The details of the proof are left to the reader. □

Theorem 4.5. *Let* $B := \{w \in \mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_{s_\epsilon}) \mid \|w\|_{\mathscr{C}^{2,\alpha}} \leqslant 2c_\kappa\varepsilon^2\}$. *Let* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *as above, and let* $\Theta = (\theta_1, \theta_2)$ *with* $|\theta_1| + |\theta_2| \leqslant \varepsilon$. *Then the nonlinear mapping* $S(\Theta, \varphi, \cdot)$ *defined above has a unique fixed point* v *in* B.

Proof. The previous proposition shows that, if ε is chosen small enough, the nonlinear mapping S is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^2$ in $\mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_{s_\epsilon})$ into itself. This value follows from the estimate of the norm of $S(\Theta, \varphi, 0)$. Consequently by the Schauder fixed point theorem, $S(\Theta, \varphi, \cdot)$ has a unique fixed point v in this ball. □

On the set $B_{2s_\varepsilon} - B_{s_\varepsilon}$, the function $U = v + w_\varphi$ is the solution of Equation (19). Using the vertical translation c_0e_3, we can fix the value $c_0 + \varphi$ at the boundary, obtaining $U = c_0 + w_\varphi + v$.

The function v depends nonlinearly on φ. Using the Schauder estimate for the equation on a fixed bounded domain, we find

$$\|v(\varphi_1) - v(\varphi_2)\|_{\mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_{s_\epsilon})} \leqslant c_\kappa\varepsilon\|\varphi_1 - \varphi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}.$$

This can be done uniformly in (θ_1, θ_2). Now we want to obtain the parametrization of the surface close to the annulus with linear growth ends (from which we have removed D_{s_ϵ}) in a neighbourhood of ∂D_{s_ϵ}. We recall that D_{s_ϵ} corresponds to B_{s_ϵ} by a conformal mapping. From now on, φ will be considered as the boundary data for ∂D_{s_ϵ}. We will denote its harmonic extension by $w_\varphi = \tilde{H}_{s_\epsilon,\varphi}$. We observe that near $\partial D_{s_\varepsilon}$ the function U grows logarithmically. The hypothesis that φ is orthogonal to 1 implies that the function w_φ is also and is bounded. This is not the case for v, which can be seen as the sum of a bounded function that is orthogonal to 1 and of a function of the form $c\ln(r/s_\varepsilon)$, where $c = c(|T|, \theta_1, \theta_2)$, defined in a neighborhood of $\partial D_{s_\varepsilon}$. We can determine c using a flux formula.

Let γ_1 and γ_2 be two closed curves in $\overline{\Sigma}/T$ chosen to correspond by conformal mapping to the boundaries of two circular neighborhoods N_1 and N_2 of the punctures corresponding to the ends with linear growth. Let $\mathscr{S} = \overline{\mathbb{C}} - (B_{s_\varepsilon} \cup N_1 \cup N_2)$.

Now $\int_{\mathcal{S}} \Delta X = 0$ since X is the parameterization of a minimal surface. By the divergence theorem, if $\Gamma = \partial\mathcal{S}$, then

$$0 = \int_{\mathcal{S}} \Delta X = \int_{\Gamma} \frac{\partial X}{\partial \eta} ds = \int_{\gamma_1} \frac{\partial X}{\partial \eta} ds + \int_{\gamma_2} \frac{\partial X}{\partial \eta} ds + \int_{\partial D_{s_\varepsilon}} \frac{\partial X}{\partial \eta} ds,$$

where η denotes the conormal along Γ. This equality implies

$$\int_{\partial D_{s_\varepsilon}} \frac{\partial U}{\partial \eta} ds = \sin\theta_1 |T| + \sin\theta_2 |T|.$$

By integration we can conclude that

$$U = \frac{|T|}{2\pi}(\sin\theta_1 + \sin\theta_2)\ln(r/s_\varepsilon) + c_0 + w_\varphi + v^\perp \quad \text{on } D_{2s_\varepsilon} - D_{s_\varepsilon}, \text{ with } v^\perp \perp 1.$$

We observe that if $\theta_2 = \theta_1 = 0$, there exists an infinite family of minimal surfaces that are close to the surface $\Sigma - D_{s_\varepsilon}$. Let $S_m(\varphi)$ be one such surface. It can be seen as the graph about $D_{2s_\varepsilon} - D_{s_\varepsilon}$ of the function

$$\overline{U}_m(r,\theta) = c_0 + \tilde{H}_{s_\varepsilon,\varphi}(r,\theta) + \overline{V}_m(r,\theta),$$

where $V_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon)$, and it satisfies

$$\|\overline{V}_m(\varphi_1) - \overline{V}_m(\varphi_2)\|_{\mathcal{C}^{2,\alpha}(D_{2s_\varepsilon} - D_{s_\varepsilon})} \leqslant c_\kappa \varepsilon \|\varphi_1 - \varphi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)} \tag{20}$$

for $\varphi_2, \varphi_1 \in C^{2,\alpha}(\mathbb{S}^1)$.

If $(\theta_2, \theta_1) \neq 0$, we choose $|T|$ so that $(|T|/2\pi)(\sin\theta_1 + \sin\theta_2) = 1$. There exists an infinite family of minimal surfaces that are close to the periodic Scherk-type example. After a vertical translation, any such surface can be seen as the graph about $D_{2s_\varepsilon} - D_{s_\varepsilon}$ of the function

$$\overline{U}_t(r,\theta) = \ln(2r) + c_0 + \tilde{H}_{s_\varepsilon,\varphi}(r,\theta) + \overline{V}_t(r,\theta) \tag{21}$$

where $\overline{V}_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon)$, and it satisfies

$$\|\overline{V}_t(\varphi_1) - \overline{V}_t(\varphi_2)\|_{\mathcal{C}^{2,\alpha}(D_{2s_\varepsilon} - D_{s_\varepsilon})} \leqslant c_\kappa \varepsilon \|\varphi_1 - \varphi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}, \tag{22}$$

for $\varphi_2, \varphi_1 \in C^{2,\alpha}(\mathbb{S}^1)$.

Remark 4.6. If the boundary data φ is an even function, it is clear the surfaces we have just described are symmetric across the vertical plane $\{x_2 = 0\}$. However, if the boundary data φ is an odd function and $\theta_1 = \theta_2$, the surfaces are symmetric across the plane $\{x_1 = 0\}$.

5. KMR examples

Here we briefly present the *KMR examples* $M_{\sigma,\alpha,\beta}$ studied in [Karcher 1988; 1989; Meeks and Rosenberg 1989; Rodríguez 2007] — these are also called *toroidal half-plane layers* — which are the only properly embedded, doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient; see [Pérez et al. 2005].

For each $\sigma \in (0, \pi/2)$, $\alpha \in [0, \pi/2]$ and $\beta \in [0, \pi/2]$ with $(\alpha, \beta) \neq (0, \sigma)$, consider the rectangular torus $\Sigma_\sigma = \{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z^2 + \lambda^2)(z^2 + \lambda^{-2})\}$, where $\lambda = \lambda(\sigma) = \cot(\sigma/2) > 1$. By means of the Weierstrass representation, the KMR example $M_{\sigma,\alpha,\beta}$ is determined by its Gauss map g and the differential of its height function h, which are defined on Σ_σ and given by

$$g(z, w) = \frac{az+b}{i(\bar{a}-\bar{b}z)} \quad \text{and} \quad dh = \mu \frac{dz}{w},$$

with

$$a = a(\alpha, \beta) = \cos\tfrac{1}{2}(\alpha+\beta) + i\cos\tfrac{1}{2}(\alpha-\beta),$$
$$b = b(\alpha, \beta) = \sin\tfrac{1}{2}(\alpha-\beta) + i\sin\tfrac{1}{2}(\alpha+\beta), \qquad \mu = \mu(\sigma) = \frac{\pi\csc\sigma}{\mathcal{K}(\sin^2\sigma)},$$

where $\mathcal{K}(m) = \int_0^{\pi/2} 1/(1 - m\sin^2 u)^{1/2} du$ for $0 < m < 1$ is the complete elliptic integral of first kind. Such μ has been chosen so that the vertical part of the flux of $M_{\sigma,\alpha,\beta}$ along any horizontal level section equals 2π.

Remark 5.1. These statements give us a better understanding of the geometrical meaning of a and b:

(i) $b \to 0$ if and only if $\alpha \to 0$ and $\beta \to 0$, in which case $a \to 1 + i$.

(ii) $|b|^2 + |a|^2 = 2$.

(iii) If $\beta = 0$, then $a = (1 + i)\cos(\alpha/2)$ and $b = (1 + i)\sin(\alpha/2)$, and $b = \mathcal{O}(\alpha)$.

(iv) If $\alpha = 0$, then $a = (1 + i)\cos(\beta/2)$ and $b = (-1 + i)\sin(\beta/2)$, and $b = \mathcal{O}(\beta)$.

(v) In general, $|b/a| = \tan(\varphi/2)$, where φ is the angle between the north pole $(0, 0, 1) \in \mathbb{S}^2$ and the pole of g seen in $\mathbb{S}^2$ via the inverse of the stereographic projection.

The ends of $M_{\sigma,\alpha,\beta}$ correspond to the punctures $\{A, A', A'', A'''\} = g^{-1}(\{0, \infty\})$, and the branch values of g are those with $w = 0$, that is,

$$D = (-i\lambda, 0), \quad D' = (i\lambda, 0), \quad D'' = (i/\lambda, 0), \quad D''' = (-i/\lambda, 0). \tag{23}$$

Seen in $\mathbb{S}^2$, these points form two pairs $D'' = -D$ and $D''' = -D'$ of antipodal points, and each KMR example can be given in terms of the branch values of its Gauss map; see [Rodríguez 2007].

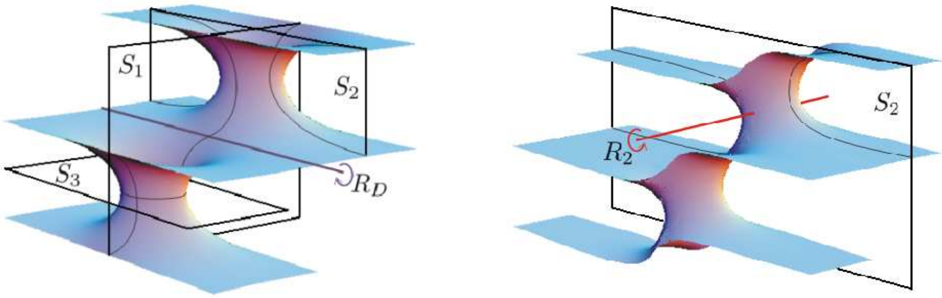

Figure 3. Left: $M_{\sigma,0,0}$, with $\sigma = \pi/4$. Right: $M_{\sigma,\alpha,0}$, with $\sigma = \alpha = \pi/4$.

Denote by T the period of $M_{\sigma,\alpha,\beta}$ at its ends. We focus on two more symmetric subfamilies of KMR examples:

$$\{M_{\sigma,\alpha,0} \mid 0 < \sigma < \tfrac{1}{2}\pi,\ 0 \leqslant \alpha \leqslant \tfrac{1}{2}\pi\} \quad \text{and} \quad \{M_{\sigma,0,\beta} \mid 0 < \sigma < \tfrac{1}{2}\pi,\ 0 \leqslant \beta < \sigma\}.$$

(1) When $\alpha = \beta = 0$, $M_{\sigma,0,0}$ contains four straight lines parallel to the x_1 axis. The isometry group of $M_{\sigma,0,0}$ is generated by the π-rotation R_D around one of the four straight lines contained in the surface, and by three reflection symmetries S_1, S_2, S_3, where each S_i is across the $\{x_i = 0\}$ plane; see Figure 3 left. In this case, $T = (0, \pi\mu, 0)$.

(2) When $0 < \alpha < \pi/2$, the isometry group of $M_{\sigma,\alpha,0}$ is generated by $\mathfrak{D}$ (corresponding to the deck transformation $(z, w) \mapsto (z, -w)$), which represents in $\mathbb{R}^3$ a central symmetry about any of the four branch points of the Gauss map of $M_{\sigma,\alpha,0}$; the reflection S_2 across the $\{x_2 = 0\}$ plane; and the π-rotation R_2 around a line parallel to the x_2 axis that cuts $M_{\sigma,\alpha,0}$ orthogonally; see Figure 3 right. Now $T = (0, \pi\mu t_\alpha, 0)$, with $t_\alpha = \sin\sigma/(\sin^2\sigma\cos^2\alpha + \sin^2\alpha)^{1/2}$.

(3) Suppose that $0 < \beta < \sigma$. Then $M_{\sigma,0,\beta}$ contains four straight lines parallel to the x_1 axis, and the isometry group of $M_{\sigma,0,\beta}$ is generated by the reflection S_1 across the $\{x_1 = 0\}$ plane; the π-rotation R_1 around a line parallel to the x_1 axis that cuts the surface orthogonally; and the π-rotation R_D around any one of the straight lines contained in the surface; see Figure 4. Moreover, $T = (0, \pi\mu t^\beta, 0)$, where $t^\beta = \sin\sigma/(\sin^2\sigma - \sin^2\beta)^{1/2}$.

Finally, it will be useful to see Σ_σ as a branched 2-covering of $\overline{\mathbb{C}}$ through the map $(z, w) \mapsto z$. Thus Σ_σ can be seen as two copies $\overline{\mathbb{C}}_1$ and $\overline{\mathbb{C}}_2$ of $\overline{\mathbb{C}}$ glued along two common cuts γ_1 and γ_2, which can be taken along the imaginary axis: γ_1 from D to D', and γ_2 from D'' to D'''.

5.1. ***$M_{\sigma,\alpha,\beta}$ as a graph over $\{x_3 = 0\}/T$.*** The KMR examples $M_{\sigma,\alpha,\beta}$ converge as $(\sigma, \alpha, \beta) \to (0, 0, 0)$ to a vertical catenoid, since Σ_σ converges to two pinched

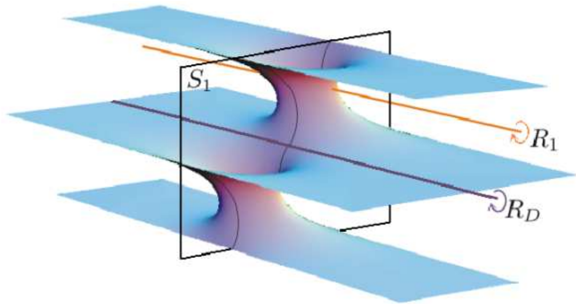

Figure 4. $M_{\sigma,0,\beta}$, where $\sigma = \pi/4$ and $\beta = \pi/8$.

spheres, $g(z) \to z$ and $dh \to \pm dz/z$ as $(\sigma, \alpha, \beta) \to (0, 0, 0)$. In fact, we can obtain two catenoids in the limit, depending on the choice of branch for w (for each copy of $\overline{\mathbb{C}}$ in Σ_σ, we obtain one catenoid in the limit). One of our aims for this paper is to take KMR examples $M_{\sigma,\alpha,0}$ or $M_{\sigma,0,\beta}$ near this catenoidal limit and glue them to a convenient compact piece of the surface $M_k(\varepsilon/2)$. In this subsection, we express part of $M_{\sigma,\alpha,\beta}$ as a vertical graph over the $\{x_3 = 0\}$ plane when σ, α, β are small.

Consider $M_{\sigma,\alpha,\beta}$ near the catenoidal limit, that is, σ, α, β close to zero. Without lost of generality, we can assume $dh \sim -dz/z$ in $\overline{\mathbb{C}}_1$. We are studying the surface in an annulus about one of its ends, say a zero of its Gauss map.

Lemma 5.2. *Consider $\alpha + \beta + \sigma \leqslant \varepsilon$ small. Up to translations, $M_{\sigma,\alpha,\beta}$ can be parameterized in the annulus* $\{(z, w) \in \Sigma_\sigma \mid z \in \overline{\mathbb{C}}_1,\ |b/a| < |z| < \nu\}$, *for* $\nu \in (|b/a|, 1)$ *small, by*

$$X_1 + iX_2 = \tfrac{1}{2}\,(z + 1/\bar{z}) + \frac{(1+i)\bar{b}}{4\bar{z}^2} + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}),$$
$$X_3 = -\ln|z| + \mathcal{O}(\varepsilon^2 z^{-2}),$$

Proof. Recall we have assumed $dh \sim -dz/z$ in the annulus we are working on. More precisely, we have

$$dh = -\frac{\mu\, dz}{\sqrt{(z^2+\lambda^2)(z^2+\lambda^{-2})}} = -\frac{\mu}{\lambda\sqrt{1+\lambda^{-2}z^2+\lambda^{-2}z^{-2}+\lambda^{-4}}}\,\frac{dz}{z}.$$

Since $\mu/\lambda = \pi/((1+\cos(\sigma))\mathcal{K}(\sin^2\sigma)) = 1 + \mathcal{O}(\sigma^4)$, and $\lambda^{-1} = \tan(\sigma/2) = \mathcal{O}(\varepsilon)$, we get

$$dh = -\frac{dz}{z}(1 + \mathcal{O}(\varepsilon^4))(1 + \mathcal{O}(\varepsilon^2 z^2 + \varepsilon^2 z^{-2} + \varepsilon^4)).$$

Since $|z| < \nu < 1$, we have $dh = -(dz/z)(1 + \mathcal{O}(\varepsilon^2 z^{-2}))$. Fix any point $z_0 \in \mathbb{C}_1$, with $z_0 \notin \{-b/a, \bar{a}/\bar{b}\}$ (which correspond to two ends of the KMR example), and recall that $g = -i(az + b)/(\bar{a} - \bar{b}z)$. Straightforward computations give, for

$|b/a| < |z| < 1$,

$$\int_{z_0}^{z} \frac{dh}{g} = \frac{i\bar{b}}{a} \ln z + \frac{2i}{a^2 z} - \frac{2ib}{a^3 z^2} - C_1 + \mathbb{O}(\varepsilon^2 z^{-3}),$$
$$\int_{z_0}^{z} g\, dh = \frac{ib}{\bar{a}} \ln z + \frac{2i}{\bar{a}^2} z - C_2 + \mathbb{O}(\varepsilon^2 z^{-1}),$$

where $C_1, C_2 \in \mathbb{C}$ satisfy $\frac{1}{2}(\overline{C_1} - C_2) = \frac{1}{2}(z_0 + 1/\overline{z_0}) + \mathbb{O}(\varepsilon)$. Taking into account that $a = (1+i) + \mathbb{O}(\varepsilon)$, we obtain

$$\begin{aligned} X_1 + iX_2 &= \frac{1}{2}\Big(\overline{\int_{z_0}^{z} \frac{dh}{g}} - \int_{z_0}^{z} g\, dh\Big) \\ &= -\frac{i}{\bar{a}^2}\Big(z + \frac{1}{\bar{z}}\Big) - \frac{ib}{\bar{a}} \ln|z| + \frac{i\bar{b}}{\bar{a}^3 \bar{z}^2} - \frac{1}{2}\Big(z_0 + \frac{1}{\overline{z_0}}\Big) + \mathbb{O}(\varepsilon^2 z^{-3}) \\ &= \frac{1}{2}\Big(z + \frac{1}{\bar{z}}\Big) + \frac{(1+i)\bar{b}}{4\bar{z}^2} - \frac{1}{2}\Big(z_0 + \frac{1}{\overline{z_0}}\Big) + \mathbb{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}). \end{aligned}$$

Similarly, $\int_{z_0}^{z} dh = -\ln z + \ln z_0 + \mathbb{O}(\varepsilon^2 z^{-2})$; hence

$$X_3 = \operatorname{Re} \int_{z_0}^{z} dh = -\ln|z| + \ln|z_0| + \mathbb{O}(\varepsilon^2 z^{-2}). \qquad \square$$

By suitably translating $M_{\sigma,\alpha,\beta}$, we can assume its coordinate functions are as in Lemma 5.2.

Lemma 5.3. *Let (r, θ) denote the polar coordinates in the $\{x_3 = 0\}$ plane, and let $r_\varepsilon = 1/(2\sqrt{\varepsilon})$. If $\alpha + \beta + \sigma \leqslant \varepsilon$ small, then an annular piece of $M_{\sigma,\alpha,\beta}$ can be written as a vertical graph of the function*

$$\widetilde{U}(r,\theta) = \ln(2r) + r(-\kappa_1 \cos\theta + \kappa_2 \sin\theta) + \mathbb{O}(\varepsilon),$$

for $(r,\theta) \in (r_\varepsilon/2, 2r_\varepsilon) \times [0, 2\pi)$, where $\kappa_1 = \operatorname{Re}(b) + \operatorname{Im}(b)$ and $\kappa_2 = \operatorname{Re}(b) - \operatorname{Im}(b)$.

We denote by $M_{\sigma,\alpha,\beta}(\gamma, \xi)$ the KMR example $M_{\sigma,\alpha,\beta}$ dilated by $1+\gamma$ for some small $\gamma \leqslant 0$, and translated by a vector $\xi = (\xi_1, \xi_2, \xi_3)$. Then an annular piece of $M_{\sigma,\alpha,\beta}(\gamma, \xi)$ can be written as a vertical graph of

$$\widetilde{U}_{\gamma,\xi}(r,\theta) =$$
$$(1+\gamma)\ln(2r) + r\,(-\kappa_1 \cos\theta + \kappa_2 \sin\theta) - \frac{1+\gamma}{r}(\xi_1 \cos\theta + \xi_2 \sin\theta) + d + \mathbb{O}(\varepsilon),$$

for $(r,\theta) \in (r_\varepsilon/2, 2r_\varepsilon) \times [0, 2\pi)$, where $d = \xi_3 - (1+\gamma)\ln(1+\gamma)$.

Remark 5.4. Recall that $b = \sin\frac{1}{2}(\alpha - \beta) + i \sin\frac{1}{2}(\alpha + \beta)$. Here are some special cases:

- When $\beta = 0$, we have $\kappa_1 = 2\sin\frac{1}{2}\alpha$ and $\kappa_2 = 0$, so

$$\widetilde{U}_{\gamma,\xi}(r,\theta) = (1+\gamma)\ln(2r) - 2r\sin\tfrac{1}{2}(\alpha)\cos\theta - \frac{1+\gamma}{r}(\xi_1\cos\theta + \xi_2\sin\theta) + d + \mathcal{O}(\varepsilon).$$

- When $\alpha = 0$, we have $\kappa_1 = 0$ and $\kappa_2 = 2\sin\frac{1}{2}(\beta)$, so

$$\widetilde{U}_{\gamma,\xi}(r,\theta) = (1+\gamma)\ln(2r) + 2r\sin\tfrac{1}{2}\beta\sin\theta - \frac{1+\gamma}{r}(\xi_1\cos\theta + \xi_2\sin\theta) + d + \mathcal{O}(\varepsilon).$$

In Section 7, we will consider $\xi_1 = 0$ when $\alpha = 0$, and $\xi_2 = 0$ when $\beta = 0$.

Proof. Suppose $|b/a| < |z| < \nu$, with $\nu > |b/a|$ small. From Lemma 5.2, we know the coordinate functions (X_1, X_2, X_3) of the perturbed KMR example $M_{\sigma,\alpha,\beta}(\gamma,\xi)$ are given by

$$\begin{aligned} X_1 + iX_2 &= \tfrac{1}{2}(1+\gamma)(z + 1/\bar{z}) + A(z), \\ X_3 &= -(1+\gamma)\ln|z| + \xi_3 + \mathcal{O}(\varepsilon^2 z^{-2}), \end{aligned} \tag{24}$$

where

$$\begin{aligned} A(z) &= \frac{(1+\gamma)(1+i)\bar{b}}{4\bar{z}^2} + (\xi_1 + i\xi_2) + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}) \\ &= \frac{(1+\gamma)(\kappa_1 + i\kappa_2)}{4\bar{z}^2} + (\xi_1 + i\xi_2) + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}). \end{aligned}$$

If we set $z = |z|e^{i\psi}$ and $X_1 + iX_2 = re^{i\theta}$, then $z + 1/\bar{z} = (|z| + 1/|z|)\,e^{i\psi}$ and

$$\begin{aligned} r\cos\theta &= \tfrac{1}{2}(1+\gamma)\,(|z| + 1/|z|)\cos\psi + A_1, \\ r\sin\theta &= \tfrac{1}{2}(1+\gamma)\,(|z| + 1/|z|)\sin\psi + A_2, \end{aligned}$$

where $A_1 = \mathrm{Re}(A)$ and $A_2 = \mathrm{Im}(A)$. Therefore,

$$r^2 = \tfrac{1}{4}(1+\gamma)^2\Big(|z| + \frac{1}{|z|}\Big)^2\bigg(1 + \frac{4|z|}{(1+\gamma)(|z|^2+1)}(A_1\cos\psi + A_2\sin\psi) + \frac{4|z|^2}{(1+\gamma)^2(|z|^2+1)^2}(A_1^2 + A_2^2)\bigg). \tag{25}$$

When $\sqrt{\varepsilon}/R \leqslant |z| \leqslant R\sqrt{\varepsilon}$ for some $R > 0$, the functions A_i are bounded, and we get

$$r = \tfrac{1}{2}(1+\gamma)\Big(|z| + \frac{1}{|z|}\Big)(1 + \mathcal{O}(\sqrt{\varepsilon})) = \frac{1+\gamma}{2|z|} + \mathcal{O}(\sqrt{\varepsilon}). \tag{26}$$

In particular, $r = \mathcal{O}(1/\sqrt{\varepsilon})$. We consider $R > 0$ large enough so that

$$\{r_\varepsilon/2 \leqslant r \leqslant 2r_\varepsilon\} \subset \{\sqrt{\varepsilon}/R \leqslant |z| \leqslant R\sqrt{\varepsilon}\}.$$

From (26), we get $r\big/\big(\frac{1}{2}(1+\gamma)\,(|z| + 1/|z|)\big) = 1 + \mathcal{O}(\sqrt{\varepsilon})$, which gives

$$\frac{X_1 + iX_2}{\frac{1}{2}(1+\gamma)\,(|z| + 1/|z|)} = e^{i\theta}(1 + \mathcal{O}(\sqrt{\varepsilon})).$$

On the other hand,

$$\frac{X_1 + i X_2}{\frac{1}{2}(1+\gamma)\,(|z| + 1/|z|)} = e^{i\psi} + \frac{2|z|A}{(1+\gamma)(1+|z|^2)} = e^{i\psi} + \mathbb{O}(\sqrt{\varepsilon}).$$

Thus $e^{i\psi} = e^{i\theta}(1 + \mathbb{O}(\sqrt{\varepsilon}))$.

From (25) and (26) we can deduce

$$\frac{(1+\gamma)^2(1+|z|^2)^2}{4|z|^2} = r^2\big(1 - (2/r)\,(A_1 \cos\psi + A_2 \sin\psi) + \mathbb{O}(\varepsilon)\big),$$

from which we obtain

$$\begin{aligned}\frac{1}{|z|^2} &= \Big(\frac{2r}{1+\gamma}\Big)^2\big(1 - (2/r)\,(A_1\cos\psi + A_2 \sin\psi) + \mathbb{O}(\varepsilon)\big)(1 + \mathbb{O}(\varepsilon))\\ &= \Big(\frac{2r}{1+\gamma}\Big)^2\big(1 - (2/r)\,(A_1\cos\psi + A_2\sin\psi) + \mathbb{O}(\varepsilon)\big),\end{aligned}$$

and then

$$-\ln|z| = \ln\frac{2r}{1+\gamma} - \frac{1}{r}(A_1\cos\psi + A_2\sin\psi) + \mathbb{O}(\varepsilon). \tag{27}$$

Finally, it is not difficult to prove that

$$\begin{aligned}A_1 &= \frac{1+\gamma}{4|z|^2}(\kappa_1\cos(2\psi) - \kappa_2\sin(2\psi)) + \xi_1 + \mathbb{O}(\sqrt{\varepsilon}),\\ A_2 &= \frac{1+\gamma}{4|z|^2}(\kappa_1\sin(2\psi) + \kappa_2\cos(2\psi)) + \xi_2 + \mathbb{O}(\sqrt{\varepsilon}).\end{aligned}$$

Therefore,

$$\begin{aligned}A_1\cos\psi + A_2\sin\psi &= \frac{1+\gamma}{4|z|^2}\,(\kappa_1\cos\psi - \kappa_2\sin\psi) + \xi_1\cos\psi + \xi_2\sin\psi + \mathbb{O}(\sqrt{\varepsilon})\\ &= \frac{r^2}{1+\gamma}\,(\kappa_1\cos\theta - \kappa_2\sin\theta) + \xi_1\cos\theta + \xi_2\sin\theta + \mathbb{O}(\sqrt{\varepsilon}).\end{aligned}$$

From here, (27) and (24), Lemma 5.3 follows. □

5.2. ***Parameterization of the KMR example on the cylinder.*** In this subsection we want to parameterize the KMR example $M_{\sigma,\alpha,\beta}$ on a cylinder. Recall its conformal compactification Σ_σ only depends on σ. The parameter $\sigma \in (0, \pi/2)$ will remain fixed along this subsection, and we will omit the dependence on σ of the functions we are introducing. Also recall that Σ_σ can be seen as a branched 2-covering of $\overline{\mathbb{C}}$ by gluing $\overline{\mathbb{C}}_1$, $\overline{\mathbb{C}}_2$ along two common cuts γ_1 and γ_2 along the imaginary axis joining the branch points D, D' and D'', D''', respectively; see (23).

We introduce the spheroconal coordinates (x, y) on the annulus $\mathbb{S}^2 - (\gamma_1 \cup \gamma_2)$ as in [Jansen 1977]: For any $(x, y) \in \mathbb{S}^1 \times (0, \pi) \equiv [0, 2\pi) \times (0, \pi)$, we define

$$F(x, y) = (\cos x \sin y, \sin x\, m(y), l(x)\, \cos y) \in \mathbb{S}^2 - (\gamma_1 \cup \gamma_2),$$

where

$$m(y) = (1 - \cos^2\sigma \cos^2 y)^{1/2} \quad \text{and} \quad l(x) = (1 - \sin^2\sigma \sin^2 x)^{1/2}.$$

Geometrically, $\{x = \text{const}\}$ and $\{y = \text{const}\}$ correspond to two closed curves on $\mathbb{S}^2$ that are the intersection of the sphere with two elliptic cones (one with horizontal axis, the other one with vertical axis) having as vertex the center of the sphere.

If we compose $F(x, y)$ with the stereographic projection and enlarge the domain of definition of the function, we obtain a differentiable map $\mathbf{z}$ defined on the torus $\mathbb{S}^1 \times \mathbb{S}^1 \equiv [0, 2\pi) \times [0, 2\pi) \to \overline{\mathbb{C}}$ and given by

$$\mathbf{z}(x, y) = \frac{\cos x \sin y + i \sin x\, m(y)}{1 - l(x) \cos y}, \tag{28}$$

which is a branch 2-covering of $\overline{\mathbb{C}}$ with branch values in the four points whose spheroconal coordinates are $(x, y) \in \{\pm\pi/2\} \times \{0, \pi\}$; these correspond to D, D', D'' and D'''. Moreover, $\mathbf{z}$ maps $\mathbb{S}^1 \times (0, \pi)$ onto $\overline{\mathbb{C}} - (\gamma_1 \cup \gamma_2)$. Hence we can parameterize the KMR example by $\mathbf{z}$, via its Weierstrass data

$$g(\mathbf{z}) = \frac{a\mathbf{z} + b}{i(\bar{a} - \bar{b}\mathbf{z})}, \qquad dh = \mu \frac{d\mathbf{z}}{\sqrt{(\mathbf{z}^2 + \lambda^2)(\mathbf{z}^2 + \lambda^{-2})}},$$

We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{R}^3/T$ by forgetting its nonhorizontal period (that is, its period in homology, $\widetilde{T}$). We can then parameterize $\widetilde{M}_{\sigma,\alpha,\beta}$ on $\mathbb{S}^1 \times \mathbb{R}$ by extending $\mathbf{z}$ to $[0, 2\pi) \times \mathbb{R}$ periodically. But such a parameterization is not conformal, since the spheroconal coordinates $(x, y) \mapsto F(x, y)$ of the sphere are not conformal. As the stereographic projection is a conformal map, it suffices to find new conformal coordinates (u, v) of the sphere defined on the cylinder. In particular, we look for a change of variables $(x, y) \mapsto (u, v)$ for which $|\widetilde{F}_u| = |\widetilde{F}_v|$ and $\langle \widetilde{F}_u, \widetilde{F}_v \rangle = 0$, where $\widetilde{F}(u, v) = F(x(u, v), y(u, v))$.

We observe that

$$|F_x| = \sqrt{k(x, y)}/l(x) \quad \text{and} \quad |F_y| = \sqrt{k(x, y)}/m(y),$$

with $k(x, y) = \sin^2\sigma \cos^2 x + \cos^2\sigma \sin^2 y$. Then it is natural to consider the change of variables $(x, y) \in [0, 2\pi) \times \mathbb{R} \mapsto (u, v) \in [0, U_\sigma) \times \mathbb{R}$ defined by

$$u(x) = \int_0^x \frac{dt}{l(t)} \quad \text{and} \quad v(y) = \int_{\pi/2}^y \frac{dt}{m(t)}, \tag{29}$$

where

$$U_\sigma = u(2\pi) = \int_0^{2\pi} \frac{dt}{\sqrt{1 - \sin^2\sigma \sin^2 t}}.$$

Note that U_σ is a function on σ that goes to 2π as σ approaches to zero, and that the change of variables above is well defined because $\sigma \in (0, \pi/2)$.

In these variables (u, v), $\mathbf{z}$ is v-periodic with period

$$V_\sigma = v(2\pi) - v(0) = \int_0^{2\pi} \frac{dt}{\sqrt{1-\cos^2\sigma\,\cos^2 t}}.$$

The period V_σ goes to $+\infty$ as σ goes to zero (see the proof of Lemma 5.5), which is made clear by taking into account the limits of $M_{\sigma,\alpha,\beta}$ as σ tends to zero.

From all this, we can deduce that $\widetilde{M}_{\sigma,\alpha,\beta}$ (respectively $M_{\sigma,\alpha,\beta}$) is conformally parameterized on $(u, v) \in I_\sigma \times \mathbb{R}$ (respectively $(u, v) \in I_\sigma \times J_\sigma$), where $I_\sigma = [0, U_\sigma]$ and $J_\sigma = [v(0), v(2\pi)]$. In Section 6, which is devoted to the study of the mapping properties of the Jacobi operator of $\widetilde{M}_{\sigma,\alpha,\beta}$, we will use the (u, v) variables.

In Lemma 5.3, an appropriate piece of $\widetilde{M}_{\sigma,\alpha,\beta}$ has been written as a vertical graph over the annulus $\{r_\varepsilon/2 \leqslant r \leqslant 2r_\varepsilon\} \subset \{x_3 = 0\}$. The boundary curve of $\widetilde{M}_{\sigma,\alpha,\beta}$ along which we will glue a piece of the Costa–Hoffman–Meeks surface corresponds to $\{r = r_\varepsilon\}$. Equation (26) says that if r is near r_ε, then z is in a neighborhood of $\{|\mathbf{z}| = \sqrt{\varepsilon}\}$. Next lemma gives us the values of v corresponding to such a neighborhood.

Lemma 5.5. *Consider $\sigma \leqslant \varepsilon$. If $\sqrt{\varepsilon}/R \leqslant |\mathbf{z}| \leqslant R\sqrt{\varepsilon}$, for $R > 0$, then*

$$-\tfrac{1}{2}\ln\varepsilon + c_1 \leqslant v \leqslant -\tfrac{1}{2}\ln\varepsilon + c_2,$$

where c_1 and c_2 are constant. Under the same assumptions, $V_\sigma = -4\ln\varepsilon + \mathbb{O}(1)$.

Proof. Using Equation (28), we can show that, if $\sqrt{\varepsilon}/R \leqslant |\mathbf{z}(x, y)| \leqslant R\sqrt{\varepsilon}$, then $\pi - d_1\sqrt{\varepsilon} \leqslant y \leqslant \pi - d_2\sqrt{\varepsilon}$, where $d_1 > d_2 > 0$ are constant. This means, since v is increasing function of y, that $v(\pi - d_1\sqrt{\varepsilon}) \leqslant v(y) \leqslant v(\pi - d_2\sqrt{\varepsilon})$. Let us compute $v(\pi - d_i\sqrt{\varepsilon})$ for $i = 1, 2$. We have

$$\begin{aligned} v(y) - v(0) &= \int_0^y \frac{ds}{\sqrt{1-\cos^2\sigma\,\cos^2 s}} = \int_0^y \frac{ds}{\sqrt{1-\cos^2\sigma+\cos^2\sigma\,\sin^2 s}} \\ &= \frac{1}{\sin\sigma}\int_0^y \frac{ds}{\sqrt{1+\cot^2\sigma\,\sin^2 s}} = \frac{1}{\sin\sigma}\mathcal{F}(y, -\cot^2\sigma), \end{aligned}$$

where $\mathcal{F}(y, m) = \int_0^y (1 - m\sin^2 s)^{-1/2} ds$ is the incomplete elliptic integral of first kind. $\mathcal{F}(y, m)$ is an odd function in y and, if $k \in \mathbb{Z}$,

$$\mathcal{F}(y + k\pi, m) = \mathcal{F}(y, m) + 2k\,\mathcal{K}(m),$$

where $\mathcal{K}(m) = \mathcal{F}(\pi/2, m)$ is the complete elliptic integral of first kind. Since $\sigma = \mathbb{O}(\varepsilon)$, we have

$$\frac{1}{\sin\sigma}\mathcal{F}(d\sqrt{\varepsilon}, -\cot^2\sigma) = -\tfrac{1}{2}\ln\varepsilon + \mathbb{O}(1).$$

On the other hand, if $|m|$ is sufficiently big, then

$$\mathcal{K}(m) = \frac{1}{\sqrt{-m}}\left(\ln 4 + \tfrac{1}{2}\ln(-m)\right)\left(1 + \mathbb{O}(1/m)\right).$$

It follows that

$$\frac{1}{\sin\sigma}\mathcal{K}(-\cot^2\sigma) = -\ln\sigma + \ln 4 + \mathbb{O}(\sigma^2) = -\ln\varepsilon + \mathbb{O}(1).$$

Then, for $i = 1, 2$,

$$\begin{aligned} v(\pi - d_i\sqrt{\varepsilon}) &= \frac{1}{\sin\sigma}\left(\mathcal{F}(\pi - d_i\sqrt{\varepsilon}, -\cot^2\sigma) - \mathcal{K}(-\cot^2\sigma)\right) \\ &= \frac{1}{\sin\sigma}\left(\mathcal{F}(-d_i\sqrt{\varepsilon}, -\cot^2\sigma) + 2\mathcal{K}(-\cot^2\sigma) - \mathcal{K}(-\cot^2\sigma)\right) \\ &= \frac{1}{\sin\sigma}\left(-\mathcal{F}(d_i\sqrt{\varepsilon}, -\cot^2\sigma) + \mathcal{K}(-\cot^2\sigma)\right) = -\tfrac{1}{2}\ln\varepsilon + \mathbb{O}(1). \end{aligned}$$

Hence there exist constants c_1 and c_2 such that $v(\pi - d_1\sqrt{\varepsilon}) \geqslant -\frac{1}{2}\ln\varepsilon + c_1$ and $v(\pi - d_2\sqrt{\varepsilon}) \leqslant -\frac{1}{2}\ln\varepsilon + c_2$.

The result concerning $V_\sigma = v(2\pi) - v(0)$ follows once it is observed that $v(2\pi) = (3/\sin\sigma)\mathcal{K}(-\cot^2\sigma)$ and $v(0) = -(1/\sin\sigma)\mathcal{K}(-\cot^2\sigma)$. □

From Lemma 5.5 it follows that the value of the v corresponding to $|\mathbf{z}| = \sqrt{\varepsilon}$ is $v_\varepsilon = -\frac{1}{2}\ln\varepsilon + \mathbb{O}(1)$.

6. The Jacobi operator about KMR examples

The Jacobi operator for $M_{\sigma,\alpha,\beta}$ is given by $\mathcal{J} = \Delta_{ds^2} + |A|^2$, where $|A|^2$ is the squared norm of the second fundamental form on $M_{\sigma,\alpha,\beta}$ and Δ_{ds^2} is the Laplace–Beltrami operator with respect to the metric $ds^2 = \frac{1}{4}(|g| + |g|^{-1})^2|dh|^2$ on the surface. We consider the metric on the torus Σ_σ obtained as pull-back of the standard metric ds_0^2 on the sphere $\mathbb{S}^2$ by the Gauss map $N : M_{\sigma,\alpha,\beta} \to \mathbb{S}^2$; that is, $dN^*(ds_0^2) = -K\,ds^2$, where $K = -\frac{1}{2}|A|^2$ denotes the Gauss curvature of $M_{\sigma,\alpha,\beta}$. Hence $\Delta_{ds^2} = -K\,\Delta_{ds_0^2}$, and so $\mathcal{J} = -K(\Delta_{ds_0^2} + 2)$. From [Jansen 1977] and taking into account the parameterization of $M_{\sigma,\alpha,\beta}$ on the cylinder given in Section 5.2, we can deduce that, in the (x, y) variables,

$$\Delta_{ds_0^2} := \frac{l(x)m(y)}{k(x, y)}\left(\partial_x\left(\frac{l(x)}{m(y)}\partial_x\right) + \partial_y\left(\frac{m(y)}{l(x)}\partial_y\right)\right).$$

Recall $k(x, y) = \sin^2\sigma\cos^2 x + \cos^2\sigma\sin^2 y$. In the (u, v) variables defined by (29), we have $\mathcal{J} = -(K/k(u, v))\mathcal{L}_\sigma$, where $k(u, v) = k(x(u), y(v))$ and

$$\mathcal{L}_\sigma := \partial^2_{uu} + \partial^2_{vv} + 2k(u, v) \tag{30}$$

is the Lamé operator [Jansen 1977].

Remark 6.1. In Proposition 6.5, we will take $\sigma \to 0$. For such a limit, the torus Σ_σ degenerates into a Riemann surface with nodes consisting of two spheres joined at two common points p_0 and p_1, and the corresponding Jacobi operator equals the Legendre operator on $\mathbb{S}^2 - \{p_0, p_1\}$ given by $\mathcal{L}_0 = \partial^2_{xx} + \sin y\, \partial_y(\sin y\, \partial_y) + 2\sin^2 y$ in the (x, y) variables. When $\sigma = 0$, the change of variables $(x, y) \mapsto (u, v)$ given in (29) is not well defined.

The mapping properties of the Jacobi operator. From now on, we consider the conformal parameterization of $\widetilde{M}_{\sigma,\alpha,\beta}$ on the cylinder $\mathbb{S}^1 \times \mathbb{R} \equiv I_\sigma \times \mathbb{R}$ described in Section 5.2. In this subsection, we study the mapping properties of the operator $\mathcal{J}$. It is clear that it suffices to study the simpler operator $\mathcal{L}_\sigma$ defined by (30), so we will study the problem

$$\begin{cases} \mathcal{L}_\sigma w = f & \text{in } I_\sigma \times [v_0, +\infty[, \\ \quad w = \varphi & \text{on } I_\sigma \times \{v_0\} \end{cases}$$

with $v_0 \in \mathbb{R}$ and consider convenient normed functional spaces for w, f, φ so that the norm of w is bounded by that of f.

We will work in two different functional spaces to solve the Dirichlet problem above. To explain the reason, we recall that the isometry group of $\widetilde{M}_{\sigma,\alpha,\beta}$ depends on the values of the three parameters σ, α, β. When $\beta = 0$, $\widetilde{M}_{\sigma,\alpha,\beta}$ is invariant by reflection about the $\{x_2 = 0\}$ plane; when $\alpha = 0$, it is invariant about the $\{x_1 = 0\}$ plane. We want show there exist families of minimal surfaces close to $\widetilde{M}_{\sigma,\alpha,0}$ and $\widetilde{M}_{\sigma,0,\beta}$ and having the same symmetry properties. Thus the surfaces in the family about $\widetilde{M}_{\sigma,\alpha,0}$ (respectively $\widetilde{M}_{\sigma,0,\beta}$) will be defined as normal graphs of functions defined in $I_\sigma \times \mathbb{R}$ that are even (respectively odd) in the first variable. We will solve the Dirichlet problem above in the first case. The second one follows similarly.

Definition 6.2. Given $\sigma \in (0, \pi/2)$, $\ell \in \mathbb{N}$, $\alpha \in (0, 1)$, $\mu \in \mathbb{R}$, and an interval I, we define $\mathcal{C}^{\ell,\alpha}_\mu(I_\sigma \times I)$ as the space of functions $w = w(u, v)$ in $\mathcal{C}^{\ell,\alpha}_{\mathrm{loc}}(I_\sigma \times I)$ that are even and U_σ-periodic in the variable u and for which the following norm is finite:

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_\mu(I_\sigma \times I)} := \sup_{v \in I} \left(e^{-\mu v} \|w\|_{\mathcal{C}^{\ell,\alpha}(I_\sigma \times [v, v+1])} \right).$$

We observe that the Jacobi operator $\mathcal{L}_\sigma$ becomes a Fredholm operator when restricted to $\mathcal{C}^{2,\alpha}_\mu(I_\sigma \times I)$. Moreover, $\mathcal{L}_\sigma$ has separated variables. Then we consider the operator $L_\sigma = \partial^2_{uu} + 2\sin^2\sigma \cos^2(x(u))$ defined on the space of U_σ-periodic and even functions in I_σ. This operator L_σ is uniformly elliptic and selfadjoint. In particular, L_σ has discrete spectrum $(\lambda_{\sigma,i})_{i \geqslant 0}$, which we assume is arranged so that $\lambda_{\sigma,i} < \lambda_{\sigma,i+1}$ for every i. Each eigenvalue $\lambda_{\sigma,i}$ is simple because we only consider even functions. We denote by $e_{\sigma,i}$ the even eigenfunction associated to $\lambda_{\sigma,i}$ and normalized so that

$$\int_0^{U_\sigma} (e_{\sigma,i}(u))^2\,du = 1.$$

Lemma 6.3. *For every $i \geqslant 0$, the eigenvalue $\lambda_{\sigma,i}$ of the operator L_σ and its associated eigenfunctions $e_{\sigma,i}$ satisfy*

$$-2\sin^2\sigma \leqslant \lambda_{\sigma,i} - i^2 \leqslant 0 \quad \textit{and} \quad \|e_{\sigma,i} - e_{0,i}\|_{\mathscr{C}^2(I_\sigma)} \leqslant c_i \sin^2\sigma,$$

where $e_{0,i}(u) := \cos(i x(u))$ for every $u \in I_\sigma$, and the constant $c_i > 0$ depends only on i (it does not depend on σ).

Proof. The bound for $\lambda_{\sigma,i} - i^2$ comes from the variational characterization of the eigenvalue $\lambda_{\sigma,i}$ as

$$\lambda_{\sigma,i} = \sup_{\operatorname{codim} E = i} \left(\inf_{e\in E,\, \|e\|_{L^2}=1} \int_0^{U_\sigma} \left((\partial_u e)^2 - 2\sin^2\sigma\,\cos^2(x(u))e^2\right)\,du \right),$$

where E is a subset of the space of U_σ-periodic and even functions in $L^2(I_\sigma)$, since it always holds $0 \leqslant 2\sin^2\sigma\,\cos^2(x(u)) \leqslant 2\sin^2\sigma$. The bound for the eigenfunctions follows from standard perturbation theory [Kato 1980]. □

The Hilbert basis $\{e_{\sigma,i}\}_{i\in\mathbb{N}}$ of the space of U_σ-periodic and even functions in $L^2(I_\sigma)$ introduced above induces the Fourier decomposition

$$g(u,v) = \sum_{i\geqslant 0} g_i(v)\, e_{\sigma,i}(u)$$

of functions $g = g(u,v)$ in $L^2(I_\sigma \times \mathbb{R})$ that are U_σ-periodic and even in the variable u. From this, we deduce that the operator $\mathscr{L}_\sigma$ can be decomposed as $\mathscr{L}_\sigma = \sum_{i\geqslant 0} L_{\sigma,i}$, where

$$L_{\sigma,i} = \partial^2_{vv} + 2\cos^2\sigma\,\sin^2(y(v)) - \lambda_{\sigma,i} \quad \text{for every } i \geqslant 0.$$

Since $0 \leqslant 2\cos^2\sigma\,\sin^2(y(v)) \leqslant 2\cos^2\sigma = 2 - 2\sin^2\sigma$, Lemma 6.3 gives us

$$P_{\sigma,i} := 2\cos^2\sigma\,\sin^2(y(v)) - \lambda_{\sigma,i} \leqslant 2 - i^2. \tag{31}$$

This fact allows us to prove the following lemma, which ensures that $\mathscr{L}_\sigma$ is injective when restricted to the set of functions that in the variable u are even and L^2-orthogonal to $e_{\sigma,0}$ and $e_{\sigma,1}$.

Lemma 6.4. *Given $v_0 < v_1$, let w be a solution of $\mathscr{L}_\sigma w = 0$ on $I_\sigma \times [v_0, v_1]$ that is U_σ-periodic and even in the variable u and satisfies*

(i) $w(\,\cdot\,, v_0) = w(\,\cdot\,, v_1) = 0$;

(ii) $\int_0^{U_\sigma} w(u,v)e_{\sigma,i}(u)\,du = 0$ *for every $v \in [v_0, v_1]$ and every $i \in \{0, 1\}$.*

Then $w = 0$.

Proof. By (ii), $w = \sum_{i\geqslant 2} w_i(v)e_{\sigma,i}(u)$. Since the potential $P_{\sigma,i}$ of the operator $L_{\sigma,i}$ is negative for every $i \geqslant 2$ (see (31)) and the operator $L_{\sigma,i}$ is elliptic, the maximum principle holds. We can then conclude that $w = 0$ from (i). □

To study the mapping properties of the Jacobi operator $\mathscr{L}_\sigma$, we need to give a description of the Jacobi fields associated to $M_{\sigma,\alpha,0}$, which are defined as the solutions of $\mathscr{L}_\sigma v = 0$. Since $M_{\sigma,\alpha,0}$ is invariant by reflection across the $\{x_2 = 0\}$ plane, there are only four independent Jacobi fields:

- Two Jacobi fields induced by vertical translations and by horizontal translations in the x_1 direction. These Jacobi fields are clearly periodic and hence bounded.
- A third Jacobi field generated by the 1-parameter group of dilations, which is not bounded (it grows linearly).
- A last Jacobi field obtained by considering the 1-parameter family of minimal surfaces induced by changing the parameter σ. This Jacobi field is not periodic and grows linearly.

The Jacobi fields induced by translation along the x_3 axis and by dilatation are solutions of $\mathscr{L}_\sigma u = 0$ that are collinear to the eigenfunction $e_{\sigma,0}$. The Jacobi fields induced by the horizontal translation and by the variation of the parameter σ are collinear to $e_{\sigma,1}$.

The Jacobi fields of $M_{\sigma,0,\beta}$, which is invariant by reflection across the plane $\{x_1 = 0\}$, are the same as those of $M_{\sigma,\alpha,0}$, with the exception that the one induced by horizontal translations in the x_1 direction is to be replaced by the field induced by horizontal translations in the x_2 direction.

The next proposition states that for an appropriately chosen parameter μ and interval I, there exists a right inverse for $\mathscr{L}_\sigma : \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times I) \to \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times I)$ whose norm is uniformly bounded.

Proposition 6.5. *Given* $\mu \in (-2,-1)$, *there exists a* $\sigma_0 \in (0, \pi/2)$ *such that, for every* $\sigma \in (0, \sigma_0)$ *and* $v_0 \in \mathbb{R}$, *there exists an operator*

$$G_{\sigma,v_0} : \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty)) \to \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$$

such that for every $f \in \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$, *the function* $w := G_{\sigma,v_0}(f)$ *solves*

$$\begin{cases} \mathscr{L}_\sigma w = f & \text{in } I_\sigma \times [v_0, +\infty), \\ \quad w \in \operatorname{Span}\{e_{\sigma,0}, e_{\sigma,1}\} & \text{on } I_\sigma \times \{v_0\}. \end{cases}$$

Moreover $\|w\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant c\|f\|_{\mathscr{C}^{0,\alpha}_\mu}$ *for some constant* $c > 0$ *that depends neither on* $\sigma \in (0, \sigma_0)$ *nor on* $v_0 \in \mathbb{R}$.

Proof. Every $f \in \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$ can be decomposed as

$$f = f_0 e_{\sigma,0} + f_1 e_{\sigma,1} + \bar{f},$$

where $\bar{f}(\,\cdot\,, v)$ is L^2-orthogonal to $e_{\sigma,0}$ and to $e_{\sigma,1}$ for each $v \in [v_0, +\infty)$.

Step 1. First, let's prove Proposition 6.5 for functions $\bar{f} \in \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$ that are L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$. By Lemma 6.4, $\mathscr{L}_\sigma$ acts injectively on such a function space. Hence, the Fredholm alternative ensures that there exists for each $v_1 > v_0 + 1$ a unique $\overline{w} \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_0, v_1])$ in which $\overline{w}(\,\cdot\,, v)$ is L^2-orthogonal to $e_{\sigma,0}, e_{\sigma,1}$ and satisfies

$$\begin{cases} \mathscr{L}_\sigma \overline{w} = \bar{f} & \text{on } I_\sigma \times [v_0, v_1], \\ \overline{w}(\,\cdot\,, v_0) = \overline{w}(\,\cdot\,, v_1) = 0. \end{cases} \tag{32}$$

Claim 6.6. *There exist $c \in \mathbb{R}$ and $\sigma_0 \in (0, \pi/2)$ such that, for every $\sigma \in (0, \sigma_0)$, $v_0 \in \mathbb{R}$, $v_1 > v_0 + 1$ and $\bar{f} \in \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, v_1])$, there exists $\overline{w} \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_0, v_1])$ that is L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$ and satisfies* (32) *and*

$$\sup_{I_\sigma \times [v_0, v_1]} (e^{-\mu v} |\overline{w}|) \leqslant c \sup_{I_\sigma \times [v_0, v_1]} (e^{-\mu v} |\bar{f}|). \tag{33}$$

Proof. Suppose by contradiction that Claim 6.6 is false. Then, for every $n \in \mathbb{N}$ there exists $\sigma_n \in (0, 1/n)$, $v_{1,n} > v_{0,n} + 1$ and $\bar{f}_n, \overline{w}_n$ satisfying (32) (but with $\sigma_n, v_{0,n}, v_{1,n}$ instead of σ, v_0, v_1) such that

$$\sup_{I_{\sigma_n} \times [v_{0,n}, v_{1,n}]} (e^{-\mu v} |\bar{f}_n|) = 1 \qquad \text{and}$$

$$A_n := \sup_{I_{\sigma_n} \times [v_{0,n}, v_{1,n}]} (e^{-\mu v} |\overline{w}_n|) \to +\infty \quad \text{as } n \to \infty.$$

Since $I_{\sigma_n} \times [v_{0,n}, v_{1,n}]$ is a compact set, A_n is achieved at a point (u_n, v_n) in it.

After passing to a subsequence, the intervals $I_n = [v_{0,n} - v_n, v_{1,n} - v_n]$ converge to a set I_∞. Elliptic estimates imply that

$$\sup_{I_{\sigma_n} \times [v_{0,n}, v_{0,n}+1/2]} (e^{-\mu v} |\nabla \overline{w}_n|) \leqslant c \left(\sup_{I_{\sigma_n} \times [v_{0,n}, v_{0,n}+1]} (e^{-\mu v} |\bar{f}_n|) + \sup_{I_{\sigma_n} \times [v_{0,n}, v_{0,n}+1]} (e^{-\mu v} |\overline{w}_n|) \right).$$

Hence the supremum of $(e^{-\mu v} |\nabla \overline{w}_n|)$ over $I_{\sigma_n} \times [v_{0,n}, v_{0,n} + 1/2]$ is $\leqslant c(1 + A_n)$. From this estimate for the gradient of $\overline{w}_n$ near $v = v_{0,n}$, it follows that v_n cannot be too close to $v_{0,n}$, where $\overline{w}_n$ vanishes. More precisely, $v_{0,n} - v_n$ remains bounded away from 0, and then it converges to some $\bar{v}_0 \in [-\infty, 0)$. By similar arguments, it is possible to show that $\nabla \overline{w}_n$ is bounded near $v_{1,n}$, and consequently $v_{1,n} - v_n$ converges to some $\bar{v}_1 \in (0, +\infty]$. Then we can conclude that $I_\infty = [\bar{v}_0, \bar{v}_1]$.

We define

$$\tilde{\omega}_n(u,v) := \frac{e^{-\mu v_n}}{A_n}\overline{\omega}_n(u, v+v_n) \quad \text{for } (u,v) \in I_{\sigma_n} \times I_n.$$

We observe that

$$|\tilde{\omega}_n(u,v)| \leqslant e^{\mu v}\frac{e^{-\mu(v+v_n)}|\overline{\omega}_n(u,v+v_n)|}{A_n} \leqslant e^{\mu v},$$

$$\sup_{I_{\sigma_n}\times I_n} (e^{-\mu v}|\tilde{\omega}_n|) = 1.$$

Using the above estimate for $e^{-\mu v}|\nabla\overline{\omega}_n|$, we obtain

$$|\nabla\tilde{\omega}_n| \leqslant c\,\frac{1+A_n}{A_n}e^{\mu v} < 2c\,e^{\mu v}.$$

Since the sequences $\{\tilde{\omega}_n\}_n$ and $\{\nabla\tilde{\omega}_n\}_n$ are uniformly bounded, the Ascoli–Arzelà theorem ensures that, if $n \to +\infty$, a subsequence of $\{\tilde{\omega}_n\}_n$ converges on compact sets of $I_0 \times I_\infty$ to a function $\tilde{\omega}_\infty$ that vanishes on $I_0 \times \partial I_\infty$ when $\partial I_\infty \neq \varnothing$, and such that $\tilde{\omega}_\infty(\,\cdot\,, v)$ is L^2-orthogonal to $\{e_{0,0}, e_{0,1}\}$ for each $v \in I_\infty$. Moreover,

$$\sup_{I_0\times I_\infty} (e^{-\mu v}|\tilde{\omega}_\infty|) = 1. \tag{34}$$

Since $\sigma_n \to 0$ as $n \to \infty$, we can conclude that $\tilde{\omega}_\infty$ satisfies

$$\begin{cases} \mathscr{L}_0\tilde{\omega}_\infty = 0 & \text{in } I_0 \times I_\infty, \\ \quad\tilde{\omega}_\infty = 0 & \text{on } I_0 \times \partial I_\infty \text{ (if } \partial I_\infty \neq \varnothing). \end{cases}$$

If I_∞ is bounded, the maximum principle allows us to conclude that $\tilde{\omega}_\infty = 0$ on $I_0 \times I_\infty$, which contradicts (34). Hence I_∞ is an unbounded interval.

Recall $\mathscr{L}_0$ is given in terms of the (x, y) variables. The equation $\mathscr{L}_0\tilde{\omega}_\infty = 0$ becomes

$$\partial^2_{xx}\tilde{\omega}_\infty + \sin y\,\partial_y(\sin y\,\partial_y\tilde{\omega}_\infty) + 2\sin^2 y\,\tilde{\omega}_\infty = 0.$$

Now we consider $\tilde{\omega}_\infty$ decomposed into eigenfunctions as

$$\tilde{\omega}_\infty(x,y) = \sum_{j\geqslant 2} a_j(y)\cos(j\,x).$$

Each coefficient a_j with $j \geqslant 2$ must satisfy the associated Legendre differential equation (see Appendix C)

$$\sin y\,\partial_y(\sin y\,\partial_y a_j) - j^2 a_j + 2\sin^2 y\,a_j = 0.$$

We obtain that $a_j(y)$ is the associated Legendre functions of second kind, that is, $a_j(y) = Q_1^j(\cos y)$ for $j \geqslant 2$.

We obtain from (29) that

$$u(x) \to x \quad \text{and} \quad v(y) \to \tfrac{1}{2}\ln|\tan(y/2)| \quad \text{as } \sigma \to 0.$$

In particular, define $y(v) = 2\arctan(e^{2v})$ for $\sigma = 0$. Then

$$\cos(y(v)) = \frac{1-e^{4v}}{1+e^{4v}},$$
$$\tilde{w}_\infty(u,v) = \sum_{j\geqslant 2} Q_1^j\Big(\frac{1-e^{4v}}{1+e^{4v}}\Big)\cos(ju).$$

One can show that $|a_j|$ tends to $+\infty$ as the function $e^{2j|v|}$ does. Since the interval I_∞ is unbounded, we reach a contradiction with (34), proving Claim 6.6. □

Let $c \in \mathbb{R}$ and σ_0 satisfy Claim 6.6. Choose $\sigma \in (0,\sigma_0)$, $v_0 \in \mathbb{R}$ and then an $\bar{f} \in \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0,+\infty))$. Then, for every $v_1 > v_0+1$, there exists a function $\overline{w}$ that is L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$ and satisfies (32) and (33). Let's take the limit as $v_1 \to \infty$. Clearly

$$e^{-\mu v}|\overline{w}| \leqslant \|\overline{w}\|_{\mathscr{C}^{0,\alpha}_\mu(I_\sigma\times[v_0,v_1])} \leqslant c\|\bar{f}\|_{\mathscr{C}^{0,\alpha}_\mu(I_\sigma\times[v_0,v_1])}.$$

And using Schauder estimates, we get

$$\begin{aligned} e^{-\mu v}|\nabla\overline{w}| &\leqslant \|\overline{w}\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma\times[v_0,v_1])} \\ &\leqslant c_1\big(\|\bar{f}\|_{\mathscr{C}^{0,\alpha}_\mu(I_\sigma\times[v_0,v_1])} + \|\overline{w}\|_{\mathscr{C}^{0,\alpha}_\mu(I_\sigma\times[v_0,v_1])}\big) \leqslant c_2\|\bar{f}\|_{\mathscr{C}^{0,\alpha}_\mu(I_\sigma\times[v_0,v_1])}.\end{aligned}$$

Hence the Ascoli–Arzelà theorem ensures that a subsequence of $\{\overline{w}_{v_1}\}_{v_1>v_0+1}$ converges to a function $\overline{w}$ defined on $I_\sigma \times [v_0,+\infty)$, which satisfies

$$\sup_{I_\sigma\times[v_0,+\infty)} e^{-\mu v}|\overline{w}| \leqslant c \sup_{I_\sigma\times[v_0,+\infty)} e^{-\mu v}|\bar{f}|.$$

Using again elliptic estimates we can conclude that $\overline{w}$ satisfies the statement of Proposition 6.5. The uniqueness of the solution follows from Lemma 6.4.

Step 2. Let's now consider $f \in \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0,+\infty))$ in $\operatorname{Span}\{e_{\sigma,0}, e_{\sigma,1}\}$, that is,

$$f(u,v) = f_0(v)e_{\sigma,0}(u) + f_1(v)e_{\sigma,1}(u).$$

We extend the functions $f_0(v)$ and $f_1(v)$ for $v \leqslant v_0$ by $f_0(v_0)$ and $f_1(v_0)$, respectively. Given $v_1 > v_0+1$, consider the problem

$$\begin{cases} L_{\sigma,j}w_j = f_j & \text{in } (-\infty, v_1], \\ w_j(v_1) = \partial_v w_j(v_1) = 0. \end{cases} \tag{35}$$

The Cauchy–Lipschitz theorem and the linearity of the equation ensure the existence and the uniqueness of the solution w_j. We aim to prove the following result.

Claim 6.7. *For some constant c that does not depend on v_1,*

$$\sup_{(-\infty,v_1]}(e^{-\mu v}|w_j|) \leqslant c \sup_{(-\infty,v_1]}(e^{-\mu v}|f_j|).$$

Proof. Suppose to the contrary that for every $n \in \mathbb{N}$ there exists $\sigma_n \in (0, 1/n)$, $v_{1,n} > v_{0,n} + 1$ and $f_{j,n}$, $w_{j,n}$ satisfying (35) such that

$$\sup_{(-\infty, v_{1,n}]} (e^{-\mu v} |\bar{f}_{j,n}|) = 1,$$

$$A_n := \sup_{(-\infty, v_{1,n}]} (e^{-\mu v} |w_{j,n}|) \to +\infty \quad \text{as } n \to \infty.$$

The solution $w_{j,n}$ of the previous equation is a linear combination of the two solutions of the homogeneous problem $L_{\sigma_n, j} w = 0$. They grow at most linearly at ∞ (recall that the Jacobi fields have this rate of growth). Hence the supremum A_n is achieved at a point $v_n \in (-\infty, v_{1,n}]$. We define on $I_n := (-\infty, v_{1,n} - v_n]$ the function

$$\tilde{w}_{j,n}(v) := \frac{e^{-\mu v_n}}{A_n} w_{j,n}(v_n + v).$$

As in Step 1, one shows that the sequence $\{v_{1,n} - v_n\}_n$ remains bounded away from 0 and, after passing to a subsequence, it converges to $\bar{v}_1 \in (0, +\infty]$, and $\{\tilde{w}_{j,n}\}_n$ converges on compact subsets of $I_\infty = (-\infty, \bar{v}_1]$ to a nontrivial function $\tilde{w}_j$ such that

$$\sup_{I_\infty} (e^{-\mu v} |\tilde{w}_j|) = 1 \tag{36}$$

and $\tilde{w}_j(\bar{v}_1) = \partial_v w_j(\bar{v}_1) = 0$ if $\bar{v}_1 < +\infty$. The function $\tilde{w}_j$ solves a second order ordinary differential equation given, in terms of the (x, y) variables, by

$$\sin y\, \partial_y (\sin y\, \partial_y \tilde{w}_j) - j^2\, \tilde{w}_j + 2 \sin^2 y\, \tilde{w}_j = 0. \tag{37}$$

If $\bar{v}_1 < +\infty$, then $\tilde{w}_j = 0$, and this contradicts (36). In the case $\bar{v}_1 = +\infty$ we will try to reach a contradiction by determining the solution of (37). This is again the associated Legendre differential equation; see Appendix C. The solutions of (37) are linear combinations of the associated Legendre functions of first and second kind: $P_1^j(\cos y)$ and $Q_1^j(\cos y)$ for $j = 0, 1$. Specifically, $P_1^0(\cos y) = \cos y$ and $P_1^1(\cos y) = -\sin y$. We change coordinates to express $\tilde{w}_j$ in terms of the (u, v) variables. As $v \to \pm\infty$, one can show that $|Q_1^1(\cos y(v))|$ and $|Q_1^0(\cos y(v))|$ tend to ∞ as $e^{2|v|}$ and $|v|$, respectively. We conclude that the functions $\tilde{w}_1$ and $\tilde{w}_0$ do not satisfy (36) with $\mu \in (-2, -1)$, a contradiction. □

Therefore, $\sup_{(-\infty, v_1]} (e^{-\mu v} |w_j|) \leqslant c \sup_{(-\infty, v_1]} (e^{-\mu v} |f_j|)$. Taking $v_1 \to +\infty$, we get a solution of $L_{\sigma, j}\, w_j = f_j$ defined in $[v_0, +\infty)$ that satisfies

$$\sup_{[v_0, +\infty)} (e^{-\mu v} |w_j|) \leqslant c \sup_{[v_0, +\infty)} (e^{-\mu v} |f_j|).$$

Elliptic estimates allow us to obtain the desired estimates for the derivatives. To prove the uniqueness of solution, it suffices to observe that no solution of $\mathcal{L}_\sigma v = 0$

that is collinear with $e_{\sigma,0}$ and $e_{\sigma,1}$ decays exponentially at ∞. This fact follows from the behavior of the Jacobi fields. $\square$

Remark 6.8. The results proved in this section also follow from considering not $\widetilde{M}_{\sigma,\alpha,0}$ but $\widetilde{M}_{\sigma,0,\beta}$: It is invariant by reflection about the $\{x_1 = 0\}$ plane. To keep such a symmetry, we work with functions that are odd (and not even) in the variable u. Hence $\mathscr{C}^{\ell,\alpha}_{\mu}(I_\sigma \times I)$ will be, in this case, the space of functions $w = w(u, v)$ in $\mathscr{C}^{\ell,\alpha}_{\mathrm{loc}}(I_\sigma \times I)$ that are odd and U_σ-periodic in the variable u, and for which the norm $\|w\|_{\mathscr{C}^{\ell,\alpha}_{\mu}(I_\sigma \times I)}$ is finite. Also, we replace in the above results $e_{0,i}(u) = \cos(ix(u))$ by $\widetilde{e}_{0,i}(u) = \sin(ix(u))$, and $e_{\sigma,i}$ by the normalized odd eigenfunction $\widetilde{e}_{\sigma,i}$ associated to the eigenvector $\lambda_{\sigma,i}$ of the operator L_σ.

7. A family of minimal surfaces close to $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,\alpha,0}$

The aim of this section is to find a family of minimal surfaces near conveniently translated and dilated pieces of $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,\alpha,0}$, with given Dirichlet data on the boundary.

We denote by Z the immersion of the surface $\widetilde{M}_{\sigma,\alpha,\beta}$. The following proposition, proved in Appendix B, states that the linearization of the mean curvature operator about $\widetilde{M}_{\sigma,\alpha,\beta}$ is the Lamé operator $\mathscr{L}_\sigma$ introduced in Section 5.2; see (30).

Proposition 7.1. *The surface parameterized by $Z_f := Z + fN$ is minimal if and only if the function f is a solution of*

$$\mathscr{L}_\sigma f = Q_\sigma(f)$$

where Q_σ is a nonlinear operator that satisfies

$$\begin{aligned}(38)\quad & \|Q_\sigma(f_2) - Q_\sigma(f_1)\|_{\mathscr{C}^{0,\alpha}(I_\sigma\times[v,v+1])} \\ & \leqslant c\Big(\sup_{i=1,2} \|f_i\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\Big)\|f_2 - f_1\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\end{aligned}$$

for all functions f_1, f_2 such that $\|f_i\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])} \leqslant 1$. Here the constant $c > 0$ depends neither on $v \in \mathbb{R}$ nor on $\sigma \in (0, \pi/2)$.

In Section 5.1 (see Lemma 5.3) we have written annular pieces of $M_{\sigma,\alpha,0}(\gamma, \xi)$ and $M_{\sigma,0,\beta}(\gamma, \xi)$ as vertical graphs over an annular neighborhood of $\{r = r_\varepsilon\}$ in $\{x_3 = 0\}$ of the functions

$$(39)\quad \widetilde{U}^{\alpha}_{\gamma,\xi_1}(r,\theta) = (1+\gamma)\ln(2r) - 2r\sin\tfrac{1}{2}\alpha\cos\theta - \frac{1+\gamma}{r}\xi_1\cos\theta + d + \mathbb{O}(\varepsilon),$$

$$(40)\quad \widetilde{U}^{\beta}_{\gamma,\xi_2}(r,\theta) = (1+\gamma)\ln(2r) + 2r\sin\tfrac{1}{2}\beta\sin\theta - \frac{1+\gamma}{r}\xi_2\sin\theta + d + \mathbb{O}(\varepsilon),$$

respectively, where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\gamma, \xi_1, \xi_2, \xi_3$ are small. We now truncate the surfaces $\widetilde{M}_{\sigma,\alpha,0}(\gamma, \xi)$ and $\widetilde{M}_{\sigma,0,\beta}(\gamma, \xi)$ at their respective graph curves over

$\{r = r_\varepsilon\}$. We only consider the upper half of these surfaces, which we call M_1 and M_2, respectively. We are interested those minimal normal graphs over M_1 and M_2 that are asymptotic to them, and whose boundary is prescribed.

As a consequence of the dilation of the surfaces by the factor $1+\gamma$, the minimal surface equation becomes

$$\mathcal{L}_\sigma w = \frac{1}{1+\gamma} Q_\sigma\left((1+\gamma)w\right). \tag{41}$$

That is, the normal graph of a function w over the dilated $\widetilde{M}_{\sigma,\alpha,\beta}$ is minimal if and only if w is a solution of (41).

Two more modifications are required: In Lemma 5.5 we showed that the value of the variable v corresponding to $r = r_\varepsilon$ is $v_\varepsilon = -\frac{1}{2}\ln\varepsilon + \mathbb{O}(1)$. Since we are working in the (u, v) variables, we would like to parameterize M_i in $I_\sigma \times [v_\varepsilon, +\infty]$ for $i = 1, 2$. But the boundary of M_i does not correspond to the curve $\{v = v_\varepsilon\}$. We therefore modify the parameterization so that it remains fixed for $v \geqslant v_\varepsilon + \ln 4$, while requiring, in a small annular neighborhood of $\{v = v_\varepsilon\}$, that the curves $\{v = \text{const}\}$ correspond to the vertical graphs of curves $\{r = \text{const}\}$ by the corresponding function (39) or (40). We also want the normal vector field relative to M_i to be vertical near its boundary. This can be achieved by modifying the normal vector field into a transverse vector field $\tilde{N}$ that agrees with N when $v \geqslant v_\varepsilon + \ln 4$, and with e_3 when $v \in [v_\varepsilon, v_\varepsilon + \ln 2]$.

We consider a graph of some function w over M_i, using the modified vector field $\tilde{N}$. This graph will be minimal if and only if the function w is a solution of a nonlinear elliptic equation related to (41). To get the new equation, we take into account the effects of the change of parameterization and the change of the vector field N into $\tilde{N}$. The new minimal surface equation is

$$\mathcal{L}_\sigma w = \tilde{L}_\varepsilon w + \tilde{Q}_\sigma(w). \tag{42}$$

Here $\tilde{Q}_\sigma$ enjoys the same properties as Q_σ, since it is obtained by a slight perturbation from it. The operator $\tilde{L}_\varepsilon$ is a linear second order operator whose coefficients are supported in $I_\sigma \times [v_\varepsilon, v_\varepsilon + \ln 4]$ and are bounded in the $\mathscr{C}^\infty$ topology by a constant multiplied by $\sqrt{\varepsilon}$, where partial derivatives are computed with respect to the vector fields ∂_u and ∂_v. In fact, if we take into account the effect of the change of the normal vector field, we would obtain by applying the result of [Hauswirth and Pacard 2007, Appendix B] a similar formula in which the coefficients of the corresponding operator $\tilde{L}_\varepsilon$ are bounded by a constant multiplied by ε, since

$$\tilde{N}_\varepsilon \cdot N_\varepsilon = 1 + \mathbb{O}_{\mathscr{C}_b^{2,\alpha}}(\varepsilon) \quad \text{when } v \in [v_\varepsilon, v_\varepsilon + \ln 2].$$

If we take into account the effect of the change in the parameterization, we would obtain a similar formula in which the coefficients of the corresponding operator $\tilde{L}_\varepsilon$

are bounded by a constant multiplied by $\sqrt{\varepsilon}$. The estimate of the coefficients of $\tilde{L}_\varepsilon$ follows from these considerations.

Now we will give a detailed proof of the existence of a family of minimal graphs about M_1 and asymptotic to it. Recall that M_1 is invariant by reflection across the $\{x_2 = 0\}$ plane. The normal graph of the function $w = w(u, v)$ over M_1 inherits the same symmetry property if w is even in the u variable. The corresponding results for M_2 are obtained similarly, considering odd functions instead of even ones.

We consider a function $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ that is even and L^2-orthogonal to $e_{0,0}$, $e_{0,1}$ and that satisfies

$$\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon, \tag{43}$$

where $\kappa > 0$ is a constant. We define $w_\varphi(u', v) := \overline{\mathscr{H}}_{v_\varepsilon,\varphi}$, where $\overline{\mathscr{H}}_{v_\varepsilon,\varphi}$ is the harmonic extension introduced in Proposition A.5. If $u = (2\pi/U_\sigma)u'$, then $w_\varphi(u, v)$ belongs to $\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$, and $w_\varphi(\,\cdot\,, v_\varepsilon) \in \mathscr{C}^{2,\alpha}(I_\sigma)$ is even and L^2-orthogonal to $e_{\sigma,0}$, $e_{\sigma,1}$. To solve Equation (42), we choose $\mu \in (-2, -1)$ and look for $w \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$ of the form $w = w_\varphi + g$ for some $g \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$. Using Proposition 6.5, we can rephrase this problem as a fixed point problem

$$g = S(\varphi, g) := G_{\varepsilon,v_\varepsilon}\big(\tilde{L}_\varepsilon(w_\varphi + g) - \mathscr{L}_\sigma w_\varphi + \tilde{Q}_\sigma(w_\varphi + g)\big). \tag{44}$$

where the nonlinear mapping S depends on σ, ε, γ, and operator $G_{\varepsilon,v_\varepsilon}$ is as defined in Proposition 6.5. To prove the existence of a fixed point for (44), we need the next lemma. We will abbreviate by writing $\|\cdot\|_{\mathscr{C}^{2,\alpha}_\mu}$ instead of $\|\cdot\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))}$.

Proposition 7.2. *Let* $0 < \sigma \leqslant \varepsilon$, $\mu \in (-2, -1)$. *Suppose* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *satisfies* (43) *and enjoys the properties given above. Then there exist some constants* $c_\kappa > 0$ *and* $\varepsilon_\kappa > 0$ *such that*

$$\|S(\varphi, 0)\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))} \leqslant c_\kappa \varepsilon^{(3+\mu)/2} \tag{45}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\|S(\varphi, g_2) - S(\varphi, g_1)\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))} \leqslant \tfrac{1}{2}\|g_2 - g_1\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))},$$

$$\|S(\varphi_2, g) - S(\varphi_1, g)\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))} \leqslant c\varepsilon^{\frac{1}{2}+\mu/2}\|\varphi_2 - \varphi_1\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \tag{46}$$

for all $g, g_1, g_2 \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$ *such that* $\|g_i\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant 2c_\kappa\, \varepsilon^{(3+\mu)/2}$, *and all* $\varphi_1, \varphi_2 \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *enjoying the same properties as* φ.

Proof. We know from Proposition 6.5 that $\|G_{\varepsilon,v_\varepsilon}(f)\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant c\|f\|_{\mathscr{C}^{0,\alpha}_\mu}$ for some $c > 0$ (throughout the proof, c will denote an arbitrary positive constant). Then

$$\begin{aligned}\|S(\varphi, 0)\|_{\mathscr{C}^{2,\alpha}_\mu} &\leqslant c\|\tilde{L}_\varepsilon w_\varphi - \mathscr{L}_\sigma\, w_\varphi + \tilde{Q}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} \\ &\leqslant c\big(\|\tilde{L}_\varepsilon w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} + \|\mathscr{L}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} + \|\tilde{Q}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu}\big).\end{aligned}$$

So we need to estimate the three terms above.

In the proof of Proposition A.5 we obtain that, for every $v \in [v_\varepsilon, +\infty)$,

$$\|w_\varphi\|_{\mathscr{C}^{2,\alpha}(I_\sigma \times [v,v+1])} \leqslant c e^{-2(v-v_\varepsilon)} \|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}.$$

Therefore,

$$\begin{aligned}\|w_\varphi\|_{\mathscr{C}^{2,\alpha}_\mu} &= \sup_{v\in[v_\varepsilon,+\infty)} \left(e^{-\mu v}\|w_\varphi\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\right)\\ &\leqslant c \sup_{v\in[v_\varepsilon,+\infty)} (e^{-\mu v-2(v-v_\varepsilon)})\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant c e^{-\mu v_\varepsilon}\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa c \varepsilon^{1+\mu/2}.\end{aligned}$$

From this inequality and the estimates of the coefficients of $\tilde{L}_\varepsilon$, it follows that

$$\|\tilde{L}_\varepsilon(w_\varphi)\|_{\mathscr{C}^{0,\alpha}_\mu} \leqslant c\varepsilon^{1/2}\|w_\varphi\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant \kappa c \varepsilon^{(3+\mu)/2}.$$

Since w_φ is an harmonic function, the definition of $\mathscr{L}_\sigma$ in (30) gives the equality

$$\mathscr{L}_\sigma w_\varphi = 2k(u,v) w_\varphi.$$

Recall (see Lemma 5.5) that if $v \geqslant v_\varepsilon$, then $y(v) \geqslant \pi - a_\varepsilon$, where $a_\varepsilon = \mathbb{O}(\sqrt{\varepsilon})$. From the facts that if $|y(v) - \pi| \leqslant a_\varepsilon$, then

$$k(u,v) = \sin^2\sigma\cos^2(x(u)) + \cos^2\sigma\sin^2(y(v)) \leqslant \sin^2\sigma + \sin^2(a_\varepsilon) \leqslant c\varepsilon$$

and that w_φ is an exponentially decaying function, we conclude that

$$\|\mathscr{L}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} \leqslant c\varepsilon\|w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} \leqslant \kappa c \varepsilon^{2+\mu/2}.$$

Finally, $\|\tilde{Q}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}(I_\sigma\times[v,v+1])} \leqslant c\|w_\varphi\|^2_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}$, so

$$\|\tilde{Q}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} \leqslant c \sup_{v\in[v_\varepsilon,+\infty)} \left(e^{-\mu v}\|w_\varphi\|^2_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\right) \leqslant c\,\|w_\varphi\|^2_{\mathscr{C}^{2,\alpha}_{\mu/2}} \leqslant \kappa^2 c\,\varepsilon^{2+\mu/2}.$$

Putting together these estimates, we get (45). The details of other the estimates are left to the reader. □

Theorem 7.3. *Consider* $0 < \sigma \leqslant \varepsilon$, $\mu \in (-2,-1)$ *and* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *as above. We define* $B := \{g \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty)) : \|g\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant 2c_k\varepsilon^{(3+\mu)/2}\}$. *Then the nonlinear mapping* $S(\varphi,\cdot)$ *has a unique fixed point* g *in* B.

Proof. The previous proposition shows that if ε is chosen small enough, the nonlinear mapping $S(\varphi,\cdot)$ is a contraction mapping from B into itself. Hence Schauder's theorem ensures that $S(\varphi,\cdot)$ has a fixed point g in B. □

Theorem 7.3 provides, for each even function $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ L^2-orthogonal to $e_{0,0}, e_{0,1}$ with $\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon$, a minimal surface $S_{t,\alpha,\gamma,\xi,d}(\varphi)$ close to M_1 (the subindex t reflects the fact we are considering the upper half of $\widetilde{M}_{\sigma,\alpha,0}(\gamma,\xi)$). In a neighborhood of its boundary, this surface can be written as a vertical graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon} \subset \{x_3 = 0\}$ of the function

$$\begin{aligned}(47)\quad \overline{U}_{t,1}(r,\theta) = (1+\gamma)\ln(2r) - 2r\sin\tfrac{1}{2}(\alpha)\cos\theta - \frac{1+\gamma}{r}\xi\cos\theta \\ + d + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \overline{V}_{t,1}(r,\theta).\end{aligned}$$

The function $\overline{V}_{t,1} = \overline{V}_{t,1}(\gamma,\varphi)$ depends nonlinearly on γ and φ, and there exists a $c > 0$ such that

$$\begin{aligned}&\|\overline{V}_{t,1}(\gamma,\varphi)(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\varepsilon,\\ (48)\qquad &\|\overline{V}_{t,1}(\gamma,\varphi_1)(r_\varepsilon\,\cdot\,,\cdot) - \overline{V}_{t,1}(\gamma,\varphi_2)(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_1)}\\ &\qquad\qquad \leqslant c\varepsilon^{1/2}\|\varphi_1 - \varphi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)},\end{aligned}$$

for all even functions $\varphi, \varphi_1, \varphi_2 \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ that are L^2-orthogonal to $e_{0,0}, e_{0,1}$ and whose $\mathscr{C}^{2,\alpha}$-norms are bounded above by $\kappa\varepsilon$. The latter estimate follows from estimate (46) and

$$\begin{aligned}&\|\overline{V}_{t,1}(\gamma,\varphi_1)(r_\varepsilon\,\cdot\,,\cdot) - \overline{V}_{t,1}(\gamma,\varphi_2)(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_1)}\\ &\qquad\qquad \leqslant e^{\mu v_\varepsilon}\|S(\varphi_1,\overline{V}_{t,1}) - S(\varphi_2,\overline{V}_{t,1})\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}.\end{aligned}$$

The boundary of $S_{t,\alpha,\gamma,\xi,d}(\varphi)$ corresponds to the image by $\overline{U}_{t,1}$ of $\{r = r_\varepsilon\}$.

Similar arguments can be followed for the lower half of $\widetilde{M}_{\sigma,\alpha,0}(\gamma,\xi)$, and we obtain a minimal surface $S_{b,\alpha,\gamma,\xi,d}(\varphi)$ close to such a half of $\widetilde{M}_{\sigma,\alpha,0}(\gamma,\xi)$, which can be written in a neighborhood of its boundary as a vertical graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the function

$$\begin{aligned}(49)\quad \overline{U}_{b,1}(r,\theta) = -(1+\gamma)\ln(2r) - 2r\sin\tfrac{1}{2}\alpha\cos\theta - \frac{1+\gamma}{r}\xi\cos\theta \\ + d + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \overline{V}_{b,1}(r,\theta),\end{aligned}$$

where the function $\overline{V}_{b,1} = \overline{V}_{b,1}(\gamma,\varphi)$ enjoys the same properties as $\overline{V}_{t,1}$. The boundary of $S_{b,\alpha,\gamma,\xi,d}(\varphi)$ corresponds to the image by $\overline{U}_{b,1}$ of $\{r = r_\varepsilon\}$.

Analogously, for an odd function $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ that is L^2-orthogonal to $\widetilde{e}_{0,0}, \widetilde{e}_{0,1}$ (see Remark 6.8) and that satisfies $\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon$, we obtain minimal surfaces $\widetilde{S}_{t,\beta,\gamma,\xi,d}(\varphi)$ and $\widetilde{S}_{b,\beta,\gamma,\xi,d}(\varphi)$ near the upper and lower half of $\widetilde{M}_{\sigma,0,\beta}(\gamma,\xi)$ that can be written in a neighborhood of their boundary as vertical graphs over the annulus

$B_{2r_\varepsilon} - B_{r_\varepsilon}$ respectively of the functions

$$\begin{aligned}\overline{U}_{t,2}(r,\theta) &= (1+\gamma)\ln(2r) + 2r\sin\tfrac{1}{2}\beta\sin\theta - \frac{1+\gamma}{r}\xi\sin\theta \\ &\qquad + d + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \overline{V}_{t,2}(r,\theta),\\ \overline{U}_{b,2}(r,\theta) &= -(1+\gamma)\ln(2r) + 2r\sin\tfrac{1}{2}\beta\sin\theta - \frac{1+\gamma}{r}\xi\sin\theta \\ &\qquad + d + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \overline{V}_{b,2}(r,\theta),\end{aligned}$$

where the functions $\overline{V}_{t,2} = \overline{V}_{t,2}(\gamma,\varphi)$ and $\overline{V}_{b,2} = \overline{V}_{b,2}(\gamma,\varphi)$ enjoy the same properties as $\overline{V}_{t,1}$. Their respective boundaries correspond to the image by $\overline{U}_{t,2}$ and $\overline{U}_{b,2}$ of $\{r = r_\varepsilon\}$.

8. The matching of Cauchy data

In this section we shall complete the proof of Theorems 1.1, 1.2 and 1.3.

8.1. ***Proof of Theorem 1.2.*** The proof is articulated in two distinct parts: the proof of the existence of the family $\mathcal{K}_1$ and of the existence of the family $\mathcal{K}_2$.

We start with the second. Its proof is based on an analytical gluing procedure. The surfaces in the family $\mathcal{K}_2$ are symmetric about the plane $\{x_2 = 0\}$, so all the surfaces involved in the proof must have the same property. We will show how to glue a compact piece of a Costa–Hoffman–Meeks-type surface with bent catenoidal end to two halves of the KMR example $\widetilde{M}_{\sigma,\alpha,0}$ along the upper and lower boundaries and to a horizontal periodic flat annulus with a disk removed along the middle boundary. All the surfaces just mentioned have the desired symmetry, as do the surfaces obtained from them by slight perturbation. We recall below the necessary results proved in previous sections.

As we have seen in Section 3, we can construct a minimal surface $\overline{M}^T_{k,\varepsilon}(\varepsilon/2,\Psi)$, with $\Psi = (\psi_t,\psi_b,\psi_m)$, that is close to a truncated genus k Costa–Hoffman–Meeks surface M_k and has three boundaries. The functions $\psi_t,\psi_b,\psi_m \in C^{2,\alpha}(\mathbb{S}^1)$ are even. Also, ψ_m is L^2-orthogonal to 1, and ψ_t and ψ_b are L^2-orthogonal to 1 and to $\cos\theta$. Close to its upper, lower and middle boundaries, the surface $\overline{M}^T_{k,\varepsilon}(\varepsilon/2,\Psi)$ is a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$, respectively, of the functions

$$\begin{aligned}U_t(r,\theta) &= \sigma_t + \ln(2r) - \tfrac{1}{2}\varepsilon r\cos\theta + H_{\psi_t}(s_\varepsilon - \ln(2r),\theta) + \mathbb{O}_{C^{2,\alpha}_b}(\varepsilon),\\ U_b(r,\theta) &= -\sigma_b - \ln(2r) - \tfrac{1}{2}\varepsilon r\cos\theta + H_{\psi_b}(s_\varepsilon - \ln(2r),\theta) + \mathbb{O}_{C^{2,\alpha}_b}(\varepsilon),\\ U_m(r,\theta) &= \tilde{H}_{\rho_\varepsilon,\psi_m}(1/r,\theta) + \mathbb{O}_{C^{2,\alpha}_b}(\varepsilon),\end{aligned}$$

where $s_\varepsilon = -\ln\sqrt{\varepsilon}$ and $\rho_\varepsilon = 2\sqrt{\varepsilon}$; see Equations (14), (15) and (16).

Using the results of Section 7 we can show the existence of a minimal surface $S_{t,\alpha_t,\gamma_t,\xi_t,d_t}(\varphi_t)$ near the upper half of the KMR example $\widetilde{M}_{\sigma_t,\alpha_t,0}$, and asymptotic to it. Near its boundary, this surface can be parameterized over $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of (see (47))

$$\bar{U}_t(r,\theta) = (1+\gamma_t)\ln(2r) - 2r\sin\tfrac{1}{2}\alpha_t\cos\theta - \frac{1+\gamma_t}{r}\xi_t\cos\theta \\ + d_t + \bar{\mathcal{H}}_{v_\varepsilon,\varphi_t}(\ln 2r,\theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon).$$

We recall that $\varphi_t \in C^{2,\alpha}(\mathbb{S}^1)$ is an even function L^2-orthogonal to 1 and to $\cos\theta$. The surface $S_{t,\alpha_t,\gamma_t,\xi_t,d_t}(\varphi_t)$ will be glued to the upper boundary of $\bar{M}^T_{k,\varepsilon}(\varepsilon/2,\Psi)$.

Near its boundary, the surface $S_{b,\alpha_b,\gamma_b,\xi_b,d_b}(\varphi_b)$ that will be glued along the lower boundary of $\bar{M}^T_{k,\varepsilon}(\varepsilon/2,\Psi)$ can be parameterized in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\bar{U}_b(r,\theta) = -(1+\gamma_b)\ln(2r) - 2r\sin\tfrac{1}{2}\alpha_b\cos\theta - \frac{1+\gamma_b}{r}\xi_b\cos\theta \\ + d_b + \bar{\mathcal{H}}_{v_\varepsilon,\varphi_b}(\ln 2r,\theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon);$$

see (49). Recall that we assumed $\varphi_b \in C^{2,\alpha}(\mathbb{S}^1)$ to be an even function that is L^2-orthogonal to 1 and to $\cos\theta$.

Using the results of Section 4, we can construct a minimal graph $S_m(\varphi_m)$ close to a horizontal periodic flat annulus with a disk removed. Here $\varphi_m \in C^{2,\alpha}(\mathbb{S}^1)$ is an even function L^2-orthogonal to 1. In a neighborhood of its boundary, it can be parameterized (see (21)) as the vertical graph over $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of $\bar{U}_m(r,\theta) = \tilde{H}_{r_\varepsilon,\varphi_m}(r,\theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$.

The functions $\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$ in the formulas above replace the functions V_t, V_b, V_m, $\bar{V}_t$, $\bar{V}_b$ and $\bar{V}_m$ that appear in Equations (14), (15), (16), (47), (49) and (21). They depend nonlinearly on the different parameters and boundary data, but they are bounded by a constant times ε in the $C_b^{2,\alpha}$ topology, where partial derivatives are taken with respect to the vector fields $r\partial_r$ and ∂_θ.

We assume that the parameters and the boundary functions are chosen so that

$$(50)\quad |\gamma_t| + |\gamma_b| + |-\gamma_t\ln\sqrt{\varepsilon} + \eta_t| + |\gamma_b\ln\sqrt{\varepsilon} + \eta_b| \\ + (4\sqrt{\varepsilon})^{-1}\big|-4\sin(\alpha_t/2)+\varepsilon\big| + (4\sqrt{\varepsilon})^{-1}\big|-4\sin(\alpha_b/2)+\varepsilon\big| \\ + 2\sqrt{\varepsilon}(|(1+\gamma_t)\xi_t| + |(1+\gamma_b)\xi_b|) \\ + \|\varphi_t\|_{C^{2,\alpha}(\mathbb{S}^1)} + \|\varphi_b\|_{C^{2,\alpha}(\mathbb{S}^1)} + \|\varphi_m\|_{C^{2,\alpha}(\mathbb{S}^1)} \\ + \|\psi_t\|_{C^{2,\alpha}(\mathbb{S}^1)} + \|\psi_b\|_{C^{2,\alpha}(\mathbb{S}^1)} + \|\psi_m\|_{C^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon,$$

where $\eta_t = d_t - \sigma_t$ and $\eta_b = d_b + \sigma_b$ for some fixed constant $\kappa > 0$ large enough.

It remains to show that, for all ε small enough, it is possible to choose the parameters and boundary functions so that the surface

$$M^T_k(\varepsilon/2,\Psi) \cup S_{t,\alpha_t,\gamma_t,\xi_t,d_t}(\varphi_t) \cup S_{b,\alpha_b,\gamma_b,\xi_b,d_b}(\varphi_b) \cup S_m(\varphi_m)$$

is a C^1 surface across the boundaries of the different summands. Regularity theory will then ensure that this surface is in fact smooth, and then by construction it has the desired properties. This will therefore complete the proof of the existence of the family of examples $\mathcal{K}_2$.

It is necessary to fulfill the following system of equations on $\mathbb{S}^1$:

$$\begin{cases} U_t(r_\varepsilon,\cdot) = \overline{U}_t(r_\varepsilon,\cdot), & \partial_r U_t(r_\varepsilon,\cdot) = \partial_r \overline{U}_t(r_\varepsilon,\cdot), \\ U_b(r_\varepsilon,\cdot) = \overline{U}_b(r_\varepsilon,\cdot), & \partial_r U_b(r_\varepsilon,\cdot) = \partial_r \overline{U}_b(r_\varepsilon,\cdot), \\ U_m(r_\varepsilon,\cdot) = \overline{U}_m(r_\varepsilon,\cdot), & \partial_r U_m(r_\varepsilon,\cdot) = \partial_r \overline{U}_m(r_\varepsilon,\cdot). \end{cases}$$

The left three equations lead to the system

$$(51)\qquad \begin{cases} \gamma_t \ln(2r_\varepsilon) + \eta_t - (1+\gamma_t)(\xi_t/r_\varepsilon)\cos\theta \\ \qquad + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_t + \tfrac{1}{2}\varepsilon)\cos\theta + \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), \\ -\gamma_b \ln(2r_\varepsilon) + \eta_b - (1+\gamma_b)(\xi_b/r_\varepsilon)\cos\theta \\ \qquad + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_b + \tfrac{1}{2}\varepsilon)\cos\theta + \varphi_b - \psi_b = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \qquad \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon). \end{cases}$$

The right three equations give the system

$$(52)\qquad \begin{cases} \gamma_t + (1+\gamma_t)(\xi_t/r_\varepsilon)\cos\theta \\ \qquad + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_t + \tfrac{1}{2}\varepsilon)\cos\theta + \partial_\theta^*(\varphi_t + \psi_t) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon), \\ -\gamma_b + (1+\gamma_b)(\xi_b/r_\varepsilon)\cos\theta \\ \qquad + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_b + \tfrac{1}{2}\varepsilon)\cos\theta + \partial_\theta^*(\varphi_b + \psi_b) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \qquad \partial_\theta^*(\varphi_m + \psi_m) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon). \end{cases}$$

Here ∂_θ^* denotes the operator that associates to $\phi = \sum_{i\geqslant 1}\phi_i\cos(i\theta)$ the function $\partial_\theta^*\phi = \sum_{i\geqslant 1} i\phi_i\cos(i\theta)$. To obtain this system, we applied the results of Lemmas A.6 and A.7. The functions $\mathcal{O}_{C^{l,\alpha}}(\varepsilon)$ in the above expansions depend nonlinearly on the different parameters and boundary data functions, but they are bounded in the $C^{l,\alpha}$ topology by a constant times ε. The projection of the first two equations of each system over the L^2-orthogonal complement of $\mathrm{Span}\{1,\cos\theta\}$, together with the remaining two equations, gives the system

$$(53)\qquad \begin{cases} \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*\varphi_t + \partial_\theta^*\psi_t = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \varphi_b - \psi_b = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*\varphi_b + \partial_\theta^*\psi_b = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*\varphi_m + \partial_\theta^*\psi_m = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon). \end{cases}$$

Lemma 8.1 [Fakhi and Pacard 2000]. *The operator*

$$h : C^{2,\alpha}(\mathbb{S}^1) \to C^{1,\alpha}(\mathbb{S}^1), \quad \varphi \mapsto \partial_\theta^*\varphi$$

is invertible when acting on functions that are even and L^2-orthogonal to 1.

Proof. If we decompose $\varphi = \sum_{j\geqslant 1} \varphi_j \cos(j\theta)$, then

$$h(\varphi) = \sum_{j\geqslant 1} j\varphi_j \cos(j\theta),$$

is clearly invertible from $H^1(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$. Elliptic regularity theory implies that this is still true when this operator is defined between Hölder spaces. $\square$

Using this result, the system (53) can be rewritten as

$$(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m) = \mathbb{O}_{C^{2,\alpha}}(\varepsilon). \tag{54}$$

Recall that the right hand side depends nonlinearly on $\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m$ and also on the parameters $\gamma_t, \gamma_b, \eta_t, \eta_b, \xi_t, \xi_b, \alpha_t, \alpha_b$. We look at this equation as a fixed point problem and fix κ large enough. Thanks to estimates (48), (20),(22), (17) and (18), we can use a fixed point theorem for contracting mappings in the ball of radius $\kappa\varepsilon$ in $(C^{2,\alpha}(\mathbb{S}^1))^6$ to obtain, for all ε small enough, a solution $(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m)$ of (54). Since this solution is a fixed point for a contraction mapping and since the right hand side of (54) is continuous with respect to all data, we see that this fixed point $(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m)$ depends continuously (and in fact smoothly) on the parameters $\gamma_t, \gamma_b, \eta_t, \eta_b, \xi_t, \xi_b, \alpha_t, \alpha_b$. Inserting this solution into (51) and (52), we see that it only remains to solve a system of the form

$$\begin{cases} \gamma_t \ln(2r_\varepsilon) + \eta_t + \Big(-(1+\gamma_t)\dfrac{\xi_t}{r_\varepsilon} + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_t + \tfrac{1}{2}\varepsilon)\Big)\cos\theta = \mathbb{O}(\varepsilon), \\ -\gamma_b \ln(2r_\varepsilon) + \eta_b + \Big(-(1+\gamma_b)\dfrac{\xi_b}{r_\varepsilon} + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_b + \tfrac{1}{2}\varepsilon)\Big)\cos\theta = \mathbb{O}(\varepsilon), \\ \gamma_t + \Big((1+\gamma_t)\dfrac{\xi_t}{r_\varepsilon} + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_t + \tfrac{1}{2}\varepsilon)\Big)\cos\theta = \mathbb{O}(\varepsilon), \\ -\gamma_b + \Big((1+\gamma_b)\dfrac{\xi_b}{r_\varepsilon} + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_b + \tfrac{1}{2}\varepsilon)\Big)\cos\theta = \mathbb{O}(\varepsilon), \end{cases}$$

where this time the right hand sides only depend nonlinearly on γ_t, γ_b, η_t, η_b, ξ_t, ξ_b, α_t, α_b. There are eight equations that are obtained by projecting this system over 1 and $\cos\theta$. If we set

$$(\overline{\eta}_t, \overline{\eta}_b) = (\gamma_t \ln(2r_\varepsilon) + \eta_t, -\gamma_b \ln(2r_\varepsilon) + \eta_b),$$
$$(\overline{\xi}_t, \overline{\xi}_b) = r_\varepsilon^{-1}((1+\gamma_t)\xi_t, (1+\gamma_b)\xi_b), \qquad (\overline{\alpha}_t, \overline{\alpha}_b) = r_\varepsilon(2\sin\tfrac{1}{2}\alpha_t, 2\sin\tfrac{1}{2}\alpha_t),$$

the previous system can be rewritten as

$$(\gamma_t, \gamma_b, \overline{\xi}_t, \overline{\xi}_b, \overline{\eta}_t, \overline{\eta}_b, \overline{\alpha}_t, \overline{\alpha}_b) = \mathbb{O}(\varepsilon). \tag{55}$$

This time, provided κ has been fixed large enough, we can use the Leray–Schauder fixed point theorem in the ball of radius $\kappa\varepsilon$ in $\mathbb{R}^8$ to solve (55), for all ε small enough. This provides a set of parameters and a set of boundary data such that (51) and (52) hold. Equivalently, we have proved the existence of a solution of systems (51) and (52). So the proof of the first part of Theorem 1.2 is complete.

The proof of the second part uses the same arguments as above, so we will omit most of the details. We wish to show the existence of the family of surfaces $\mathcal{K}_1$, which are symmetric about the plane $\{x_1 = 0\}$. It is important to observe in this proof that the KMR example is obtained by slight perturbation of $\widetilde{M}_{\sigma,0,\beta}$. The symmetry properties of this surface differ from those of the surface close to $\widetilde{M}_{\sigma,\alpha,0}$ involved in the previous gluing procedure. In particular $\widetilde{M}_{\sigma,0,\beta}$ is symmetric about the plane $\{x_1 = 0\}$, whereas the Costa–Hoffman–Meeks-type surface from before is symmetric about the plane $\{x_2 = 0\}$. Thus $\widetilde{M}_{\sigma,0,\beta}$ is not appropriate for gluing with a KMR example of the type described above. To obtain a surface with the desired symmetry about $\{x_1 = 0\}$, we rotate the Costa–Hoffman–Meeks surface with bent catenoidal ends described in Section 3 counterclockwise by $\pi/2$ about the x_3 axis. In the parameterizations of the top and bottom ends, the cosine function is replaced by the sine function, that is,

$$\begin{aligned} U_t(r,\theta) &= \quad \sigma_t + \ln(2r) - \tfrac{1}{2}\varepsilon r \sin\theta + H_{\psi_t}(s_\varepsilon - \ln(2r), \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \\ U_b(r,\theta) &= -\sigma_b - \ln(2r) - \tfrac{1}{2}\varepsilon r \sin\theta + H_{\psi_b}(s_\varepsilon - \ln(2r), \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \end{aligned}$$

where $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$ and $(r,\theta) \in B_{r_\varepsilon} - B_{r_\varepsilon/2}$. As for the planar middle end, the form of its parameterization remains unchanged; see the first part of the proof. Another important remark concerns the Dirichlet boundary data ψ_t, ψ_b, ψ_m. Before, to preserve the symmetry about the plane $\{x_2 = 0\}$, it was required that these were even functions and that ψ_t and ψ_b were orthogonal to 1 and to $\cos\theta$. Now these must be odd functions and ψ_t and ψ_b must be orthogonal to 1 and to $\sin\theta$. Then all results shown in Section 3 continue to hold (see Remark 6.8).

Now we parameterize the surface $\widetilde{S}_{t,\beta_t,\gamma_t,\xi_t,d_t}(\varphi_t)$, the minimal surface obtained by perturbation from the KMR example $M_{\sigma,0,\beta}$ and asymptotic to it. This surface can be parameterized in the neighborhood $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\begin{aligned} \overline{U}_t(r,\theta) = (1+\gamma_t)\ln(2r) + 2r \sin\tfrac{1}{2}\beta_t \sin\theta - (1+\gamma_t)/r\xi_t \sin\theta \\ + d_t + \overline{\mathcal{H}}_{v_\varepsilon,\varphi_t}(\ln 2r, \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon). \end{aligned}$$

The parameterization of $\widetilde{S}_{b,\beta_b,\gamma_b,\xi_b,d_b}(\varphi_b)$, the surface that we will glue to the Costa–Hoffman–Meeks-type surface along its lower boundary, is given by

$$\begin{aligned} \overline{U}_b(r,\theta) = -(1+\gamma_b)\ln(2r) + 2r \sin\tfrac{1}{2}\beta_t \sin\theta - (1+\gamma_b)/r\xi_b \sin\theta \\ + d_b + \overline{\mathcal{H}}_{v_\varepsilon,\varphi_b}(\ln 2r, \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \end{aligned}$$

where $(r,\theta) \in B_{2r_\varepsilon} - B_{r_\varepsilon}$.

To prove the theorem it is necessary to show there is solution to the system

$$\begin{cases} U_t(r_\varepsilon,\cdot) = \overline{U}_t(r_\varepsilon,\cdot), & \partial_r U_b(r_\varepsilon,\cdot) = \partial_r \overline{U}_b(r_\varepsilon,\cdot), \\ U_b(r_\varepsilon,\cdot) = \overline{U}_b(r_\varepsilon,\cdot), & \partial_r U_t(r_\varepsilon,\cdot) = \partial_r \overline{U}_t(r_\varepsilon,\cdot), \\ U_m(r_\varepsilon,\cdot) = \overline{U}_m(r_\varepsilon,\cdot), & \partial_r U_m(r_\varepsilon,\cdot) = \partial_r \overline{U}_m(r_\varepsilon,\cdot) \end{cases}$$

on $\mathbb{S}^1$, under the assumption (50) for the parameters and the boundary functions. It is clear that the existence proof for this system is based on the same arguments seen before. Note that the role played before by the functions $\cos(i\theta)$ is now played by the functions $\sin(i\theta)$. This completes the proof of Theorem 1.2. $\square$

8.2. ***The proof of Theorem 1.1.*** We will glue a compact piece of the surface $M_k^T(\xi)$ with $\xi = 0$ described in Section 3 to two halves of a Scherk-type surface along the upper and lower boundary and to a horizontal periodic flat annulus along the middle boundary. The construction of these surfaces was shown in Section 4. In particular, we showed the existence of a minimal graph close to half of a Scherk-type example whose ends have asymptotic directions given by $\cos\theta_1 e_1 + \sin\theta_1 e_3$ and $-\cos\theta_2 e_1 + \sin\theta_2 e_3$. These surfaces, in the neighborhood $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the boundary, admit the parameterization

$$\overline{U}_t = d_t + \ln(2r) + \tilde{H}_{r_\varepsilon,\varphi_t}(r,\theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon),$$
$$\overline{U}_b = d_b - \ln(2r) + \tilde{H}_{r_\varepsilon,\varphi_b}(r,\theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon),$$

where the Dirichlet boundary data $\tilde{H}_{r_\varepsilon,\varphi_i} \in C^{2,\alpha}(\mathbb{S}^1)$ for $i = t, b$ is required to be even and orthogonal to 1, and $\tilde{H}_{r_\varepsilon,\varphi_i}$ denotes their harmonic extensions. The other surfaces in the gluing procedure have been described in Section 8.1.

The proof is similar to the one given for Theorem 1.2, so we will give only the essentials. We must show there is a solution to the system

$$\begin{cases} U_t(r_\varepsilon,\cdot) = \overline{U}_t(r_\varepsilon,\cdot), & \partial_r U_b(r_\varepsilon,\cdot) = \partial_r \overline{U}_b(r_\varepsilon,\cdot), \\ U_b(r_\varepsilon,\cdot) = \overline{U}_b(r_\varepsilon,\cdot), & \partial_r U_t(r_\varepsilon,\cdot) = \partial_r \overline{U}_t(r_\varepsilon,\cdot), \\ U_m(r_\varepsilon,\cdot) = \overline{U}_m(r_\varepsilon,\cdot), & \partial_r U_m(r_\varepsilon,\cdot) = \partial_r \overline{U}_m(r_\varepsilon,\cdot) \end{cases}$$

on $\mathbb{S}^1$, under an assumption similar to (50). See Section 8.1 for the expressions of $U_t, U_b, U_m, \overline{U}_m$. We point out that here we consider the more symmetric example (with $\xi = 0$) in the family $(M_k^T(\xi))_\xi$, so we must replace $\varepsilon/2$ by 0 in the expressions of the functions U_t and U_b of the top and bottom ends.

The boundary data for the surfaces we will glue together do not all share the same orthogonality properties. All are orthogonal to the constant function, but only ψ_t and ψ_b are orthogonal to $\cos\theta$. The functions denoted by $\mathbb{O}_{\mathscr{C}_b^{2,\alpha}}(\varepsilon)$, appearing in the expressions of $\overline{U}_i$ and U_i with $i = t, b, m$, have a Fourier series decomposition containing a term collinear to $\cos\theta$ only if the corresponding boundary data is assumed to be orthogonal only to the constant function. Furthermore the fact that $\xi = 0$ (which reflects that the catenoidal ends are not bent) implies that the functions parameterizing the top and bottom end of $M_k^T(0)$ are orthogonal to $\cos\theta$. In other words, in contrast to the Scherk-type surfaces, we are not able this time to prescribe the coefficients of the eigenfunction $\cos\theta$ for the catenoidal ends of $M_k^T(0)$, because they are required to vanish in this more symmetric setting.

The left three equations lead to the system

$$\begin{cases} \eta_t + \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), \\ \eta_b + \varphi_b - \psi_b = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), \\ \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), \end{cases}$$

where $\eta_t = d_t - \sigma_t$, $\eta_b = d_b + \sigma_b$. The right three equations give the system

$$\begin{cases} \partial_\theta^*(\varphi_t + \psi_t) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \partial_\theta^*(\varphi_b + \psi_b) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \partial_\theta^*(\varphi_m + \psi_m) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon). \end{cases}$$

The proof is completed by the arguments of Section 8.1. $\square$

8.3. ***The proof of Theorem 1.3.*** To prove this theorem, we treat separately the cases $k = 0$ and $k \geqslant 1$.

The case $k = 0$. We will glue half of a Scherk example with half of a KMR example with $\alpha = \beta = 0$. We observe that this surface is symmetric about the $\{x_1 = 0\}$ and $\{x_2 = 0\}$ planes. The Scherk example is symmetric about the $\{x_2 = 0\}$ plane. To preserve this property of symmetry in the surface obtained by the gluing procedure, we will consider the perturbation of $\tilde{M}_{\sigma,0,0}$ that has the same mirror symmetry. This is the surface denoted by $S_{t,0,\gamma_t,\xi_t,d_t}(\varphi_t)$ with $\gamma_t = \xi_t = 0$ and $d_t = d$. It can be parameterized in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\overline{U}(r,\theta) = \ln(2r) + \overline{d} + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

The Scherk example is parameterized as the vertical graph of

$$U_t(r,\theta) = \ln(2r) + d + \tilde{H}_{r_\varepsilon,\psi}(r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

As for the Dirichlet boundary data, we assume φ to be an even function orthogonal to the constant function and to $\cos\theta$, and we assume ψ to be even and orthogonal to 1.

To prove the theorem in the case $k = 0$, we must show there is a solution to the system

$$\begin{cases} U(r_\varepsilon, \cdot) = \bar{U}_t(r_\varepsilon, \cdot), \\ \partial_r U(r_\varepsilon, \cdot) = \partial_r \bar{U}_t(r_\varepsilon, \cdot) \end{cases}$$

on $\mathbb{S}^1$, under appropriate assumptions on the norms of the Dirichlet boundary data and the parameters ξ, d, $\bar{d}$.

These equations lead to the system

$$\begin{cases} \eta + \varphi - \psi = \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \\ \partial_\theta^*(\varphi + \psi) = \mathbb{O}_{C_b^{1,\alpha}}(\varepsilon). \end{cases}$$

where $\eta = \bar{d} - d$. The proof is completed by the arguments of Section 8.1.

The case $k \geqslant 1$. The proof in this case is similar the proof of Theorem 1.1. In fact three of the surfaces we are going to glue are ones we used there: a compact piece of the Costa–Hoffman–Meeks example M_k, half of a Scherk-type example, and a horizontal periodic flat annulus. The fourth surface is half of a KMR example, of the type we used in the $k = 0$ case. The surfaces are parameterized as vertical graphs over $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the following functions:

$$\bar{U}_b(r, \theta) = -\ln(2r) + d_b + \tilde{H}_{r_\varepsilon, \varphi_b}(r, \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$$

for the Scherk-type example;

$$\bar{U}_m(r, \theta) = \tilde{H}_{r_\varepsilon, \varphi_m}(r, \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$$

for the horizontal periodic flat annulus;

$$\bar{U}_t(r, \theta) = \ln(2r) + d_t + \bar{\mathcal{H}}_{v_\varepsilon, \varphi_t}(\ln 2r, \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$$

for the KMR example; and

$$\begin{aligned} U_t(r, \theta) &= \quad \sigma_t + \ln(2r) + H_{\psi_t}(s_\varepsilon - \ln(2r), \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \\ U_b(r, \theta) &= -\sigma_b - \ln(2r) + H_{\psi_b}(s_\varepsilon - \ln(2r), \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \\ U_m(r, \theta) &= \tilde{H}_{\rho_\varepsilon, \varphi_m}(1/r, \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon) \end{aligned}$$

for the compact piece of the Costa–Hoffman–Meeks example. We require the Dirichlet boundary data to consist of even functions. The functions ψ_t and ψ_b are orthogonal to 1 and to $\cos\theta$, but ψ_m, φ_t, φ_b and φ_m are orthogonal only to 1. In this case the system of equations to solve is

$$\begin{cases} \eta_t + \varphi_t - \psi_t = \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*(\varphi_t + \psi_t) = \mathbb{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \eta_b + \varphi_b - \psi_b = \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*(\varphi_b + \psi_b) = \mathbb{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \varphi_m - \psi_m = \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*(\varphi_m + \psi_m) = \mathbb{O}_{C_b^{1,\alpha}}(\varepsilon), \end{cases}$$

where $\eta_t = d_t - \sigma_t$ and $\eta_b = d_b + \sigma_b$. The details are left to the reader.

Appendix A

Definition A.1. For $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, the space $\mathscr{C}^{\ell,\alpha}_{\nu}(B_{\rho_0}(0))$ is defined to be the space of functions in $\mathscr{C}^{\ell,\alpha}_{\mathrm{loc}}(B_{\rho_0}(0))$ for which the norm $\|\rho^{-\nu}\, w\|_{\mathscr{C}^{\ell,\alpha}(B_{\rho_0}(0))}$ is finite.

Proposition A.2. *There exists an operator $\tilde{H} : C^{2,\alpha}(\mathbb{S}^1) \to C^{2,\alpha}_{-1}([\bar\rho, +\infty) \times \mathbb{S}^1)$, such that for each even function $\varphi(\theta) \in C^{2,\alpha}(\mathbb{S}^1)$ that is L^2-orthogonal to* 1, *the function $w_\varphi = \tilde{H}_{\bar\rho,\varphi}$ solves*

$$\begin{cases} \Delta w_\varphi = 0 & \textit{on } [\bar\rho, +\infty) \times \mathbb{S}^1, \\ w_\varphi = \varphi & \textit{on } \{\bar\rho\} \times \mathbb{S}^1. \end{cases}$$

Moreover, $\|\tilde{H}_{\bar\rho,\varphi}\|_{C^{2,\alpha}_{-1}([\bar\rho,+\infty)\times\mathbb{S}^1)} \leqslant c\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^1)}$ *for some constant $c > 0$.*

Remark A.3. Following the arguments of the proof below, it is possible to state a similar proposition but with the hypothesis that φ is odd.

Proof. We decompose the function φ in the basis $\{\cos(i\theta)\}$ as $\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(i\theta)$. Then the solution w_φ is given by

$$w_\varphi(\rho, \theta) = \sum_{i=1}^{\infty} \left(\frac{\bar\rho}{\rho}\right)^i \varphi_i \cos(i\theta).$$

Because $\bar\rho/\rho \leqslant 1$, we have $(\bar\rho/\rho)^i \leqslant (\bar\rho/\rho)$. Thus $|w(r,\theta)| \leqslant c\rho^{-1}|\varphi(\theta)|$ and then $\|w_\varphi\|_{C^{2,\alpha}_{-1}} \leqslant c\|\varphi\|_{C^{2,\alpha}}$. □

Now we state a useful result; for a proof see [Fakhi and Pacard 2000].

Proposition A.4. *There exists an operator $H : \mathscr{C}^{2,\alpha}(\mathbb{S}^1) \to \mathscr{C}^{2,\alpha}_{-2}([0, +\infty) \times \mathbb{S}^1)$, such that, for all $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ that are even and L^2-orthogonal to* 1 *and* $\cos\theta$, *the function $w = H_\varphi$ solves*

$$\begin{cases} (\partial_s^2 + \partial_\theta^2) w = 0 & \textit{in } [0, +\infty) \times \mathbb{S}^1, \\ w = \varphi & \textit{on } \{0\} \times \mathbb{S}^1. \end{cases}$$

Moreover $\|H_\varphi\|_{\mathscr{C}^{2,\alpha}_{-2}} \leqslant c\|\varphi\|_{\mathscr{C}^{2,\alpha}}$ *for some constant $c > 0$.*

Proposition A.5. *There exists an operator*

$$\overline{\mathscr{H}}_{v_0} : C^{2,\alpha}(\mathbb{S}^1) \to C^{2,\alpha}_{\mu}([v_0, +\infty) \times \mathbb{S}^1)$$

for $\mu \in (-2,-1)$ *such that, for every function* $\varphi(u) \in C^{2,\alpha}(\mathbb{S}^1)$ *that is even and* L^2*-orthogonal to* $e_{0,i}(u)$ *with* $i = 0, 1$, *the function* $w_\varphi = \overline{\mathcal{H}}_{v_0,\varphi}$ *solves*

$$\begin{cases} \partial^2_{uu} w_\varphi + \partial^2_{vv} w_\varphi = 0 & \text{on } [v_0,+\infty) \times \mathbb{S}^1, \\ w_\varphi = \varphi & \text{on } \{v_0\} \times \mathbb{S}^1. \end{cases}$$

Moreover, $\|\overline{\mathcal{H}}_{v_0,\varphi}\|_{C^{2,\alpha}_\mu([v_0,+\infty)\times\mathbb{S}^1)} \leqslant c\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^1)}$ *for some constant* $c > 0$.

Proof. We decompose of the function φ in the basis $\{e_{0,i}(u)\}$ as $\varphi = \sum_{i=2}^{\infty} \varphi_i e_{0,i}(u)$. Then the solution w_φ is given by

$$w_\varphi(u,v) = \sum_{i=2}^{\infty} e^{-i(v-v_0)} \varphi_i e_{0,i}(u).$$

We recall that $\mu \in (-2,-1)$, so we have $-i \leqslant \mu$, from which it follows that

$$\begin{aligned} \|w_\varphi\|_{\mathscr{C}^{2,\alpha}([v,v+1]\times\mathbb{S}^1)} &\leqslant e^{\mu(v-v_0)}\|\varphi\|_{C^{2,\alpha}}, \\ \|w_\varphi\|_{\mathscr{C}^{2,\alpha}_\mu} &= \sup_{v\in[v_0,\infty]} e^{-\mu v}\|w_\varphi\|_{\mathscr{C}^{2,\alpha}([v,v+1]\times\mathbb{S}^1)} \\ &\leqslant \sup_{v\in[v_0,\infty]} e^{-\mu v} e^{\mu(v-v_0)}\|\varphi\|_{C^{2,\alpha}} \leqslant e^{-\mu v_0}\|\varphi\|_{C^{2,\alpha}}. \end{aligned}$$ □

Lemma A.6. *Let* $u(r,\theta)$ *be the harmonic extension defined on* $[r_0,+\infty)\times\mathbb{S}^1$ *of the even function* $\varphi = \sum_{i\geqslant 0}\varphi_i\cos(i\theta) \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$, *and suppose* $u(r_0,\theta) = \varphi(\theta)$. *Then* $\partial^*_\theta\varphi(\theta) = r_0\partial_r u(r,\theta)|_{r=r_0}$.

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0}\varphi_i\cos(i\theta)$, then the function u is given by

$$u(r,\theta) = \sum_{i\geqslant 0}\varphi_i\left(\frac{r}{r_0}\right)^i\cos(i\theta).$$

Then $\partial_r u(r,\theta) = \sum_{i\geqslant 1}\varphi_i\,(r/r_0)^i\,i\cos(i\theta)/r$, and $\partial^*_\theta\varphi(\theta) = r_0\partial_r u(r,\theta)|_{r=r_0}$. □

Lemma A.7. *Let* $u(r,\theta)$ *be the harmonic extension defined on* $[0,r_0]\times\mathbb{S}^1$ *of the even function* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$, *with* $u(r_0,\theta) = \varphi(\theta)$. *Then* $\partial^*_\theta\varphi(\theta) = -r_0\partial_r u(r,\theta)_{|r=r_0}$.

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0}\varphi_i\cos(i\theta)$, then

$$u(r,\theta) = \sum_{i\geqslant 0}\varphi_i(r_0/r)^i\cos(i\theta).$$

Then $\partial_r u(r,\theta) = -\sum_{i\geqslant 1}\varphi_i\,(r_0/r)^i\,i\cos(i\theta)/r$, and the result follows. □

Appendix B

Proof of Proposition 7.1. Let Z be the immersion of the surface $\widetilde{M}_{\sigma,\alpha,\beta}$ and N its normal vector. We want to find the differential equation a function f must satisfy so that the surface parameterized by $Z_f = Z + fN$ is minimal. In Section 5.2 we parameterized the surface $\widetilde{M}_{\sigma,\alpha,\beta}$ on the cylinder $\mathbb{S}^1 \times \mathbb{R}$. We introduced the map $\mathbf{z}(x, y) : \mathbb{S}^1 \times [0, \pi[\to \overline{\mathbb{C}}$ where x, y denote the spheroconal coordinates. We start with the conformal variables p and q, defined to be as the real and the imaginary part of $\mathbf{z}$. We have

$$\begin{aligned} |Z_p|^2 &= |Z_q|^2 = \Lambda, & |N_p|^2 &= |N_q|^2 = -K\Lambda, \\ \langle N_p, N\rangle &= \langle N_q, N\rangle = 0, & \langle Z_p, Z_q\rangle &= 0, \quad \langle N_p, N_q\rangle = 0, \\ \langle N_q, Z_q\rangle &= -\langle N_p, Z_p\rangle, & \langle N_q, Z_p\rangle &= \langle N_p, Z_q\rangle, \end{aligned}$$

so

$$\begin{aligned} \langle N_p, Z_p\rangle &= |N_p||Z_p| \cos\gamma_1 = \sqrt{-K}\Lambda \cos\gamma_1, \\ \langle N_p, Z_q\rangle &= |N_p||Z_q| \cos\gamma_2 = \sqrt{-K}\Lambda \cos\gamma_2. \end{aligned}$$

Here K denotes the Gauss curvature, Z_p, Z_q and N_p, N_q denote the partial derivatives of the vectors Z and N, γ_1 is the angle between the vectors N_p and Z_p, and γ_2 is the angle between the vectors N_p and Z_q.

The proof of Proposition 7.1 is articulated through some lemmas. We denote by E_f, F_f, G_f the coefficients of the second fundamental form for the surface parameterized by Z_f. The first lemma expresses the area energy functional.

Lemma B.1. $A(f) := \int (E_f G_f - F_f^2)^{1/2}\, dp\, dq$, *with*

$$\begin{aligned} E_f G_f - F_f^2 = \Lambda^2 + \Lambda(f_p^2 + f_q^2) + 2K\Lambda^2 f^2 + 2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1 \\ - 4ff_pf_q\sqrt{-K}\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2. \end{aligned}$$

Proof. The coefficients of the second fundamental form are

$$\begin{aligned} E_f &= |\partial_p Z_f|^2 = |Z_p|^2 + f_p^2 + f^2|N_p|^2 + 2f\langle N_p, Z_p\rangle, \\ G_f &= |\partial_q Z_f|^2 = |Z_q|^2 + f_q^2 + f^2|N_q|^2 + 2f\langle N_q, Z_q\rangle, \\ F_f &= |\partial_p Z_f \cdot \partial_q Z_f| = f_p f_q + f(\langle Z_p, N_q\rangle + \langle Z_q, N_p\rangle). \end{aligned}$$

Then

$$\begin{aligned} E_f G_f = |Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2) \\ + f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2 + 4f^2(\langle N_pZ_p\rangle)(\langle N_qZ_q\rangle) \\ + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle) + f_p^2f_q^2 \\ + 2f(\langle N_q, Z_q\rangle|Z_p|^2 + \langle N_p, Z_p\rangle|Z_q|^2) + 2f^3(\langle N_q, Z_q\rangle|Z_p|^2 + \langle N_p, Z_p\rangle|Z_q|^2). \end{aligned}$$

Since $\langle N_q, Z_q\rangle + \langle N_p, Z_p\rangle = 0$ and $|Z_p|^2 = |Z_q|^2$, we can conclude that the last two terms of the previous expression are zero. Since $\langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle$, we have $F_f = f_p f_q + 2f\langle N_p, Z_q\rangle$. Then

$$F_f^2 = f_p^2 f_q^2 + 4f^2(\langle N_p, Z_q\rangle)^2 + 4ff_pf_q\langle N_p, Z_q\rangle.$$

So the expression for $E_f G_f - F_f^2$ is

$$\begin{aligned}|Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2)\\ + f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2 + 4f^2\langle N_p, Z_p\rangle\langle N_q, Z_q\rangle\\ + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle) - 4f^2(\langle N_p, Z_q\rangle)^2 - 4ff_pf_q\langle N_p, Z_q\rangle.\end{aligned}$$

Ordering the terms, we get

$$\begin{aligned}|Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2)\\ - 4f^2\langle N_p, Z_q\rangle^2 + 4f^2\langle N_p, Z_p\rangle\langle N_q, Z_q\rangle + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle)\\ - 4ff_pf_q\langle N_p, Z_q\rangle + f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2.\end{aligned}$$

The expression for $E_f G_f - F_f^2$ becomes

$$\begin{aligned}\Lambda^2 + \Lambda(f_p^2 + f_q^2) - 2K\Lambda^2 f^2 + 4f^2K\Lambda^2(\cos^2\gamma_1 + \cos^2\gamma_2)\\ + 2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1 - 4ff_pf_q\sqrt{-K}\Lambda\cos\gamma_2\\ - K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2.\end{aligned}$$

Using the relations $\langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle$ and $\langle N_q, Z_q\rangle = -\langle N_p, Z_p\rangle$, one can see that vectors are pointed so that $\gamma_2 = \pi/2 \pm \gamma_1$. So $\cos^2\gamma_2 = \cos^2(\pi/2 \pm \gamma_1) = \sin^2\gamma_1$ and $\cos^2\gamma_1 + \cos^2\gamma_2 = 1$. Then we can write

$$\begin{aligned}\Lambda^2 + \Lambda(f_p^2 + f_q^2) + 2K\Lambda^2 f^2 + 2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1\\ - 4ff_pf_q\sqrt{-K}\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2. \quad \square\end{aligned}$$

The next lemma completes the proof of Proposition 7.1.

Lemma B.2. *The surface whose immersion is given by $Z + fN$, is minimal if and only if f satisfies*

$$\mathcal{L}_\sigma f + Q_\sigma(f) = 0,$$

where $\mathcal{L}_\sigma$ is the Lamé operator and Q_σ is a second order differential operator that satisfies

$$\begin{aligned}\|Q_\sigma(f_2) - Q_\sigma(f_1)\|_{\mathscr{C}^{0,\alpha}(I_\sigma\times[v,v+1])}\\ \leqslant c\sup_{i=1,2}\|f_i\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\|f_2 - f_1\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}.\end{aligned}$$

Proof. The surface parameterized by $Z_f = Z + fN$ is minimal if and only the first variation of $A(f)$ is 0. That is,

$$2DA(g) = \int \frac{1}{(E_f G_f - F_f^2)^{1/2}}\Big|_{f=0} D_f(E_f G_f - F_f^2)(g)\, dp\, dq = 0.$$

By the previous lemma, the integrand above is equal to

$$\begin{aligned}\frac{1}{\Lambda}\Big(&2\Lambda(f_p g_p + f_q g_q) + 4K\Lambda^2 fg \\ &\qquad + 2\sqrt{-K}\Lambda\cos\gamma_1\big(2ff_q g_q + gf_q^2 - 2ff_p g_p - gf_p^2\big) \\ &\qquad - 4\sqrt{-K}\Lambda\cos\gamma_2\big(ff_q g_p + fg_q f_p + gf_p f_q\big) \\ &\qquad\quad - 2K\Lambda\big(fgf_p^2 + f_p g_p f^2 + fgf_q^2 + f_q g_q f^2\big) + 4K^2\Lambda^2 f^3 g\Big),\end{aligned}$$

which, by reordering the summands, becomes

$$\begin{aligned}2\Big(&f_p g_p + f_q g_q + 2K\Lambda fg \\ &\qquad + \sqrt{-K}\cos\gamma_1\big(2f(f_q g_q - f_p g_p) + g(f_q^2 - f_p^2)\big) \\ &\qquad - 2\sqrt{-K}\cos\gamma_2\big(f(f_q g_p + g_q f_p) + gf_p f_q\big) \\ &\qquad\quad - K\big(fg(f_p^2 + f_q^2) + f^2(f_p g_p + f_q g_q)\big) + 2K^2\Lambda f^3 g\Big).\end{aligned}$$

In the next computation we skip the overall factor of 2 in this expression. We find

$$f_p g_p + f_q g_q + 2K\Lambda fg + Q_1(f, f_p, f_q)g - Q_2(f, f_p, f_q)g_p - Q_3(f, f_p, f_q)g_q = 0,$$

where

$$\begin{aligned}Q_1(f, f_p, f_q) &= -(f_p^2 - f_q^2)\sqrt{-K}\cos\gamma_1 - 2f_p f_q\sqrt{-K}\cos\gamma_2 \\ &\qquad\qquad - Kf(f_p^2 + f_q^2) + 2K^2\Lambda f^3, \\ Q_2(f, f_p, f_q) &= 2ff_p\sqrt{-K}\cos\gamma_1 + 2ff_q\sqrt{-K}\cos\gamma_2 + Kf^2 f_p, \\ Q_3(f, f_p, f_q) &= -2ff_q\sqrt{-K}\cos\gamma_1 + 2ff_p\sqrt{-K}\cos\gamma_2 + Kf^2 f_q.\end{aligned}$$

An integration by parts and a change of sign give us the equation

$$\begin{aligned}\big(&f_{pp} + f_{qq} - 2K\Lambda f - Q_1(f, f_p, f_q) \\ &\qquad + P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) + P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq})\big)g = 0,\end{aligned}$$

where

$$\begin{aligned}P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) &= \partial_p Q_2(f, f_p, f_q), \\ P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) &= \partial_q Q_3(f, f_p, f_q).\end{aligned}$$

That is,

$$
\begin{aligned}
&P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = \\
&\quad 2(f_p^2 + f f_{pp})\sqrt{-K}\cos\gamma_1 + 2(f_p f_q + f f_{pq})\sqrt{-K}\cos\gamma_2 + K(2 f f_p^2 + f^2 f_{pp}) \\
&\qquad + 2f\big(f_p(\sqrt{-K}\cos\gamma_1)_p + f_q(\sqrt{-K}\cos\gamma_2)_p\big) + f^2 f_p K_p, \\
&P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = \\
&\quad -2(f_q^2 + f f_{qq})\sqrt{-K}\cos\gamma_1 + 2(f_p f_q + f f_{pq})\sqrt{-K}\cos\gamma_2 + K(2 f f_q^2 + f^2 f_{qq}) \\
&\qquad + 2f\big(-f_q(\sqrt{-K}\cos\gamma_1)_q + f_p(\sqrt{-K}\cos\gamma_2)_q\big) + f^2 f_q K_q.
\end{aligned}
$$

Now we want to understand how differential equation above changes when passing from the (p, q) to the (u, v) variables. We recall that p and q are the real and imaginary part of the variable $\mathbf{z}$ that is expressed in terms of the spheroconal coordinates x, y in (28). The metric $\bar{g}$ induced on a surface whose immersion Z is given by the Weierstrass representation on a domain of the complex $\mathbf{z}$-plane can be expressed in terms of the metric $d\bar{s}^2 = dp^2 + dq^2$ by $\bar{g} = \Lambda(dp^2 + dq^2)$, where $\Lambda = |Z_p|^2 = |Z_q|^2$. The Laplace–Beltrami operators written with respect to the metrics $d\bar{s}^2$ and $\bar{g}$ are related by $\Delta_{d\bar{s}^2} = (1/\Lambda)\Delta_{\bar{g}}$, that is, they differ by the conformal factor $1/\Lambda$. In Proposition 7.1, we observed that the conformal factor related to the change of coordinates $(x, y) \to (u, v)$ is $-K/k$. So the conformal factor induced by the change $(p, q) \to (u, v)$ is the product of the conformal factors described above. Summarizing, we have

$$f_{pp} + f_{qq} = \frac{-K\Lambda}{k}(f_{uu} + f_{vv}).$$

So we can write

$$\frac{-K\Lambda}{k}(f_{uu} + f_{vv}) + 2(-K\Lambda) f + R_1 + R_2 + R_3 = 0,$$

where

$$
\begin{aligned}
&R_1(f, f_u, f_v) = \\
&\quad -\frac{-K\Lambda}{k}\big(-(f_u^2 - f_v^2)\sqrt{-K}\cos\gamma_1 - 2 f_u f_v\sqrt{-K}\cos\gamma_2 - K f(f_u^2 + f_v^2)\big) - 2K^2\Lambda f^3 \\
&\quad = \frac{-K\Lambda}{k}\big((f_u^2 - f_v^2)\sqrt{-K}\cos\gamma_1 + 2 f_u f_v\sqrt{-K}\cos\gamma_2 + K f(f_u^2 + f_v^2) - 2Kk f^3\big) \\
&\quad = \frac{-K\Lambda}{k}\bar{P}_1(f, f_u, f_v),
\end{aligned}
$$

and

$$
\begin{aligned}
R_2(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) &= \frac{-K\Lambda}{k} P_2(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}), \\
R_3(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) &= \frac{-K\Lambda}{k} P_3(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}).
\end{aligned}
$$

Simplifying the notation, we can write

$$\frac{-K\Lambda}{k}\big(f_{uu}+f_{vv}+2k(u,v)f+\bar{P}_1(f)+P_2(f)+P_3(f)\big)=0.$$

We can recognize the Lamé operator in

$$\mathscr{L}_\sigma f = f_{uu}+f_{vv}+2(\sin^2\sigma\cos^2 x(u)+\cos^2\sigma\sin^2 y(v))f;$$

then, if we set $Q_\sigma=\bar{P}_1(f)+P_2(f)+P_3(f)$, the equation can be written

$$\mathscr{L}_\sigma f+Q_\sigma(f)=0.$$

To show the estimate of Q_σ, it suffices to show that all its coefficients are bounded. In particular we will show that the Gauss curvature K and its derivatives K_u and K_v are bounded. We start observing that $-K/k(x(u),y(v))$ is bounded. It is well known that the Gauss curvature can be expressed in terms of the Weierstrass data g, dh as

$$K=-16\Big(|g|+\frac{1}{|g|}\Big)^{-4}\Big|\frac{dg}{g}\Big|^2|dh|^{-2}$$

We recall that $dh=\mu\,dz/\sqrt{(z^2+\lambda^2)(z^2+\lambda^{-2})}$. Now $|z^2+\lambda^2||z^2+\lambda^{-2}|$ and $k(x,y)=\sin^2\sigma\cos^2 x(u)+\cos^2\sigma\sin^2 y(v)$ have the same zeros, that is, the points D, D', D'', D''' are given by (23), so $-K/k$ is bounded, as are its derivatives.

We estimate the derivatives of K and $\sqrt{-K}$. We can write $\sqrt{-K}=\sqrt{k}\sqrt{-K/k}$. From the observations made above, it follows that to show that the derivatives of $\sqrt{-K}$ are bounded, it suffices to study the derivatives of $\sqrt{k}$.

We recall that

$$l(x)=\sqrt{1-\sin^2\sigma\sin^2 x}\quad\text{and}\quad m(y)=\sqrt{1-\cos^2\sigma\cos^2 y}.$$

From the expression of k, it is easy to get from (29) that

$$\frac{\partial}{\partial u}\sqrt{k}=-\frac{\sin^2\sigma\sin 2x(u)}{2\sqrt{k}}l(x(u))\quad\text{and}\quad\frac{\partial}{\partial v}\sqrt{k}=\frac{\cos^2\sigma\sin 2y(v)}{2\sqrt{k}}m(y(v)).$$

Then

$$\Big|\frac{\partial}{\partial u}\sqrt{k}\Big|=\frac{\sin^2\sigma|\sin 2x(u)|l(x(u))}{2\sqrt{\sin^2\sigma\cos^2 x(u)+\cos^2\sigma\sin^2 y(v)}}\leqslant\frac{\sin^2\sigma|\sin 2x(u)|}{2\sin\sigma|\cos x(u)|}\leqslant\sin\sigma,$$

$$\Big|\frac{\partial}{\partial v}\sqrt{k}\Big|=\frac{\cos^2\sigma|\sin 2y(v)|m(y(v))}{2\sqrt{\sin^2\sigma\cos^2 x(u)+\cos^2\sigma\sin^2 y(v)}}\leqslant\frac{\cos^2\sigma|\sin 2y(v)|}{2\cos\sigma|\sin y(v)|}\leqslant\cos\sigma.$$

Thus the derivatives of $\sqrt{k}$ (and consequently those of $\sqrt{-K}$) are bounded. □

Appendix C

The differential equation

$$\sin y\, \partial_y(\sin y\, \partial_y f) - j^2 f + 2\sin^2 y\, f = 0 \tag{56}$$

is the $l = 1$ case of the associated Legendre differential equation

$$\sin y\, \partial_y(\sin y\, \partial_y f) - j^2 f + l(l+1)\sin^2 y\, f = 0,$$

where $l, j \in \mathbb{N}$. The family of the solutions of (56) (see [Abramowitz and Stegun 1964]) is $c_1 P_l^j(\cos y) + c_2 Q_l^j(\cos y)$ for $l = 1$, where $P_l^j(t)$ and $Q_l^j(t)$ are respectively the associated Legendre functions of first and second kind. If $l = 1$, these functions are defined as follows:

$$P_1^j(t) = \begin{cases} t & \text{if } j = 0, \\ -\sqrt{1-t^2} & \text{if } j = 1, \\ 0 & \text{if } j \geqslant 2, \end{cases}$$

$$Q_1^j(t) = (-1)^j \sqrt{(1-t^2)^j}\, \frac{d^j Q_1^0(t)}{dt^j}$$

$$Q_1^0(t) = \tfrac{1}{2} t \ln\Big(\frac{1+t}{1-t}\Big) - 1.$$

References

[Abramowitz and Stegun 1964] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series **55**, US Government Printing Office, Washington, D.C., 1964. MR 29 #4914 Zbl 0171.38503

[Costa 1982] C. J. Costa, *Imersões minimas en* $\mathbb{R}^3$ *de gênero un e curvatura total finita*, thesis, Instituto Nacional De Mathemática Pura e Aplicada, Rio de Janeiro, 1982.

[Costa 1984] C. J. Costa, "Example of a complete minimal immersion in $\mathbf{R}^3$ of genus one and three embedded ends", *Bol. Soc. Brasil. Mat.* **15**:1-2 (1984), 47–54. MR 87c:53111 Zbl 0613.53002

[Fakhi and Pacard 2000] S. Fakhi and F. Pacard, "Existence result for minimal hypersurfaces with a prescribed finite number of planar ends", *Manuscripta Math.* **103**:4 (2000), 465–512. MR 2002d:53017 Zbl 0992.53011

[Hauswirth and Pacard 2007] L. Hauswirth and F. Pacard, "Higher genus Riemann minimal surfaces", *Invent. Math.* **169**:3 (2007), 569–620. MR 2008i:58013 Zbl 1129.53009

[Hauswirth and Traizet 2002] L. Hauswirth and M. Traizet, "The space of embedded doubly-periodic minimal surfaces", *Indiana Univ. Math. J.* **51**:5 (2002), 1041–1079. MR 2003k:53010 Zbl 1039.53012

[Hoffman and Meeks 1985] D. A. Hoffman and W. Meeks, III, "A complete embedded minimal surface in $\mathbf{R}^3$ with genus one and three ends", *J. Differential Geom.* **21**:1 (1985), 109–127. MR 87d:53008 Zbl 0604.53002

[Hoffman and Meeks 1989] D. Hoffman and W. H. Meeks, III, "The asymptotic behavior of properly embedded minimal surfaces of finite topology", *J. Amer. Math. Soc.* **2**:4 (1989), 667–682. MR 90f:53010 Zbl 0683.53005

[Hoffman and Meeks 1990] D. Hoffman and W. H. Meeks, III, "Embedded minimal surfaces of finite topology", *Ann. of Math.* (2) **131**:1 (1990), 1–34. MR 91i:53010 Zbl 0695.53004

[Jansen 1977] J. K. M. Jansen, *Simple-periodic and non-periodic Lamé functions*, Mathematical Centre Tracts **72**, Mathematisch Centrum, Amsterdam, 1977. MR 57 #3476

[Jleli 2004] M. Jleli, *Constant mean curvature hypersurfaces*, thesis, University Paris 12, 2004.

[Karcher 1988] H. Karcher, "Embedded minimal surfaces derived from Scherk's examples", *Manuscripta Math.* **62**:1 (1988), 83–114. MR 89i:53009 Zbl 0658.53006

[Karcher 1989] H. Karcher, *Construction of minimal surfaces*, University of Tokyo, 1989. Also, Lecture Notes n. 12, SFB256, Berlin (1989).

[Kato 1980] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1980. MR 96a:47025 Zbl 0836.47009

[Kusner et al. 1996] R. Kusner, R. Mazzeo, and D. Pollack, "The moduli space of complete embedded constant mean curvature surfaces", *Geom. Funct. Anal.* **6**:1 (1996), 120–137. MR 97b:58022 Zbl 0966.58005

[Martín and Ramos Batista 2006] F. Martín and V. Ramos Batista, "The embedded singly periodic Scherk–Costa surfaces", *Math. Ann.* **336**:1 (2006), 155–189. MR 2007d:53011 Zbl 1103.53003

[Mazet and Traizet 2008] L. Mazet and M. Traizet, "A quasi-periodic minimal surface", *Comment. Math. Helv.* **83**:3 (2008), 573–601. MR 2410781 Zbl 05311215

[Mazet et al. 2007] L. Mazet, M. M. Rodríguez, and M. Traizet, "Saddle towers with infinitely many ends", *Indiana Univ. Math. J.* **56**:6 (2007), 2821–2838. MR 2375703 Zbl 05254001

[Meeks and Rosenberg 1989] W. H. Meeks, III and H. Rosenberg, "The global theory of doubly periodic minimal surfaces", *Invent. Math.* **97**:2 (1989), 351–379. MR 90m:53017

[Melrose 1993] R. B. Melrose, *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics **4**, Peters, Wellesley, MA, 1993. MR 96g:58180 Zbl 0796.58050

[Morabito 2008a] F. Morabito, "About a family of deformation of the Costa–Hoffmann–Meeks surfaces", preprint, 2008. To appear in *Bull. Bras. Math. Soc.*

[Morabito 2008b] F. Morabito, "Index and nullity of the Gauss map of the Costa–Hoffman–Meeks surfaces", preprint, 2008. To appear in *Indiana Univ. Math. J.* arXiv 0806.1836v1

[Nayatani 1992] S. Nayatani, "Morse index of complete minimal surfaces", pp. 181–189 in *The problem of Plateau*, edited by T. M. Rassias, World Sci. Publ., River Edge, NJ, 1992. MR 94e:58027 Zbl 0794.58011

[Nayatani 1993] S. Nayatani, "Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space", *Comment. Math. Helv.* **68**:4 (1993), 511–537. MR 95b:58039 Zbl 0797.58018

[Pérez et al. 2005] J. Pérez, M. M. Rodríguez, and M. Traizet, "The classification of doubly periodic minimal tori with parallel ends", *J. Differential Geom.* **69**:3 (2005), 523–577. MR 2006j:53010 Zbl 1094.53007

[Ramos Batista 2005] V. Ramos Batista, "Singly periodic Costa surfaces", *J. London Math. Soc.* (2) **72**:2 (2005), 478–496. MR 2006m:53018 Zbl 1083.53012

[Rodríguez 2007] M. M. Rodríguez, "The space of doubly periodic minimal tori with parallel ends: Standard examples", *Michigan Math. J.* **55**:1 (2007), 103–122. MR 2008i:53009 Zbl 1127.53006

[Rossman et al. 2000] W. Rossman, E. C. Thayer, and M. Wohlgemuth, "Embedded, doubly periodic minimal surfaces", *Experiment. Math.* **9**:2 (2000), 197–219. MR 2001g:53016 Zbl 1031.53014

[Scherk 1835] H. Scherk, "Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen", *J. R. Angew. Math.* **13** (1835), 185–208.

[Wei 1992] F. S. Wei, "Some existence and uniqueness theorems for doubly periodic minimal surfaces", *Invent. Math.* **109**:1 (1992), 113–136. MR 93c:53004 Zbl 0773.53005

Received June 10, 2008.

LAURENT HAUSWIRTH
UNIVERSITÉ PARIS-EST
LABORATOIRE D'ANALYSE ET MATHÉMATIQUES APPLIQUÉES
5 BLVD DESCARTES
77454 CHAMPS-SUR-MARNE
FRANCE
laurent.hauswirth@univ-mlv.fr

FILIPPO MORABITO
UNIVERSITÉ PARIS-EST
LABORATOIRE D'ANALYSE ET MATHÉMATIQUES APPLIQUÉES
5 BLVD DESCARTES
77454 CHAMPS-SUR-MARNE
FRANCE
and
UNIVERSITÀ ROMA TRE
DIPARTIMENTO DI MATEMATICA
LARGO S.L. MURIALDO 1
00146 ROMA
ITALY
morabito@mat.uniroma3.it

M. MAGDALENA RODRÍGUEZ
UNIVERSIDAD COMPLUTENSE DE MADRID
DEPARTAMENTO DE ÁLGEBRA
PLAZA DE LAS CIENCIAS 3
28040 MADRID
SPAIN
magdalena@mat.ucm.es

PACIFIC JOURNAL OF MATHEMATICS
Vol. 241, No. 1, 2009

SYMMETRIC SURFACES OF CONSTANT MEAN CURVATURE IN $\mathbb{S}^3$

RYAN HYND, SUNG-HO PARK AND JOHN MCCUAN

We introduce two notions of symmetry for surfaces in $\mathbb{S}^3$. The first, special spherical symmetry, generalizes the notion of rotational symmetry, and we classify all complete surfaces of constant mean curvature having this symmetry. These surfaces turn out to also be rotationally symmetric, so our characterization answers a question first posed by Hsiang in 1982 and also considered by several authors since. From this point of view, these are the Delaunay surfaces of $\mathbb{S}^3$.

Our second notion of symmetry, spherical symmetry, is a substantial, and we believe important, technical generalization of special spherical symmetry. We classify all compact surfaces of constant mean curvature having this symmetry. We show in particular that the only compact embedded minimal surfaces possessing this kind of symmetry are the great spheres and the Clifford torus.

We derive from our classification theorem a special case of Lawson's conjecture that the only embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus.

Introduction

We consider surfaces in the three-dimensional sphere $\mathbb{S}^3 = \{\mathbf{x} \in \mathbb{R}^4 : |\boldsymbol{x}| = 1\}$. Perhaps the simplest notion of symmetry for surfaces in $\mathbb{S}^3$ is that of rotational symmetry, as exemplified by the Clifford torus

$$\mathscr{C} = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 = 1/2 = z^2 + w^2\}.$$

It is usual to explain the symmetry of such a surface by saying it is invariant under an $\mathbb{S}^1$ action that fixes a geodesic. We will take a somewhat different approach

MSC2000: 53A10.

Keywords: Delaunay surfaces, constant mean curvature, Willmore surfaces, Clifford torus, minimal surfaces, submanifolds in space forms, Lawson's conjecture.

Hynd was supported in part by the Georgia Tech School of Mathematics VIGRE program and a Georgia Tech Summer Research Assistantship. Park was supported by the Postdoctoral Fellowship Program of the Korean Science & Engineering Foundation (KOSEF). McCuan was supported in part by NSF grant DMS-0103848, and portions of the work were completed at the Max-Planck-Institut für Mathematik in den Naturwissenschaften.

suggested by the decomposition of rotational symmetry into some sufficient number of invariances under reflection maps. Such a decomposition was first used by Alexandrov [1962] and modified in the direction of our interest in [Wente 1980] and [McCuan 1997]. Each of these papers consider surfaces in $\mathbb{R}^3$, and though the symmetry conditions considered here are described directly in $\mathbb{S}^3$ without reference to particular coordinates, we find it easiest to think about and describe these results in terms of stereographic projections into $\mathbb{R}^3$ of the geometric objects involved.

The surface $\mathscr{C}$ mentioned above stereographically projects to the anchor ring

$$\left\{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - \sqrt{2}\right)^2 + z^2 = 1\right\},$$

which is rotationally symmetric in $\mathbb{R}^3$. The anchor ring is invariant under the orientation-reversing transformation of reflection through each vertical plane containing the z-axis. If we apply a preliminary rotation R of $\mathbb{S}^3$ to obtain the congruent torus $R(\mathscr{C})$, the stereographic projection of $R(\mathscr{C})$ will not, in general, be invariant under the orientation-preserving isometry of rotation in $\mathbb{R}^3$. Properly generalized, however, invariance under reflections (Kelvin transforms) will be preserved.

Our first symmetry condition, *special spherical symmetry*, is more general than, but includes, the usual condition of being invariant under an $\mathbb{S}^1$ action that fixes a geodesic. Roughly speaking, the stereographic projection must be invariant under a continuous one-parameter family of reflections (that is, Kelvin transforms) through spheres, and these reflections must fit together in a nice way, that is, satisfy a coherence condition. The case of rotational symmetry (of Alexandrov and Wente and the standard projection of the Clifford torus) mentioned above appears as a kind of degenerate case in which each of the spheres is a plane, and the coherence condition requires that the planes all contain a common line. In the more general case, it is required that the spheres of symmetry all contain a common circle.

Notice also that there are certainly surfaces in $\mathbb{S}^3$ with stereographic projection rotationally symmetric in $\mathbb{R}^3$ but that do not satisfy the usual definition of rotational symmetry in $\mathbb{S}^3$. The fact that our new notions of symmetry do not depend on isometries of $\mathbb{S}^3$ is one of the reasons that they are more general and one of the main reasons they are of interest.

In Section 2 we classify all complete, connected constant mean curvature surfaces having this symmetry; see Theorem 1 below. The classified surfaces are parameterized explicitly in terms of elliptic integrals, and we can determine completely the topology, and compactness in particular, of each surface.

Next, we introduce a more general symmetry condition, *spherical symmetry*, in which continuity of the one-parameter family of reflections is relaxed and the coherence condition is partially relaxed. More precisely, in the stereographic projection, the symmetry spheres are not required to contain a common circle, but their Euclidean centers are required to be points on a Euclidean line.

In Section 3 we show that all compact surfaces with spherical symmetry actually posses special spherical symmetry; see Theorem 2. This classification restricted to those surfaces that are embedded, compact and minimal gives a special case of Lawson's conjecture [Yau 1982, Problem 97]:

Theorem. *The Clifford torus is the only embedded minimal torus with spherical symmetry.*

A somewhat analogous result under very different symmetry assumptions was obtained by Ros [1995]. The referee brought to our attention the very beautiful paper of Kilian and Schmidt [2008], which gives a much more extensive discussion of Lawson's conjecture than found here and proves the result under the assumption that the torus has finite type.

The classification of complete constant mean curvature surfaces with spherical symmetry remains open. We suspect there are no new surfaces in this broader class, but we use compactness very strongly and cannot make that assertion with any confidence.

Ultimately, the conditions of special spherical symmetry and spherical symmetry each rely on invariance of the image of the surface considered as an immersion into $\mathbb{S}^3$ from an abstract Riemannian manifold. For this reason, we must make preliminary arguments in Section 1 to obtain an explicit local parameterization on a domain in $\mathbb{R}^2$ whose dependent variables will satisfy explicit differential equations. The existence of such parameterizations has been, as far as we can tell, simply assumed in the literature. Our basic parameterization result, Theorem 4, should naturally generalize to apply in other contexts. Once a parameterization is obtained, stereographic projection plays a key role again in providing a crucial change of variables through which the equation of constant mean curvature takes a form amenable to explicit integration.

Rossman and Sultana [2007; 2008] have recently considered classification questions similar to those considered here. The particular coordinates we have chosen are crucial to the latter developments of the paper and have other advantages as well, including making obvious the identification of these surfaces with the surfaces of Delaunay and permitting easy access to explicit integral representation. Both of these topics will be discussed further below.

0.1. ***Reflection and special spherical symmetry.*** We now describe the notion of *generalized* (or spherical) *reflection maps*, on which our symmetry conditions are based. There are two kinds of generalized reflection maps $f : \mathbb{S}^3 \to \mathbb{S}^3$. The first is

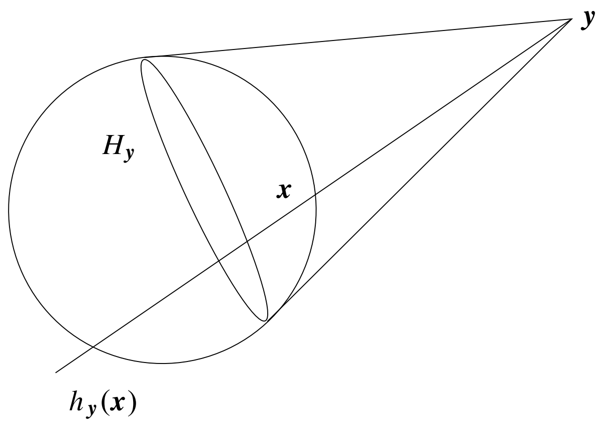

Figure 1. Cone point reflection.

the restriction to $\mathbb{S}^3$ of appropriate reflections of $\mathbb{R}^4$. To be precise, given $\boldsymbol{n} \in \mathbb{S}^3$,

$$g_{\boldsymbol{n}} : \mathbb{S}^3 \to \mathbb{S}^3, \quad \boldsymbol{x} \mapsto \boldsymbol{x} - 2(\boldsymbol{x} \cdot \boldsymbol{n})\boldsymbol{n}$$

is the restriction to $\mathbb{S}^3$ of the reflection through the (hyper)plane $\{\boldsymbol{x} \in \mathbb{R}^4 : \boldsymbol{x} \cdot \boldsymbol{n} = 0\}$. The intersection of this plane with $\mathbb{S}^3$ is the great sphere $G_{\boldsymbol{n}} = \{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot \boldsymbol{n} = 0\}$; we will call such maps great sphere reflections, and they may be identified either by the corresponding great sphere $G_{\boldsymbol{n}}$ or a corresponding normal $\boldsymbol{n}$ (determined up to a sign). The great sphere $G_{\boldsymbol{n}}$ is also called the *symmetry sphere* of $g_{\boldsymbol{n}}$.

The second kind of map we wish to consider is determined by a point $\boldsymbol{y} \in \mathbb{R}^4$ with $|\boldsymbol{y}| > 1$. Each is an orientation-reversing diffeomorphism

$$h_{\boldsymbol{y}} : \mathbb{S}^3 \to \mathbb{S}^3, \quad \boldsymbol{x} \mapsto \boldsymbol{y} + (|\boldsymbol{y}|^2 - 1)\frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^2}.$$

The formula for $h_{\boldsymbol{y}}$ also has a simple geometric interpretation. If ℓ is the line through $\boldsymbol{y}$ and $\boldsymbol{x}$, then $\ell \cap \mathbb{S}^3 = \{\boldsymbol{y}, h_{\boldsymbol{y}}(\boldsymbol{x})\}$. The fixed point set of $h_{\boldsymbol{y}}$ is the nondegenerate sphere

$$H_{\boldsymbol{y}} = \{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot \boldsymbol{y} = 1\},$$

which we call the symmetry sphere of $h_{\boldsymbol{y}}$. Note that if an observer is located at $\boldsymbol{y}$, then $H_{\boldsymbol{y}}$ appears as the horizon on $\mathbb{S}^3$; see Figure 1. Accordingly, we refer to $h_{\boldsymbol{y}}$ as the cone point reflection based at the *cone point* $\boldsymbol{y} \in \mathbb{R}^4 \setminus \overline{B_1(0)}$; the map $h_{\boldsymbol{y}}$ may be identified uniquely by either its horizon sphere $H_{\boldsymbol{y}}$ or its cone point $\boldsymbol{y}$.

Definition 1 (special spherical symmetry). A set $\mathcal{S} \subset \mathbb{S}^3$ has *special spherical symmetry* if $\mathcal{S}$ is invariant under a family of generalized reflections consisting of either all the great sphere reflections $g_{\boldsymbol{n}}$ associated to the points $\boldsymbol{n}$ of a great circle in $\mathbb{S}^3$ or all the cone point reflections $h_{\boldsymbol{y}}$ associated to a line in $\mathbb{R}^4 \setminus \overline{B_1(0)}$.

0.2. ***Classification theorem.*** Our classification by Theorem 1 of constant mean curvature (CMC) surfaces with special spherical symmetry bears a strong superficial resemblance to the classification of Delaunay surfaces (rotationally symmetric constant mean curvature surfaces in $\mathbb{R}^3$). Recall that the Delaunay surfaces form a two parameter family consisting of six qualitative types: spheres, cylinders, catenoids, unduloids, nodoids, and planes. We borrow this terminology, so we briefly recall some properties of these surfaces.

The surfaces of Delaunay are often indexed by the two parameters mean curvature and neck size (the minimum distance from the meridian curve to the axis). Neck size is positive, except for the spheres and the plane. After a rigid motion, one may assume the axis of rotation is the vertical z-axis in $\mathbb{R}^3$ and that each surface has a natural parameterization of the form

$$(t,\theta) \mapsto (r\cos\theta, r\sin\theta, u), \tag{1}$$

where $t \mapsto (r(t), u(t))$ parameterizes the surface's meridian curve. Catenoids have a unique neck, that is, point on the meridian curve closest to the axis. The meridian curve of a catenoid is an embedded graph of an unbounded convex function over the entire axis of rotation. After translating the neck to $z = 0$, the meridian is even.

The unduloids and nodoids have periodic meridians. The meridian of an unduloid is an embedded graph with one positive minimum (neck), one maximum (bulge), and one inflection per closed half period, as shown in Figure 2. The nodoids have immersed meridians of nonvanishing curvature with loops toward the axis. When considered as a limit of unduloids or nodoids (neck size tending to zero), the sphere naturally arises as a "string of pearls".

Note finally that a multiple cover of the surface arises from the parameter θ in (1). The parameterization becomes singular when r vanishes, as in the case of the sphere and plane. Our classification theorem also gives a natural parameterization (2) containing a wrapping parameter ϕ and a dependent function r whose behavior is completely analogous.

Theorem 1. *The complete CMC surfaces in $\mathbb{S}^3$ with special spherical symmetry form, up to rigid motions, a-two parameter family consisting of five qualitatively different types of surfaces.*

Generating curves and parameterization. *Associated to each surface is a parametric curve $\gamma : t \mapsto (r(t), \theta(t))$ given explicitly in terms of elliptic integrals. Given the generating curve γ, a fundamental domain on the corresponding surface is parameterized by $X : \mathrm{Dom}(\gamma) \times \mathbb{R} \to \mathbb{S}^3$, where*

$$X(t,\phi) = \frac{1}{R\sqrt{r^2+1}}\left(R\cos\theta, R\sin\theta, r\sin\phi, R\sqrt{r^2+1} - 1\right) \tag{2}$$

and $R = \sqrt{r^2+1} + r\cos\phi$.

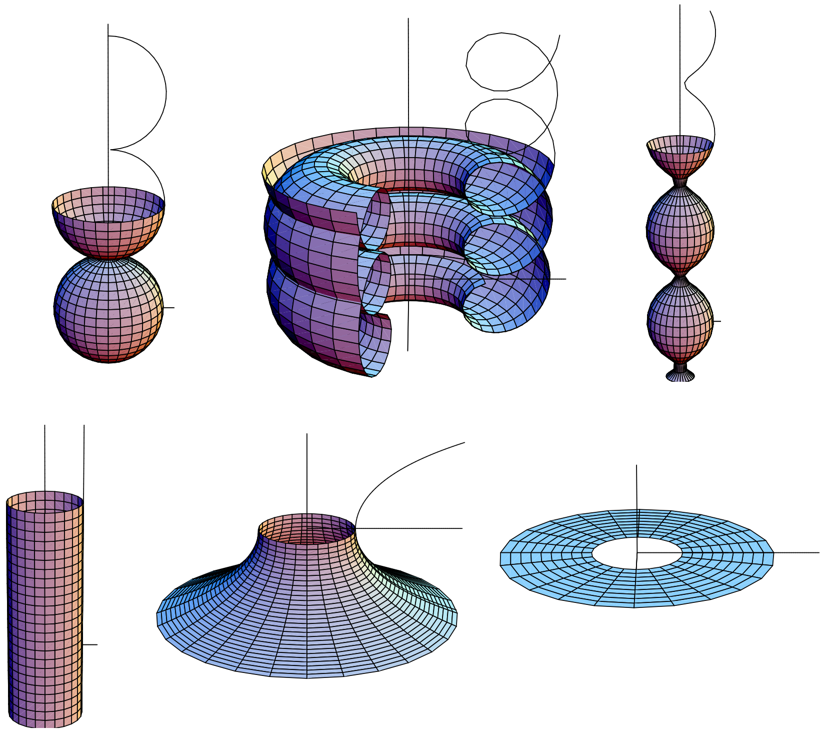

Figure 2. The Delaunay surfaces and their meridian curves and immersions: sphere, nodoid, unduloid, cylinder, catenoid, plane.

Differential equation and parameters for the space of surfaces. *With the exception of the great sphere and the standard tori* (*described under classification headings* (i) *and* (ii) *below*), *it is possible to take $t = r$ on an appropriate interval in* $[0, \infty)$ *and obtain the complete surface by taking closures* (*if necessary*) *and extending by rigid reflection. In these cases, $\theta = \theta(r)$ satisfies the ordinary differential equation*

$$r\theta'' = 2H_s r\left(\theta'^2 + \frac{1}{r^2+1}\right)^{3/2} - \frac{\theta'}{r^2+1} + (r^2-1)\theta'^3, \tag{3}$$

where H_s is the mean curvature of the corresponding surface in $\mathbb{S}^3$. A first integration of this equation yields

$$\theta' = \frac{c(r^2+1) - H_s}{\sqrt{(1+r^2)(r^2 - (c(r^2+1) - H_s)^2)}}, \tag{4}$$

where c is a constant of integration. We will use c as the second parameter to index the solution surfaces.

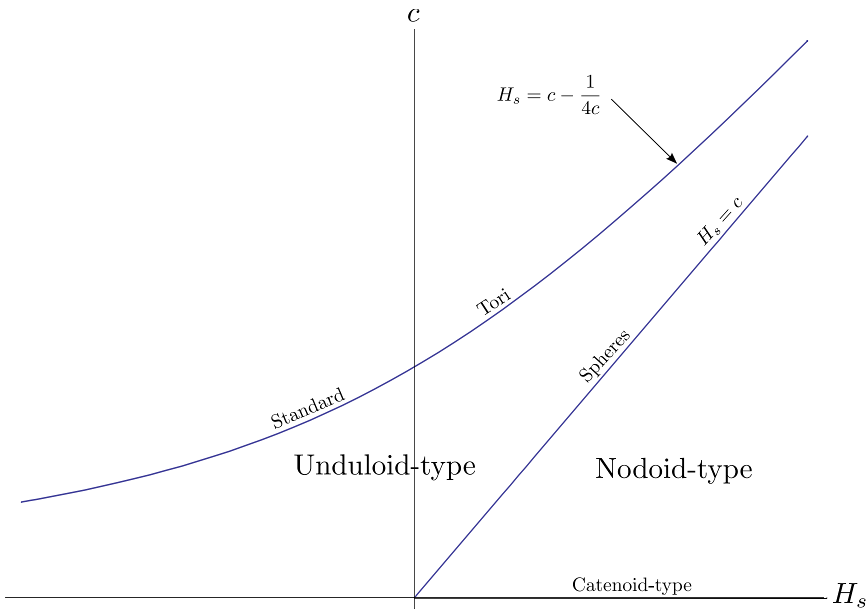

Figure 3. The parameter domain for symmetric CMC surfaces in $\mathbb{S}^3$.

Up to a choice of normal, all surfaces are represented in the parameter region

$$\{(H_s, c) : 0 < c \leq (H_s + \sqrt{H_s^2 + 1})/2\} \cup \{(H_s, 0) : H_s \geq 0\}.$$

While the expression (4) *is not defined along the curve* $c = (H_s + \sqrt{H_s^2 + 1})/2$, *these parameter values correspond naturally to the standard tori*; *see* Figure 3. *All possible complete solution surfaces are as follows.*

Classification.

(i) Spheres ($c = H_s \geq 0$): *For* $H_s \neq 0$, *we find the relation*

$$r = \sqrt{a^2 - (a^2 - 1)\sin^2\theta},$$

where $a = 1/H_s$. *If this expression is used to define* $r = r(\theta)$ *for* θ *in* $0 \leq |\theta| \leq \theta_{\max} = \sin^{-1}(1/\sqrt{H_s^2 + 1})$, *the expression* (2) *regularly parameterizes a sphere minus the two points*

$$\{X(\pm\theta_{\max}, \phi)\} = \{(\cos\theta_{\max}, \pm\sin\theta_{\max}, 0, 1)\}$$

on $(-\theta_{\max}, \theta_{\max}) \times \mathbb{R}$.

For $H_s = 0$, *we take* $t = r \in [0, \infty)$ *and* $\theta \equiv 0$; *the expression* (2) *then parameterizes the open hemisphere* $\{\boldsymbol{x} = (x, y, z, w) \in \mathbb{S}^3 : x > 0, y = 0\}$ *of the* (*minimal*) *great sphere* $\{\boldsymbol{x} \in \mathbb{S}^3 : y = 0\}$.

(ii) Standard tori ($c = (H_s + \sqrt{H_s^2+1})/2$): *In this case, we take*

$$r \equiv H_s + \sqrt{H_s^2+1} \tag{5}$$

constant. If $\theta(t) = t$ *in* (2), *we obtain a regular covering map of*

$$\{\boldsymbol{x} : x^2 + y^2 = 1/\sqrt{r^2+1},\, z^2 + w^2 = r/\sqrt{r^2+1}\},$$

which is a CMC torus.

(iii) Catenoid-type ($c = 0$, $H_s > 0$): *Integration leads to the relation*

$$\theta = -H_s \int_{H_s}^{r} \frac{1}{\sqrt{(\tau^2+1)(\tau^2-H_s^2)}} d\tau = -\cos\alpha\, F(\cos^{-1}(H_s/r), \alpha),$$

where $\alpha = \sin^{-1}(1/\sqrt{H_s^2+1})$ *and* F *is the standard elliptic integral of the first kind.*[1] *Using this formula to define* $\theta = \theta(r)$, *we can set*

$$\theta_{\max} = -\lim_{r \nearrow +\infty} \theta(r) = \cos\alpha\, K(\alpha)$$

with K *the complete elliptic integral of the first kind. Then we can let* $r = r(\theta)$ *be defined implicitly by the same formula on* $(-\theta_{\max}, 0)$. *This function has a unique (real analytic) extension to* $(-\theta_{\max}, \theta_{\max})$ *that is even, convex and unbounded; see Figure 4.*

The image of the resulting mapping (2) *restricted to* $[0, \theta_{\max})$ *is an embedded topological cylinder bounded by the circles*

$$C_0 = X(\{0\} \times \mathbb{R}) = \left\{\boldsymbol{x} \in \mathbb{S}^3 : y = 0, x = 1/\sqrt{H_s^2+1}\right\}$$

and the great circle

$$C_1 = \lim_{\theta \nearrow \theta_{\max}} X(\{\theta\} \times \mathbb{R}) = \{\boldsymbol{x} \in \mathbb{S}^3 : x = 0 = y\}.$$

The surface extends smoothly (by reflection $y \mapsto -y$*) according to the same formula on* $-\theta_{\max} < \theta \le 0$. *If we extend* $r = r(\theta)$ *to* $(\theta_{\max}, 3\theta_{\max})$ *by the formula* $r(\theta) = -r(\theta - 2\theta_{\max})$, *formula* (2) *leads to a smooth extension across* C_1 *by reflection with respect to the plane* $x + y \tan\theta_{max} = 0$. *More generally, we extend the function* $r = r(\theta)$ *to be periodic on* $\mathbb{R} \setminus \{(2k+1)\theta_{\max} : k \in \mathbb{Z}\}$ *with period* $4\theta_{\max}$. *In this way, the image*

$$\mathcal{I}_k = X(((2k-1)\theta_{\max}, (2k+1)\theta_{\max}) \times \mathbb{R})$$

under the map given in (2) *is the reflection of* $\mathcal{I}_{k+1}$ *with respect to the plane* $x + y\tan(2k+1)\theta_{\max} = 0$; *the union* $\mathcal{I}_k \cup \mathcal{I}_{k+1} \cup C_1$ *of these two embedded*

[1] See Section 2 below for our precise conventions concerning parameters in the elliptic integrals.

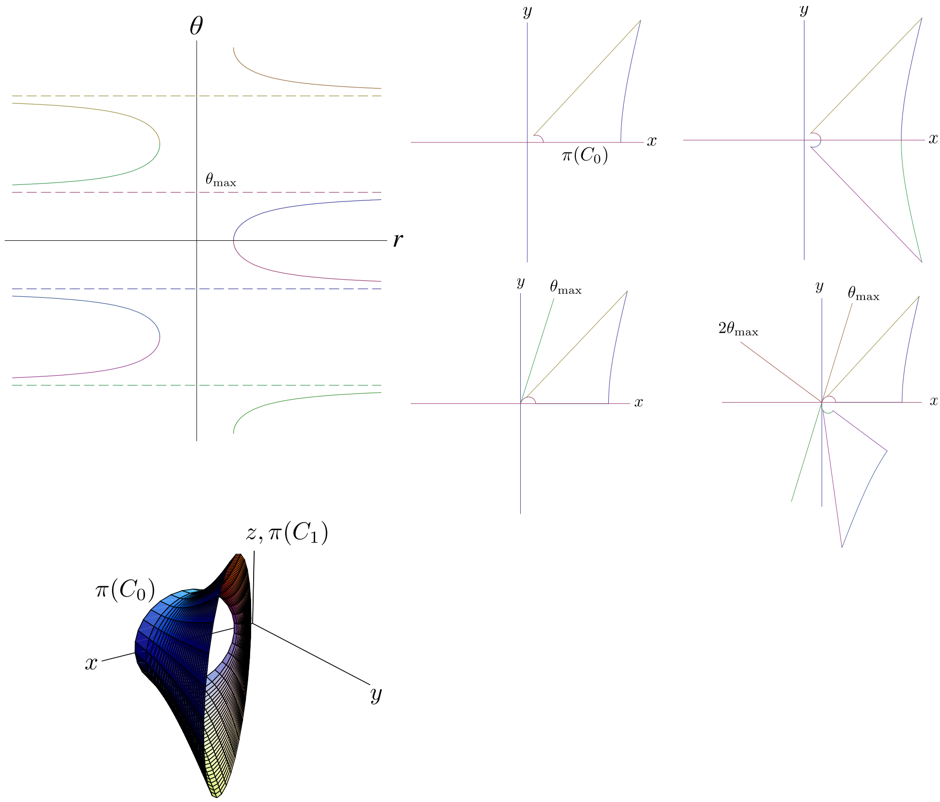

Figure 4. The generating curve for catenoid-type surfaces. Also shown are one "horn" of the stereographic projection of the catenoid-type surface with $H = \pm 1$, and projections that illustrate the extension of the surface from a fundamental domain. The horn is actually only a portion of the (stereographic projection of a) fundamental domain; the fundamental domain occupies the entire sector $0 \le \theta < \theta_{\max}$. The top row projections show first the outline of the projection into the (x, y)-plane of the horn and, second, extension by reflection across C_0. The bottom row shows extension across C_1 (which is the z-axis in the projection).

annuli and the disjoint circle C_1 form a single smooth CMC annulus (*which is embedded if* $\theta_{\max} \le \pi/4$).

The union $\bigcup_k \mathcal{I}_k \cup C_1$ *is a strictly immersed topological cylinder. As* H_s *increases from* 0 *to* ∞, *the value of* $\theta_{\max}$ *increases and takes all values between* 0 *and* $\pi/2$. *The immersion is a covering of a torus if and only if* $\theta_{\max} = n\pi/(2m)$, *where* $n, m \in \mathbb{N}$ *are relatively prime and* $n < m$. *In this case,* C_1 *is covered* $2m$ *times by the immersion for* $0 \le \theta \le 4m\theta_{\max} = 2n\pi$.

(iv) Unduloid-type ($\max\{0, H_s\} < c < (H_s + \sqrt{H_s^2+1})/2$): *For*

$$r_{\min} = \frac{1-\sqrt{1-4c(c-H_s)}}{2c} \le r \le r_{\max} = \frac{1+\sqrt{1-4c(c-H_s)}}{2c},$$

we have the relation

$$\theta = \int_{r_{\min}}^{r} \frac{c\tau^2+c-H_s}{\sqrt{(1+\tau^2)(\tau^2-(c\tau_c^2-H_s)^2)}}\,d\tau = \tag{6}$$

$$\frac{c-H_s}{cr_{\max}d_0} F\left(\sin^{-1}\sqrt{\frac{1-(r_{\min}/r)^2}{1-\mu^2}}, \alpha\right) + \frac{\mu r_{\min}}{d_0}\Pi\left(\nu, \sin^{-1}\sqrt{\frac{1-(r_{\min}/r)^2}{1-\mu^2}}, \alpha\right),$$

where

$$\mu = r_{\min}/r_{\max}, \quad d_0 = \sqrt{1+r_{\min}^2}, \quad \nu = 1-\mu^2, \quad \alpha = \sin^{-1}(\sqrt{\nu}/d_0),$$

and Π *is the standard elliptic integral of the third kind.*[2] *This relation determines an interval*

$$\begin{aligned} 0 \le \theta \le \theta_{\max} &= \int_{r_{\min}}^{r_{\max}} \frac{c\tau^2+c-H_s}{\sqrt{(1+\tau^2)[\tau^2-(c\tau_c^2-H_s)^2]}}\,d\tau \\ &= \frac{c-H_s}{cr_{\max}d_0}K(\alpha) + \frac{\mu r_{\min}}{d_0}\Pi(\nu, \pi/2, \alpha). \end{aligned} \tag{7}$$

We may then use (6) *to define* $r = r(\theta)$ *on* $[0, \theta_{\max}]$ *and extend* r *to be even and periodic with period* $2\theta_{\max}$; *see Figure 5. The resulting immersion* (2) *has an immersed topological cylinder as image.*

For these unduloid-type surfaces, $\theta_{\max}$ *as a function of* H_s *and* c *is a mapping onto the interval* $(0, \pi)$.

A given unduloid-type immersion is a covering of an immersed torus if and only if $\theta_{\max}$ *is a rational multiple of* π. *Such an immersed torus is embedded if and only if* $\theta_{\max} = \pi/m$ *for some* $m = 1, 2, 3, \ldots$. *This does not occur for* $H_s \le 0$ *or* $m = 1$, *but it does occur for* $H_s > 0$. *In fact, for each* $m = 2, 3, 4, \ldots$ *the relation* $\theta_{\max}(H_s, c_m) = \pi/m$ *defines a unique smooth increasing function* $c_m = c_m(H_s)$ *taking the interval* $[(\cot(\pi/m), (m^2/2-1)/\sqrt{m^2-1}]$ *onto* $[\cot(\pi/m), \sqrt{m^2-1}/2]$. *Each pair of parameters* $(H_s, c_m(H_s))$ *corresponds to an embedded unduloid-type torus with* m *bulges and* m *necks. See Figure 6.*

(v) Nodoid-type ($0 < c < H_s$): *For*

$$r_{\min} = \frac{-1+\sqrt{1-4c(c-H_s)}}{2c} \le r \le r_{\max} = \frac{1+\sqrt{1-4c(c-H_s)}}{2c},$$

[2] See Section 2 below for our precise conventions concerning parameters in the elliptic integrals.

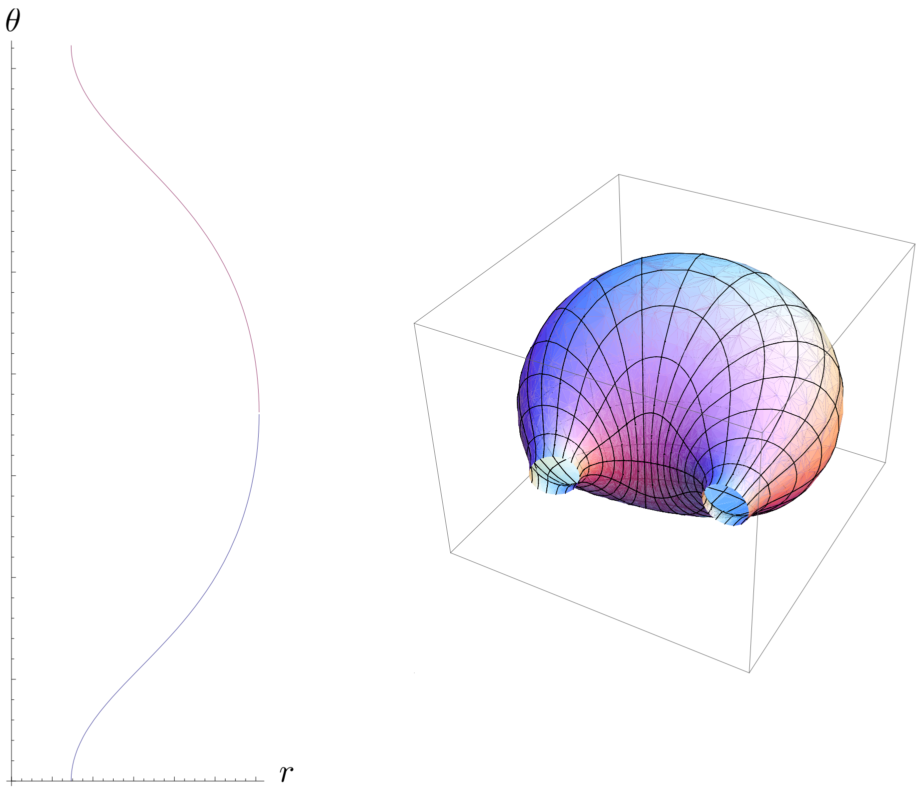

Figure 5. The generating curve for unduloid-type surfaces (one period corresponding to $0 < \theta < 2\theta_{\max}$) and the stereographic projection of two fundamental domains. The parameter values for this surface are $(H_s, c) = (.25, .5)$, and $\theta_{\max} \approx 1.8 > \pi/2$.

we have the relation (6). *The function $\theta(r)$ thereby defined is not monotone in this instance, but the inclination angle $\psi = \psi(r)$ satisfying*

$$\sin\psi = \frac{\theta'}{\sqrt{1+\theta'^2}}$$

is monotone and allows us to define $r = r(\psi)$ and $\theta = \theta(\psi)$ so that $r(\psi - \pi/2)$ is even and periodic with period 2π, and $\theta(\psi + \pi) = \theta(\psi) + \theta_{\max}$, where $\theta_{\max}$ is given by (7). *The resulting generating curve $\gamma : \psi \mapsto (r(\psi), \theta(\psi))$ resembles the meridian of a nodoid*; *it has nonvanishing curvature and loops toward the θ-axis.*

This immersed curve gives rise, *via* (2), *to a strictly immersed cylinder that covers an immersed torus if and only if $\theta_{\max}$ is a rational multiple of π. See Figure 7.*

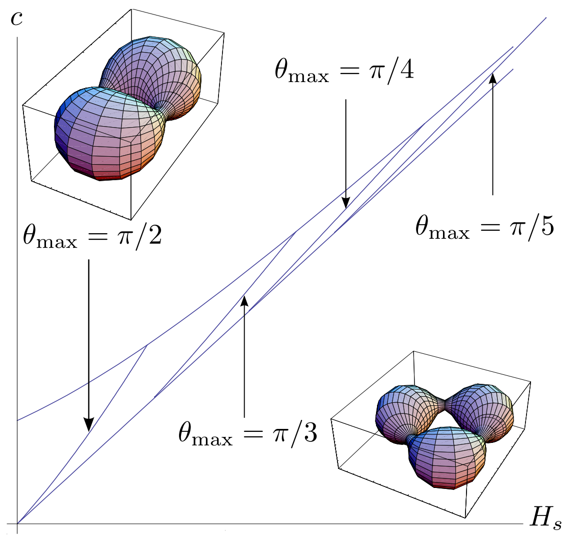

Figure 6. The parameter curves for unduloid-type embedded tori and corresponding examples for $m = 2, 3$.

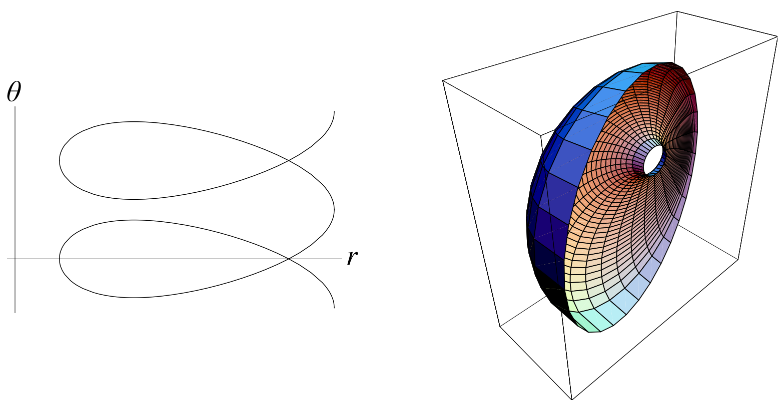

Figure 7. Two periods of the generating curve for a nodoid-type surface and the stereographic projection of one half period of the surface (one fundamental domain).

0.3. ***Radial lines and spherical symmetry.*** One drawback of Definition 1 is that it appears to simply concatenate two unrelated notions of symmetry (having a great circle's worth of great circle reflectional symmetry — which is equivalent to rotational symmetry — or a line's worth of cone point reflectional symmetry). We obtain some unification of these two kinds of symmetry by using radial lines. The

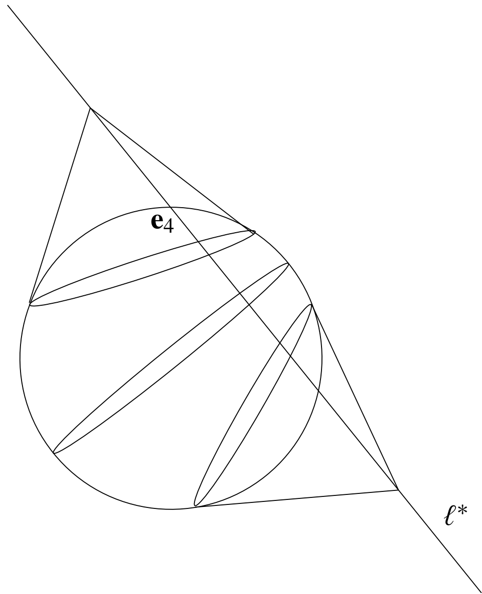

Figure 8. Reflections with the same radial line.

radial line associated to the great circle reflection $g_{\boldsymbol{n}}$ is

$$\ell^* = \{\mathbf{e}_4 + t\boldsymbol{n} : t \in \mathbb{R}\},$$

which passes through the north pole $\mathbf{e}_4 = (0, 0, 0, 1)$ in the direction of $\boldsymbol{n}$. The radial line associated to the cone point reflection $h_{\boldsymbol{y}}$ is

$$\ell^* = \{(1-t)\mathbf{e}_4 + t\boldsymbol{y} : t \in \mathbb{R}\},$$

which passes through $\mathbf{e}_4$ and $\boldsymbol{y}$. If any line $\ell^* = \{\mathbf{e}_4 + t\boldsymbol{v} : t \in \mathbb{R}\}$ through $\mathbf{e}_4$ is specified as a radial line, then a certain family of generalized reflections is specified. The family consists of cone point reflections $h_{\boldsymbol{y}}$ associated to the points in $\ell^* \backslash \overline{B_1(0)}$ and the great circle reflection $g_{\boldsymbol{n}}$, where $\boldsymbol{n} = \boldsymbol{v}/|\boldsymbol{v}|$. The family of generalized reflections associated to ℓ^* is indicated (by symmetry spheres) in Figure 8. The map that associates to a generalized (cone point) reflection a specific point $\boldsymbol{y} \in \ell^*$ is called the radial function. The terminology for radial lines and the radial function will be explained in Section 1 below. We now formulate the definition of spherical symmetry, which takes advantage of this unified viewpoint concerning reflections.

Definition 2 (spherical symmetry). A set $\mathcal{S}$ in $\mathbb{S}^3$ has spherical symmetry if there exists a family Λ of generalized reflections such that

(i) $\mathcal{S}$ is invariant under each map in Λ, and

(ii) There is some line $\Lambda^* = \{\boldsymbol{b} + t\boldsymbol{v} : t \in \mathbb{R}\} \subset \mathbb{R}^4 \setminus \overline{B_1(0)}$ such that

$$\bigcup_{\ell^* \in \mathcal{R}} \ell^* \supset \Lambda^* \cup \{\mathbf{e}_4 + \boldsymbol{v}/|\boldsymbol{v}|\}, \tag{8}$$

where $\mathcal{R}$ is the set of radial lines associated to maps in Λ.

We prove the following classification theorem in Section 3.

Theorem 2. *Any spherically symmetric surface that is compact has special spherical symmetry. Consequently, the compact spherically symmetric CMC surfaces are either*

(i) *spheres*,

(ii) *standard tori*,

(iii) *catenoid-type tori with* $\theta_{\max} = n\pi/(2m)$, *with* $m, n \in \mathbb{N}$ *relatively prime and* $n/m < 1$,

(iv) *unduloid-type tori with* $\theta_{\max} = n\pi/m$, *with* $m, n \in \mathbb{N}$ *relatively prime and* $n/m < 1$, *or*

(v) *nodoid-type tori with* $\theta_{\max} = n\pi/m$.

The embedded examples are spheres, standard tori, and countably many unduloid-type tori corresponding to $\theta_{max} = \pi/m$, *with* $m = 2, 3, 4, \dots$; *all unduloid-type examples correspond to parameters in our classification with* $H_s > 0$.

Corollary 1. *The only embedded minimal torus with spherical symmetry is the Clifford torus.*

Some authors, for example [Hsiang 1982; Jagy 1998; Park 2002; Brito and Leite 1990], have considered formally the family of rotationally symmetric CMC surfaces generated by an appropriate meridian curve. It turns out that the surfaces described in Theorem 1 are precisely these surfaces, though the parameterization (2) we have chosen does not make this apparent. In the final Section 4 we briefly describe the meridian curves associated to these surfaces.

Still other authors, for example [Ôtsuki 1970; do Carmo and Dajczer 1983], have considered in greater detail the special case of minimal surfaces in this context. Aside from the great sphere, all minimal surfaces are unduloid-type. Thus, our assertion that $\theta_{\max}$ is never π/m for $H_s \leq 0$ generalizes a result of Ôtsuki [1988] asserting that in the minimal case $\pi/2 < \theta_{\max} < \pi/\sqrt{2}$.

1. Preliminaries

Here we discuss the definitions of spherical symmetry introduced above, and we describe in particular the notion of spherical symmetry along a line in $\mathbb{R}^3$. This is an important technical tool in our proofs of the classification theorems.

Recall that stereographic projection

$$\pi : \mathbb{S}^3 \setminus \{\mathbf{e}_4\} \to \mathbb{R}^3, \quad \boldsymbol{x} = (x, y, z, w) \mapsto \frac{1}{1-w}(x, y, z) \tag{9}$$

is a conformal (angle-preserving) diffeomorphism with inverse given by

$$\pi^{-1} : \boldsymbol{x} = (x, y, z) \mapsto \frac{1}{|\boldsymbol{x}|^2+1}(2x, 2y, 2z, |\boldsymbol{x}|^2 - 1).$$

The domain of the mapping π may be extended by the same formula to $\mathbb{R}^4 \setminus \{w = 1\}$. We denote the resulting surjection by $\overline{\pi}$.

Notation. An effort will be made to denote points in $\mathbb{R}^n$ with lowercase boldface letters or by uppercase letters when the image of an immersion is under discussion (see two paragraphs below). The j-th standard basis vector (with 1 in the j-th entry and zeros elsewhere) will be denoted by $\mathbf{e}_j$. The coordinates of points may appear as (x, y, z, w) or (x_1, x_2, x_3, x_4). We will underline points to indicate that they have been projected into a lower dimensional subspace. Thus, when $\boldsymbol{x} = (x, y, z, w)$, the point $\underline{\boldsymbol{x}}$ will denote (x, y, z). Among these conventions, the context should make any ambiguities clear.

By a rotation of $\mathbb{S}^3$, we mean the restriction to $\mathbb{S}^3$ of a linear transformation of $\mathbb{R}^4$ with determinant 1. Similarly, a rotation of $\mathbb{R}^3$ is an element of $\mathrm{SL}_3(\mathbb{R})$. Our discussion could be given with little change in the context of rigid motions, that is, linear transformations with determinant ± 1. Certain special rotations will be important for the discussion below. If R is a rotation of $\mathbb{R}^3$, then the *trivial extension* of R to $\mathbb{R}^4$ is the rotation of $\mathbb{R}^4$ defined by $\mathbf{e}_j \mapsto (R(\underline{\mathbf{e}}_j), 0)$ for $j = 1, 2, 3$ and $\mathbf{e}_4 \mapsto \mathbf{e}_4$. We will denote this trivial extension by the same name R. We denote rotations of the coordinate 2-planes in $\mathbb{R}^4$ by superscripting the rotated coordinates and subscripting the angle. Thus, R_ψ^{xw} is the rotation corresponding to the matrix

$$\begin{pmatrix} \cos\psi & 0 & 0 & -\sin\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\psi & 0 & 0 & \cos\psi \end{pmatrix}.$$

With this notation, we observe the following result, whose proof may be found in [McCuan and Spietz 1998].

Theorem 3. *Any rotation R of $\mathbb{S}^3 \subset \mathbb{R}^4$ is a composition*

$$R = R_0 \circ R_\psi^{xw} \circ R_\phi^{zw} \circ R_\theta^{xy}, \tag{10}$$

where R_0 is the trivial extension to $\mathbb{R}^4$ of a rotation of $\mathbb{R}^3 = \{(x, y, z, 0)\}$.

Throughout the paper $\mathcal{S}$ will denote the image in $\mathbb{S}^3$ of a smooth immersion $X_0 : M \to \mathbb{S}^3$, where M is a complete, connected two-dimensional Riemannian manifold without boundary. It will be assumed, in general, that the immersion has constant mean curvature and that the image $\mathcal{S} = X_0(M)$ is spherically symmetric. We assume explicitly that M is second countable so that $\mathcal{S}$ has measure zero and has, in particular, dense complement [Hirsch 1994, Proposition 3.1.2].

For $\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{S}^3$ and $0 \le B \le \pi$, the sphere with center $\boldsymbol{n}$ and radius B is $\Gamma = \{\boldsymbol{x} \in \mathbb{S}^3 : \operatorname{dist}(\boldsymbol{x}, \boldsymbol{n}) = B\}$, where the distance is measured intrinsically in $\mathbb{S}^3$. Equivalently, this sphere is given by

$$\Gamma = \{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot \boldsymbol{n} = \cos B\}. \tag{11}$$

The center and radius are not unique but are determined to the extent that they lie among specific pairs $\{(\boldsymbol{n}, B), (-\boldsymbol{n}, \pi - B)\}$. More generally, we recognize (11) as the intersection with $\mathbb{S}^3$ of a hyperplane $\{\boldsymbol{x} \in \mathbb{R}^4 : \boldsymbol{x} \cdot \boldsymbol{y} = |\boldsymbol{y}| \cos B\}$. We say that Γ is nondegenerate if $0 < B < \pi$. In this case, Γ is a smooth submanifold that stereographically projects to a round sphere (or a flat plane if the north pole is in Γ). To be precise, one finds $\pi(\Gamma) = S_\rho(\boldsymbol{a}) \equiv \partial B_\rho(\boldsymbol{a})$, where

$$\boldsymbol{a} = \underline{\boldsymbol{y}}/(|\boldsymbol{y}| \cos B - y_4) \quad \text{and} \quad \rho = |\boldsymbol{y}| \sin B / |y_4 - |\boldsymbol{y}| \cos B|$$

when $\mathbf{e}_4 \notin \Gamma$ and $\pi(\Gamma \setminus \{\mathbf{e}_4\}) = \{\boldsymbol{x} : \boldsymbol{x} \cdot \underline{\boldsymbol{y}} = y_4\}$ otherwise. In this way, stereographic projection provides a one-to-one correspondence between the set of nondegenerate spheres in $\mathbb{S}^3$ and the set of nondegenerate spheres and planes in $\mathbb{R}^3$. For reference, we record the explicit formulas for the inverse stereographic projection of a sphere $S_\rho(\boldsymbol{a})$, which is

$$\{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot (-2\boldsymbol{a}, \rho^2 - |\boldsymbol{a}|^2 + 1) = \rho^2 - |\boldsymbol{a}|^2 - 1\},$$

and a plane $\{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x} \cdot \boldsymbol{n} = e\}$, which is

$$\{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot (\boldsymbol{n}, e) = e\} \setminus \{\mathbf{e}_4\}.$$

A great sphere in $\mathbb{S}^3$ is one of radius $\pi/2$. Alternatively, a great sphere is the intersection of a hyperplane subspace with $\mathbb{S}^3$. The great spheres are in one-to-one correspondence with the spheres and planes in $\mathbb{R}^3$ passing through a great circle on $\mathbb{S}^2$. A circle in $\mathbb{S}^3$ or $\mathbb{R}^3$ arises as the intersection of two spheres. Aside from degenerate cases, the collection of circles in $\mathbb{S}^3$ is in one-to-one correspondence with the collection of circles and lines in $\mathbb{R}^3$. A great circle is the intersection of two distinct great spheres.

1.1. ***Special spherical symmetry and a parameterization theorem.*** We now consider a surface $\mathcal{S}$ with special spherical symmetry and examine its stereographic projection $\mathcal{P} = \pi(\mathcal{S} \setminus \{\mathbf{e}_4\})$. We begin by obtaining several refinements of the following basic result.

Lemma 1. *If $\mathcal{S} \subset \mathbb{S}^3$ has special spherical symmetry, then there is a rotation R^s such that $\mathcal{P} = \pi(R^s(\mathcal{S}) \setminus \{\mathbf{e}_4\})$ is rotationally symmetric about an axis in $\mathbb{R}^3$.*

Proof. Let us first assume the symmetry group of $\mathcal{S}$ contains the great sphere reflections $\mathcal{G} = \{g_{\boldsymbol{n}} : \boldsymbol{n} \cdot \boldsymbol{m} = 0 = \boldsymbol{n} \cdot \tilde{\boldsymbol{m}}\}$, where $\boldsymbol{m}$ and $\tilde{\boldsymbol{m}}$ are nonparallel unit vectors. By preliminary rotation, we may assume $\tilde{\boldsymbol{m}} = \mathbf{e}_1$ and $\boldsymbol{m} = (m_1, m_2, 0, 0) \neq \pm\mathbf{e}_1$. In this situation,

$$\mathcal{G} = \{g_{\boldsymbol{n}} : \boldsymbol{n} = (0, 0, n_3, n_4) \in \mathbb{S}^3\}.$$

Our primary interest is in this position, the Apollonian position. Only one of the associated great spheres $G_{\boldsymbol{n}} = \{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot \boldsymbol{n} = 0\}$, namely $G_{\mathbf{e}_3}$, contains $\mathbf{e}_4$. To obtain a specific local parameterization for such a surface, we temporarily consider a rotation to another position in which each of the symmetry spheres contains $\mathbf{e}_4$. Let us, in particular, consider a rotation R^s of $\mathbb{S}^3$ for which $\mathbf{e}_1 \mapsto \mathbf{e}_4$, $\mathbf{e}_2 \mapsto \mathbf{e}_3$, $\mathbf{e}_3 \mapsto -\mathbf{e}_1$, and $\mathbf{e}_4 \mapsto \mathbf{e}_2$. This rotation decomposes as

$$R^s = R^{yz}_{\pi/2} \circ R^{xw}_{\pi/2} \circ R^{zw}_{\pi/2}. \tag{12}$$

The surface $\mathcal{S}_s = R^s(\mathcal{S})$ is now said to be in symmetric position, and $\mathcal{S}_s$ is invariant under the great sphere reflections in $\mathcal{G}_s = \{g_{\boldsymbol{n}} : \boldsymbol{n} = (n_1, n_2, 0, 0) \in \mathbb{S}^3\}$ with corresponding symmetry great spheres $G_{\boldsymbol{n}} = \{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot \boldsymbol{n} = 0\}$, which each contain $\mathbf{e}_4$ and stereographically project to planes

$$P_{\boldsymbol{n}} = \pi(G_{\boldsymbol{n}} \setminus \{\mathbf{e}_4\}) = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x} \cdot \underline{\boldsymbol{n}} = 0\}.$$

Each of these planes contains the z-axis, and a calculation shows that $g_{\boldsymbol{n}} \in \mathcal{G}_s$ induces a standard reflection $\psi_{\boldsymbol{n}}(\boldsymbol{p}) = \boldsymbol{p} - 2(\boldsymbol{p} \cdot \underline{\boldsymbol{n}})\underline{\boldsymbol{n}}$ about $P_{\boldsymbol{n}}$ in $\mathbb{R}^3$. It is well known (see for example [Hopf 1983, Chapter VIII §2]) that this implies $\mathcal{P}_s = \pi(\mathcal{S}_s \setminus \{\mathbf{e}_3\})$ is invariant under all rotations about the z-axis.

We turn next to the situation in which the symmetry group of $\mathcal{S}$ contains the cone point reflections h_y corresponding to the cone points $\boldsymbol{y}$ along a line in $\mathbb{R}^4 \setminus \overline{B_1(0)}$. The conclusion is substantially the same, though the calculation is somewhat more technical, and several of it details generalize the discussion above.

A line in $\mathbb{R}^4 \setminus \overline{B_1(0)}$ may be represented as $\{\boldsymbol{y}_0 + t\boldsymbol{v} : t \in \mathbb{R}\}$, where $\boldsymbol{y}_0$, with $|\boldsymbol{y}_0| > 1$, is the closest point on the line to $\mathbb{S}^3$, $\boldsymbol{v} \in \mathbb{S}^3$, and $\boldsymbol{y}_0 \cdot \boldsymbol{v} = 0$. The horizon sphere $H_{\boldsymbol{y}_0} = \{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot \boldsymbol{y}_0 = 1\}$ intersects the great sphere $G_{\boldsymbol{v}} = \{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot \boldsymbol{v} = 0\}$ in a circle C_0. It is easily checked that

$$C_0 = H_{\boldsymbol{y}_0} \cap G_{\boldsymbol{v}} = H_{\boldsymbol{y}_0 + t\boldsymbol{v}} \cap H_{\boldsymbol{y}_0} = H_{\boldsymbol{y}_0 + t\boldsymbol{v}} \cap G_{\boldsymbol{v}} \quad \text{for all } t \in \mathbb{R} \setminus \{0\},$$

and an explicit parameterization of C_0 is given by

$$\gamma(A)=\frac{\boldsymbol{y}_0}{\alpha^2}+\frac{\sqrt{\alpha^2-1}(\cos A\ \boldsymbol{w}_1+\sin A\ \boldsymbol{w}_2)}{\alpha}=\frac{1}{\alpha^2}\boldsymbol{y}_0+\frac{1}{c}(\cos A\ \boldsymbol{w}_1+\sin A\ \boldsymbol{w}_2),$$

where $\alpha=|\boldsymbol{y}_0|$, $a=1/\sqrt{\alpha^2-1}$, $c=a\alpha=\sqrt{a^2+1}$, and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{y}_0/\alpha, \boldsymbol{v}\}$ is an orthonormal basis for $\mathbb{R}^4$.

Using a preliminary rotation, we may assume $\boldsymbol{w}_1=\mathbf{e}_1$, $\boldsymbol{w}_2=\mathbf{e}_2$, $\boldsymbol{y}_0=\alpha\mathbf{e}_3$, and $\boldsymbol{v}=\mathbf{e}_4$. With this normalization, we have

$$\gamma(A)=\cos B\mathbf{e}_3+\sin B(\cos A\ \mathbf{e}_1+\sin A\ \mathbf{e}_2),$$

where $\cos B=a/c$ and $\sin B=1/c$; this is again called the Apollonian position.

We now obtain symmetric position by applying a rotation R^s so that the intersection circle $C_s=R^s(C_0)$ passes through the north pole $\mathbf{e}_4$. One rotation that does this is $R^{yz}_{\pi/2}\circ R^{xw}_{B}\circ R^{zw}_{\pi/2}$. This choice is fairly straightforward in light of the rotation decomposition theorem (Theorem 3) and the ansatz that the inverse rotation satisfies $\mathbf{e}_4\mapsto\gamma(0)=\cos B\mathbf{e}_3+\sin B\mathbf{e}_1$ and, in addition, that $\pi\circ R^s(\cos B\mathbf{e}_3-\sin B\mathbf{e}_1)=-a\mathbf{e}_1$. More generally, we may apply

$$R^s=R^{yz}_{\pi/2}\circ R^{xw}_{B}\circ R^{zw}_{\pi/2}\circ R^{xy}_{-A_0},$$

where A_0 is any fixed angle. Note that $R^{xy}_{-A_0}$ leaves C_0 invariant, but moves a specified point $\gamma(A_0)$ to $\gamma(0)=\cos B\mathbf{e}_3+\sin B\mathbf{e}_1$, so that $R^s\circ\gamma(A_0)=\mathbf{e}_4$.

In symmetric position, therefore, the intersection circle $C_s=R(C_0)$ is parameterized by

$$\begin{aligned}\gamma_s(A)&=\cos B(-\sin B\mathbf{e}_1+\cos B\mathbf{e}_4)\\&\qquad+\sin B(\cos(A-A_0)(\cos B\mathbf{e}_1+\sin B\mathbf{e}_4)+\sin(A-A_0)\mathbf{e}_3)\\&=\cos B\sin B(\cos(A-A_0)-1)\mathbf{e}_1+\sin B\sin(A-A_0)\mathbf{e}_3\\&\qquad+(\cos^2 B+\sin^2 B\cos(A-A_0))\mathbf{e}_4.\end{aligned}$$

Note $\gamma_s(A_0)=\mathbf{e}_4$, and the stereographic projection of $C_s\setminus\{\mathbf{e}_4\}$ is parameterized by

$$\begin{aligned}\pi\circ\gamma_s(A)&=-\cot B\mathbf{e}_1+\csc B\frac{\sin(A-A_0)}{1-\cos(A-A_0)}\mathbf{e}_3\\&=(-a, c\alpha\sin(A-A_0)/(1-\cos(A-A_0)), 0).\end{aligned}$$

For $A\in(0,2\pi)$, the function $f(A)=\sin A/(1-\cos A)$ is decreasing and takes all values in $\mathbb{R}$. Thus, $\pi(C_s\setminus\{\mathbf{e}_4\})$ is the vertical line L through $(-a,0,0)$ in $\mathbb{R}^3$.

One also checks that $\pi(H_{\boldsymbol{y}}\setminus\{\mathbf{e}_4\})$, where $\boldsymbol{y}=R^s(\boldsymbol{y}_0+t\boldsymbol{v})=-\alpha\sin B\mathbf{e}_1+t\mathbf{e}_2+\alpha\cos B\mathbf{e}_4=-\mathbf{e}_1/a+t\mathbf{e}_2+\mathbf{e}_4$, is the plane

$$P_{\boldsymbol{y}}=\{\boldsymbol{x}\in\mathbb{R}^3:\boldsymbol{x}\cdot(-\mathbf{e}_1/a+t\mathbf{e}_2)=1\},\tag{13}$$

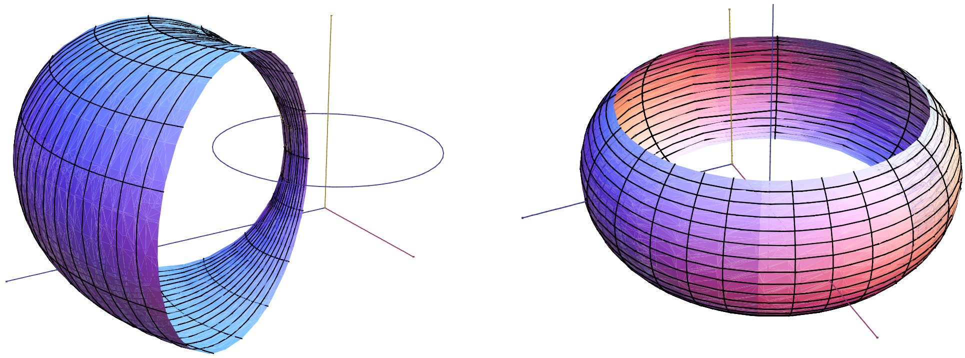

Figure 9. Stereographic projections of an annular surface. At left, Apollonian position; also shown is the image of the circle C_0. At right, symmetric position; notice the horizontal vertical axis $L = \pi(C_s) \setminus \{\mathbf{e}_4\}$ of rotational symmetry.

and h_y induces the standard reflection $\psi_y(\boldsymbol{p}) = \boldsymbol{p} - 2(\boldsymbol{p} \cdot \underline{\boldsymbol{y}} - 1)\underline{\boldsymbol{y}}/|\underline{\boldsymbol{y}}|^2$ about P_y on $\mathbb{R}^3$. All planes containing the vertical line $L = \pi(C_s \setminus \{\mathbf{e}_4\})$ are represented in (13) as t ranges over $\mathbb{R}$, except the x, z-plane. It follows that $\mathcal{P}_s = \pi(R^s(\mathcal{S}) \setminus \{\mathbf{e}_4\})$ is invariant under rotation about the vertical line L. See Figure 9. $\square$

In view of the foregoing discussion, we digress temporarily to prove a parameterization theorem for rotationally symmetric images in $\mathbb{R}^3$.

Theorem 4. *Let $Y : N \to \mathbb{R}^3$ be an immersion of a complete, second countable, two-dimensional manifold N, complete in the metric induced by the immersion and without boundary. Assume the image $Y(N)$ is rotationally symmetric with respect to the vertical axis $L = \{(-a, 0, t) : t \in \mathbb{R}\}$. Then either each connected component N_c of N is diffeomorphic to $\mathbb{R}^2$ and Y is an embedding of each N_c onto a horizontal plane, or there is a point $p_0 \in N$ whose image $\boldsymbol{p}_0 = Y(p_0)$ has the form $(-a + r_0, 0, z_0)$ with $r_0 > 0$, and there is some $\epsilon > 0$, an immersion $\jmath : \mathbb{R} \times (z_0 - \epsilon, z_0 + \epsilon) \to N$ and a smooth positive function $r = r(z)$ defined on $(z_0 - \epsilon, z_0 + \epsilon)$ such that*

(i) $Y \circ \jmath(\theta, z) = (-a + r \cos\theta, r \sin\theta, z)$,

(ii) $\jmath(0, z_0) = p_0$, *and*

(iii) *for every $\theta \in \mathbb{R}$, the restriction $\jmath : (\theta, \theta + 2\pi) \times (z_0 - \epsilon, z_0 + \epsilon) \to N$ is a local parameter chart for N.*

Proof. Evidently, the special case of the theorem when $a = 0$ and L is the z-axis implies the result as stated.

Consider the action R of rotation about the z-axis Z on $\mathbb{R}^3 \setminus Z$. We will use the notations

$$R_\theta(\boldsymbol{x}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{x} \quad \text{and} \quad \dot{R}_\theta(\boldsymbol{x}) = \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x}.$$

Note that $\tilde{N} = N \setminus Y^{-1}(L)$ is a nonempty open submanifold of N. Any point $p_0 \in \tilde{N}$ has image $\boldsymbol{p}_0 = Y(p_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0, z_0)$ where $r_0 > 0$. Let $\xi : U \to V \subset \mathbb{R}^2$ be a local coordinate chart on $\tilde{N}$ with $\xi(p_0) = (0, 0)$ and such that $Y \circ \xi^{-1}$ is an embedding of V into $\mathbb{R}^3 \setminus L$. Set $\mathcal{S}_0 = Y(U)$.

We claim that $\dot{R}_0(\boldsymbol{p}) \in T_{\boldsymbol{p}}\mathbb{R}^3$ satisfies $\dot{R}_0(\boldsymbol{p}) \in T_{\boldsymbol{p}}\mathcal{S}_0$ for every $\boldsymbol{p} \in \mathcal{S}_0$. In fact, this is obvious from the rotational symmetry, since otherwise $\dot{R}_0(\boldsymbol{p})$ is transverse to $T_{\boldsymbol{p}}\mathcal{S}_0$, and we find $Y(N) \supset \bigcup_\theta R_\theta(\mathcal{S}_0)$, which contains an open set in $\mathbb{R}^3$; this contradicts our assumption that N is second countable since Hirsch shows in [1994, Proposition 3.1.2] that all images of second countable manifolds have empty interior. We thus obtain a nonvanishing vector field

$$\boldsymbol{w}_\xi = (d(Y \circ \xi^{-1})^{-1} \dot{R}_0(Y) \quad \text{on } V \subset \mathbb{R}^2.$$

By reparametrizing V [Chern 1959, Theorem 1.4], we may assume $\boldsymbol{w} = \mathbf{e}_2$, or equivalently $\boldsymbol{w} = \partial/\partial v_2$ in terms of (v_1, v_2)-coordinates on V.

It follows that the images under $Y \circ \xi^{-1}$ of the coordinate lines $v_1 =$ constant in V lie along the circular orbits of R. In fact, $Y \circ \xi^{-1}(v_1, v_2) = R_{v_2}(Y \circ \xi^{-1}(v_1, 0))$ at least locally in some open ball about $(0, 0) \in V$, since both expressions satisfy the ODE

$$\frac{d\boldsymbol{x}}{dv_2} = \dot{R}_0(\boldsymbol{x}), \quad \boldsymbol{x}(0) = Y \circ \xi^{-1}(v_1, 0).$$

A local basis for $T_{\boldsymbol{p}}\mathcal{S}_0$, where $\boldsymbol{p} = Y \circ \xi^{-1}(v_1, v_2)$, is thus given by

$$\boldsymbol{u} := \frac{d}{dv_1}(Y \circ \xi^{-1}(v_1, v_2)) \quad \text{and} \quad \dot{R}_0(\boldsymbol{p}).$$

Since these vectors are independent, $\boldsymbol{u} \cdot \mathbf{e}_3 = 0$ if and only if $\underline{Y} = (Y_1, Y_2, 0) \in T_{\boldsymbol{p}}\mathcal{S}_0$.

Let us consider first the possibility that $\underline{Y}(0, 0) \in T_{\boldsymbol{p}_0}\mathcal{S}_0$ for every $\boldsymbol{p}_0 \in Y(N) \setminus L$. In this case, we consider

$$\eta : V \to \mathbb{R}^2, \quad (v_1, v_2) \mapsto \chi \circ Y \circ \xi^{-1}(v_1, v_2)$$

where χ is a branch of polar coordinates on $[\theta_0 - \pi, \theta_0 + \pi)$. Note that

$$D\eta(v_1, v_2) = \begin{pmatrix} (Y \cdot \boldsymbol{u})/\chi_1 & 0 \\ (\dot{R}_0(Y) \cdot \boldsymbol{u})/\chi_1^2 & 1 \end{pmatrix}, \quad \text{where } \chi_1 = |\underline{Y}|.$$

In particular, $\det D\eta(0, 0) = \boldsymbol{p}_0 \cdot \boldsymbol{u}_0 / r_0$ where $\boldsymbol{u}_0 = \boldsymbol{u}(0, 0)$. We are assuming $\boldsymbol{u}_0 \cdot \mathbf{e}_3 = 0$, so $\boldsymbol{u}_0 = (\boldsymbol{u}_0 \cdot \boldsymbol{p}_0)\underline{\boldsymbol{p}}_0 / r_0^2 + (\boldsymbol{u}_0 \cdot \dot{R}_0(\boldsymbol{p}_0))\dot{R}_0(\boldsymbol{p}_0)/r_0^2$. Since $\boldsymbol{u}_0$ and $\dot{R}_0(\boldsymbol{p}_0)$

are linearly independent, we must have $\boldsymbol{u}_0 \cdot \boldsymbol{p}_0 \neq 0$. Thus, η is invertible in some neighborhood $V_0 \subset\subset V$. Setting $U_0 = \xi^{-1}(V_0)$, we have a local coordinate chart $\zeta = \eta \circ \xi : U_0 \to \mathbb{R}^2$ for which $Y \circ \zeta^{-1}(r, \theta) = (r \cos\theta, r \sin\theta, z_0)$. The form of the first two coordinates follows from the definition of η^{-1}; the last coordinate is a consequence of the fact that $\boldsymbol{u} \cdot \mathbf{e}_3 = 0 = \dot{R}_0(Y) \cdot \mathbf{e}_3$.

There exist positive numbers $\epsilon, \delta > 0$ such that

$$(r_0 - \epsilon, r_0 + \epsilon) \times (\theta_0 - \delta, \theta_0 + \delta) \subset \eta(V_0),$$

and we have an immersion

$$\iota = \zeta^{-1} : (r_0 - \epsilon, r_0 + \epsilon) \times (\theta_0 - \delta, \theta_0 + \delta) \to N$$

satisfying $Y \circ \iota(r, \theta) = (r \cos\theta, r \sin\theta, z_0)$ and $\iota(r_0, \theta_0) = p_0$.

We first extend ι to a local diffeomorphism on all of $(0, \infty) \times \mathbb{R}$ onto all of N_c, where N_c is the component of N containing p_0. The resulting map will still satisfy $Y \circ \iota(r, \theta) = (r \cos\theta, r \sin\theta, z_0)$.

Let ι now also denote a maximal extension of $\iota = \zeta^{-1}$ to an open subset Σ of $(0, \infty) \times \mathbb{R}$ such that $Y \circ \iota(r, \theta) = (r \cos\theta, r \sin\theta, z_0)$ and for every $\theta \in \mathbb{R}$, the restriction of ι to $\Sigma \cap (0, \infty) \times (\theta, \theta + 2\pi)$ is a local parameter chart for N. We claim that $\Sigma = (0, \infty) \times \mathbb{R}$. Otherwise, there is some $(r_*, \theta_*) \in \partial\Sigma \cap (0, \infty) \times \mathbb{R}$. Since the point p_0 in the reasoning above was arbitrary in $\tilde{N}$, we may apply the same argument to

$$p_* = \lim_{\Sigma \ni (r,\theta) \to (r_*, \theta_*)} \iota(r, \theta)$$

and obtain a nontrivial extension of ι to a neighborhood of (r_*, θ_*), which contradicts the maximality of ι.

Finally, we set $q_0 = \lim_{r \to 0} \iota(r, \theta)$. Since this point is well defined, we see that $N_c = \iota(\Sigma) \cup \{q_0\}$ is diffeomorphic to $\mathbb{R}^2$.

We now turn to the alternative situation in which there is a point $p_0 \in \tilde{N}$ with

$$Y(p_0) = \boldsymbol{p}_0 = (r_0 \cos\theta_0, r_0 \sin\theta_0, z_0)$$

and $\underline{Y}(p_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0, 0) \notin T_{\boldsymbol{p}_0}\mathcal{S}_0$, where $\mathcal{S}_0 = Y(U)$ is the image of a coordinate neighborhood with local coordinate $\xi : U \to V \subset \mathbb{R}^2$ on $\tilde{N}$ with $\xi(p_0) = (0, 0)$ and $Y \circ \xi^{-1}$ an embedding much as above. Continuing the same line of reasoning, we may assume v_1 is the polar displacement in local coordinates v_1, v_2 on V. In this case

$$Y \circ \xi^{-1}(v_1, v_2) = R_{v_1 - \theta_0}(Y \circ \xi^{-1}(0, v_2)) \tag{14}$$

locally near $(0, 0) \in V$ and

$$\boldsymbol{v}(v_1, v_2) := \frac{\partial}{\partial v_2}(Y \circ \xi^{-1}(v_1, v_2))$$

satisfies $\boldsymbol{v}\cdot\mathbf{e}_3\neq 0$ (also near the origin). We consider the map

$$\begin{aligned}\eta(v_1,v_2)&=(\chi_2\circ Y\circ\xi^{-1}(v_1,v_2),\,Y\circ\xi^{-1}(v_1,v_2)\cdot\mathbf{e}_3\\&=(\theta_0+v_1,\,R_{v_1}(Y\circ\xi^{-1}(0,v_2))\cdot\mathbf{e}_3)\\&=(\theta_0+v_1,\,Y\circ\xi^{-1}(0,v_2)\cdot\mathbf{e}_3)\end{aligned}$$

and note that

$$D\eta=\begin{pmatrix}1&0\\0&\boldsymbol{v}\cdot\mathbf{e}_3\end{pmatrix}\quad\text{where } \boldsymbol{v}=\boldsymbol{v}(0,v_2). \tag{15}$$

In particular, $\det D\eta(0,0)$ is nonzero, and again we obtain a local coordinate chart $\zeta=\eta\circ\xi:U_0\to\mathbb{R}^2$ where $V_0\subset\subset V$ and $U_0=\xi^{-1}(V_0)$. This time, however, we find

$$Y\circ\zeta^{-1}(\theta,z)=(r\cos\theta,r\sin\theta,z),\quad\text{where } r=r(\theta,z)=\chi_1\circ Y\circ\zeta^{-1}(\theta,z).$$

We claim that $r=r(z)$ is independent of θ. To see this, we compute

$$\frac{\partial}{\partial\theta}(\chi_1\circ(Y\circ\xi^{-1})\circ\eta^{-1}(\theta,z_0))=\\\frac{Y\circ\zeta^{-1}(\theta,z)}{\chi_1\circ Y\circ\zeta^{-1}(\theta,z)}D(Y\circ\xi^{-1}(\eta^{-1}(\theta,z))\cdot\frac{\partial}{\partial\theta}\cdot(\eta^{-1}(\theta,z)).$$

We have from (15) that

$$D\eta^{-1}(\theta,z)=\frac{1}{\det D\eta(\theta,z)}=\begin{pmatrix}\boldsymbol{v}(\eta^{-1}(\theta,z))\cdot\mathbf{e}_3&0\\0&1\end{pmatrix}.$$

Thus,

$$\frac{\partial}{\partial\theta}(\chi_1\circ Y\circ\zeta^{-1}(\theta,z))=\mu Y\circ\zeta^{-1}(\theta,z)\cdot\frac{\partial(Y\circ\xi^{-1})}{\partial v_1}(\eta^{-1}(\theta,z)),$$

where $\mu=\boldsymbol{v}(\eta^{-1}(\theta,z))\cdot\mathbf{e}_3/(\chi_1\circ Y\circ\zeta^{-1}(\theta,z)\det D\eta(\eta^{-1}(\theta,z)))$. On the other hand, using (14),

$$\begin{aligned}\frac{\partial(Y\circ\xi^{-1})}{\partial v_1}(\eta^{-1}(\theta,z))&=\dot R_{\eta_1^{-1}(\theta,z)}(Y\circ\xi^{-1}(0,\eta_2^{-1}(\theta,z)))\\&=R_{\pi/2+\eta_1^{-1}(\theta,z)}(Y\circ\xi^{-1}(0,\eta_2^{-1}(\theta,z)))\\&=R_{\pi/2}(Y\circ\xi^{-1}\circ\eta^{-1}(\theta,z)).\end{aligned}$$

Thus,

$$\frac{\partial}{\partial\theta}(r(\theta,z))=\mu Y\circ\zeta^{-1}(\theta,z)\cdot R_{\pi/2}(Y\circ\zeta^{-1}(\theta,z))=0,$$

and $r=r(\theta)$ as claimed.

We conclude that $\jmath:=\zeta^{-1}:(\theta_0-\delta,\theta_0+\delta)\times(z_0-\epsilon,z_0+\epsilon)\to N$ is a well-defined immersion (for ϵ and δ small enough) that satisfies conditions (i) and (ii)

$$
\begin{tikzcd}
 & N \arrow[d] \arrow[dddrr, "Y_0"] & & \\
 & M \arrow[d, "X_0"'] & & \\
C_0 \arrow[r] \arrow[d, "R^s"'] & S^3 \arrow[d, "R^s"'] & & \\
C_s \arrow[r] & S^3 & S^3\backslash\{\mathbf{e}_4\} \arrow[l] \arrow[r, "\pi"] & R^3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & R\times(z_0-\epsilon, z_0+\epsilon) \arrow[dl, "\jmath"'] \arrow[dddr] & \\
 & & N \arrow[ddll] \arrow[d] \arrow[ddddrr, "Y_0"] & & \\
 & & M \arrow[d, "X_0"'] & & \\
R^3 & S^3\backslash\{\mathbf{e}_4\} \arrow[l, "\pi"'] \arrow[r] & S^3 \arrow[d, "R^s"'] & & \mathcal{P}_s \arrow[d] \\
 & & S^3 & S^3\backslash\{\mathbf{e}_4\} \arrow[l] \arrow[r, "\pi"] & R^3
\end{tikzcd}
$$

Figure 10. Mappings used to obtain an explicit parameterization. Unlabeled mappings are either identity immersions or compositions as shown.

of the theorem. Finally, we let $\jmath$ also denote an extension to a subset Σ of the strip $\mathbb{R}\times(z_0-\epsilon, z_0+\epsilon)$ that is maximal subject to these conditions:

- $\jmath$ is an immersion.
- There is a smooth positive function $r = r(z)$ defined on

  $$\Sigma_2 := \{z \text{ such that there exists a } \theta\in\mathbb{R} \text{ with } (\theta, z)\in\Sigma\}$$

  such that $Y\circ\jmath(\theta, z) = (r\cos\theta, r\sin\theta, z)$.
- For every $\theta\in\mathbb{R}$, the restriction $\jmath : \Sigma\cap(\theta, \theta+2\pi)\times(z_0-\epsilon, z_0+\epsilon)\to N$ is a local parameter chart for N.

It follows that $\jmath$ satisfies all the conditions of the theorem's second alternative. $\square$

We now return to the CMC immersion $X_0 : M\to\mathbb{S}^3$. In the discussion that follows, the diagrams of Figure 10 display the relations between various submanifolds and maps. We set $N = M\setminus(R^s\circ X_0)^{-1}(\mathbf{e}_4)$ and apply Theorem 4 to $Y_0 = \pi\circ R^s\circ X_0 : N\to\mathbb{R}^3$ to obtain the following basic result.

Lemma 2. *If $X_0 : M\to\mathbb{S}^3$ has image $\mathcal{S}$ with special spherical symmetry, then either $\mathcal{S}$ is a sphere, or (without loss of generality) there is some $\epsilon > 0$ and a nonsingular immersion $\jmath : (-\epsilon, \epsilon)\times\mathbb{R}\to M$, a positive smooth function $r = r(\theta)$ defined on $(-\epsilon, \epsilon)$, a positive constant $\rho_0 > 0$, and another constant h such that $X_0\circ\jmath$ satisfies*

$$\pi\circ X_0\circ\jmath(\theta, \phi) = (R\cos\theta, R\sin\theta, r\sin\phi + h)$$

where $R = \sqrt{r^2+\rho_0^2} + r\cos\phi$.

Proof. If Y_0 parameterizes horizontal planes, then we let N_c be any connected component of N. We know $Y_0(N_c) = \{(x, y, h)\}$ for some $h \in \mathbb{R}$, and by the formula for inverse stereographic projection,

$$R^s \circ X_0(N_c) = \{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot (\mathbf{e}_3 + h\mathbf{e}_4) = h\} \setminus \{\mathbf{e}_4\}.$$

Since the image of X_0 is complete, there is some $p_0 \in M$ for which

$$\lim_{p \in N_c, |Y_0(p)| \to \infty} p = p_0 \quad \text{and} \quad R^s \circ X_0(p_0) = \mathbf{e}_4.$$

There is a neighborhood U_0 of p_0 in M such that $U_0 \setminus \{p_0\} \subset N_c$, and it follows that $M = N_c \cup \{p_0\}$ is a sphere with $X_0(M) = R^s\{\boldsymbol{x} \in \mathbb{S}^3 : \boldsymbol{x} \cdot (\mathbf{e}_3 + h\mathbf{e}_4) = h\}$.

If the second alternative of Theorem 4 holds, there is an immersion

$$\jmath : \mathbb{R} \times (z_0 - \epsilon, z_0 + \epsilon) \to N$$

such that $Y_0 \circ \jmath(\theta_s, z_s) = (-a + r_s \cos\theta_s, r_s \sin\theta_s, z_s)$ parameterizes an embedded annulus. Since $Y_0 = \pi \circ R^s \circ X_0$, we find

$$\begin{aligned} R^s \circ X_0 \circ \jmath &= \frac{(2(-a + r_s \cos\theta_s), 2r_s \sin\theta_s, 2z_s, a^2 - 2ar_s \cos\theta_s + {r_s}^2 + {z_s}^2 - 1)}{a^2 - 2ar_s \cos\theta_s + {r_s}^2 + {z_s}^2 + 1} \\ &= \frac{(2(-a + r_s \cos\theta_s), 2r_s \sin\theta_s, 2z_s, \mu^2 - 2ar_s \cos\theta_s - 2)}{\mu^2 - 2ar_s \cos\theta_s}, \end{aligned}$$

where $\mu^2 = {r_s}^2 + {z_s}^2 + a^2 + 1$. Note that the s subscripts on r_s, θ_s, and z_s indicate that these are cylindrical coordinates specifically associated with the projection from symmetric position. For z_s fixed, the expression $R^s \circ X_0 \circ \jmath(\theta_s, z_s)$ parameterizes a circle $\Gamma_s(z_s)$ in $\mathbb{S}^3$. Since $X_0 \circ \jmath$ is an embedding (mod 2π in θ_s), we know that $\mathbf{e}_4$ can lie in at most one of the Apollonian circles $\Gamma(z_s) := (R^s)^{-1}(\Gamma_s(z_s))$. By adjusting the z_s interval if necessary, we may assume that $\mathbf{e}_4$ does not belong to $\{X_0 \circ \jmath(\mathbb{R} \times (z_0 - \epsilon, z_0 + \epsilon)\}$. Thus, $\pi(\Gamma(z_s))$ is a circle in $\mathbb{R}^3$ for every z_s.

Recalling that $c = \sqrt{a^2 + 1} = \csc B$ and $R^s = R^{yz}_{\pi/2} \circ R^{xw}_B \circ R^{zw}_{\pi/2} \circ R^{xy}_{-A_0}$, we find

$$\begin{aligned} R^{xw}_B \circ R^{zw}_{\pi/2} \circ R^{xy}_{-A_0} \circ X_0 \circ \jmath &= \frac{(2(-a + r_s \cos\theta_s), 2z_s, -2r_s \sin\theta_s, \mu^2 - 2ar_s \cos\theta_s - 2)}{\mu^2 - 2ar_s \cos\theta_s}, \\ R^{zw}_{\pi/2} \circ R^{xy}_{-A_0} \circ X_0 \circ \jmath &= \frac{(\mu^2 - 2c^2, 2cz_s, -2cr_s \sin\theta_s, a\mu^2 - 2c^2 r_s \cos\theta_s)}{c(\mu^2 - 2ar_s \cos\theta_s)}, \end{aligned}$$

since $\cos B = a/c$ and $\sin B = 1/c$. Hence,

$$R^{xy}_{-A_0} \circ X_0 \circ \jmath = \frac{(\mu^2 - 2c^2, 2cz_s, a\mu^2 - 2c^2 r_s \cos\theta_s, 2cr_s \sin\theta_s)}{c(\mu^2 - 2ar_s \cos\theta_s)},$$

If $\mu^2-2c^2\equiv 0$ on $(z_0-\epsilon, z_0+\epsilon)$, then $r_s(z_s)\equiv\sqrt{c^2-{z_s}^2}$, and the annular image $\mathcal{P}_s$ of $Y_0\circ\jmath$ is a part of the sphere $\partial B_c(-a,0,0)$. In particular, X_0 is an isometry onto a portion of a great sphere locally near the point p_0.

If μ^2-2c^2 does not vanish identically, we may again adjust the z_s interval and assume

$$\theta=\theta(z_s)=\tan^{-1}\Big(\frac{2cz_s}{\mu^2-2c^2}\Big) \tag{16}$$

is well defined for $z_s\in(z_0-\epsilon, z_0+\epsilon)$. If $\theta'\equiv 0$, then we have an ODE for $r_s=r_s(z_s)$, namely,

$$\Big(\frac{2cz_s}{\mu^2-2c^2}\Big)'\Big/\Big(1+\Big(\frac{2cz_s}{\mu^2-2c^2}\Big)^2\Big)=0,$$

that is, $2z_sr_s{r_s}'-{r_s}^2+{z_s}^2+c^2=0$. The solutions of this equation have the form $r_s(z_s)=\sqrt{c^2-{z_s}^2+2kz_s}$, and $Y_0\circ\jmath$ parameterizes a portion of the sphere $\partial B_{\sqrt{c^2+k^2}}(-a,0,k)$. Again, $X_0\circ\jmath$ parameterizes a portion of a great sphere.

Finally, if θ' does not vanish identically, we may restrict the values of z_s to an interval $(z_0-\epsilon, z_0+\epsilon)$ on which $\theta'(z_s)$ does not vanish. Furthermore, projecting, we have

$$\begin{aligned}\pi\circ R^{xy}_{-A_0}\circ X_0\circ\jmath &= \frac{(\mu^2-2c^2, 2cz_s, a\mu^2-2c^2r_s\cos\theta_s)}{c(\mu^2-2ar_s\cos\theta_s-2r_s\sin\theta_s)}\\ &= \frac{(\mu^2-2c^2, 2cz_s, a\mu^2-2c^2r_s\cos\theta_s)}{c(\mu^2-2cr_s\cos(\theta_s-B))}\\ &= \frac{\sqrt{(\mu^2-2c^2)^2+4c^2{z_s}^2}}{c(\mu^2-2cr_s\cos(\theta_s-B))}(\cos\theta,\sin\theta,0)\\ &\qquad\qquad+\frac{a\mu^2-2c^2r_s\cos\theta_s}{c(\mu^2-2cr_s\cos(\theta_s-B))}(0,0,1).\end{aligned}$$

Notice that if z_s is fixed, then $\theta=\theta(z_s)$ is constant and the last expression is a parameterization of the circle $\pi\circ R^{xy}_{-A_0}(\Gamma(z_s))$ in the plane $y=x\tan\theta$. In order to see this in more convenient coordinates, set

$$\gamma_0(\theta_s,z_s)=\frac{\sqrt{(\mu^2-2c^2)^2+4c^2{z_s}^2}}{c(\mu^2-2cr_s\cos(\theta_s-B))}\quad\text{and}\quad\sigma_0(\theta_s,z_s)=\frac{a\mu^2-2c^2r_s\cos\theta_s}{c(\mu^2-2cr_s\cos(\theta_s-B))}.$$

Again, thinking of z_s as fixed and $\gamma_0=\gamma_0(\theta_s)$, we can take a derivative to find that γ_0 has exactly one maximum ρ_+ at $\theta_s=B$ and one minimum ρ_- at $\theta_s=B+\pi$ on the interval $[0,2\pi)$. The radius of this circle must be $r=(\rho_+-\rho_-)/2$, that is,

$$\begin{aligned}r&=\frac{1}{2c}\Big(\frac{1}{\mu^2-2cr_s}-\frac{1}{\mu^2+2cr_s}\Big)\sqrt{(\mu^2-2c^2)^2+4c^2{z_s}^2}\\ &=2r_s\frac{\sqrt{(\mu^2-2c^2)^2+4c^2{z_s}^2}}{\mu^4-4c^2{r_s}^2}=\frac{2r_s}{\sqrt{\mu^4-4c^2{r_s}^2}}.\end{aligned}$$

The last equality uses the fact that $(\mu^2-2c^2)^2+4c^2{z_s}^2=\mu^4-4c^2(\mu^2-c^2-{z_s}^2)$ $=\mu^4-4c^2{r_s}^2$. Similarly, the center of this circle must be $d(\cos\theta_s,\sin\theta_s,0)+h\mathbf{e}_3$, where $h=\sigma_0(B)=\sigma_0(B+\pi)$ and $d=(\rho_+-\rho_-)/2$, that is,

$$h=\cos B \quad \text{and} \quad d=\mu^2\sin B/\sqrt{\mu^4-4c^2{r_s}^2}.$$

Next, we define γ and σ by the equations $\gamma_0=d+r\gamma$ and $\sigma_0=h+r\sigma$. That is,

$$\gamma=\frac{\mu^2\cos(\theta_s-B)-2cr_s}{\mu^2-2cr_s\cos(\theta_s-B)} \quad \text{and} \quad \sigma=\frac{\sin(B-\theta_s)\sqrt{\mu^4-4c^2{r_s}^2}}{\mu^2-2cr_s\cos(\theta_s-B)}.$$

The quantities γ and σ are the cosine and sine of the projected Apollonian angle ϕ appearing in the fundamental parameterization (2), as we will now see. Note first that $\gamma=\gamma(\theta_s)$ has maxima and minima corresponding to those of γ_0 with values 1 and -1 respectively. Thus, the assignment $\cos\phi=\gamma$ is always possible. Also, one can see directly that $\gamma^2+\sigma^2=1$. In fact,

$$\begin{aligned}(\mu^2\cos(\theta_s-B)-2cr_s)^2&+\sin^2(B-\theta_s)((\mu^2-2c^2)^2+4c^2{z_s}^2)\\ &=\mu^4-4cr_s\mu^2\cos(\theta_s-B)+4c^2({r_s}^2-(\mu^2-c^2-{z_s}^2)\sin^2(\theta_s-B))\\ &=\mu^4-4cr_s\mu^2\cos(\theta_s-B)+4c^2{r_s}^2\cos^2(\theta_s-B)\\ &=(\mu^2-2cr_s\cos(\theta_s-B))^2.\end{aligned}$$

Thus, the assignments $\cos\phi=\gamma(\theta_s,z_s)$ and $\sin\phi=\sigma(\theta_s,z_s)$, along with the definition of θ given in (16), define ϕ and θ as smooth functions of θ_s and z_s. To see that this defines a nonsingular change of variables, we compute the determinant of $D(\phi,\theta)$. In fact, we already know that $\theta=\theta(z_s)$ and $\theta'\neq 0$. From this we see also that $r=r(\theta)$ is well defined from the definition above. Recalling that $r_s=r_s(z)$, we find

$$D\phi=\frac{(4c^2{r_s}^2-\mu^4,\,2((2{r_s}^2-\mu^2){r_s}'+2r_sz_s)\sin(\theta_s-B))}{\sqrt{\mu^4-4c^2{r_s}^2}(\mu^2-2cr_s\cos(\theta_s-B))}.$$

We need only check that the first coordinate $\partial\phi/\partial\theta_s$ is nonzero. In fact,

$$\mu^4-4c^2{r_s}^2=(\mu^2-2cr_s)(\mu^2+2cr_s)=((r_s-c)^2+{z_s}^2)((r_s+c)^2+{z_s}^2)>0.$$

Thus changing variables, we obtain a local parameterization of $\pi\circ R^{xy}_{-A_0}\circ X_0(M)$ on $(\theta_0-\epsilon_0,\theta_0+\epsilon_0)\times\mathbb{R}$ given by

$$(\theta,\phi)\mapsto(d+r\cos\phi)(\cos\theta,\sin\theta,0)+(h+r\sin\phi)\mathbf{e}_3,$$

where $d=d(\theta)$, $r=r(\theta)$, and $h=\cos B$ is constant. Finally, we set

$$\rho_0=\sqrt{d^2-r^2}=\sqrt{\frac{\mu^4\sin^2 B-4{r_s}^2}{\mu^4-4c^2{r_s}^2}}=\sin B$$

and use the arbitrary rotation by A_0 in order to shift the interval for θ to $(-\epsilon_0, \epsilon_0)$; so we obtain a local parameterization $X : (-\epsilon_0, \epsilon_0) \times \mathbb{R} \to \mathbb{R}^3$ of $\mathcal{P}_0 = \pi \circ X_0(M)$ given by

$$X(\theta, \phi) = (\sqrt{\rho_0^2 + r^2} + r \cos\phi)(\cos\theta, \sin\theta, 0) + (h + r \sin\phi)\mathbf{e}_3.$$

We conclude that, aside from the case of spheres, all CMC surfaces with special spherical symmetry have stereographic projections that can be parameterized in this way. For our classification, this is the basic expression with which we will work. The one unknown function is $r = r(\theta)$, and we need to derive (and solve) the ordinary differential equation corresponding to constant mean curvature in $\mathbb{S}^3$. It will be convenient to write this expression as

$$X(\theta, \phi) = R\boldsymbol{u}_1 + (h + r \sin\phi)\mathbf{e}_3, \tag{17}$$

with $R = d + r\cos\phi$ and $d^2 = r^2 + \rho_0^2$.

Before turning to this classification, we briefly describe spherical symmetry in terms of stereographic projection and point out the key differences making it a (much) more general notion.

1.2. ***Interpreting spherical symmetry.*** Given a line $\Lambda^* = \{\boldsymbol{b} + t\boldsymbol{v} : t \in \mathbb{R}\}$, we may often assume $\boldsymbol{b} \perp \boldsymbol{v}$, and this assumption will be made below whenever possible and convenient.

If $\mathcal{S} \subset \mathbb{S}^3$ has special spherical symmetry and is invariant with respect to cone point reflections

$$h_{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{y} + (|\boldsymbol{y}|^2 - 1)\frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^2}$$

for $\boldsymbol{y} \in \Lambda^* = \{\boldsymbol{b} + t\boldsymbol{v} : t \in \mathbb{R}\}$, some line in $\mathbb{R}^4 \setminus \overline{B_1(0)}$, then a straightforward calculation shows that for $\boldsymbol{x}$ fixed

$$\lim_{t\to\infty} h_{\boldsymbol{b}+t\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{x} - 2(\boldsymbol{x}\cdot\boldsymbol{n})\boldsymbol{n} = g_{\boldsymbol{n}}(\boldsymbol{x}),$$

where $\boldsymbol{n} = \boldsymbol{v}/|\boldsymbol{v}|$ and $g_{\boldsymbol{n}}$ is a great sphere reflection. Thus, for bounded sets like $\mathcal{S}$, we have $g_{\boldsymbol{n}}(\mathcal{S}) = \lim_{t\to\infty} h_{\boldsymbol{b}+t\boldsymbol{v}}(\mathcal{S}) = \mathcal{S}$. Hence, the set Λ of all generalized reflections under which $\mathcal{S}$ is invariant contains $\{h_{\boldsymbol{b}+t\boldsymbol{v}} : t \in \mathbb{R}\} \cup \{g_{\boldsymbol{v}/|\boldsymbol{v}|}\}$. The radial line associated with $h_{\boldsymbol{b}+t\boldsymbol{v}}$ is

$$\ell^* = \{(1-\tau)\mathbf{e}_4 + \tau(\boldsymbol{b} + t\boldsymbol{v}) : \tau \in \mathbb{R}\} \ni \boldsymbol{b} + t\boldsymbol{v},$$

and the radial line associated with $g_{\boldsymbol{v}/|\boldsymbol{v}|}$ is

$$\ell^* = \{\mathbf{e}_4 + \tau\boldsymbol{v}/|\boldsymbol{v}| : \tau \in \mathbb{R}\} \ni \mathbf{e}_4 + \boldsymbol{v}/|\boldsymbol{v}|.$$

We have thus shown half of the following result.

Lemma 3. *If $\mathcal{S} \subset \mathbb{S}^3$ has special spherical symmetry, $\mathcal{S}$ has spherical symmetry.*

To see the other half of the proof, we must consider the case in which $\mathcal{S}$ is invariant with respect to $\Lambda = \{g_{\boldsymbol{n}} : \boldsymbol{n}\cdot\boldsymbol{m} = 0 = \boldsymbol{n}\cdot\tilde{\boldsymbol{m}}\}$, where $\boldsymbol{m}$ and $\tilde{\boldsymbol{m}}$ are nonparallel unit vectors. In this case, we first observe that

$$\bigcup_{\ell^*\in\mathcal{R}} \ell^* = \{\mathbf{e}_4 + \tau\boldsymbol{n} : \tau \in \mathbb{R},\ \boldsymbol{n}\cdot\boldsymbol{m} = 0 = \boldsymbol{n}\cdot\tilde{\boldsymbol{m}}\}.$$

Fixing any $g_{\boldsymbol{n}_0} \in \Lambda$, we can construct an orthonormal basis $\{\boldsymbol{m}, \tilde{\boldsymbol{m}}, \boldsymbol{n}_0, \boldsymbol{v}\}$ for $\mathbb{R}^4$. Setting $\boldsymbol{b} = \mathbf{e}_4 + 3\boldsymbol{n}_0$, we claim that

$$\Lambda^* = \{\boldsymbol{b} + t\boldsymbol{v} : t \in \mathbb{R}\}$$

satisfies requirement (ii) of Definition 2. First note that $|\boldsymbol{b}+t\boldsymbol{v}| \geq |3\boldsymbol{n}_0+t\boldsymbol{v}| - |\mathbf{e}_4| \geq 3-1 = 2$. Thus, $\Lambda^* \subset \mathbb{R}^4 \setminus \overline{B_1(0)}$. Second, $3\boldsymbol{n}_0+t\boldsymbol{v}$ is orthogonal to both $\boldsymbol{m}$ and $\tilde{\boldsymbol{m}}$. Thus, $3\boldsymbol{n}_0 + t\boldsymbol{v} = \tau\boldsymbol{n}$, where $\tau = |3\boldsymbol{n}_0 + t\boldsymbol{v}| \in \mathbb{R}$ and $\boldsymbol{n} = (3\boldsymbol{n}_0 + t\boldsymbol{v})/|3\boldsymbol{n}_0 + t\boldsymbol{v}|$ is orthogonal to both $\boldsymbol{m}$ and $\tilde{\boldsymbol{m}}$. It follows that $\boldsymbol{b} + t\boldsymbol{v} = \mathbf{e}_4 + 3\boldsymbol{n}_0 + t\boldsymbol{v} = \mathbf{e}_4 + \tau\boldsymbol{n}$, which belongs to $\bigcup_{\ell^*\in\mathcal{R}} \ell^*$. Finally, $\mathbf{e}_4 + \boldsymbol{v}/|\boldsymbol{v}| = \mathbf{e}_4 + \boldsymbol{v}$ and $\boldsymbol{v}\cdot\boldsymbol{m} = 0 = \boldsymbol{v}\cdot\tilde{\boldsymbol{m}}$. Thus, $\mathcal{S}$ has spherical symmetry. □

Having shown that special spherical symmetry is a special case of spherical symmetry, we turn our attention to a set $\mathcal{S} \subset \mathbb{S}^3$ with spherical symmetry and make some basic observations concerning the stereographic projection $\mathcal{P} = \pi(\mathcal{S} \setminus \{\mathbf{e}_4\})$.

As usual, we assume $\Lambda^* = \{\boldsymbol{b} + t\boldsymbol{v} : t \in \mathbb{R}\}$ in Definition 2 is given with $\boldsymbol{b} \perp \boldsymbol{v}$. After a preliminary rotation of $\mathbb{R}^4$, we may also assume

$$\Lambda^* = \{-a\mathbf{e}_1 + t\mathbf{e}_3 : t \in \mathbb{R}\} \subset \mathbb{R}^3 \subset \mathbb{R}^4.$$

Let $\boldsymbol{p} \in \Lambda^*$ and $f \in \Lambda$ with radial line ℓ^* passing through $\boldsymbol{p}$. It is easy to check that the projection of the symmetry sphere S of f (where $S = G_{\boldsymbol{n}}$ or $S = H_{\boldsymbol{y}}$ as in Section 0.1) passes through $\mathbf{e}_4$ only if the radial line ℓ^* of f lies in the $x_4 = 1$ hyperplane. Since Λ^* does not intersect this plane, the projection of S is a sphere $\partial B_\rho(\boldsymbol{a})$ in $\mathbb{R}^3$. It follows that $\mathcal{P} \setminus \{\boldsymbol{a}\}$ is invariant under the transformation given in (18). One can check, furthermore, that $\pi(\ell^*) = \{\boldsymbol{a}\} = \underline{\boldsymbol{p}}$. In particular, the centers of the projected spheres comprise the points along a line L in $\mathbb{R}^3$.

Note that spherical symmetry does not specify the radius ρ of $\partial B_\rho(\boldsymbol{a})$ as does special spherical symmetry since, in that case, taking $\boldsymbol{b} = -a\mathbf{e}_1$ and $\boldsymbol{a} = -a\mathbf{e}_1 + t\mathbf{e}_3$, we have $\rho = \rho(t) = \sqrt{\rho_0^2 + (h-t)^2}$, where ρ_0 and h are given constants. This is the key difference. In this regard, it is useful to note that the cone points $\boldsymbol{y}$ in a radial line (any line passing through the north pole but not lying in $x_4 = 1$) correspond to spheres in $\mathbb{S}^3$ that project to concentric spheres in $\mathbb{R}^3$. Hence, specifying the radial line ℓ^* of a reflection specifies the center of a sphere in $\mathbb{R}^3$; specifying the specific cone point $\boldsymbol{y}$ on ℓ^* specifies both the center and radius of $\partial B_\rho(\boldsymbol{a}) = \pi(H_{\boldsymbol{y}})$. With this in mind, we formulate the following property of $\mathcal{P}$.

Definition 3 (spherical symmetry along a line in $\mathbb{R}^3$). A set $\mathcal{P} \subset \mathbb{R}^3$ has spherical symmetry along a line if there is some line L in $\mathbb{R}^3$ and for each $\boldsymbol{a} \in L$, there is some radius $\rho > 0$ such that $\mathcal{P} \setminus \{\boldsymbol{a}\}$ is invariant under the map

$$\boldsymbol{p} \mapsto \rho^2 \frac{\boldsymbol{p}-\boldsymbol{a}}{|\boldsymbol{p}-\boldsymbol{a}|^2} + \boldsymbol{a}. \tag{18}$$

The following result is immediate from the discussion above.

Lemma 4. *If $\mathcal{S} \subset \mathbb{S}^3$ has spherical symmetry, there is some rotation R of $\mathbb{S}^3$ such that $\mathcal{P} = \pi(R(\mathcal{S}) \setminus \{\mathbf{e}_4\})$ has spherical symmetry along a line. Conversely, if $\mathcal{P} \subset \mathbb{R}^3$ has spherical symmetry along a line L, and $\mathcal{P} \cap L = \phi$, then $\mathcal{S} = \pi^{-1}(\mathcal{P}) \subset \mathbb{S}^3$ has spherical symmetry.*

It is not known, in general, if surfaces with spherical symmetry admit convenient parameterization. We sharpen the observations of this section in Section 3 and show that compact surfaces with spherical symmetry are well behaved.

2. CMC surfaces with special spherical symmetry

Consider a local projected immersion $\mathcal{P}$ with parameterization of the form (17):

$$X(\theta, \phi) = R\boldsymbol{u}_1 + (h + r\sin\phi)\mathbf{e}_3 \tag{19}$$

on $(-\epsilon, \epsilon) \times \mathbb{R}$ with $r = r(\theta)$ some smooth function and $R = \sqrt{r^2 + \rho_0^2} + r\cos\phi$; $\rho_0 \in (0, 1)$ and h are constants. We also have an initial condition $r(0) = r_0 > 0$. It will be convenient to let $d = d(\theta) = \sqrt{r^2 + \rho_0^2}$ as above. Calculating the mean curvature H_s of $\mathcal{S}$ at $X_0 = \pi^{-1} \circ X$ we find [Park 2002]

$$H_s = \tfrac{1}{2}(1 + |X|^2)H + X \cdot N$$

where H is the Euclidean mean curvature of $\mathcal{P}$ and N is the normal to $\mathcal{P}$ at $X = X(\theta, \phi)$. Here the s subscript indicates the mean curvature with respect to $\mathbb{S}^3$. It is easily checked that

$$\frac{\partial H}{\partial \phi}\Big|_{\phi=0} = 0.$$

Thus, we find

$$0 = \frac{\partial H_s}{\partial \phi}\Big|_{\phi=0} = (HX \cdot X_\phi + X \cdot N_\phi)|_{\phi=0}. \tag{20}$$

We recall the expression for $\boldsymbol{u}_1$ and introduce notation for its derivative:

$$\boldsymbol{u}_1 = (\cos\theta, \sin\theta, 0) \quad \text{and} \quad \boldsymbol{u}_2 = (-\sin\theta, \cos\theta, 0).$$

Of course, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\mathbf{e}_3$ form an orthonormal basis.

A somewhat lengthy calculation, outlined below, provides these formulas:

$$
\begin{aligned}
X_\phi|_{\phi=0} &= (-r\sin\phi\boldsymbol{u}_1 + r\cos\phi\mathbf{e}_3)|_{\phi=0} = r\mathbf{e}_3,\\
N|_{\phi=0} &= \frac{1}{\sqrt{r'^2+d^2}}(d\boldsymbol{u}_1 - r'\boldsymbol{u}_2),\\
N_\phi|_{\phi=0} &= \frac{d}{\sqrt{r'^2+d^2}}\mathbf{e}_3,\\
H|_{\phi=0} &= \frac{a_0+a_1+a_2}{2r(r+d)^2(r'^2+d^2)^{3/2}} \qquad (21)
\end{aligned}
$$

where

$$
\begin{aligned}
a_0 &= d(rd^2r'' - (r^2+d^2)r'^2 - d^4),\\
a_1 &= r(rd^2r'' - (4r^2+3\rho_0^3)r'^2 - 3d^4),\\
a_2 &= -2r^2d(r'^2+d^2).
\end{aligned}
$$

Substituting the first three formulas into (20), we find

$$0 = h(rH|_{\phi=0} + d/\sqrt{r'^2+d^2}).$$

If $h \neq 0$, then $H|_{\phi=0} = -d/(r\sqrt{r'^2+d^2})$. Comparing this equation with (21), we arrive at

$$(r^2+\rho_0^2)rr'' + \rho_0^2r'^2 + (r^2+\rho_0^2)^2 = 0. \qquad (22)$$

Given $r_0 = r(0) > 0$ and $v_0 = r'(0)$, there are unique values of $a > 0$ and $\theta_1 \in (-\pi, \pi)$ for which the solution is given by

$$r = \sqrt{(a^2+\rho_0^2)\cos^2(\theta-\theta_1) - \rho_0^2}, \qquad (23)$$

which correspond to a (portion of a) sphere. It can be checked that

$$a^2 = r_0^2(1 + v_0^2/(r_0^2+\rho_0^2))$$

and θ_1 is determined by

$$\cos\theta_1 = \frac{r_0^2+\rho_0^2}{\sqrt{(r_0^2+\rho_0^2)^2+r_0^2v_0^2}} \quad \text{and} \quad \sin\theta_1 = \frac{r_0v_0}{\sqrt{(r_0^2+\rho_0^2)^2+r_0^2v_0^2}}.$$

Of course, formal solutions can be obtained with θ_1 in other intervals; this does not affect the values of $r = r(\theta)$, but such an interval will not contain the normalized initial value $\theta_0 = 0$. It is instructive to write the ODE (22) as the equivalent system

$$
\begin{cases}
r' = v\\
v' = -(\rho_0^2v^2 + (r^2+\rho_0^2)^2) \,/\, (r(r^2+\rho_0^2))
\end{cases}
$$

whose phase diagram is shown in Figure 11 along with the geometrical quantities associated with the corresponding spherical solutions.

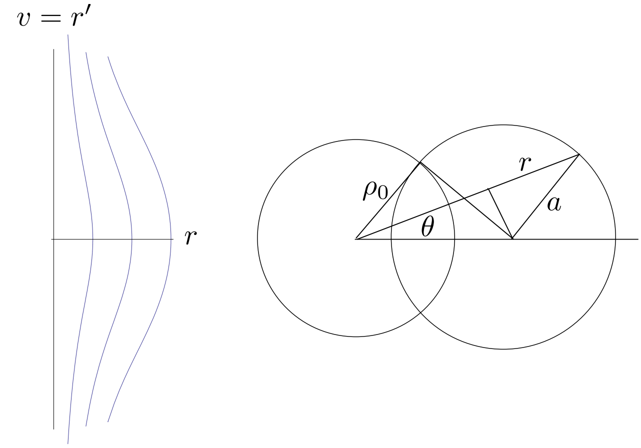

Figure 11. Spherical solutions with nonstandard projection.

From these considerations, we the following lemma is immediate.

Lemma 5. *Unless $\mathcal{S}$ is a sphere, we must have $h = 0$.*

Corollary 2. *Unless $\mathcal{S}$ is a sphere, we must also have $\rho_0 = 1$.*

Proof. This is immediate once we recall that $h = \cos B$ and $\rho_0 = \sin B$ with $B \in (0, \pi/2]$ determined by the original surface. Since $h = 0$, we must have $B = \pi/2$ and $\rho_0 = 1$. The surface must therefore fall back into the class of surfaces that may be stereographically projected (at least locally) to rotationally symmetric ones in $\mathbb{R}^3$. □

We outline below the long calculation alluded to above for the parameterization

$$X(\theta, \phi) = R\boldsymbol{u}_1 + r \sin\phi \mathbf{e}_3. \tag{24}$$

The first and second order quantities mentioned above do not depend on h; we will include the constant $\rho_0 \in (0, 1]$, though in the end, we will use Corollary 2 and specialize to the case $\rho_0 = 1$.

$$N = \frac{X_\theta \times X_\phi}{|X_\theta \times X_\phi|} = \frac{d \cos\phi\, \boldsymbol{u}_1 - r'\boldsymbol{u}_2 + d \sin\phi\, \mathbf{e}_3}{\sqrt{r'^2 + d^2}},$$

$$E = |X_\theta|^2 = r^2 \cos^2\phi + 2r(d + 1/d)\cos\phi + (1 + r^2/d^2)r'^2 + d^2,$$

$$F = X_\theta \cdot X_\phi = -(r^2 r'/d)\sin\phi,$$

$$G = |X_\phi|^2 = r^2,$$

$$e = X_{\theta\theta} \cdot N = \frac{-rd \cos^2\phi + (rr'' - r^2 r'^2/d^2 - r'^2 - d^2)\cos\phi - 2rr'^2/d + dr''}{\sqrt{r'^2 + d^2}},$$

$$f = X_{\theta\phi} \cdot N = (rr'/\sqrt{r'^2 + d^2})\sin\phi,$$

$$
\begin{aligned}
g &= X_{\phi\phi} \cdot N = -rd/\sqrt{r'^2+d^2},\\
X \cdot N &= (d/\sqrt{r'^2+d^2})(d\cos\phi + r),\\
H &= \frac{eG-2fF+gE}{2(EG-F^2)} = \frac{a_0+a_1\cos\phi+a_2\cos^2\phi}{2rR^2(r'^2+d^2)^{3/2}},\\
a_0 &= d(rd^2r'' - (r^2+d^2)r'^2 - d^4),\\
a_1 &= r(rd^2r'' - (r^2+3d^2)r'^2 - 3d^4),\\
a_2 &= -2r^2d(r'^2+d^2),\\
H_s &= \tfrac{1}{2}(1+|X|^2)H + X\cdot N = \frac{c_0+c_1\cos\phi+c_2\cos^2\phi}{4rR^2(r'^2+d^2)^{3/2}},\\
c_0 &= d(rd^2(r^2+d^2+1)r'' - (r^2+d^2+(r^2-d^2)^2)r'^2 + d^4(3r^2-d^2-1)),\\
c_1 &= r(rd^2(r^2+3d^2+1)r'' - (r^2+3d^2+(r^2-d^2)^2)r'^2 + d^4(5r^2-d^2-3)),\\
c_2 &= 2r^2d(rd^2r'' - r'^2 + (r^2-1)d^2).
\end{aligned}
$$

Finally, we obtain for $\rho_0 = 1$

$$
H_s = \frac{\sqrt{r^2+1}(r(r^2+1)r'' - r'^2 + r^4 - 1)}{2r(r'^2+r^2+1)^{3/2}}. \tag{25}
$$

It is important to note that (25) depends on the particular parameterization we have chosen, but not essentially. To be more precise, the mean curvature of a given surface depends on a choice of normal, and the opposite choice of normal results in a change in sign of the mean curvature. The expression we have obtained is for a particular choice of normal ("outward" for the local annular patch in the stereographic projection). Because we are considering all possible signs of the mean curvature, we will obtain all possible portions of surfaces with local parameterization of this form. It is possible, however (and it does happen) that two annular portions of a surface can fit together along a circle (singular with respect to (24)) to form a single smooth piece of constant mean curvature surface; one piece will have mean curvature H_s given by (25), and the other will have mean curvature $-H_s$ according to the same formula. Such surfaces all fall into the nodoid-type class, and this technicality will be discussed further below when it becomes an issue.

We are now in a position to prove Theorem 1. We begin by considering the system corresponding to (25) in the $r > 0$ halfplane:

$$
\begin{cases}
r' = v,\\
v' = 2H_s\Big(1+\dfrac{v^2}{r^2+1}\Big)^{3/2} + \dfrac{1}{r}\Big(1+\dfrac{v^2}{r^2+1}\Big) - r.
\end{cases} \tag{26}
$$

A unique equilibrium point occurs for the system (26) at $(r, v) = (r_*, 0)$, where r_* is a solution of the equation $r^2 - 2H_s r - 1 = 0$, which is easily solved to obtain (5). The Clifford torus lies in a collection of anchor ring solutions that project to

$$\left\{(\sqrt{r_*^2+1} + r_* \cos\phi)(\cos\theta, \sin\theta, 0) + r_* \sin\phi \mathbf{e}_3 : \theta, \phi \in \mathbb{R}\right\}.$$

These anchor rings become, as H_s tends to $-\infty$ due to our choice of normal, thin tubes around the great circle $\{x^2 + y^2 = 1\}$.

More generally, in any solution with initial condition $(r_0, 0)$ and $r_0 \neq r_*$, the point(s) with $v = 0$ is isolated. (If $0 < r_0 < r_*$, then $v'(\theta_0) = -(r_0^2 - 2H_s r_0 - 1) > 0$; if $r_* < r_0$, then $v'(\theta_0) < 0$.) Consequently, aside from the standard tori (anchor ring solutions), all solutions may be pieced together along circles from annular pieces that may be parameterized as

$$X(r, \phi) = (\sqrt{r^2+1} + r \cos\phi)(\cos\theta, \sin\theta, 0) + r \sin\phi \, \mathbf{e}_3, \tag{27}$$

where $\theta = \theta(r)$. This is the only fact we will use for now about the system (26), whose phase diagrams for representative values of H_s, namely, 0 and ± 1, are shown in Figure 12. We will note for future reference one important observation.

Lemma 6. *If* (r, v) *is a solution of* (26), *then* $(\tilde{r}(\theta), \tilde{v}(\theta)) = (r(-\theta), -v(-\theta))$ *is also a solution. Consequently, the phase diagram for* (26) *is symmetric with respect to the r axis; solutions satisfying* $r'(0) = 0$ *are even.*

We note concerning the phase diagrams that each nodoid trajectory (the ones asymptotic to vertical lines) having $H_s < 0$ fits together with a nodoid trajectory having $H_s > 0$ and asymptotic to the same line. One may also match the solutions indicated in the phase diagrams with those represented on two vertical lines in the parameter domain indicated in Figure 13; move downward for $H_s < 0$ and upward for $H_s > 0$. The phase diagrams of Figure 12 are numerically generated, and the precise global properties of all trajectories will not be evident until we finish the proof of Theorem 1; then they may be determined from the explicit formulas.

Before working directly with the equation for $\theta = \theta(r)$, we briefly return to Equation (25). Whenever $v = r' \neq 0$, there is a locally defined smooth function $u = u(r)$ such that $u(r) = r'$. Consequently, $r'' = u'r' = uu'$ and (25) may be rewritten as

$$\frac{ru}{(u^2+r^2+1)^{3/2}} = \frac{u^2-r^4+1}{(r^2+1)(u^2+r^2+1)^{3/2}} + \frac{2H_s r}{(r^s+1)^{3/2}}.$$

Thinking of this equation as $M(r, u)u' = N(r, u)$ and making a standard search for an integrating factor $\phi = \phi(r)$, we find that $\phi = 1/\sqrt{r^2+1}$ and

$$\left(\frac{r}{\sqrt{(u^2+r^2+1)(r^2+1)}} + \frac{H_s}{r^2+1}\right)' = 0.$$

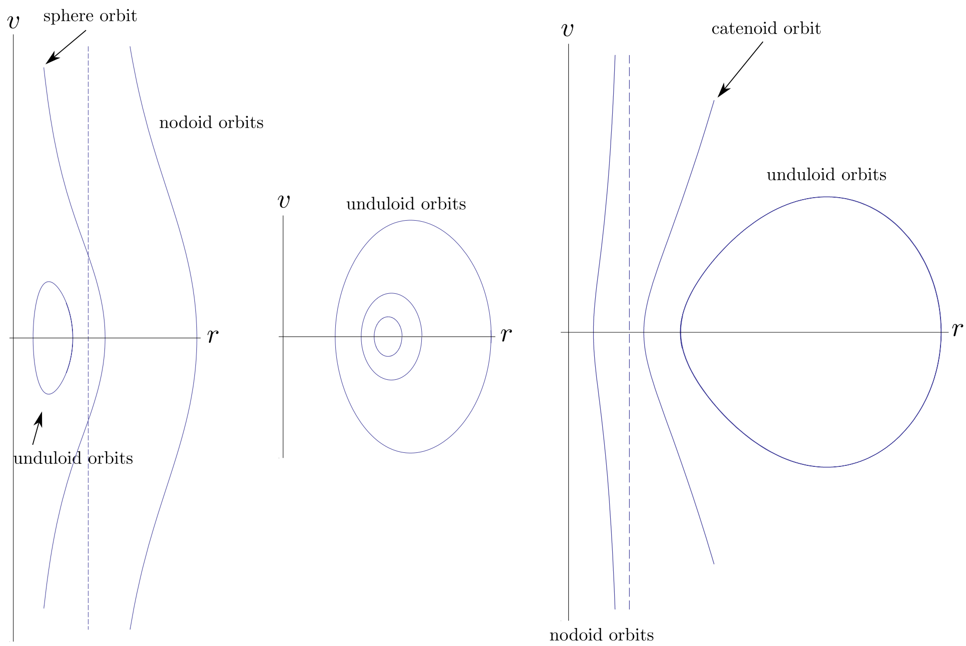

Figure 12. Phase portraits of solutions $r = r(\theta)$ for $H_s < 0$ (left), $H_s = 0$ (center) and $H_s > 0$ (right). The nodoid-type solutions shown (with asymptotes) correspond to only a portion of a fundamental domain. In this particular figure we also see the correspondence of nodoid-type solutions because the figure on the left is for $H_s = -1$, the one on the right is for $H_s = 1$, and the nodoid-type solutions indicated ($c = 3/5$) may be joined along a circle to comprise a fundamental domain; the asymptotes coincide at $r_* = 3/5$ as in the proof of Lemma 11 below.

Thus, we obtain a first integral

$$\frac{r}{\sqrt{(r'^2+r^2+1)(r^2+1)}} + \frac{H_s}{r^2+1} = c. \tag{28}$$

Changing variables in (25), we obtain

$$r\theta'' = -\operatorname{sign}(\theta')2H_s r\left(\theta'^2 + \frac{1}{r^2+1}\right)^{3/2} - \frac{1}{r^2+1}\theta' + (r^2-1)\theta'^3.$$

At this point, we recall that the formula (25) assumes the outward normal on an annular piece of surface, and if θ' changes sign at a smooth finite point of the curve traced out by (r, θ), then Equation (25) is using the opposite choice of normal on opposite sides of that sign change. If $\theta'(r) > 0$ for $r < r_0$, for example, with

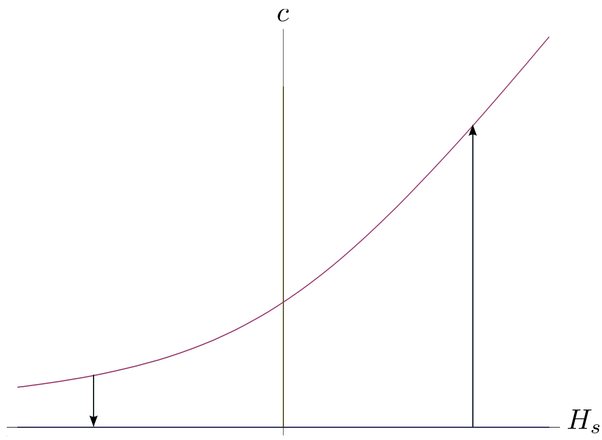

Figure 13. Piecing together solutions represented in parameter space.

$\theta'(r) > 0$ for $r > r_0$, then the equation above assumes the downward (that is, outward) normal for $r < r_0$ and the upward (that is, outward) normal for $r > r_0$. Taking this into account, we get a single equation applying to a single CMC annulus having mean curvature H_s with respect to the upward normal (that is, in the positive θ direction), which is nonsingular with respect to the parameterization (27):

$$r\theta'' = 2H_s r\left(\theta'^2 + \frac{1}{r^2+1}\right)^{3/2} - \frac{1}{r^2+1}\theta' + (r^2-1)\theta'^3. \tag{29}$$

This is the origin of Equation (3), which we could have derived from the parameterization itself via a long calculation. The first integral proved more difficult to derive for this equation as well.

The same change of variables in (28) yields

$$\theta'^2 = \frac{(cr+(c-H_s)/r)^2}{(r^2+1)(1-(cr+(c-H_s)/r)^2)}. \tag{30}$$

It is easily checked that this agrees with (4) up to a sign. In fact, consideration of both possible signs only results in obtaining geometrically congruent pieces of surface, as we will explain below.

The expression on the right in (30) is well defined on intervals where the function $f(r) = cr + (c - H_s)/r$ takes values in $(-1, 1)$. Furthermore, if we temporarily ignore the possibility of ambiguity due to a sign change when f vanishes, we can take the square root in (30) and obtain

$$\theta' = \frac{f}{\sqrt{(r^2+1)(1-f^2)}}, \tag{31}$$

where we have ignored the possible sign change of the right side, since that is equivalent to a change in sign of both c and H_s. It is straightforward to see that the right side has integrable singularities at values of r_m for which $f(r_m) = \pm 1$. A very simple analysis of this function f (see Figure 14) leads to the distinct parameter regions of Theorem 1 and a complete qualitative understanding of the (θ, r) meridian curves for solutions, as summarized in the following result, which we state for convenience under the temporary restriction $c \geq 0$.

Lemma 7. *Let $r \mapsto (\theta(r), r)$ parameterize a portion of the meridian curve of a CMC surface satisfying* (31) *with $c \geq 0$. The inclination angle ψ of such a curve with respect to the r-axis satisfies*

$$\sin\psi := \frac{\theta'}{\sqrt{1+\theta'^2}} = \frac{f}{\sqrt{1+r^2(1-f^2)}}.$$

Consequently, $\sin\psi$ *has the same monotonicity and sign of f on their common interval of definition. Furthermore*, *they both take the values ± 1 at precisely the same singular values r_m. The qualitative behavior of* $\sin\psi$ *may thus be obtained from that of f as follows*:

(i) *If $c = 0$ and $H_s > 0$*, *then* $\sin\psi$ *takes the value -1 at $r_{\min} = H_s$ and increases to* 0 *smoothly on the interval* $[r_{\min}, \infty)$. *The singularity is integrable and the resulting solution is a catenoid-type surface described by Theorem 1*(iii).

(ii) *If* $0 < c < H_s$, *then* $\sin\psi = -1$ *at* $r_{\min} = (-1 + \sqrt{1 - 4c(c - H_s)})/(2c)$ *and increases to* $+1$ *at* $r_{\max} = (1 + \sqrt{1 - 4c(c - H_s)})/(2c)$. *Both singularities are integrable*, *and the resulting solution is of nodoid-type as described by Theorem 1*(v).

(iii) *If* $0 < c = H_s$, *then* $f(r) = cr$, *and* $\sin\psi$ *is defined on* $[0, 1/c]$ *with an integrable singularity at* $1/c$. *Elementary integration leads to the spherical surfaces described in Theorem 1*(i); *the case* $c = H_s = 0$ *is also described there.*

(iv) *If* $c - 1/(4c) < H_s < c$, *then* $\sin\psi$ *is positive and well defined precisely between the singular points* $r_{\min} = (1 - \sqrt{1 - 4c(c - H_s)})/(2c)$ *and* $r_{\max} = (1 + \sqrt{1 - 4c(c - H_s)})/(2c)$; *at both of these points* $\sin\psi = +1$. *Again*, *both singularities are integrable*, *and one obtains a solution with profile curve of unduloid-type as in Theorem 1*(iv).

This lemma is essentially self-explanatory. One can almost obtain Theorem 1 directly from this result by simply expressing the integrals for θ in terms of standard elliptic integrals. The only ambiguities arise from various questions concerning signs, which we now discuss.

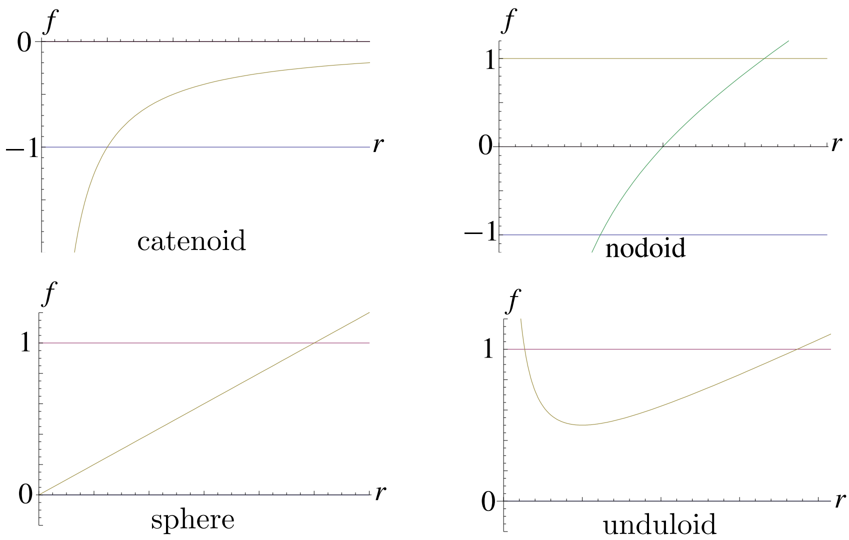

Figure 14. Profiles of inclination angle/indicator function f.

We recall that the only ambiguity in taking the square root in (30) is when f changes sign. This only occurs in the case of nodoid-type surfaces. When that sign change occurs, one has $\theta'' \neq 0$, that is, nodoid-type meridians have no inflections. Thus, the uniqueness theorem for ODEs applied to Equation (29) shows that one must keep the same sign across the singularity. This justifies only consideration of (31) as long as we consider all possible signs for c and H_s.

Our classification does not, in fact, consider all possible signs for c and H_s, because the surface corresponding to (H_s, c) is geometrically congruent to the $(-H_s, -c)$ surface. For example, if $c = 0 > H_s$, we obtain a surface geometrically congruent to that for $c = 0 < H_s$, but with stereographic projection reflected across a plane through the z-axis; this is simply a change of sign for H_s corresponding to a reversal of normal as described in connection with Equation (29). The same remarks apply to all pairs of surfaces determined by the correspondence $(H_s, c) \longleftrightarrow (-H_s, -c)$.

We conclude this section with some remarks on the reduction to standard elliptic integrals and the resulting period conditions.

For catenoid-type surfaces ($c = 0$, $H_s > 0$), we have

$$\theta(r) = -H_s \int_{H_s}^{r} \frac{1}{\sqrt{(\tau^2+1)(\tau^2-H_s^2)}}\, d\tau.$$

(Technically, $|\theta'(r_0)| < \infty$ implies $r(0) = r_0 > H_s$, but since the singularity at H_s is integrable, we may apply a rotation R^{xy} to obtain the expression above.) The change of variables $\tau = H_s \sec t$ yields

$$\begin{aligned} \theta(r) &= -H_s \int_0^{\cos^{-1}(H_s/r)} \frac{\sec t}{\sqrt{H_s^2 \sec^2 t + 1}}\, dt \\ &= -\cos\alpha\, F(\cos^{-1}(H_s/r), \alpha), \end{aligned} \tag{32}$$

where $\alpha = \sin^{-1}(1/\sqrt{H_s^2+1})$ and $F(\phi, \alpha) = \int_0^\phi 1/\sqrt{1-\sin^2\alpha \sin^2 t}\, dt$ is the standard elliptic integral of the first kind.

Thus one finds, as described in Theorem 1(iii), that $\theta_{\max} = \cos\alpha\, K(\alpha)$, where $K(\alpha) = F(\pi/2, \alpha)$ is the complete elliptic integral of the first kind. One can show that $\theta_{\max}$ increases as a function of H_s, taking all values between 0 and $\pi/2$. While the general properties of elliptic integrals are well known, it can be somewhat involved to verify statements like these. For brevity, we will only show how one such result is proved and leave the rest to precisely stated lemmas involving one-dimensional calculus, accompanied by illustrative numerical plots.

Lemma 8. *The function* $\cos\alpha\, K(\alpha)$ *is decreasing in* α *with*

$$\lim_{\alpha \searrow 0} \cos\alpha\, K(\alpha) = \pi/2 \quad \textit{and} \quad \lim_{\alpha \nearrow \pi/2} \cos\alpha\, K(\alpha) = 0.$$

Proof. Note first that $(d/d\alpha)K = \sec\alpha \csc\alpha\, E - \cot\alpha\, K$, where

$$E(\phi, \alpha) = \int_0^\phi \sqrt{1-\sin^2\alpha \sin^2 t}\, dt$$

is the standard elliptic integral of the second kind and (here) $E = E(\alpha) = E(\pi/2, \alpha)$ is the complete elliptic integral of the second kind [Whittaker and Watson 1996, page 521]. Thus,

$$\begin{aligned} \frac{d}{d\alpha}(\cos\alpha\, K) &= -\sin\alpha\, K + \csc\alpha\, E - \cos\alpha \cot\alpha\, K = \frac{1}{\sin\alpha}(E-K) \\ &= -\int_0^{\pi/2} \frac{\sin\alpha \sin^2 t}{\sqrt{1-\sin^2\alpha \sin^2 t}}\, dt < 0. \end{aligned}$$

The first limit is immediate. To see the second, observe that for any $\epsilon \in (0, 1)$, there is some $\delta = \delta(\epsilon) > 0$ such that $\sin^2 t \le 1-(1-\epsilon)(t-\pi/2)^2$ for $\pi/2-\delta \le t \le \pi/2$. Consequently,

$$\begin{aligned} \cos\alpha\, K &= \cos\alpha \Big(\int_0^{\pi/2-\delta} \frac{1}{\sqrt{1-\sin^2\alpha \sin^2 t}}\, dt + \int_{\pi/2-\delta}^{\pi/2} \frac{1}{\sqrt{1-\sin^2\alpha \sin^2 t}}\, dt \Big) \\ &\le \cos\alpha \int_0^{\pi/2-\delta} \sec t\, dt + \int_{\pi/2-\delta}^{\pi/2} \frac{\cos\alpha}{\sqrt{1-\sin^2\alpha(1-(1-\epsilon)(t-\pi/2)^2)}}\, dt \end{aligned}$$

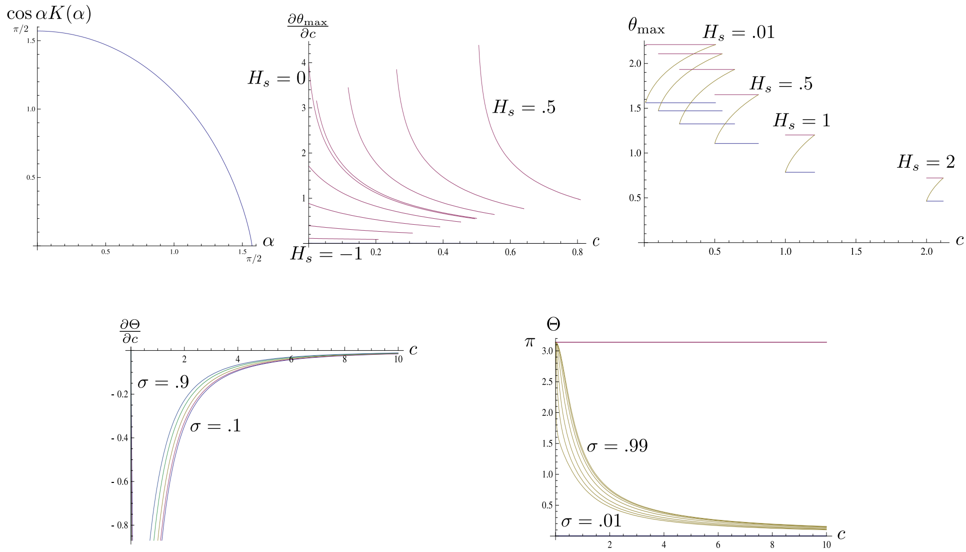

Figure 15. Properties of elliptic integrals.

$$= \cos\alpha \int_0^{\pi/2-\delta} \sec t \, dt + \int_0^{\delta} \frac{\cos\alpha}{\sqrt{\cos^2\alpha + \sin^2\alpha(1-\epsilon)t^2}} \, dt$$

$$= \cos\alpha \int_0^{\pi/2-\delta} \sec t \, dt + \int_0^{\delta} \frac{\cot\alpha/\sqrt{1-\epsilon}}{\sqrt{\cot^2\alpha/(1-\epsilon)+t^2}} \, dt$$

$$= \cos\alpha \int_0^{\pi/2-\delta} \sec t \, dt + \frac{\cot\alpha}{\sqrt{1-\epsilon}} \left(\ln\left(\delta + \sqrt{\frac{\cot^2\alpha}{1-\epsilon} + \delta^2}\right) - \ln\left(\sqrt{\frac{\cot^2\alpha}{1-\epsilon}}\right) \right).$$

Since ϵ is fixed in $(0, 1)$ and δ is fixed and positive,

$$\lim_{\alpha \nearrow \pi/2} \cos\alpha K = - \lim_{\alpha \nearrow \pi/2} \cot\alpha \ln(\cos\alpha)/\sqrt{1-\epsilon} = 0. \qquad \square$$

It is essentially the same for reductions of unduloid-type $(c - 1/(4c) < H_s < c)$ and nodoid-type $(0 < c < H_s)$. In each case, we may take $r_0 = r_{\min}$, so that

$$\theta(r) = \frac{1}{c} \int_{r_{\min}}^{r} \frac{c(\tau^2+1) - H_s}{\sqrt{(1+\tau^2)(r_{\max}^2 - \tau^2)(\tau^2 - r_{\min}^2)}} \, d\tau$$

$$= \frac{c - H_s}{c} \int_{r_{\min}}^{r} \frac{1}{\sqrt{(1+\tau^2)(r_{\max}^2 - \tau^2)(\tau^2 - r_{\min}^2)}} \, d\tau$$

$$+ \int_{r_{\min}}^{r} \frac{\tau^2}{\sqrt{(1+\tau^2)(r_{\max}^2 - \tau^2)(\tau^2 - r_{\min}^2)}} \, d\tau.$$

In each integral we substitute

$$t = \sin^{-1}\sqrt{1-(r_{\min}/\tau)^2/(1-\mu^2)},$$

where $\mu = r_{\min}/r_{\max} \in (0, 1)$ and obtain

$$\tau = \frac{r_{\min}}{\sqrt{1-(1-\mu^2)\sin^2 t}}, \qquad d\tau = \frac{(1-\mu^2) r_{\min} \sin t \cos t}{(1-(1-\mu^2)\sin^2 t)^{3/2}},$$

$$\begin{aligned}\frac{1}{\sqrt{1+\tau^2}} &= \frac{\sqrt{1-(1-\mu^2)\sin^2 t}}{\sqrt{1-(1-\mu^2)\sin^2 t + r_{\min}^2}}\\ &= \frac{1}{\sqrt{1+r_{\min}^2}}\cdot\frac{\sqrt{1-(1-\mu^2)\sin^2 t}}{\sqrt{1-(1-\mu^2)\sin^2 t/(1+r_{\min}^2)}},\end{aligned}$$

$$\frac{1}{\sqrt{r_{\max}^2-\tau^2}} = \frac{\sqrt{1-(1-\mu^2)\sin^2 t}}{r_{\max}\cos t\sqrt{1-\mu^2}}, \qquad \frac{1}{\sqrt{\tau^2-r_{\min}^2}} = \frac{\sqrt{1-(1-\mu^2)\sin^2 t}}{r_{\min}\sin t\sqrt{1-\mu^2}},$$

so that

$$\begin{aligned}\theta(r) &= \frac{c-H_s}{c r_{\max}\sqrt{1+r_{\min}^2}}\int_0^A \frac{1}{\sqrt{1-(1-\mu^2)\sin^2 t/(1+r_{\min}^2)}}\,dt\\ &\quad+\frac{\mu r_{\min}}{\sqrt{1+r_{\min}^2}}\int_0^A \frac{1}{(1-(1-\mu^2)\sin^2 t)\sqrt{1-(1-\mu^2)\sin^2 t/(1+r_{\min}^2)}}\,dt\\ &= \frac{c-H_s}{c r_{\max} d_0}F(A,\alpha) + \frac{\mu r_{\min}}{d_0}\Pi(\nu, A, \alpha)\end{aligned}$$

where

$$A = \sin^{-1}\sqrt{\frac{1-(r_{\min}/r)^2}{1-\mu^2}}, \qquad d_0 = \sqrt{1+r_{\min}^2},$$

$$\alpha = \sin^{-1}\frac{\sqrt{1-\mu^2}}{d_0} = \sin^{-1}(\sqrt{\nu}/d_0), \qquad \nu = 1-\mu^2,$$

and

$$\Pi(\nu,\phi,\alpha) = \int_0^\phi \frac{1}{(1-\nu\sin^2 t)\sqrt{1-\sin^2\alpha\sin^2 t}}\,dt$$

is the standard elliptic integral of the third kind.

For both the unduloid and nodoid-type surfaces, the half-period

$$\theta(r_{\max}) = \frac{c-H_s}{c r_{\max} d_0}K(\alpha) + \frac{\mu r_{\min}}{d_0}\Pi(\nu, \pi/2, \alpha) \tag{33}$$

is of interest. Let us first consider the unduloid-type region by fixing H_s and restricting attention to vertical segments $\max\{0, H_s\} < c < (H_s + \sqrt{H_s^2+1})/2$.

Lemma 9. *For fixed H_s, the function $\theta_{\max} = \theta(r_{\max})$ is increasing as a function of c on* $(\max\{0, H_s\}, (H_s + \sqrt{H_s^2+1})/2)$ *with*

$$\lim_{c\searrow \max\{0,H_s\}} \theta_{\max} = \begin{cases} \dfrac{\pi}{2} - \dfrac{H_s}{\sqrt{H_s^2+1}} K\big(\sin^{-1}(1/\sqrt{H_s^2+1})\big) & \textit{if } H_s < 0, \\ \sin^{-1}(1/\sqrt{H_s^2+1}) & \textit{if } H_s \geq 0 \end{cases}$$

and

$$\lim_{c\nearrow \left(H_s+\sqrt{H_s^2+1}\right)/2} \theta_{\max} = \frac{\sqrt{H_s^2+1} - H_s}{\sqrt{2\big(H_s^2+1-H_s\sqrt{H_s^2+1}\big)}}\,\pi.$$

Notes on proof. The derivative $\partial\theta_{\max}/\partial c$ is a (complicated) expression of the form $AK(\alpha)+BE(\alpha)$, where K and E are the complete elliptic integrals of the first and second kinds, and A and B are rational functions of c, H_s, and $\sqrt{1-4c(c-H_s)}$. The second plot in Figure 15 shows the values of this derivative as a function of c in the unduloid-type region for $H_s = -1, -.5, -.25, -.1, -.01, 0, .1, .25, .5$.

In order to see the limits, it is convenient to set $\lambda = \sqrt{1-4c(c-H_s)}$ and write, for example, $r_{\min}$ and d_0 in the nonsingular forms

$$r_{\min} = \frac{2(c-H_s)}{1+\lambda} \quad \text{and} \quad d_0 = \frac{\sqrt{(1+\lambda)^2+4(c-H_s)^2}}{1+\lambda} = \frac{\sqrt{2}\sqrt{1+\lambda-2H_s(c-H_s)}}{1+\lambda}.$$

Making these substitutions, it is not difficult to see that

$$\frac{c-H_s}{cr_{\max}d_0}K(\alpha) = \frac{\sqrt{2}(c-H_s)}{\sqrt{1+\lambda-2H_s(c-H_s)}}K\left(\sin^{-1}\sqrt{\frac{2\lambda}{1+\lambda-2H_s(c-H_s)}}\right)$$

and

$$\frac{\mu r_{\min}}{d_0}\Pi(\nu,\pi/2,\alpha) = \\ \frac{\sqrt{2}(c-H_s)(1-\lambda)}{(1+\lambda)\sqrt{1+\lambda-2H_s(c-H_s)}}\Pi\left(\frac{4\lambda}{(1+\lambda)^2},\pi/2,\sin^{-1}\sqrt{\frac{2\lambda}{1+\lambda-2H_s(c-H_s)}}\right).$$

The sum of these two expressions is, of course, $\theta(r_{\max})$; it is convenient to refer to the first one as the "K part" and the second one as the "Π part."

When $H_s < 0$ is fairly straightforward to see that

$$\lim_{c\searrow 0}\frac{c-H_s}{cr_{\max}d_0}K(\alpha) = -\frac{H_s}{\sqrt{H_s^2+1}}K\left(\sin^{-1}\left(\frac{1}{\sqrt{H_s^2+1}}\right)\right).$$

The term in (33) involving an elliptic integral of the third kind (the Π part) is more difficult for this limit. One finds, from the fact that $H_s < 0$, that for c small the integral falls into the circular classification with $\sin^2\alpha < \nu < 1$ according to Milne-Thompson [Abramowitz and Stegun 1964]. It follows that

$$\Pi(\nu,\pi/2,\alpha) = K(\alpha) + (\pi/2)\delta_2(1-\Lambda_0(\phi,\alpha)),$$

where $\delta_2 = \sqrt{\nu/((1-\nu)(\nu - \sin^2\alpha))}$ and Λ_0 is Heuman's lambda function with $\phi = \sin^{-1}\sqrt{(1-\nu)/\cos^2\alpha}$. Thus, from the expression above, one sees that it is only necessary to compute

$$\lim_{c\searrow 0}(1-\lambda)(K(\alpha) + (\pi/2)\delta_2(1-\Lambda_0(\phi,\alpha))).$$

It is easily checked that $(1-\lambda)/\cos\alpha$ has a finite limit, so that Lemma 8 applies to the first term. One can next see that the limit in the last argument of Λ_0 is nonsingular, so that one need only consider

$$\lim_{\lambda\nearrow 1}(1-\lambda)\delta_2(1-\Lambda_0(\phi,\alpha_0))\pi \quad \text{where } \alpha_0 = \sin^{-1}(1/\sqrt{H_s^2+1}).$$

Since Λ_0 is finite valued at $\phi = 0$ for all fixed α_0, one only needs to check that the finite value taken by δ_2 in the limit (with $\nu = 4\lambda/(1+\lambda)^2$) is the correct one.

It is interesting that the limit of $\theta_{\max}$ is singular for $H_s < 0$ and $c \searrow 0$: Comparing to the catenoid-type surfaces that correspond to $H_s < 0 = c$, we might expect the value $-\cos\alpha_0 K(\alpha_0)$ in accord with (32), and this is attained by the limit of the K part. An additional contribution of $\pi/2$ arises from the limit of the Π part. Thus, considering the convergence of the generating curves, the portion of the generating curve left of the inflection is dominated by the K part and that to the right is dominated by the Π part.

The second and third limits are both represented in the third graph of Figure 15. As the unduloid-type surfaces approach spheres, the limit is nonsingular; the K part vanishes in the limit and the Π part gives the value $\sin^{-1}(1/\sqrt{H_s^2+1})$ indicated.

The third (upper) limit is nonsingular and straightforward since λ tends to zero as the unduloid-type surfaces approach the standard tori. There is a peculiarity to our expressions in Theorem 1: In this limit, our expression for $r_{\min}$ (and the one for $r_{\max}$) tends to $(\sqrt{H_s^2+1} - H_s)/2$, apparently at odds with Theorem 1(iv). For convenience, we considered the tori with respect to the outward normal and the unduloid-type surfaces with respect to the inward normal; note that reversing the sign of H_s harmonizes the classification. □

Next, we consider the same region by fixing $\sigma \in (0, 1)$ and restricting attention to the curve

$$H_s = c - (1-\sigma)/(4c),$$

as indicated in Figure 16. Thus, we write $\Theta(\sigma, c) := \theta_{\max}(c - (1-\sigma)/(4c), c)$.

Lemma 10. *For $\sigma \in (0, 1)$ fixed, Θ is a decreasing function of c with*

$$\lim_{c\searrow 0}\Theta = \pi \quad \textit{and} \quad \lim_{c\nearrow\infty}\Theta = 0.$$

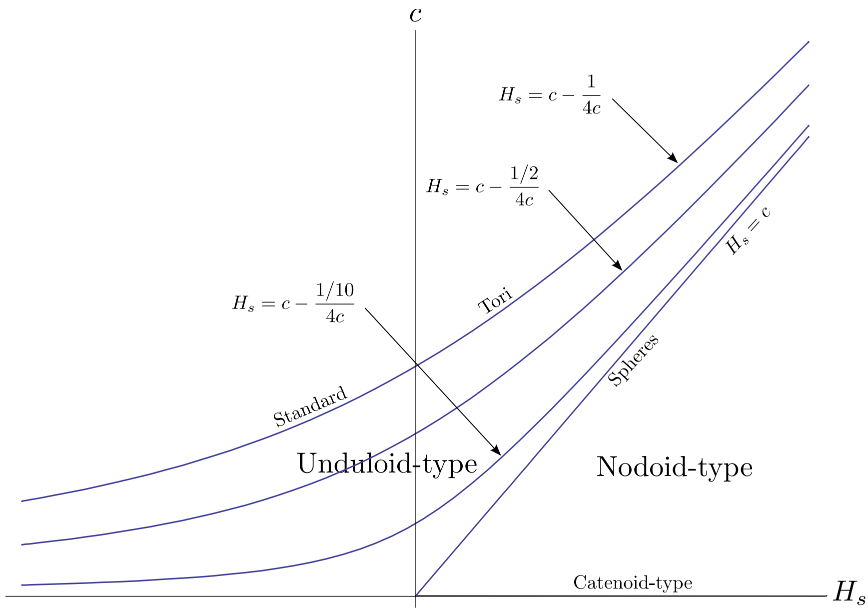

Figure 16. Curves that foliate the parameter region $c - 1/(4c) < H_s < c$.

Notes on the proof. We see immediately that $\lambda = \sqrt{\sigma}$ so that

$$\frac{c-H_s}{cr_{\max}d_0}K(\alpha) = \frac{1-\sqrt{\sigma}}{\sqrt{4c^2+(1-\sqrt{\sigma})^2}}K\left(\sin^{-1}\left(\frac{4c}{1+\sqrt{\sigma}}\sqrt{\frac{\sqrt{\sigma}}{4c^2+(1-\sqrt{\sigma})^2}}\right)\right)$$

and

$$\frac{\mu r_{\min}}{d_0}\Pi(\nu,\pi/2,\alpha) = \frac{(1-\sqrt{\sigma})^2}{(1+\sqrt{\sigma})\sqrt{4c^2+(1-\sqrt{\sigma})^2}} \times\Pi\left(\frac{4\sigma}{(1+\sqrt{\sigma})^2},\pi/2,\sin^{-1}\left(\frac{4c}{1+\sqrt{\sigma}}\sqrt{\frac{\sqrt{\sigma}}{4c^2+(1-\sqrt{\sigma})^2}}\right)\right).$$

Note that the c dependence is not as complicated in this case. The derivative $\partial\Theta/\partial c$ has the form $AK(\alpha) + BE(\alpha)$, with A and B rational functions of σ, c, and $\sqrt{4c^2 + (1-\sqrt{\sigma})^2}$. The fourth graph in Figure 15 gives $\partial\Theta/\partial c$ for various values of σ. The fifth graph illustrates the limits. $\square$

Corollary 3. (i) *For each $m = 2, 3, \ldots$ and $\sigma \in (0, 1)$ fixed, there is a unique $c = c_m(\sigma)$ for which $\Theta(\sigma, c_m) = \pi/m$.*

(ii) $\dfrac{\partial\theta_{\max}}{\partial H_s} < 0$.

(iii) *The condition $\theta_{\max}(H_s, c) = \pi/m$ defines smooth curves, as indicated in Theorem 1.*

Proof. The first claim is immediate from the monotonicity of the preceding lemma. The second follows from differentiation:

$$\frac{\partial \Theta}{\partial \sigma}(\sigma, c) = \frac{\partial \theta_{\max}}{\partial H_s}\Big(c - \frac{1-\sigma}{4c}, c\Big) \cdot \frac{1}{4c}.$$

The third follows from the second, since we obtain the ODE

$$\frac{\partial \theta_{\max}}{\partial H_s}(H_s, c_m) + \frac{\partial \theta_{\max}}{\partial c}(H_s, c_m) c_m' = 0. \qquad \square$$

Qualitatively, we observe for nodoid-type surfaces that the loops in the generating curve always face the θ-axis. More precise is the following result, whose proof is similar to the one found in [Hynd and McCuan 2006].

Lemma 11. *If* $0 < c < H_s$, *then* $\theta(r_{\max}) > \theta(r_{\min})$.

Proof. With the normalization $\theta_{\min} = 0$ as above, this assertion is equivalent to $\theta(r_{\max}) > 0$. To see this, we return to the expression (31). Notice that $f(r) = cr + (c - H_s)/r$ is increasing and concave on $[r_{\min}, r_{\max}]$, taking values -1 and 1 at the endpoints. We let $r_* = \sqrt{H_s/c - 1}$ denote the unique zero of f. Setting $\tau(r) = (H_s - c)/(cr)$, which is the unique solution of $f(\tau) = -f(r)$, we find

$$\begin{aligned}\theta(r_{\max}) &= \int_{r_{\min}}^{r_*} \frac{f}{\sqrt{(r^2+1)(1-f^2)}}\,dr + \int_{r_*}^{r_{\max}} \frac{f}{\sqrt{(r^2+1)(1-f^2)}}\,dr \\ &= \int_{r_*}^{r_{\max}} \frac{1}{r}\left(\frac{r}{\sqrt{r^2+1}} - \frac{(H_s-c)/(cr)}{\sqrt{((H_s-c)/(cr))^2+1}}\right)\frac{f}{\sqrt{1-f^2}}\,dr.\end{aligned}$$

Notice that $r/\sqrt{r^2+1}$ is increasing, and that $r \geq (H_s - c)/(cr)$ when $r \geq r_* = \sqrt{(H_s - c)/c}$. Thus, $\theta_{\max} > 0$. $\square$

3. Spherically symmetric compact surfaces

We now consider the more general condition of spherical symmetry and prove Theorem 2. Let $\mathcal{S} \subset \mathbb{S}^3$ be a compact surface with spherical symmetry.

Recall our discussion of spherical symmetry in Section 1.2 and the fact that stereographic projection extends naturally to $\mathbb{R}^4 \setminus \{x_4 = 1\}$. We begin with the preliminary rotation described there, which resulted in $\Lambda^* = \{-a_0\mathbf{e}_1 + t\mathbf{e}_3 : t \in \mathbb{R}\}$ with $a_0 > 1$. If, having made this normalization, we have that $\mathbf{e}_4 \notin \mathcal{S}$, then

$\mathcal{P} = \pi(\mathcal{S})$ is a compact surface in $\mathbb{R}^3$ with spherical symmetry along a line.

Otherwise, we seek to find a preliminary rotation R so that this assertion is true of $\mathcal{P} = \pi(R(\mathcal{S}))$. Using once again [Hirsch 1994, Proposition 3.1.2], we know that any neighborhood of $\mathbf{e}_4$ contains points of $\mathbb{S}^3 \setminus \mathcal{S}$. Taking such a point $\boldsymbol{p}$ and denoting its fourth coordinate by $\cos\epsilon$, we see that for some (small) rotation R_0

of $\mathbb{R}^3$, we can arrange to have $R_0^{-1} \circ R_{-\epsilon}^{xw}\mathbf{e}_4 = \boldsymbol{p}$. Thus, applying $R_\epsilon^{xw} \circ R_0$ as an additional preliminary rotation, we may assume

$$\Lambda^* = \{R_\epsilon^{xw}(-a_0 R_0(\mathbf{e}_1) + t R_0(\mathbf{e}_3)) : t \in \mathbb{R}\}.$$

We may write

$$R_0(\mathbf{e}_1) = \sum a_{1j}\mathbf{e}_j \quad \text{and} \quad R_0(\mathbf{e}_3) = \sum a_{3j}\mathbf{e}_j$$

so that

$$\begin{aligned}\lambda(t) &= R_\epsilon^{xw}(-a_0 R_0(\mathbf{e}_1) + t R_0(\mathbf{e}_3)) \\ &= (-a_0 a_{11} + t a_{31})\cos\epsilon\,\mathbf{e}_1 + \sum_{j=2}^{3}(-a_0 a_{1j} + t a_{3j})\mathbf{e}_j + (-a_0 a_{11} + t a_{31})\sin\epsilon\,\mathbf{e}_4.\end{aligned}$$

Since $|a_{11}| \leq 1$ and is fixed, we may assume ϵ is small enough so $-a_0 a_{11}\sin\epsilon \neq 1$. It follows that there is at most one value of t for which $\lambda(t)$ can intersect the plane $\{x_4 = 1\}$. More precisely, if $a_{31} = 0$, then $\Lambda^* \cap \{x_4 = 1\} = \phi$ and the reasoning of Section 1.2 yields that $\pi(\mathcal{S} \setminus \{\mathbf{e}_4\}) = \pi(\mathcal{S})$ is a compact surface with spherical symmetry along the line $L = \pi(\Lambda^*)$. If $a_{31} \neq 0$, then $\lambda(t_0) \in \{x_4 = 1\}$ for the unique value $t_0 = (1 + a_0 a_{11}\sin\epsilon)/(a_{31}\sin\epsilon)$. We pause here to relabel so that $\Lambda^* = \{\boldsymbol{b} + t\boldsymbol{v} : t \in \mathbb{R}\}$ with $(\boldsymbol{b} + t\boldsymbol{v}) \cdot \mathbf{e}_4 \neq 1$ unless $t = t_0$. Setting $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$, $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$, $\underline{\boldsymbol{b}} = (b_1, b_2, b_3)$ and $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$ as usual, we have for $t \neq t_0$

$$\pi(\boldsymbol{b} + t\boldsymbol{v}) = \frac{\underline{\boldsymbol{b}} + t\underline{\boldsymbol{v}}}{1 - (b_4 + t v_4)} = \frac{1}{1 - b_4}\underline{\boldsymbol{b}} + \frac{t}{1 - (b_4 + t v_4)}\left(\frac{v_4}{1 - b_4}\underline{\boldsymbol{b}} + \underline{\boldsymbol{v}}\right).$$

Thus, $\mathcal{P} = \pi(\mathcal{S})$ is a compact immersed surface in $\mathbb{R}^3$ which, by our discussion of radial lines, is invariant under maps

$$\psi(\boldsymbol{x}) = \rho^2 \frac{\boldsymbol{x} - \boldsymbol{a}}{|\boldsymbol{x} - \boldsymbol{a}|^2} + \boldsymbol{a},$$

where

$$\boldsymbol{a} = \boldsymbol{a}(t) = \frac{1}{1 - b_4}\underline{\boldsymbol{b}} + \frac{t}{1 - (b_4 + t v_4)}\left(\frac{v_4}{1 - b_4}\underline{\boldsymbol{b}} + \underline{\boldsymbol{v}}\right) \quad \text{for } t \in \mathbb{R} \setminus \{t_0\}$$

and $\rho = \rho(t) > 0$. The centers of reflection $\boldsymbol{a}(t)$ include all points on a line L in $\mathbb{R}^3$ except $\boldsymbol{a}_1 = \lim_{t\to\infty}\boldsymbol{a}(t) = -\underline{\boldsymbol{v}}/v_4$. We now recall that $\mathbf{e}_4 + \boldsymbol{v}/|\boldsymbol{v}| \in \bigcup_{\ell^* \in \mathfrak{R}} \ell^*$. Therefore, some radial line contains $\mathbf{e}_4 + \boldsymbol{v}/|\boldsymbol{v}|$, and there is some sphere $\partial B_\rho(\boldsymbol{a}) \subset \mathbb{R}^3$ with center $\boldsymbol{a} = \pi(\mathbf{e}_4 + \boldsymbol{v}/|\boldsymbol{v}|) = -\underline{\boldsymbol{v}}/v_4 = \boldsymbol{a}_1$ about which $\mathcal{P}$ is symmetric. Thus, in all cases, it is possible after preliminary rotation to project a compact surface $\mathcal{S} \subset \mathbb{S}^3$ so that $\mathcal{P} = \pi(\mathcal{S})$ is compact in $\mathbb{R}^3$ and so that $\mathcal{P}$ has spherical symmetry along a line.

The reasoning above admits an additional preliminary rotation of $\mathbb{R}^3$, which we now use to again pause and relabel:

Theorem 5. *Given a compact, connected surface $\mathcal{P}$ immersed in $\mathbb{R}^3$ and with spherical symmetry along a line $L = \{-a\mathbf{e}_1 + t\mathbf{e}_3 : t \in \mathbb{R}\} \subset \mathbb{R}^3$, the surface $\mathcal{S} = \pi^{-1}(\mathcal{P}) \subset \mathbb{S}^3$ has special spherical symmetry.*

A number of lemmas follow, which together prove this result. For all of them, we fix notation as follows: The symmetry spheres along L are denoted by $\partial B_\rho(\boldsymbol{x}_0)$ with $\rho = \rho(t)$ corresponding to $\boldsymbol{x}_0 = -a\mathbf{e}_1 + t\mathbf{e}_3$. The collection of all such spheres is denoted by Σ, and we denote by Ψ the set of associated reflection maps

$$\psi(\boldsymbol{x}) = \rho^2 \frac{\boldsymbol{x} - \boldsymbol{x}_0}{|\boldsymbol{x} - \boldsymbol{x}_0|^2} + \boldsymbol{x}_0.$$

The claim of Theorem 5 is equivalent to showing that each such sphere passes through a particular horizontal circle $\{(x, y, h) : (x+a)^2 + y^2 = \rho_0^2\}$.

Lemma 12. $\mathcal{P} \cap L = \phi$, *and hence* $\operatorname{dist}(\mathcal{P}, L) > 0$.

Proof. If $\boldsymbol{x}_0 \in \mathcal{P} \cap L$, then there is a sequence of points $\boldsymbol{x}_j \in \mathcal{P} \setminus L$ with $\boldsymbol{x}_j \to \boldsymbol{x}_0$. Since

$$\lim_{j\to\infty} |\psi(\boldsymbol{x}_j)| = \lim_{j\to\infty} \left|\rho^2 \frac{\boldsymbol{x}_j - \boldsymbol{x}_0}{|\boldsymbol{x}_j - \boldsymbol{x}_0|^2} + \boldsymbol{x}_0\right| \geq \lim_{j\to\infty} \left(\rho^2 \frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_0|} - |\boldsymbol{x}_0|\right) = \infty,$$

this contradicts compactness. □

Lemma 13. *For every $\boldsymbol{x}_0 \in L$, we must have $\mathcal{P} \cap \overline{B_\rho(\boldsymbol{x}_0)} \neq \phi$ and $\mathcal{P} \setminus B_\rho(\boldsymbol{x}_0) \neq \phi$.*

Proof. If $\boldsymbol{x} \in \mathcal{P} \setminus \overline{B_\rho(\boldsymbol{x}_0)}$, then $\psi(\boldsymbol{x}) \in \mathcal{P} \cap B_\rho(\boldsymbol{x}_0)$. □

Lemma 14. *$\mathcal{P}$ is symmetric with respect to a unique horizontal plane $L^\perp$ orthogonal to L.*

Proof. By compactness, there is some $R > 0$ such that $\mathcal{P} \subset B_R(-a\mathbf{e}_1)$. Consider $-a\mathbf{e}_1 + t_j\mathbf{e}_3$ with $t_j \nearrow +\infty$. When $t_j > R$, we must have $-a\mathbf{e}_1 + (t_j - \rho(t_j))\mathbf{e}_3$ on the segment connecting $-a\mathbf{e}_1 - R\mathbf{e}_3$ and $-a\mathbf{e}_1 + R\mathbf{e}_3$. Consequently, $-R \leq t_j - \rho(t_j) \leq R$. Taking a subsequence, we may assume $t_j - \rho(t_j) \to b_0 \in \mathbb{R}$ as $j \to +\infty$. For $\boldsymbol{x}$ in any compact set, such as $\mathcal{P}$,

$$\lim_{j\to\infty} \psi_j(\boldsymbol{x}) = \lim_{j\to\infty} \rho(t_j)^2 \frac{\boldsymbol{x} + a\mathbf{e}_1 - t_j\mathbf{e}_3}{|\boldsymbol{x} + a\mathbf{e}_1 - t_j\mathbf{e}_3|^2} - a\mathbf{e}_1 + t_j\mathbf{e}_3 = \boldsymbol{x} + 2(b_0 - \boldsymbol{x}\cdot\mathbf{e}_3)\mathbf{e}_3.$$

The last expression will be recognized as standard reflection with respect to the horizontal plane $L^\perp = \{x_3 = b_0\}$. This gives existence.

If there were another such plane of symmetry, the composition of the two associated reflections would provide a vertical translation to which $\mathcal{P}$ is invariant. This again contradicts the fact that $\mathcal{P}$ is bounded. □

Lemma 15. *Let $\psi, \tilde{\psi} \in \Psi$ with associated symmetry spheres $\partial B_\rho(\boldsymbol{x}_0)$, $\partial B_{\tilde{\rho}}(\tilde{\boldsymbol{x}}_0)$, both in Σ.*

(i) *$\partial B_\rho(\boldsymbol{x}_0)$ and $\partial B_{\tilde{\rho}}(\tilde{\boldsymbol{x}}_0)$ have nontrivial intersection outside of L.*

(ii) *$\partial B_\rho(\boldsymbol{x}_0)$ and $L^\perp$ have nontrivial intersection outside of L.*

(iii) *If $\boldsymbol{x}_0 \in \partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$, then $\psi(\partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0) \setminus \{\boldsymbol{x}_0\}) = L^\perp$.*

(iv) *If $\boldsymbol{x}_0 \notin \partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$, then $\psi(\partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)) \in \Sigma$.*

Proof. For the first claim, we proceed by contradiction. If $\partial B_\rho(\boldsymbol{x}_0) \cap \partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0) \subset L$, then either $B_\rho(\boldsymbol{x}_0) \subset B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$, $B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0) \subset B_\rho(\boldsymbol{x}_0)$, or $B_\rho(\boldsymbol{x}_0) \cap B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0) = \phi$. The second possibility is (by relabeling) the same as the first. If the last possibility obtains, then a calculation shows that $\widetilde{\psi}(\partial B_\rho(\boldsymbol{x}_0))$ is a sphere $\partial B_{\bar\rho}(\bar{\boldsymbol{x}}_0))$ with $B_{\bar\rho}(\bar{\boldsymbol{x}}_0)) \subset B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$. Also, a calculation shows that the reflection $\overline{\psi}$ associated with $\partial B_{\bar\rho}(\bar{\boldsymbol{x}}_0))$ is given by $\widetilde{\psi} \circ \psi \circ \widetilde{\psi}$. Thus, $\partial B_{\bar\rho}(\bar{\boldsymbol{x}}_0)) \in \Sigma$ and again the situation reduces to the first possibility $B_\rho(\boldsymbol{x}_0) \subset B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$. We note for future reference that

$$\bar\rho = \frac{\tilde\rho^2 \rho}{|(t-\tilde t)^2 - \rho^2|} \quad \text{and} \quad \bar{\boldsymbol{x}}_0 = \tilde{\boldsymbol{x}}_0 + \frac{\tilde\rho^2}{(t-\tilde t)^2 - \rho^2}\mathbf{e}_3.$$

We begin with the special case $\boldsymbol{x}_0 = \tilde{\boldsymbol{x}}_0$. By the reasoning above (with $\boldsymbol{x}_0$ and $\tilde{\boldsymbol{x}}_0$ reversed), we obtain $\partial B_{\rho_1}(\boldsymbol{x}_0) \in \Sigma$ with $\rho_1 = \rho^2/\tilde\rho$. Repeating this construction with $B_{\rho_1}(\boldsymbol{x}_0)$ and $B_{\tilde\rho}(\boldsymbol{x}_0)$, we obtain $\partial B_{\rho_j}(\boldsymbol{x}_0) \in \Sigma$ with $\rho_j = \tilde\rho(\rho/\tilde\rho)^{2^j} \to 0$ since $\rho/\tilde\rho < 1$. According to Lemma 13, we must have points in $\mathcal{P}$ converging to $\boldsymbol{x}_0 \in L$. This contradicts Lemma 12. This special case has this corollary:

Corollary 4. *For each $\partial B_\rho(\boldsymbol{x}) \in \Sigma$, the radius $\rho = \rho(t)$ is uniquely determined.*

More generally, if $\partial B_\rho(\boldsymbol{x}_0)$ and $\partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$ are not assumed to have the same center, but it is assumed that $\overline{B_\rho(\boldsymbol{x}_0)} \subset B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$ so that $\delta = \operatorname{dist}(\partial B_\rho(\boldsymbol{x}_0), \partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)) > 0$, then the sequence of nested spheres $\partial B_{\rho_j}(\boldsymbol{x}_j) \in \Sigma$ may still be constructed as above, and from the formula for the radius, we see that

$$\rho_1 = \rho^2 \frac{\tilde\rho}{\tilde\rho + |\tilde t - t|} \cdot \frac{1}{\tilde\rho - |\tilde t - t|} \le \frac{\rho^2}{\rho + \delta} \le \rho.$$

Noting that $\overline{B_{\rho_1}(\boldsymbol{x}_1)} \subset B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$ and $\delta_1 = \operatorname{dist}(\partial B_{\rho_1}(\boldsymbol{x}_1), \partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)) > \delta$, we find by induction (as follows) that

$$\begin{aligned}
\rho_j &\le \rho_{j-1} \frac{\rho_{j-1}}{\tilde\rho - |\tilde t - t_{j-1}|} \\
&\le \rho\left(\frac{\rho}{\rho+\delta}\right)^{j-1} \frac{\rho_{j-1}}{\rho_{j-1}+\delta} \le \rho\left(\frac{\rho}{\rho+\delta}\right)^{j} \to 0 \quad \text{as } j \to +\infty.
\end{aligned}$$

We again obtain arbitrarily small spheres and the same contradiction.

Finally, if $B_\rho(\boldsymbol{x}_0) \subset B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0)$ with $\partial B_\rho(\boldsymbol{x}_0) \cap \partial B_{\tilde\rho}(\tilde{\boldsymbol{x}}_0) \neq \phi$, then making the same construction yields

$$\rho_1 = \frac{\rho^2 \tilde\rho}{\tilde\rho^2 - (\tilde\rho - \rho)^2} = \frac{\rho\tilde\rho}{2(\tilde\rho - \rho) + \rho}.$$

By induction in this case, we have

$$\rho_j = \frac{\rho\tilde{\rho}}{2^j(\tilde{\rho}-\rho)+\rho} \to 0 \quad \text{as } j \to +\infty,$$

and the contradiction is the same one. We have established the first assertion.

The second claim follows via contradiction from the first. If $B_\rho(\boldsymbol{x}_0) \cap L^\perp \subset L$, then $\psi(L^\perp) = \partial B_{\rho_1}(\boldsymbol{x}_1) \setminus \{\boldsymbol{x}_0\}$ with $\rho_1 < \rho$ and $B_{\rho_1}(\boldsymbol{x}_1) \subset B_\rho(\boldsymbol{x}_0)$. Also, letting ψ_0 denote reflection in $L^\perp$, and ψ_1 reflection in $\partial B_{\rho_1}(\boldsymbol{x}_1)$, we have $\psi_1 = \psi \circ \psi_0 \circ \psi$, at least outside of L. Thus, $\partial B_{\rho_1}(\boldsymbol{x}_1) \in \Sigma$ with $\partial B_{\rho_1}(\boldsymbol{x}_1) \cap \partial B_\rho(\boldsymbol{x}_0) \subset L$.

That $\psi(\partial B_{\tilde{\rho}}(\tilde{\boldsymbol{x}}_0) \setminus \{\boldsymbol{x}_0\})$ is a horizontal plane if $\boldsymbol{x}_0 \in \partial B_{\tilde{\rho}}(\tilde{\boldsymbol{x}}_0)$ follows from a calculation. Another shows that reflection with respect to that plane is given by $\psi \circ \widetilde{\psi} \circ \psi$ at all points in $\mathbb{R}^3 \setminus \{\boldsymbol{x}_0\}$. Since $\boldsymbol{x}_0 \notin \mathcal{P}$ by Lemma 12, we see that $\psi(\partial B_{\tilde{\rho}}(\tilde{\boldsymbol{x}}_0) \setminus \{\boldsymbol{x}_0\})$ is a symmetry plane for $\mathcal{P}$. This plane must be $L^\perp$ of course.

The last claim follows via a similar, and now familiar, reasoning. $\square$

In view of the previous lemma, for each $\boldsymbol{x}_0 = -a\mathbf{e}_1 + t\mathbf{e}_3 \in L$ the radius $\rho_0(t)$ of the intersection circle of $\partial B_\rho(\boldsymbol{x}_0)$ with $L^\perp$ is well defined. The properties of this quantity are the key to the rest of the proof of Theorem 5.

Lemma 16. $\rho_0(t \pm \rho(t)) = \rho_0(t)$.

Proof. Let $\psi_\pm \in \Psi$ be the reflection associated to $\partial B_{\rho(t\pm\rho(t))}(\boldsymbol{x}_0 \pm \rho(t)\mathbf{e}_3) \in \Sigma$. Note that $\psi_\pm(\partial B_{\rho(t)}(\boldsymbol{x}_0) \setminus \{\boldsymbol{x}_0 \pm \rho(t)\})$ is a horizontal plane of symmetry for $\mathcal{P}$. By Lemma 14 that plane must be $L^\perp$. Moreover, the intersection circle $C_\pm = \partial B_{\rho(t\pm\rho(t))}(\boldsymbol{x}_0 \pm \rho(t)\mathbf{e}_3) \cap \partial B_{\rho(t)}(\boldsymbol{x}_0)$ is a nontrivial horizontal circle.

On the other hand, $C_\pm$ is fixed by $\psi_\pm$, and

$$\psi_\pm(C_\pm) \subset \psi_\pm(\partial B_{\rho(t)}(\boldsymbol{x}_0) \setminus \{\boldsymbol{x}_0 \pm \rho(t)\}) = L^\perp.$$

Thus $C_\pm \subset L^\perp$, and it follows that $C_\pm = \partial B_{\rho(t)}(\boldsymbol{x}_0) \cap L^\perp$. $\square$

Lemma 17. $\rho_0(t) = \rho(b_0)$ *for every* $t \in \mathbb{R}$.

Proof. We again argue by contradiction. If for some t, we have $\rho_0(t_0) < \rho(b_0)$, then $\psi_{b_0}(\partial B_{\rho(t_0)}(-a\mathbf{e}_1 + t_0\mathbf{e}_3))$ is a sphere $\partial B_{\rho_1}(\boldsymbol{x}_1)$. As usual, $\partial B_{\rho_1}(\boldsymbol{x}_1) \in \Sigma$ and $\rho_0(t_1) > \rho(b_0)$.

Now we restrict attention to $t \in \mathbb{R}$ for which $\rho_0(t) = \rho_0(t_1)$. The previous lemma gives us many such t. We first observe that $\mathcal{B} = \{|t - b_0| : \rho_0(t) = \rho_0(t_1)\}$ is bounded away from zero. In fact, by the triangle inequality

$$|t - b_0| > \rho(t) - \rho_0(t) > \rho(t) - \rho(b_0) > 0.$$

On the other hand, setting $t_j = t_{j-1} + \rho(t_{j-1})$ for $j = 2, 3, \ldots$, we obtain a sequence with $\rho_0(t_j) = \rho_0(t_1)$ such that $|t_j - b_0| \in \mathcal{B}$. Also, since $\rho(t) > |t - b_0|$, we may assume $t_1 > b_0$. Then for $j \geq 2$, we have $t_j > 2t_{j-1} - b_0$, so that inductively we find $t_j > 2^{j-1}t_1 - (2^{j-1}-1)b_0 = t_1 + (2^{j-1}-1)(t_1 - b_0) \to +\infty$ as $j \to +\infty$. It follows that $\rho(t_j) \to +\infty$ as $j \to +\infty$.

Finally, setting $\tau_j = t_j - \rho(t_j)$ we obtain another sequence of points with $\rho_0(\tau_j) = \rho_0(t_1) > \rho(b_0)$. These points satisfy

$$|\tau_j - b_0| = t_j - \tau_j - t_j + b_0 = \rho(t_j) - \sqrt{\rho(t_j)^2 - \rho_0(t_1)^2} \to 0 \quad \text{as } j \to +\infty.$$

This contradicts the fact that $\mathcal{B}$ is bounded away from zero. □

We have shown that every sphere $\partial B_\rho(\boldsymbol{x}_0)$ passes through the horizontal circle $\{(x, y, h) : (x+a)^2 + y^2 = \rho_0^2\}$ where $h = b_0$ and $\rho_0 = \rho(b_0)$. Thus, $\mathcal{S} = \pi^{-1}(\mathcal{P})$ has special spherical symmetry and Theorem 5 is proved. □

4. Rotational symmetry

If a nonspherical surface $\mathcal{S} \in \mathbb{S}^3$ stereographically projects to a surface of rotation about the projection of a geodesic, then we may assume the axis of symmetry in $\mathbb{R}^3$ is the z-axis and the meridian curve is given locally by $x = x(z)$. In this case, a natural parameterization for $\mathcal{P} = \pi(\mathcal{S})$ is

$$Y(\vartheta, z) = (x \cos\vartheta, x \sin\vartheta, z). \tag{34}$$

Stereographic projection of the expression in (2) yields

$$X(\theta, \phi) = (R\cos\theta, R\sin\theta, r\sin\phi), \quad \text{where } R = \sqrt{r^2+1} + r\cos\phi, \tag{35}$$

which does not, in general, have rotational symmetry in $\mathbb{R}^3$. Nevertheless, for an appropriate rotation R of $\mathbb{S}^3$, we find that $Y = \pi \circ R \circ \pi^{-1} \circ X$ does indeed have the form (34). In fact, taking $R = R^{yz}_{\pi/2} R^{xw}_{\pi/2}$, one checks easily that the resulting projected surface has all the planes passing through the z-axis as planes of symmetry. It follows that the surface is rotationally symmetric. Setting $\phi = 0$ and

$$z = \frac{(\sqrt{r^2+1}+r)(\sqrt{r^2+1}+r^2+1)\sin\theta}{\sqrt{r^2+1}+r^2+1+(\sqrt{r^2+1}+r)\cos\theta},$$

where $r = r(\theta, 0)$, we find the meridian curve is given by

$$x = \frac{(\sqrt{r^2+1}+r^2)(\sqrt{r^2+1}+r^2+1)}{\sqrt{r^2+1}+r^2+1+(\sqrt{r^2+1}+r)\cos\theta}.$$

We thus obtain the expression (34) parametrically in ϑ and θ:

$$Y = (x(\theta)\cos\vartheta, x(\theta)\sin\vartheta, z(\theta)).$$

Due to the spatial inhomogeneity of the spherical metric in $\mathbb{R}^3 = \pi(\mathbb{S}^3 \setminus \{\mathbf{e}_4\})$, it is not surprising that the meridian curves do not take a simple form. It is therefore

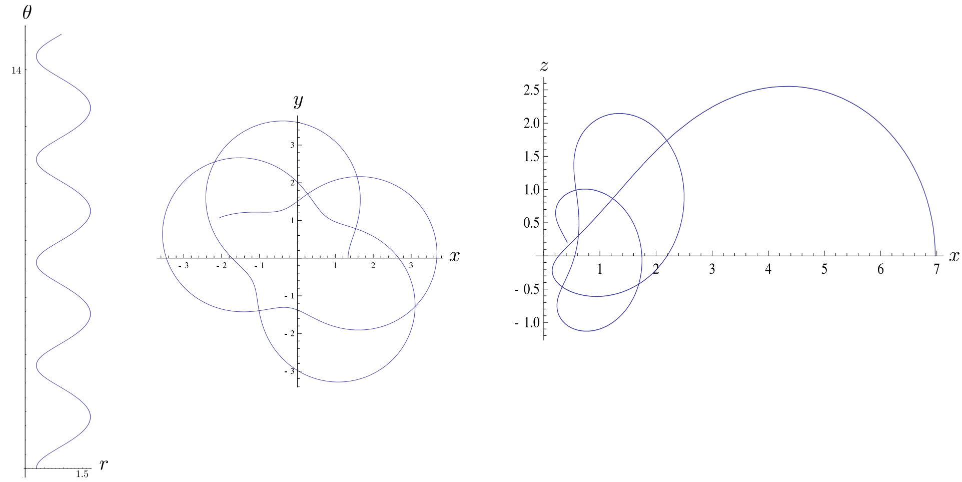

Figure 17. The generating curve of an unduloid-type surface, the stereographic projection of the curve $\phi = 0$ in Apollonian position, and the meridian of the same surface in symmetric position.

fortuitous that these surfaces are easily understood in terms of the (sometimes periodic) function r; the expression (35) might be called the *Apollonian form*.

We conclude with Figures 17–20, which are galleries of meridian curves of representative surfaces in rotationally symmetric form.

References

[Abramowitz and Stegun 1964] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series **55**, Government Printing Office, 1964. MR 29 #4914 Zbl 0171.38503

[Alexandrov 1962] A. D. Alexandrov, "A characteristic property of spheres", *Ann. Mat. Pura Appl.* (4) **58** (1962), 303–315. MR 26 #722

[Brito and Leite 1990] F. Brito and M. L. Leite, "A remark on rotational hypersurfaces of S^n", *Bull. Soc. Math. Belg. Sér. B* **42**:3 (1990), 303–318. MR 91j:53028 Zbl 0734.53010

[do Carmo and Dajczer 1983] M. do Carmo and M. Dajczer, "Rotation hypersurfaces in spaces of constant curvature", *Trans. Amer. Math. Soc.* **277** (1983), 685–709. MR 85b:53055 Zbl 0518.53059

[Chern 1959] S. S. Chern, *Differentiable manifolds*, Textos de Matemática **4**, Instituto de Física e Matemática, Universidade do Recife, 1959. MR 24 #A3566 Zbl 0099.37402

[Hirsch 1994] M. W. Hirsch, *Differential topology*, Graduate Texts in Mathematics **33**, Springer, New York, 1994. MR 96c:57001

[Hopf 1983] H. Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics **1000**, Springer, Berlin, 1983. MR 85b:53001 Zbl 0526.53002

[Hsiang 1982] W.-y. Hsiang, "Generalized rotational hypersurfaces of constant mean curvature in the Euclidean spaces, I", *J. Differential Geom.* **17** (1982), 337–356. MR 84h:53009 Zbl 0493.53043

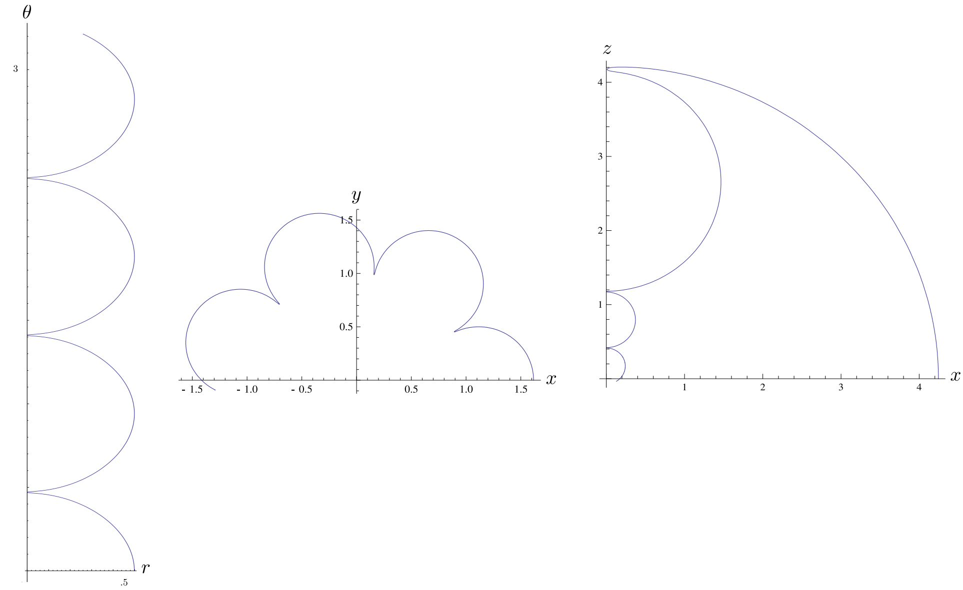

Figure 18. The generating curve of spheres, the stereographic projection of the curve $\phi = 0$ in Apollonian position, and the meridian of the same spheres in symmetric position. Note that the generating curve does not consist of arcs of circles, though the projection curves are arcs of circles.

[Hynd and McCuan 2006] R. Hynd and J. McCuan, "On toroidal rotating drops", *Pacific J. Math.* **224**:2 (2006), 279–289. MR 2007h:53010 Zbl 1125.49001

[Jagy 1998] W. C. Jagy, "Sphere-foliated constant mean curvature submanifolds", *Rocky Mountain J. Math.* **28**:3 (1998), 983–1015. MR 99j:53026 Zbl 0979.53072

[Kilian and Schmidt 2008] M. Kilian and M. U. Schmidt, "On the moduli of constant mean curvature cylinders of finite type in the 3-sphere", 2008. arXiv 0712.0108v2

[McCuan 1997] J. McCuan, "Symmetry via spherical reflection and spanning drops in a wedge", *Pacific J. Math.* **180**:2 (1997), 291–323. MR 98m:53013 Zbl 0885.53009

[McCuan and Spietz 1998] J. McCuan and L. Spietz, "Rotations of the three-sphere and symmetry of the Clifford torus", Mathematical Sciences Research Institute, 1998, Available at http://www.msri.org/publications/preprints/online/1998-052.html.

[Ôtsuki 1970] T. Ôtsuki, "Minimal hypersurfaces in a Riemannian manifold of constant curvature.", *Amer. J. Math.* **92** (1970), 145–173. MR 41 #9157 Zbl 0196.25102

[Otsuki 1988] T. Otsuki, "A nonlinear ordinary differential equation of order 2 in differential geometry", pp. 1001–1025 in *Topics in differential geometry, II* (Debrecen, 1984), edited by J. Szenthe and L. Tamássy, Colloq. Math. Soc. János Bolyai **46**, North-Holland, Amsterdam, 1988. MR 89g:53009 Zbl 0646.53054

[Park 2002] S.-H. Park, "Sphere-foliated minimal and constant mean curvature hypersurfaces in space forms and Lorentz–Minkowski space", *Rocky Mountain J. Math.* **32**:3 (2002), 1019–1044. MR 2003j:53037 Zbl 1043.53055

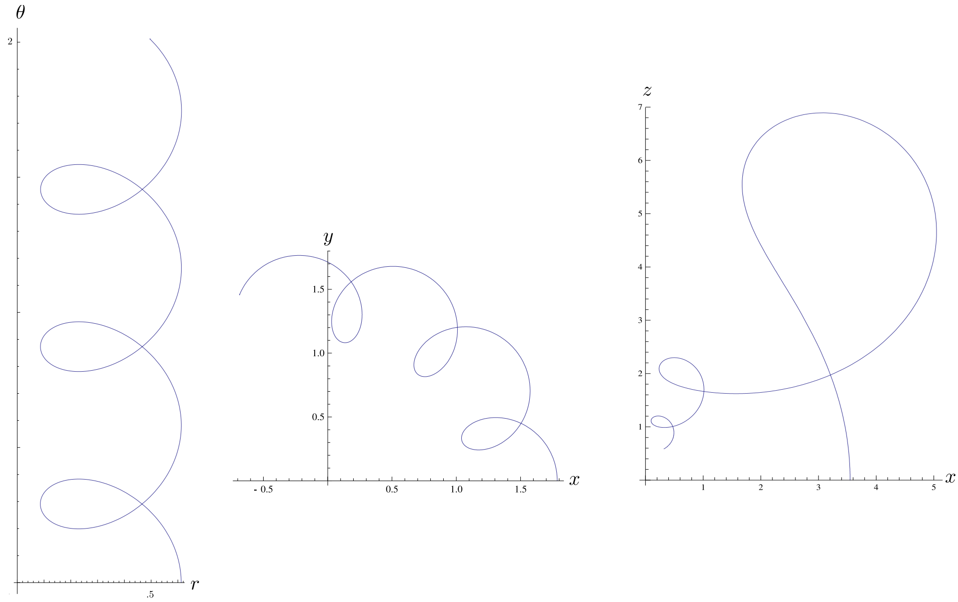

Figure 19. The generating curve of a nodoid-type surface, the stereographic projection of the curve $\phi = 0$ in Apollonian position, and the meridian of the same surface in symmetric position.

[Ros 1995] A. Ros, "A two-piece property for compact minimal surfaces in a three-sphere", *Indiana Univ. Math. J.* **44**:3 (1995), 841–849. MR 97g:53008 Zbl 0861.53009

[Rossman and Sultana 2007] W. Rossman and N. Sultana, "Morse index of constant mean curvature tori of revolution in the 3-sphere", *Illinois J. Math.* **51**:4 (2007), 1329–1340. MR 2417430

[Rossman and Sultana 2008] W. Rossman and N. Sultana, "The spectra of Jacobi operators for constant mean curvature tori of revolution in the 3-sphere", *Tokyo J. Math.* **31**:1 (2008), 161–174. MR 2009b:53010

[Wente 1980] H. C. Wente, "The symmetry of sessile and pendent drops", *Pacific J. Math.* **88**:2 (1980), 387–397. MR 83j:49042a Zbl 0473.76086

[Whittaker and Watson 1996] E. T. Whittaker and G. N. Watson, *A course of modern analysis: An introduction to the general theory of infinite processes and of analytic functions*, Cambridge University Press, 1996. MR 97k:01072 Zbl 0951.30002

[Yau 1982] S. T. Yau (editor), *Seminar on Differential Geometry*, Annals of Mathematics Studies **102**, Princeton University Press, 1982. MR 83a:53002

Received March 12, 2008. Revised January 19, 2009.

RYAN HYND
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720-3840
UNITED STATES
ryanhynd@math.berkeley.edu

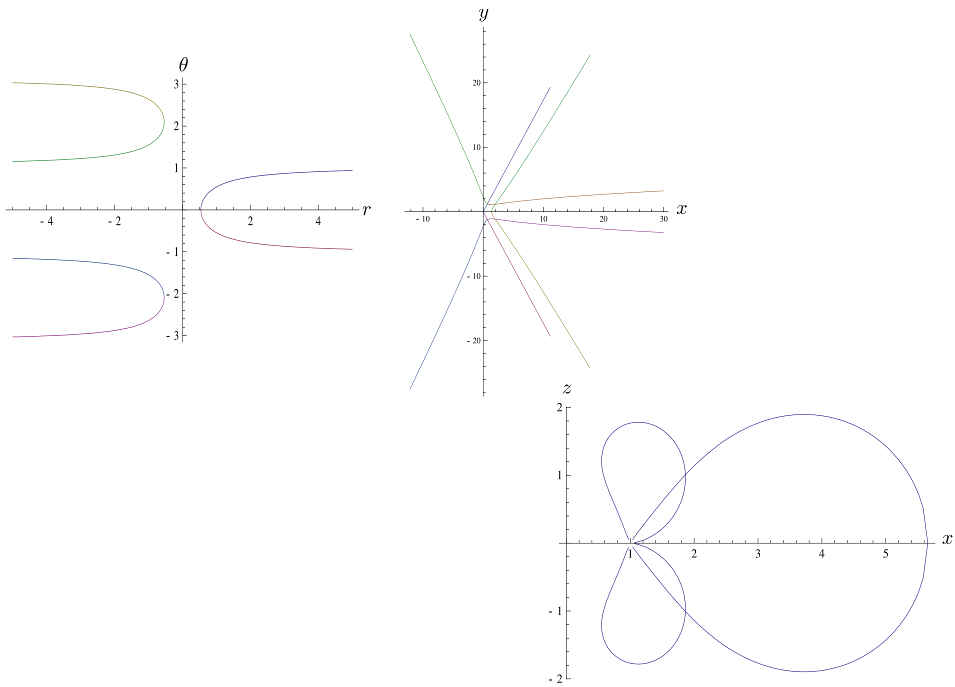

Figure 20. The generating curve of a catenoid-type surface, the stereographic projection of the curve $\phi = 0$ in Apollonian position, and the meridian of the same surface in symmetric position. This is the catenoid-type surface with $\theta_{\text{max}} = \pi/3$, so the surface is compact. Three loops are shown in the meridian; there are three more, but it is not easy to guess how they will look. In contrast, the three missing from the $\phi = 0$ curve are easy to draw in.

SUNG-HO PARK
KIAS HOEGIRO
87(207-43 CHEONGNYANGNI 2-DONG)
DONGDAEMUN-GU
SEOUL 130-722
SOUTH KOREA
shubuti@kias.re.kr

JOHN MCCUAN
SCHOOL OF MATHEMATICS
GEORGIA TECH
686 CHERRY STREET
ATLANTA, GA 30332
UNITED STATES
mccuan@math.gatech.edu
http://www.math.gatech.edu/~mccuan

PACIFIC JOURNAL OF MATHEMATICS
Vol. 241, No. 1, 2009

WIENER TAUBERIAN THEOREMS FOR $L^1(K\backslash G/K)$

E. K. NARAYANAN

We prove a Wiener-type Tauberian theorem for L^1 spherical functions on a semisimple Lie group of arbitrary real rank.

1. Introduction

Let $f \in L^1(\mathbb{R})$ and let $\tilde{f}$ be its Fourier transform. The celebrated Wiener Tauberian theorem says that the ideal generated by f is dense in $L^1(\mathbb{R})$ if and only if $\tilde{f}$ is a nowhere vanishing function on the real line. Ehrenpreis and Mautner [1955] observed that the corresponding result is not true for the commutative algebra of K-biinvariant functions on the semisimple Lie group SL(2, $\mathbb{R}$). Here K is the maximal compact subgroup SO(2). However, in the same paper it was also proved that an additional condition of not-too-rapid decay on the spherical Fourier transform of a function suffices to prove an analogue of the Wiener Tauberian theorem. That is, if f is a K-biinvariant integrable function on $G = \mathrm{SL}(2, \mathbb{R})$ and its spherical Fourier transform $\hat{f}$ does not vanish anywhere on the maximal ideal space (which can be identified with a certain strip on the complex plane) then the function f generates a dense subalgebra of $L^1(K\backslash G/K)$ provided $\hat{f}$ does not vanish too fast at ∞. See [Ehrenpreis and Mautner 1955] for precise statements.

There have been a number of attempts to generalize these results to $L^1(K\backslash G/K)$ or $L^1(G/K)$ where G is a noncompact connected semisimple Lie group with finite center. Almost complete results have been obtained when G is a group of real rank one. We refer the reader to [Benyamini and Weit 1992; Ben Natan et al. 1996; Sarkar 1998; Sitaram 1988] for results on the rank-one case. See also [Sarkar 1997] for a result on the whole group SL(2, $\mathbb{R}$).

Sitaram [1980] proved that under suitable conditions on the spherical Fourier transform of a single function f, an analogue of the Wiener Tauberian theorem holds for $L^1(K\backslash G/K)$, with no assumptions on the rank of G. The purpose of this paper is to prove such a theorem for an arbitrary family of functions with suitable conditions on the spherical Fourier transforms.

MSC2000: primary 43A20; secondary 43A30, 43A80.
Keywords: Wiener Tauberian theorems, ideal, Schwartz space.
The author was supported in part by a grant from UGC via DSA-SAP..

Notation and preliminaries. For convenience, we follow the notation in [Sitaram 1980], so we essentially reproduce its introduction. For unexplained terminology, refer to [Helgason 1994]. Let G be a connected noncompact semisimple Lie group with finite center and K a fixed maximal compact subgroup of G. Fix an Iwasawa decomposition $G = KAN$ and let $\mathbf{a}$ be the Lie algebra of A. Let $\mathbf{a}^*$ be the real dual of $\mathbf{a}$ and $\mathbf{a}^*_{\mathbb{C}}$ its complexification. Let ρ be the half sum of positive roots for the adjoint action of $\mathbf{a}$ on $\mathbf{g}$, the Lie algebra of G. The Killing form induces a positive definite form $\langle\, .\, ,\, .\,\rangle$ on $\mathbf{a}^* \times \mathbf{a}^*$. Extend this form to a bilinear form on $\mathbf{a}^*_{\mathbb{C}}$. We will use the same notation for the extension as well. Let W be the Weyl group of the symmetric space G/K. Then there is a natural action of W on $\mathbf{a}$, $\mathbf{a}^*$ and $\mathbf{a}^*_{\mathbb{C}}$, and $\langle\, .\, ,\, .\,\rangle$ is invariant under this action.

For each $\lambda \in \mathbf{a}^*_{\mathbb{C}}$, let φ_λ be the elementary spherical function associated with λ. Recall that φ_λ is given by the formula

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}\, dk, \quad x \in G.$$

It is known that $\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda' = s\lambda$ for some $s \in W$. Let l be the dimension of $\mathbf{a}$ and let $F \subset \mathbb{C}^l$ denote the set

$$F = \mathbf{a}^* + iC_\rho, \quad \text{where } C_\rho \text{ is the convex hull of } \{s\rho : s \in W\}.$$

A well-known theorem of Helgason and Johnson states that φ_λ is bounded if and only if $\lambda \in F$.

Let $I(G)$ be the set of all complex valued spherical functions on G:

$$I(G) = \{f : f(k_1xk_2) = f(x), k_1, k_2 \in K, x \in G\}.$$

Fix a Haar measure dx on G and let $I_1(G) = I(G) \cap L^1(G)$. Then it is well known that $I_1(G)$ is a commutative Banach algebra under convolution and that the maximal ideal space of $I_1(G)$ can be identified with F/W.

For $f \in I_1(G)$, define its spherical Fourier transform, $\hat{f}$ on F by

$$\hat{f}(\lambda) = \int_G f(x)\, \varphi_{-\lambda}(x)\, dx.$$

Then $\hat{f}$ is a W-invariant bounded function on F which is holomorphic in the interior F^0 of F, and continuous on F. Also $\widehat{f * g} = \hat{f}\hat{g}$ where the convolution of f and g is defined by

$$f * g(x) = \int_G f(xy^{-1})g(y)\, dy.$$

Next, we define the L^1-Schwartz space of K-biinvariant functions on G, which will be denoted by $S(G)$. Let $x \in G$. Then $x = k \exp X,\ k \in K,\ X \in \mathbf{p}$, where $\mathbf{g} = \mathbf{k} + \mathbf{p}$ is the Cartan decomposition of the Lie algebra $\mathbf{g}$ of G. Put $\sigma(x) = \|X\|$, where $\|\,.\,\|$

is the norm on $\mathbf{p}$ induced by the Killing form. For any left-invariant differential operator D on G and any integer $r \geq 0$, we define for a smooth K-biinvariant function f

$$p_{D,r}(f) = \sup_{x \in G} (1 + \sigma(x))^r \, |\varphi_0(x)|^{-2} \, |Df(x)|$$

where φ_0 is the elementary spherical function corresponding to $\lambda = 0$. Define

$$S(G) = \{f : p_{D,r}(f) < \infty \text{ for all } D, r\}.$$

Then $S(G)$ becomes a Frechet space when equipped with the topology induced by the family of seminorms $p_{D,r}$.

Let $P = P(\mathbf{a}^*_{\mathbb{C}})$ be the symmetric algebra over $\mathbf{a}^*_{\mathbb{C}}$. Then each $u \in P$ gives rise to a differential operator $\partial(u)$ on $\mathbf{a}^*_{\mathbb{C}}$. Let $Z(F)$ be the space of functions f on F satisfying the following conditions:

(i) f is holomorphic in F^0 (interior of F) and continuous on F.

(ii) If $u \in P$ and $m \geq 0$ is any integer, then

$$q_{u,m}(f) = \sup_{\lambda \in F^0} (1 + \|\lambda\|^2)^m \, |\partial(u) f(\lambda)| < \infty.$$

(iii) f is W-invariant.

Then $Z(F)$ is an algebra under pointwise multiplication and a Frechet space when equipped with the topology induced by the seminorms $q_{u,m}$.

If $a \in Z(F)$ we define the "wave packet" ψ_a on G by

$$\psi_a(x) = \frac{1}{|W|} \int_{\mathbf{a}^*} a(\lambda) \, \varphi_\lambda(x) \, |c(\lambda)|^{-2} \, d\lambda,$$

where $c(\lambda)$ is the well known Harish-Chandra c-function. By the Plancherel theorem due to Harish-Chandra we also know that the map $f \to \hat{f}$ extends to a unitary map from $L^2(K\backslash G/K)$ onto $L^2(\mathbf{a}^*, |c(\lambda)|^{-2} d\lambda)$.

Theorem 1.1 [Trombi and Varadarajan 1971]. (i) *If $f \in S(G)$ then $\hat{f} \in Z(F)$.*

(ii) *If $a \in Z(F)$ then the integral defining the "wave packet" ψ_a converges absolutely and $\psi_a \in S(G)$. Moreover, $\widehat{\psi_a} = a$.*

(iii) *The map $f \to \hat{f}$ is a topological linear isomorphism of $S(g)$ onto $Z(F)$.*

If G is of real rank one, the maximal ideal space of $I_1(G)$ is given by a certain strip domain in the complex plane that is biholomorphically equivalent to the unit disc in $\mathbb{C}$. Hence function theory in the unit disc (in particular the Beurling–Rudin theorem) can be used to study the ideals in $I_1(G)$. See [Benyamini and Weit 1992] for more details of this method. However, when the real rank of G exceeds one we need different methods. Using the Trombi–Varadarajan theorem just quoted, Sitaram [1980] proved that under certain conditions a single function will generate all of $I_1(G)$. We extend this result to an arbitrary family. Our method is as follows:

(a) From an arbitrary family of functions whose spherical Fourier transforms have no common zero, we manufacture a finite family with the same property. Here we need to use the fact that a complex analytic set admits a stratification.

(b) Next we use a result of Hörmander to generate an appropriate ring of holomorphic functions. Here the not-too-rapid decay of the Fourier transform is crucial.

(c) Finally, as in [Sitaram 1980], the Trombi–Varadarajan isomorphism result (Theorem 1.1) can be applied.

We end this section with a lemma and a proposition that will be needed later.

Lemma 1.2 [Sitaram 1980, Lemma 3.2]. *Let k be a fixed nonnegative integer and let $\varphi_k(z) = e^{\langle z,z\rangle^k}$, $z \in F$. Let X be defined by $X = \{h : h, h\varphi_k \in Z(F)\}$. Then X is a linear dense subspace of $Z(F)$.*

Proposition 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a connected domain and $\{f_\alpha\}_{\alpha\in I}$ be an arbitrary family of bounded holomorphic functions defined on Ω. Suppose that there is no $z \in \Omega$ such that $f_\alpha(z) = 0$ for all $\alpha \in I$. Then there exists functions $g_0, g_1, \dots, g_n$ such that*:

(a) *Each g_i, $i = 0, 1, \dots, n$, is an infinite linear combination of f_α's. More precisely, $g_i = \sum_{k=1}^{\infty} c_k(i) f^i_{\alpha_k}$ with $\sum_k |c_k(i)| < \infty$.*

(b) *There exists no $z \in \Omega$ such that $g_i(z) = 0$ for all $i = 0, 1, \dots, n$.*

Proof. We modify the proof of Proposition 5.7 in [Chirka 1989, page 63]. After multiplying by suitable constants we may assume that each f_α is bounded by 1. Choose a function arbitrarily from the given collection and name it g_n. The zero set Z_{g_n} of g_n is an analytic subset of Ω and so admits a stratification (page 60 of the same reference). Let M^{n-1} denote the $(n-1)$-dimensional stratum of Z_{g_n} (since Ω is connected there is no n-dimensional stratum). Using [Chirka 1989, Theorem 5.4, page 57], write M^{n-1} as a union of its irreducible components: $M^{n-1} = \bigcup_{j=1}^{k} M_j^{n-1}$, where k can be infinite. Choose $a_j \in M_j^{n-1}$ arbitrarily. By the hypothesis there exists f_1 in the given family such that $f_1(a_1) \neq 0$. We define f_j, for $j \geq 2$, as follows: If $f_{j-1}(a_j) \neq 0$ then $f_j = f_{j-1}$. Otherwise, by the hypothesis there exists a function f in the given family such that $f(a_j) \neq 0$. Define f_j to be this function f. Then $f_j(a_j) \neq 0$ for any j. Next, define constants c_j by $c_1 = \frac{1}{4}$,

$$c_j = 4^{-j} |f_1(a_1)| \cdots |f_{j-1}(a_{j-1})|, \quad j \geq 2.$$

Then $0 < c_j \le 4^{-j}$. Now define

$$g_{n-1}(z) = \sum_{k=1}^{\infty} c_k f_k(z).$$

Since $|f_k| \le 1$ the series converges uniformly, so g_{n-1} is holomorphic in Ω. Also,

$$g_{n-1}(a_j) = \left(\sum_{k=l}^{j} c_k\right) f_j(a_j) + c_{j+1} f_{j+1}(a_j) + \cdots$$

for some $1 \le l \le j$. Therefore

$$|g_{n-1}(a_j)| \ge 4^{-j} |f_1(a_1)||f_2(a_2)| \cdots |f_j(a_j)| - \sum_{m=j+1}^{\infty} c_m.$$

But

$$\sum_{m=j+1}^{\infty} c_m \le \frac{4^{-j}}{3} |f_1(a_1)| \cdots |f_j(a_j)|,$$

so $|g_{n-1}(a_j)| > 0$. It follows that $Z_{g_n} \cap Z_{g_{n-1}}$ is an analytic subset of Ω whose dimension is at most $n-2$, as in the proof of Proposition 5.7 on [Chirka 1989, page 63]. We repeat this procedure and finish the proof. □

2. A Wiener Tauberian theorem for $L^1(K\backslash G/K)$

In this section we prove, after some preliminaries, a Wiener Tauberian theorem for K-biinvariant integrable functions on G (Theorem 2.2).

Let p be a plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$. Let $A_p(\Omega)$ denote the ring of holomorphic functions f on Ω such that

$$|f(z)| \le C_1 \exp(C_2 p(z)), \quad z \in \Omega,$$

for some constants C_1 and C_2 possibly depending on f.

Theorem 2.1 [Hörmander 1967]. *Let p be a plurisubharmonic function in the open set $\Omega \subset \mathbb{C}^n$ such that*

(i) *all polynomials belong to $A_p(\Omega)$;*

(ii) *there exist constants $K_1, \ldots, K_4$ such that $z \in \Omega$ and the inequality $|z - \xi| \le \exp(-K_1 p(z) - K_2)$ implies $\xi \in \Omega$ and $p(\xi) \le K_3 p(z) + K_4$.*

Then $f_1, \ldots, f_N \in A_p(\Omega)$ generate $A_p(\Omega)$ if and only if there are positive constants c_1 and c_2 such that

$$|f_1(z)| + \cdots + |f_N(z)| \ge c_1 \exp(-c_2 p(z)), \quad z \in \Omega.$$

Let $l = \dim \mathbf{a} = \dim \mathbf{a}^*$. Write $\rho = (\rho_1, \dots \rho_l)$. We may assume that each $\rho_i \geq 0$. If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_l)$ is an l vector with $\varepsilon_i > 0$ we denote by F_ε the set

$$F_\varepsilon = \mathbf{a}^* + iC_{\rho+\varepsilon}, \quad \text{where } C_{\rho+\varepsilon} \text{ is the convex hull of } \{s(\rho+\varepsilon) : s \in W\}.$$

Theorem 2.2. *Let $\{f_\alpha : \alpha \in I\}$ be a family of functions in $I_1(G)$ such that the spherical Fourier transform $\hat{f}_\alpha$ extend to F_ε^0 as bounded holomorphic functions. Suppose that the collection $\{\hat{f}_\alpha : \alpha \in I\}$ does not have a common zero in F_ε^0. Assume further that there is an $\alpha_0 \in I$ such that*

$$|\hat{f}_{\alpha_0}(z)| \geq \exp\biggl(-c\sum_{j=1}^{l} |z_j|^m\biggr)$$

for some $c > 0$, $0 < m \in \mathbb{N}$ and for all large $z \in F_\varepsilon^0$. Then the family $\{f_\alpha : \alpha \in I\}$ generates a dense subset of $I_1(G)$.

Proof. Let Λ denote the given family of functions in $I_1(G)$ and $\hat{\Lambda}$ denote the collection of its spherical Fourier transforms. We may assume that $\|f_\alpha\|_1 \leq 1$ and $\|\hat{f}_\alpha\|_\infty \leq 1$ for all $\alpha \in I$ where $\|g\|_\infty = \sup_{z \in F_\varepsilon} |g(z)|$. Applying Proposition 1.3 we obtain finitely many functions $f_1, \dots, f_N$ in the ideal generated by Λ (in $I_1(G)$) such that $\hat{f}_1, \dots, \hat{f}_N$ have no common zero in F_ε^0. Let $\delta = (\delta_1, \dots, \delta_l)$ be an l vector such that $0 < \delta_i < \varepsilon_i$ for $i = 1, \dots, l$. Consider the domain F_δ^0. Then, by the hypothesis we have

$$|\hat{f}_{\alpha_0}(z)| + |\hat{f}_1(z)| + \cdots + |\hat{f}_N(z)| \geq c_1 \exp\biggl(-c_2 \sum_{j=1}^{l} |z_j|^m\biggr), \quad z \in F_\delta \tag{2-1}$$

for some $c_1, c_2 > 0$ and $l = \dim A$.

Next, we will apply Theorem 2.1 to these $N+1$ functions. For this, consider

$$p(z) = \log(1/d(z)) + \sum_{j=1}^{l} |z_j|^m, \quad z \in F_\delta^0,$$

where $d(z)$ is the distance of z to the boundary of F_δ^0. Adding a constant to p if necessary, we may assume that p is nonnegative. Since F_δ^0 is a convex domain the function $\log(1/d(z))$ is plurisubharmonic [Hörmander 1990, Theorem 2.6.5]. Hence the function p is plurisubharmonic. It is known that $\log(1/d(z))$ satisfies condition (ii) in Theorem 2.1; see the remarks on [Hörmander 1967, page 944]. Now, it is easy to check that the same holds for our function p defined above and that $A_p(F_\delta^0)$ contains polynomials. Since each of the functions $\hat{f}_1, \dots, \hat{f}_N$ and $\hat{f}_{\alpha_0}$ is bounded (see the construction in Proposition 1.3) they too belong to $A_p(F_\delta^0)$.

The left-hand side of (2-1) is at least $c_3 \exp(-c_4 p(z))$ in the interior of the domain F_δ, for some positive constants c_3 and c_4. Applying Theorem 2.1 we obtain holomorphic functions $g_1, \dots, g_N$ and g_0 in $A_p(F_\delta^0)$ such that

$$g_1(z)\hat{f}_1(z) + \cdots + g_N(z)\hat{f}_N(z) + g_0(z)\hat{f}_{\alpha_0}(z) = 1, \quad z \in F_\delta^0. \tag{2-2}$$

We may assume that the functions g_j are W-invariant. Let $\eta = (\eta_1, \dots, \eta_l)$ be another l vector such that $0 < \eta_j < \delta_j$ for $j = 1, \dots, l$. By the l-dimensional version of Cauchy's formula, all the derivatives of g_j, $j = 0, 1, \dots, N$, satisfy the same growth conditions as the g_j in the domain F_η^0. Hence, if k is a large enough positive integer, $g_j(z)\phi_k(z)$ will belong to $Z(F)$, where

$$\phi_k(z) = e^{-\langle z,z\rangle^k}.$$

That is (by Theorem 1.1), there are functions $\psi_j \in S(G)$ for $j = 0, 1, \dots, N$ such that $\hat{\psi}_j = \phi_k g_j$. Hence if f is any function in the L^1 Schwartz space $S(G)$, from (2-2) we have

$$\hat{f}\phi_k = (\hat{f}\psi_1)\hat{f}_1 + \cdots + (\hat{f}\psi_N)\hat{f}_N + (\hat{f}\psi_0)\hat{f}_{\alpha_0}.$$

Now the proof can be completed as in [Sitaram 1980] using Theorem 1.1 and Lemma 1.2. □

Remarks. (1) The generating family is assumed to have spherical Fourier transforms defined on a larger domain than the maximal ideal space. Even for the case of real rank one this assumption was crucial. See [Benyamini and Weit 1992; Sarkar 1998]. The condition of not-too-rapid decay is assumed on the whole domain F_ε^0; it is a stronger condition than in the rank-one case.

(2) Results similar to Theorem 2.2 can be proved for $L^p(K\backslash G/K)$; see [Sitaram 1980, Theorem 4.1].

3. Rank-one symmetric spaces revisited

In this section we assume that the real rank of G is one. Let G/K be the associated Riemannian symmetric space of noncompact type. Our aim is to derive a Wiener Tauberian theorem for the space $L^1(G/K)$ with the aid of a similar theorem for biinvariant functions and the simplicity criterion (for λ's), under certain decay conditions on the generating functions (instead of the condition of not-too-rapid decay on the Fourier transform). Although a similar result appears in [Sarkar 1998], our proof is simple and different from the one given there, which requires constructing Schwartz class functions on the whole group G with prescribed properties on the Fourier transform. We use the simplicity criterion and averaging over K instead. Moreover, our method is valid for higher-rank cases too, and a strengthening of Theorem 2.2 will readily imply a Wiener Tauberian theorem for $L^1(G/K)$. We shall state our result in terms of the Helgason Fourier transform.

The Helgason Fourier transform of a suitable function f on G/K is the function on $\mathbf{a}^* \times K/M$ defined by

$$\tilde{f}(\lambda, k) = \tilde{f}(\lambda, kM) = \int_{G/K} f(x) e^{(i\lambda-\rho)H(x^{-1}k)}\, dx$$

where $\lambda \in \mathbf{a}^*$ and $k \in K$. Here dx denotes the essentially unique left G-invariant measure on G/K. We have a Plancherel theorem, which reads

$$\int_X |f(x)|^2\, dx = \frac{1}{|W|} \int_{\mathbf{a}^* \times K/M} |\tilde{f}(\lambda, kM)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dk.$$

For other properties of this transform we refer to [Helgason 1994].

The domain F^0 defined in the previous sections becomes a horizontal strip in the complex plane (since G is of real rank one) and F^0_ε is an enlarged strip. We shall use the following result for K-biinvariant functions:

Theorem 3.1 [Benyamini and Weit 1992; Sarkar 1998]. *Let $\{f_\alpha : \alpha \in I\}$ be a family of functions in $I_1(G)$ such that the spherical Fourier transform $\hat{f}_\alpha$ extends holomorphically to the strip F^0_ε for some $\varepsilon > 0$. Suppose the collection $\{\hat{f}_\alpha : \alpha \in I\}$ does not vanish simultaneously on any point in F^0_ε. Assume further that there is an $\alpha_0 \in I$ such that $\hat{f}_{\alpha_0}$ satisfies the decay condition* $\limsup_{|\lambda|\to\infty} |\hat{f}_{\alpha_0}(\lambda)| \cdot |\exp(ke^{|\lambda|})| > 0$ *for all $k > 0$. Then the given collection generates a dense subset of $I_1(G)$.*

Our result is as follows:

Theorem 3.2. *Let f be a function on G/K that satisfies the decay assumption $|f(x)| \le Ce^{-\beta\sigma(x)^2}$, for some $\beta > 0$. Assume further that there is no $\lambda \in F^0_\varepsilon$ such that $\tilde{f}(\lambda, k)$ is identically zero as a function on K/M. Then the left G-translates of f span a dense subset of $L^1(G/K)$.*

Proof. Since f has exponential decay, $\tilde{f}(\lambda, k)$ extends as a holomorphic function to all of $\mathbf{a}^*_{\mathbb{C}}$ and is a C^∞ function in the k variable. Let V_f denote the closed span of left G-translates of the given function f. It suffices to show that $L^1(K\backslash G/K) \subset V_f$.

For each $g \in G$, define a K-biinvariant function $f_g(x) = \int_K f(gkx)\, dx$. Then $f_g \in V_f$ for all $g \in G$ and each f_g satisfies a decay estimate similar to that of f (with a smaller β). Also, the spherical Fourier transform of f_g is

$$\hat{f}_g(\lambda) = f * \varphi_\lambda(g) = \int_K \tilde{f}(\lambda, k)\, e^{(i\lambda-\rho)(H(g^{-1}k))}\, dk,$$

which is the Poisson transform of the function $k \to \tilde{f}(\lambda, k)$ [Helgason 1994]. The Poisson transform is injective if and only if λ is simple, which is the case when $\mathrm{Re}(i\lambda) \ge 0$ [Helgason 1994].

Now consider the collection of K-biinvariant functions $\{f_g : g \in G\}$. For any $\lambda \in F^0_\varepsilon$ with $\mathrm{Re}(i\lambda) \ge 0$ it is not possible to have $\hat{f}_g(\lambda) = 0$ for all $g \in G$, as this will contradict the simplicity of λ. Since $\hat{f}_g$ are even functions, it follows that there is no $\lambda \in F_\varepsilon^0$ such that $\hat{f}_g(\lambda) = 0$ for all $g \in G$. The decay condition for the spherical Fourier transform will be satisfied because of the Hardy uncertainty principles [Sitaram and Sundari 1997; Sarkar 1998, page 356]. By Theorem 3.1 it follows that $I_1(G) \subset V_f$, which finishes the proof. □

Remarks. (1) Using Proposition 4.1 in [Helgason 1994] it is easy to see that this method works well for the higher-rank case too, so long as an analogue of Theorem 3.1 is true. This amounts to weakening the decay condition in Theorem 2.2.

(2) Theorem 3.2 can also be formulated for a family of functions.

Acknowledgments

We thank Kaushal Verma and A. Sitaram for some useful discussions.

References

[Ben Natan et al. 1996] Y. Ben Natan, Y. Benyamini, H. Hedenmalm, and Y. Weit, "Wiener's Tauberian theorem for spherical functions on the automorphism group of the unit disk", *Ark. Mat.* **34**:2 (1996), 199–224. MR 98b:43009 Zbl 0858.43003

[Benyamini and Weit 1992] Y. Benyamini and Y. Weit, "Harmonic analysis of spherical functions on SU(1, 1)", *Ann. Inst. Fourier* (*Grenoble*) **42**:3 (1992), 671–694. MR 94d:43009 Zbl 0763.43006

[Chirka 1989] E. M. Chirka, *Complex analytic sets*, Math. and its Appl. (Soviet Ser.) **46**, Kluwer, Dordrecht, 1989. MR 92b:32016 Zbl 0683.32002

[Ehrenpreis and Mautner 1955] L. Ehrenpreis and F. I. Mautner, "Some properties of the Fourier transform on semisimple Lie groups. I", *Ann. of Math.* (2) **61** (1955), 406–439. MR 16,1017b Zbl 0066.35701

[Helgason 1994] S. Helgason, *Geometric analysis on symmetric spaces*, Math. Surveys and Monogr. **39**, Amer. Math. Soc., Providence, RI, 1994. MR 96h:43009 Zbl 0809.53057

[Hörmander 1967] L. Hörmander, "Generators for some rings of analytic functions", *Bull. Amer. Math. Soc.* **73** (1967), 943–949. MR 37 #1977 Zbl 0172.41701

[Hörmander 1990] L. Hörmander, *An introduction to complex analysis in several variables*, 3rd ed., North-Holland Math. Library **7**, North-Holland, Amsterdam, 1990. MR 91a:32001

[Sarkar 1997] R. P. Sarkar, "Wiener Tauberian theorems for $SL_2(\mathbf{R})$", *Pacific J. Math.* **177**:2 (1997), 291–304. MR 98m:43003 Zbl 0868.43010

[Sarkar 1998] R. P. Sarkar, "Wiener Tauberian theorem for rank one symmetric spaces", *Pacific J. Math.* **186**:2 (1998), 349–358. MR 99k:43004 Zbl 0931.43009

[Sitaram 1980] A. Sitaram, "An analogue of the Wiener–Tauberian theorem for spherical transforms on semisimple Lie groups", *Pacific J. Math.* **89**:2 (1980), 439–445. MR 82e:43008 Zbl 0455.43008

[Sitaram 1988] A. Sitaram, "On an analogue of the Wiener Tauberian theorem for symmetric spaces of the noncompact type", *Pacific J. Math.* **133**:1 (1988), 197–208. MR 89e:43024 Zbl 0628.43011

[Sitaram and Sundari 1997] A. Sitaram and M. Sundari, "An analogue of Hardy's theorem for very rapidly decreasing functions on semisimple Lie groups", *Pacific J. Math.* **177**:1 (1997), 187–200. MR 99a:22018 Zbl 0882.43005

[Trombi and Varadarajan 1971] P. C. Trombi and V. S. Varadarajan, "Spherical transforms of semisimple Lie groups", *Ann. of Math.* (2) **94** (1971), 246–303. MR 44 #6913 Zbl 0218.43010

Received July 7, 2008. Revised November 18, 2008.

E. K. NARAYANAN
INDIAN INSTITUTE OF SCIENCE
DEPARTMENT OF MATHEMATICS
BANGALORE, KARNATAKA 560012
INDIA

naru@math.iisc.ernet.in
www.math.iisc.ernet.in/∼naru

PACIFIC JOURNAL OF MATHEMATICS
Vol. 241, No. 1, 2009

BORING SPLIT LINKS

SCOTT A. TAYLOR

Boring is an operation that converts a knot or two-component link in a 3-manifold into another knot or two-component link. It generalizes rational tangle replacement and can be described as a type of 2-handle attachment. Sutured manifold theory is used to study the existence of essential spheres and planar surfaces in the exteriors of knots and links obtained by boring a split link. It is shown, for example, that if the boring operation is complicated enough, a split link or unknot cannot be obtained by boring a split link. Particular attention is paid to rational tangle replacement. If a knot is obtained by rational tangle replacement on a split link, and a few minor conditions are satisfied, the number of boundary components of a meridional planar surface is bounded below by a number depending on the distance of the rational tangle replacement. This result is used to give new proofs of two results of Eudave-Muñoz and Scharlemann's band sum theorem.

1. Introduction

Refilling and boring. Given a genus 2 handlebody W embedded in a 3-manifold M, a knot or two component link can be created by choosing an essential disc $\alpha \subset W$ and boundary-reducing W along α. Then $W - \mathring{\eta}(\alpha)$ is the regular neighborhood of a knot or link L_α. We say that the exterior $M[\alpha]$ of this regular neighborhood is obtained by *refilling* the meridian disc α. Similarly, given a knot or link $L_\alpha \subset M$, we can obtain another knot or link L_β by the following process:

(1) Attach an arc to L_α forming a graph.

(2) Thicken the graph to form a genus 2 handlebody W.

(3) Choose a meridian β for W and refill β.

Refilling the meridian α of the attached arc returns L_α. Any two knots in S^3 can be related by such a move if we allow α and β to be disjoint: Just let W be a neighborhood of the wedge of the two knots. Therefore we'll restrict attention to

---

MSC2000: 57M50, 57N10.

Keywords: 3-manifold, sutured manifold, knot theory, tunnel number one, crossing change, rational tangle, 2-handle addition, handlebody.

This research was partially supported by a grant from the National Science Foundation.

meridians of W that cannot be isotoped to be disjoint. If a knot or link L_β can be obtained from L_α by this operation, we say L_β is obtained by *boring* L_α. Since the relation is symmetric we may also say L_α and L_β are *related by boring*.

Boring generalizes several well-known operations in knot theory. Band sums, crossing changes, generalized crossing changes, and, more generally, rational tangle replacement can all be realized as boring. The band move from the Kirby calculus [Fenn and Rourke 1979; Kirby 1978] is also a type of boring. If W is the standard genus 2 handlebody in S^3 and L_α is the unlink of two components, then all tunnel number 1 knots can be obtained by boring L_α using W.

If L_α and L_β are related by boring, it is natural to ask under what circumstances both links can be split, both the unknot, both composite, and so on. Many of these questions have been effectively addressed for special types of boring, such as rational tangle replacement [Eudave-Muñoz 1988]. This paper, following [Scharlemann 2008], will focus on the exteriors $M[\alpha]$ and $M[\beta]$ of the knots L_α and L_β, respectively. There, Scharlemann conjectured that, with certain restrictions (discussed in Section 6), if $M[\alpha]$ and $M[\beta]$ are both reducible or boundary reducible, then either W is an unknotted handlebody in S^3 or α and β are positioned in a particularly nice way in W. He was able to prove his conjecture (with slightly varying hypotheses and conclusions) when $M - \mathring{W}$ is boundary reducible, when $|\alpha \cap \beta| \leq 2$, or when one of the discs is separating.

This paper looks again at these questions and completes, under stronger hypotheses, the proof of Scharlemann's conjecture except when $M = S^3$ and $M[\alpha]$ and $M[\beta]$ are solid tori. With these stronger hypotheses, however, we reach conclusions that are stronger than those obtained in [Scharlemann 2008]. Even in the one situation that is not completed, we do gain significant insight. The remaining case is finally completed in [Taylor 2008]. Here is a simplified version of one of the main theorems:

Simplified Theorem 6.1. *Suppose that M is S^3 or the exterior of a link in S^3 and that $M - \mathring{W}$ is irreducible and boundary irreducible. If α and β cannot be isotoped to be disjoint, then at least one of $M[\alpha]$ or $M[\beta]$ is irreducible. Also, if one is boundary reducible (for example, a solid torus), then the other is not reducible.*

The conclusions of Theorem 6.1 are an "arc version" of the conclusions of the main theorem of [Scharlemann 1990], which considers surgeries on knots producing reducible 3-manifolds. The methods of this paper are similar in outline to those of [Scharlemann 1990] but differ in detail.

Perhaps the most interesting application of these techniques to rational tangle replacement is the following theorem, which generalizes some results of Eudave-Muñoz and Scharlemann:

Theorem 7.3. *Suppose that L_β is a knot or link in S^3 and that $B' \subset S^3$ is a 3-ball intersecting L_β so that $(B', B' \cap L_\beta)$ is a rational tangle. Let (B', r_α) be another rational tangle of distance $d \geq 1$ from $r_\beta = B' \cap L_\beta$, and let L_α be the knot obtained by replacing r_β with r_α. Let $(B, \tau) = (S^3 - \mathring{B}', L_\beta - \mathring{B}')$. Suppose that L_α is a split link and that (B, τ) is prime. Then L_β is not a split link or unknot. Furthermore, if L_β has an essential properly embedded meridional planar surface with m boundary components, it contains such a surface $\overline{Q}$ with $|\partial \overline{Q}| \leq m$ such that either*

$$\overline{Q} \subset B \quad \text{or} \quad |\overline{Q} \cap \partial B|(d-1) \leq |\partial \overline{Q}| - 2.$$

One consequence of this is a new proof of Scharlemann's band sum theorem: If the unknot is obtained by attaching a band to a split link, then the band sum is the connected sum of unknots. This and other rational tangle replacement theorems are proved in Section 7.

The main tool in this paper is Scharlemann's combinatorial version of Gabai's sutured manifold theory. The relationship of this paper to [Scharlemann 2008], where he states his conjecture about refilling meridians, is similar to the relationship between Gabai's and Scharlemann's proofs of the band sum theorem. Earlier, Scharlemann [1985] proved that the band sum of two knots is unknotted only if it is the connect sum of two unknots. Later Scharlemann and Gabai simultaneously and independently proved that

$$\text{genus}(K_1 \#_b K_2) \geq \text{genus}(K_1) + \text{genus}(K_2),$$

where $\#_b$ denotes a band sum. Gabai [1987] used sutured manifold theory to give a particularly simple proof. Scharlemann's proof [1989] uses a completely combinatorial version of sutured manifold theory. Since rational tangle replacement is a special type of boring, a similar relationship also holds between this paper and some of Eudave-Muñoz's extensions [1988] of the original band sum theorem. The techniques of this paper can be specialized to rational tangle replacement to recapture and extend some, but not all, of his results. In [2008], Scharlemann suggests that sutured manifold theory might contribute to a solution to his conjecture. This paper vindicates that idea.

The paper [Taylor 2008] uses sutured manifold theory in a different way. The two approaches are often useful in different circumstances. For example, the one used in this paper is more effective for studying the existence of certain reducing spheres in a manifold obtained by refilling meridians and for studying non-separating surfaces that are not homologous to a surface with interior disjoint from W. The approach of [Taylor 2008] is more effective for studying essential discs and separating surfaces.

Notation. We work in the PL or smooth categories. All manifolds and surfaces will be compact and orientable, except where indicated. $|A|$ denotes the number of components of A. If A and B are embedded curves on a surface, $|A \cap B|$ will generally be assumed to be minimal among all curves isotopic to A and B. For a subcomplex $B \subset A$, we denote by $\eta(B)$ a closed regular neighborhood of B in A. Both $\mathring{B}$ and $\operatorname{int} B$ denote the interior of B, and $\operatorname{cl}(B)$ denotes the closure of B. ∂B denotes the boundary of B. All homology groups have $\mathbb{Z}$ (integer) coefficients.

2. Sutured manifold theory

We begin by reviewing a few relevant concepts from combinatorial sutured manifold theory [Scharlemann 1989].

2.1. ***Definitions.*** A *sutured manifold* is a triple (N, γ, ψ), where N is a compact, orientable 3-manifold, γ is a collection of oriented simple closed curves on ∂N, and ψ is a properly embedded 1-complex. $T(\gamma)$ denotes a collection of torus components of ∂N. The curves γ divide $\partial N - T(\gamma)$ into two surfaces that intersect along γ. Removing $\mathring{\eta}(\gamma)$ from these surfaces creates the surfaces $R_+(\gamma)$ and $R_-(\gamma)$. Let $A(\gamma) = \eta(\gamma)$.

For an orientable, connected surface $S \subset N$ in general position with respect to ψ, we define
$$\chi_\psi(S) = \max\{0, |S \cap \psi| - \chi(S)\}.$$
If S is disconnected, $\chi_\psi(S)$ is the sum of $\chi_\psi(S_i)$ for each component S_i. For a class $[S] \in H_2(N, X)$, we define $\chi_\psi([S])$ be the minimum of $\chi_\psi(S)$ over all embedded surfaces S representing $[S]$. If $\psi = \varnothing$, then $\chi_\psi(\cdot)$ is the Thurston norm.

Of utmost importance is the notion of ψ-tautness for both surfaces in a sutured manifold (N, γ, ψ) and for a sutured manifold itself. Let S be a properly embedded surface in N.

- S is ψ-minimizing in $H_2(N, \partial S)$ if $\chi_\psi(S) = \chi_\psi[S, \partial S]$.
- S is ψ-incompressible if $S - \psi$ is incompressible in $N - \psi$.
- S is ψ-taut if it is ψ-incompressible, ψ-minimizing in $H_2(N, \partial S)$, and each edge of ψ intersects S with the same sign. If $\psi = \varnothing$, we say either that S is $\varnothing$-taut or that S is taut in the Thurston norm.

A sutured manifold (N, γ, ψ) is ψ-taut if

- $\partial\psi$ (that is, valence one vertices) is disjoint from $A(\gamma) \cup T(\gamma)$;
- $T(\gamma)$, $R_+(\gamma)$, and $R_-(\gamma)$ are all ψ-taut; and
- $N - \psi$ is irreducible.

The final important notion is the concept of a conditioned surface. A *conditioned surface* $S \subset N$ is an oriented properly embedded surface such that

- if T is a component of $T(\gamma)$, then $\partial S \cap T$ consists of coherently oriented parallel circles;
- if A is a component of $A(\gamma)$, then $S \cap A$ consists of either circles parallel to γ and oriented the same direction as γ or arcs all oriented in the same direction;
- no set of simple closed curves of $\partial S \cap R(\gamma)$ is trivial in $H_1(R(\gamma), \partial R(\gamma))$;
- each edge of ψ that intersects $S \cup R(\gamma)$ does so always with the same sign.

Conditioned surfaces, along with product discs and annuli, are the surfaces along which a taut sutured manifold is decomposed to form a taut sutured manifold hierarchy. A hierarchy can be taken to be "adapted" to a parameterizing surface, that is, a surface $Q \subset N - \mathring{\eta}(\psi)$ no component of which is a disc disjoint from $\gamma \cup \eta(\psi)$. The index of a parameterizing surface is a certain number associated to Q that does not decrease as Q is modified during the hierarchy.

2.2. ***Satellite knots have property P.*** It will be helpful to review the essentials of the proof of [Scharlemann 1989, Theorem 9.1], where it is shown that satellite knots have property P.

In that theorem, which considers a 3-manifold N with ∂N a torus, it is assumed that $H_1(N)$ is torsion-free and that $k \subset N$ is a knot in N such that $(N, \varnothing)$ is a k-taut sutured manifold. Suppose that some nontrivial surgery on k creates a manifold that has a boundary-reducing disc $\overline{Q}$ and still has torsion-free first homology. The main goal is to show that $(N, \varnothing)$ is $\varnothing$-taut. The surface $Q = \overline{Q} - \mathring{\eta}(k)$ acts as a parameterizing surface for a k-taut sutured manifold hierarchy of N. At the end of the hierarchy, there is at least one component containing pieces of k. A combinatorial argument using the assumption that $H_1(N)$ is torsion-free shows that, in fact, the last stage of the hierarchy is $\varnothing$-taut. Sutured manifold theory then shows that the original manifold N is $\varnothing$-taut, as desired. This argument is extended in [Scharlemann 1990] to study surgeries on knots in 3-manifolds that produce reducible 3-manifolds. In that paper, the surface $\overline{Q}$ can be either a ∂-reducing disc or a reducing sphere.

This paper extends these techniques in two other directions. First, we use an arc $\overline{\alpha} \subset M[\alpha]$ in place of the knot $k \subset N$. Second, we develop criteria that allow the surface $\overline{Q} \subset M[\beta]$ to be any of a variety of surfaces, including essential spheres and discs. Section 5 shows how to construct a useful surface $\overline{Q}$. Section 4 discusses the placement of sutures on $\partial M[\alpha]$. This allows theorems about sutured manifolds to be phrased without reference to sutured manifold terminology. Section 6 applies the sutured manifold results in order to (partially) answer Scharlemann's conjecture about refilling meridians of genus 2 handlebodies. Section 7 uses the technology to reprove three classical theorems about rational tangle replacement and prove a new theorem about essential meridional surfaces in the exterior of a knot or link obtained by boring a split link. Finally, Section 8 shows how the sutured manifold theory results of this paper can significantly simplify certain combinatorial arguments.

3. Attaching a 2-handle

Let N be a compact orientable 3-manifold containing a component $F \subset \partial N$ of genus at least two. Let $a \subset F$ be an essential closed curve and let $\mathcal{B} = \{b_1, \dots, b_{|\mathcal{B}|}\}$ be a collection of disjoint, pairwise nonparallel essential closed curves in F isotoped so as to intersect a minimally. Suppose that $\gamma \subset \partial N$ is a collection of simple closed curves, disjoint from a, such that $(N, \gamma \cup a)$ is a taut sutured manifold and γ intersects the curves of $\mathcal{B}$ minimally. Let $\Delta_i = |b_i \cap a|$ and $\nu_i = |b_i \cap \gamma|$.

Suppose that $Q \subset N$ is a surface with q_i boundary components parallel to the curve b_i, for each $1 \leq i \leq |\mathcal{B}|$. Let $\partial_0 Q$ be the components of ∂Q that are not parallel to any b_i. Assume that ∂Q intersects $\gamma \cup a$ minimally. Define $\Delta_\partial = |\partial_0 Q \cap a|$ and $\nu_\partial = |\partial_0 Q \cap \gamma|$. We need two definitions. The first defines a specific type of boundary compression and the second (as we shall see) is related to the notion of “Scharlemann cycle”.

Definition. An *a-boundary compressing disc* for Q is a boundary compressing disc D for which $\partial D \cap F$ is a subarc of some essential circle in $\eta(a)$.

Definition. An *a-torsion $2g$-gon* is a disc $D \subset N$ with $\partial D \subset F \cup Q$ consisting of $2g$ arcs labeled around ∂D as $\delta_1, \epsilon_1, \dots, \delta_g, \epsilon_g$. The labels are chosen so that $\partial D \cap Q = \bigcup \delta_i$ and $\partial D \cap F = \bigcup \epsilon_i$. We require that each ϵ_i arc is a subarc of some essential simple closed curve in $\eta(a)$ and that the ϵ_i arcs are all mutually parallel as oriented arcs in $F - \partial Q$. Furthermore we require that attaching to Q a rectangle in $F - \partial Q$ containing all the ϵ_i arcs produces an orientable surface.

Example. Figure 1 shows a hypothetical example. The surface outlined with dashed lines is Q. It has boundary components on F. There are two such boundary components pictured. The curve running through Q and F could be the boundary of an a-torsion 4-gon. Notice that the arcs ϵ_1 and ϵ_2 are parallel and oriented in the same direction. Attaching the rectangle containing those arcs as two of its edges to Q produces an orientable surface.

Remark. Notice that an a-torsion 2-gon is an a-boundary compressing disc.

If 2-handles are attached to each curve b_i for $1 \leq i \leq |\mathcal{B}|$, a 3-manifold $N[\mathcal{B}]$ is obtained. Each component of $\partial Q - \partial_0 Q$ bounds a disc in $N[\mathcal{B}]$. Let $\overline{Q}$ be the result of attaching a disc to each component of $\partial Q - \partial_0 Q$. Then $\partial \overline{Q} = \partial_0 Q$. We will usually also attach 3-balls to spherical components of $\partial N[\mathcal{B}]$. Throughout the paper, if a 2-handle $\beta \times I$ is attached to a curve b, the cocore of the 2-handle will be denoted $\overline{\beta}$. Thus, the notations $\overline{\alpha}$, $\overline{\beta}$, and $\overline{\beta}^*$ all refer to arcs in certain 3-manifolds.

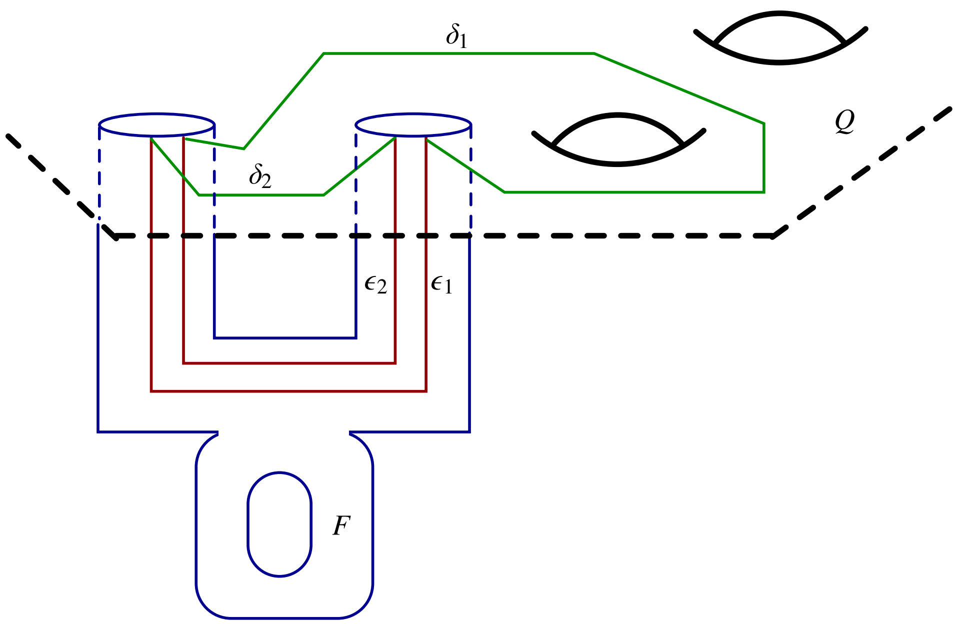

Figure 1. The boundary of an a-torsion 4-gon.

Remark. The term "a-torsion $2g$-gon" is chosen because in certain cases (but not all) the presence of an a-torsion $2g$-gon with $g \geq 2$ guarantees that $N[\mathcal{B}]$ has torsion in its first homology.

Define

$$K(\overline{Q}) = \sum_{i=1}^{|\mathcal{B}|} q_i(\Delta_i - 2) + \Delta_\partial - \nu_\partial.$$

We will use the surface Q to study the effects of attaching a 2-handle $\alpha \times I$ to a regular neighborhood of the curve $a \subset F$. Let $N[a]$ denote the resulting 3-manifold. Perform the attachment so that the 2-disc α has boundary a. Let $\overline{\alpha}$ denote the arc that is the cocore of the 2-handle $\alpha \times I$.

We can now state our main sutured manifold theory result. It is an adaptation of [Scharlemann 1989, Theorem 9.1]; see also [Scharlemann 1990, Proposition 4.1].

Main Theorem. *Suppose that $(N[a], \gamma)$ is $\overline{\alpha}$-taut, that Q is incompressible, and that Q contains no disc or sphere component disjoint from $\gamma \cup a$. Suppose that one of the following holds*:

- *$N[a]$ is not $\varnothing$-taut.*
- *There is a conditioned $\overline{\alpha}$-taut surface $S \subset N[a]$ that is not $\varnothing$-taut.*
- *$N[a]$ is homeomorphic to a solid torus $S^1 \times D^2$ and $\overline{\alpha}$ cannot be isotoped so that its projection to the S^1 factor is monotonic.*

Then at least one of the following holds:

- *There is an a-torsion $2g$-gon for Q for some $g \in \mathbb{N}$.*
- *$H_1(N[a])$ contains nontrivial torsion.*
- $-2\chi(\overline{Q}) \geq K(\overline{Q})$.

Remark. If $\overline{\alpha}$ can be isotoped to be monotonic in the solid torus $N[a]$ then it is, informally, a "braided arc". The contrapositive of this aspect of the theorem is similar to the conclusion in [Gabai 1989] and [Scharlemann 1990] that if a nontrivial surgery on a knot with nonzero wrapping number in a solid torus produces a solid torus, then the knot is a 0 or 1-bridge braid.

The rest of this section proves the theorem. Following [Scharlemann 1990], define a *Gabai disc* for Q to be an embedded disc $D \subset N[a]$ such that

- $|\overline{\alpha} \cap \mathring{D}| > 0$ and all points of intersection have the same sign of intersection,
- $|Q \cap \partial D| < |\partial Q \cap \eta(a)|$.

The next proposition points out that the existence of a Gabai disc guarantees the existence of an a-boundary compressing disc or an a-torsion $2g$-gon.

Proposition 3.1. *If there is a Gabai disc for Q, then there is an a-torsion $2g$-gon.*

Proof. Let D be a Gabai disc for Q. The intersection of Q with D produces a graph Λ on D. The vertices of Λ are ∂D and the points $\overline{\alpha} \cap D$. The latter are called the *interior vertices* of Λ. The edges of Λ are the arcs $Q \cap D$. A *loop* is an edge in Λ with initial and terminal points at the same vertex. A loop is *trivial* if it bounds a disc in D with interior disjoint from Λ.

To show that there is an a-torsion $2g$-gon for Q, we will show that the graph Λ contains a "Scharlemann cycle" of length g. The interior of the Scharlemann cycle will be the a-torsion $2g$-gon. In our situation, Scharlemann cycles will arise from a labeling of Λ that is slightly nonstandard. Traditionally, when $\overline{\alpha}$ is a knot instead of an arc, the labels on the endpoints of edges in Λ, which are used to define "Scharlemann cycles", are exactly the components of ∂Q. In our case, since each component of ∂Q likely intersects a more than once, we need to use a slightly different labeling. After defining the labeling and the revised notion of "Scharlemann cycle", it will be clear to those familiar with the traditional situation that the new Scharlemann cycles give rise to the same types of topological conclusions as in the traditional setting. The discussion is modeled on [Culler et al. 1987, Section 2.6].

A Scharlemann cycle of length 1 is defined to be a trivial loop at an interior vertex of Λ bounding a disc with interior disjoint from Λ. We now work toward a definition of Scharlemann cycles of length $g > 1$. Without loss of generality, we may assume that $|\overline{\alpha} \cap D| \geq 2$. Recall that the arc $\overline{\alpha}$ always intersects the disc D with the same sign. There is, in F, a regular neighborhood A of a such that $D \cap F \subset A$. We may choose A so that $\partial A \subset D \cap F$. Let $\partial_\pm A$ be the two boundary components of A. The boundary components of Q all have orientations arising from the orientation of $\overline{Q}$. We may assume by an isotopy that all the arcs $\partial Q \cap A$ are fibers in the product structure on A. Cyclically around A, label the arcs of $\partial Q \cap A$ with labels $c_1, \dots, c_\mu$. Let $\mathscr{C}$ be the set of labels. Being a submanifold of ∂Q, each arc is oriented. Say that two arcs are *parallel* if they run through A in the same direction (that is, both from $\partial_- A$ to $\partial_+ A$ or both from $\partial_+ A$ to $\partial_- A$). Call two arcs *antiparallel* if they run through A in opposite directions. Note that since the orientations of $\mathring{D} \cap F$ in A are all the same, an arc intersects each component of $\mathring{D} \cap F$ with the same algebraic sign.

Call an edge of Λ with at least one endpoint on ∂D a *boundary edge*, and call all other edges *interior edges*. As each edge of Λ is an arc and as all vertices of Λ are parallel oriented curves on ∂W, an edge of Λ must have endpoints on arcs of $\mathscr{C} = \{c_1, \dots, c_\mu\}$ that are antiparallel. As in [Culler et al. 1987], we call this the *parity principle*. Label each endpoint of an edge in Λ with the arc in $\mathscr{C}$ on which the endpoint lies.

We will occasionally orient an edge e of Λ; in which case, let $\partial_- e$ be the tail and $\partial_+ e$ the head. A *cycle* in Λ is a subgraph homeomorphic to a circle. An x-cycle is a cycle which, when each edge e in the cycle is given a consistent orientation, has $\partial_- e$ labeled with $x \in \mathscr{C}$. Let Λ' be a subgraph of Λ, and let x be a label in $\mathscr{C}$. We say that Λ' satisfies condition $P(x)$ if, for each vertex v of Λ', there exists an edge of Λ' incident to v with label x connecting v to an interior vertex.

Lemma 3.2 [Culler et al. 1987, Lemma 2.6.1]. *Suppose that Λ' satisfies $P(x)$. Then each component of Λ' contains an x-cycle.*

Proof. The proof is the same as in [Culler et al. 1987]. □

A *Scharlemann cycle* is an x-cycle σ where the interior of the disc in D bounded by σ is disjoint from Λ. See Figure 2. Since each intersection point of $D \cap \overline{\alpha}$ has the same sign, the set of labels on a Scharlemann cycle contains x and precisely one other label y, a component of $\mathscr{C}$ adjacent to x in A. The arc y and the arc x are antiparallel by the parity principle. The *length* of the Scharlemann cycle is the number of edges in the x-cycle.

Lemma 3.3 [Culler et al. 1987, Lemma 2.6.2]. *If Λ contains an x-cycle, then* (*possibly after a trivial* 2*-surgery on D*), *Λ contains a Scharlemann cycle.*

Proof. The proof is again the same as in [Culler et al. 1987]. It uses the assumption that Q is incompressible to eliminate circles of intersection on the interior of an innermost x-cycle. □

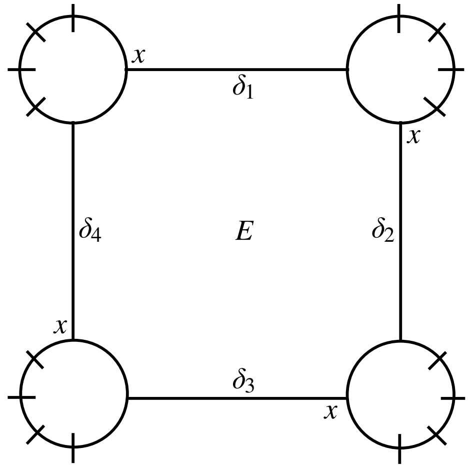

Figure 2. A Scharlemann cycle of length 4 bounding an a-torsion 8-gon.

Remark. The presence of any such disc D with Λ containing a Scharlemann cycle is good enough for our purposes. So, henceforth, we assume that all circles in Λ have been eliminated using the incompressibility of Q.

Remark. In [Culler et al. 1987], there is a distinction between x-cycles and so-called great x-cycles. We do not need this here because all components of $D \cap F$ are parallel in $\eta(\partial\alpha)$ as oriented curves.

The next corollary explains the necessity of considering Scharlemann cycles.

Corollary 3.4 [Culler et al. 1987]. *If ∂D intersects fewer than $|\partial Q \cap A|$ edges of Λ, then Λ contains a Scharlemann cycle.*

Proof. As ∂D contains fewer than $|\partial Q \cap A|$ endpoints of boundary edges in Λ, there is some $x \in \mathscr{C}$ that does not appear as a label on a boundary edge. As every interior vertex of Λ contains an edge with label x at that vertex, none of those edges can be a boundary edge. Consequently, Λ satisfies $P(x)$. Hence, by Lemmas 3.2 and 3.3, Λ contains a Scharlemann cycle of length g (for some g). □

In A there is a rectangle R with boundary consisting of the arcs x and y and subarcs of ∂A. See Figure 3. Because $\overline{\alpha}$ always intersects D with the same sign, ∂D always crosses R in the same direction. This shows that the arcs ϵ_i are all mutually parallel in F. The arcs x and y are antiparallel, so attaching R to Q produces an orientable surface. Hence, the interior of the Scharlemann cycle is an a-torsion $2g$-gon. □

We now proceed with proving the contrapositive of the theorem. Suppose that none of the three possible conclusions of the theorem hold.

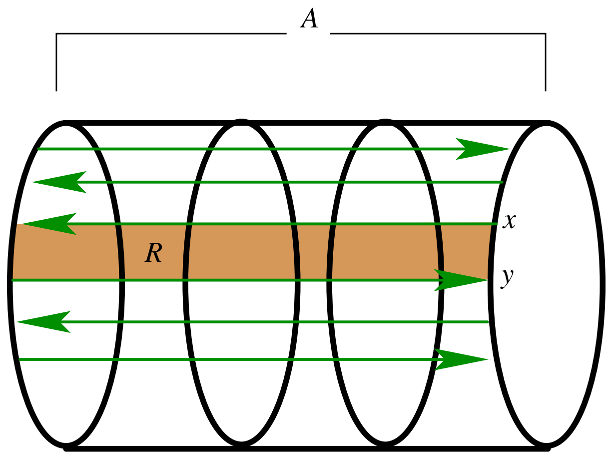

Figure 3. The rectangle R.

First, Q is a parameterizing surface for the $\overline{\alpha}$-taut sutured manifold $(N[a], \gamma)$. Let

$$(N[a], \gamma) = (N_0, \gamma_0) \xrightarrow{S_1} (N_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (N_n, \gamma_n)$$

be an $\overline{\alpha}$-taut sutured manifold hierarchy for $(N[a], \gamma)$ that is adapted to Q. The surface S_1 may be obtained from the surface S by performing the double-curve sum of S with k copies of R_+ and l copies of R_- [Scharlemann 1989, Theorem 2.6].

The index $I(Q_i)$ is defined to be

$$I(Q_i) = |\partial Q_i \cap \partial \eta(\overline{\alpha}_i)| + |\partial Q_i \cap \gamma_i| - 2\chi(Q_i),$$

where Q_i is the parameterizing surface in N_i and $\overline{\alpha}_i$ is the remnant of $\overline{\alpha}$ in N_i. Since $-2\chi(\overline{Q}) < K(\overline{Q})$, simple arithmetic shows that $I(Q) < 2|\partial Q \cap \eta(a)|$. Since there is no a-torsion $2g$-gon for Q, by the previous proposition, there is no Gabai disc for Q. The proof of [Scharlemann 1989, Theorem 9.1] shows that (N_n, γ_n) is also $\varnothing$-taut, after substituting the assumption that there are no Gabai discs for Q in N wherever [Scharlemann 1989, Lemma 9.3] was used, as in [Scharlemann 1990, Proposition 4.1]. To prove 3, 4, and 11 of [Scharlemann 1989, Theorem 9.1], use the inequality $I(Q) < 2|\partial Q \cap A|$ to derive a contradiction rather than the inequalities stated in the proofs of those claims.

Hence, the hierarchy is $\varnothing$-taut, $(N[a], \gamma)$ is a $\varnothing$-taut sutured manifold, and S_1 is a $\varnothing$-taut surface. Suppose that S is not $\varnothing$-taut. Then there is a surface S' with the same boundary as S but with smaller Thurston norm. Then the double-curve sum of S' with k copies of R_+ and l copies of R_- has smaller Thurston norm than S_1, showing that S_1 is not $\varnothing$-taut. Hence, S is $\varnothing$-taut.

The proof of [Scharlemann 1989, Theorem 9.1] concludes by noting that at the final stage of the hierarchy, there is a canceling or (nonself) amalgamating disc for each remnant of $\overline{\alpha}$. When $N[a]$ is a solid torus the only $\varnothing$-taut conditioned surfaces are unions of discs. If S is chosen to be a single disc, then S_1 is isotopic to S. To see this, notice that $R_\pm$ is an annulus and so the double-curve sum of S with $R_\pm$ is isotopic to S. Hence, the hierarchy has length one and the cancelling and (nonself) amalgamating discs show that $\overline{\alpha}$ is braided in $N[a]$. □

Remark. The proof proves more than the theorem states. It is actually shown that at the end of the hierarchy, $\overline{\alpha} \cap N_n$ consists of unknotted arcs in 3-balls. This may be useful in future work.

For this theorem to be useful, we need to discuss the placement of sutures γ on ∂N and the construction of a surface Q without a-torsion $2g$-gons. The next two sections address these issues. In each of them, we restrict F to being a genus 2 surface.

4. Placing sutures

Let N be a compact, orientable, irreducible 3-manifold with $F \subset \partial N$ a component containing an essential simple closed curve a. Suppose that $\partial N - F$ is incompressible in N. For effective application of the main theorem, we need to choose curves γ on $\partial N[a]$ so that $(N[a], \gamma)$ is $\overline{\alpha}$-taut. With our applications in mind, we restrict our attention to the situation when the boundary component F containing a has genus 2. Define $\partial_1 N[a] = \partial N - F$ and $\partial_0 N[a] = \partial N[a] - \partial_1 N[a]$.

For the moment, we consider only the choice of sutures $\hat{\gamma}$ on $\partial_0 N[a]$. If a is separating, so that $\partial_0 N[a]$ consists of two tori joined by the arc $\overline{\alpha}$, we do not place any sutures on $\partial_0 N[a]$, that is, $\hat{\gamma} = \varnothing$. See Figure 4A. If a is nonseparating, choose $\hat{\gamma}$ to be a pair of disjoint parallel loops on $F - \eta(a)$ that separate the endpoints of $\overline{\alpha}$. See Figure 4B.

If we are in the special situation of "refilling meridians", we will want to choose the curves $\hat{\gamma}$ more carefully. Recall that in this case $N \subset M$ and F bounds a genus 2 handlebody $W \subset (M - \mathring{N})$. The curves a and b bound in W discs α and β respectively.

Assuming that the discs β and α have been isotoped to intersect minimally and nontrivially, the intersection $\alpha \cap \beta$ is a collection of arcs. An arc of $\alpha \cap \beta$ that is outermost on β cobounds with a subarc ψ of b a disc with interior disjoint from α. This disc is a meridional disc of a (solid torus) component of $W - \mathring{\eta}(\alpha)$. The arc ψ has both endpoints on the same component of $\partial\eta(a) \subset F$. We therefore define a *meridional arc* of $b - a$ to be any arc of $b - \mathring{\eta}(a)$ that together with an arc in $\partial\eta(\alpha) \cap \mathring{W}$ bounds a meridional disc of $W - \mathring{\eta}(\alpha)$. If a is nonseparating, then the existence of meridional arcs shows that every arc of $b - \mathring{\eta}(a)$ with endpoints

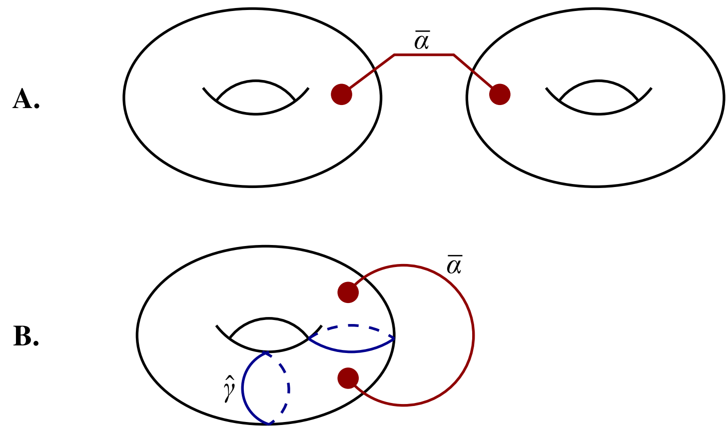

Figure 4. Choosing $\hat{\gamma}$.

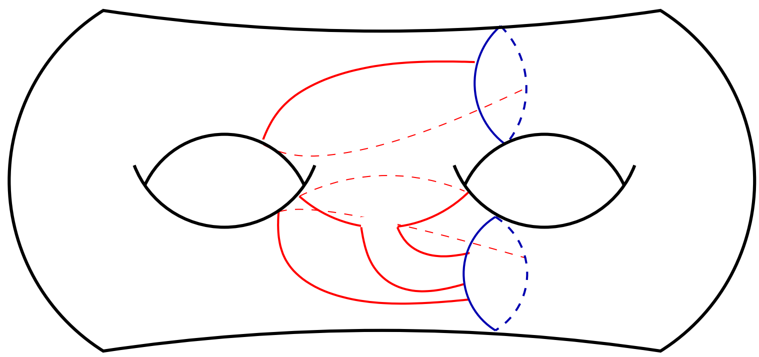

Figure 5. Some meridional arcs on ∂W.

on the same component of $\partial\eta(a) \subset F$ is a meridional arc of $b - a$. An easy counting argument shows that if a is nonseparating, then there are equal numbers of meridional arcs of $b-a$ based at each component of $\partial\eta(a) \subset F$. Hence, when a is nonseparating, the number of meridional arcs of $b-a$, denoted $\mathcal{M}_a(b)$, is even. Some meridional arcs are depicted in Figure 5.

Returning to the definition of the sutures $\hat{\gamma}$, we insist that when "refilling meridians" and when α is nonseparating, the curves $\hat{\gamma}$ be meridional curves of the solid torus $W - \mathring{\eta}(\alpha)$ that separate the endpoints of $\overline{\alpha}$ and that are disjoint from the meridional arcs of $b-a$ for a specified b.

We now show how to define sutures $\tilde{\gamma}$ on nontorus components of $\partial_1 N[a]$. Let $T(\gamma)$ be all the torus components of $\partial_1 N[a]$. If $\partial_1 N = T(\gamma)$, then $\tilde{\gamma} = \varnothing$. Otherwise, the next lemma demonstrates how to choose $\tilde{\gamma}$ so that, under certain hypotheses, $(N, \gamma \cup a)$ is taut, where $\gamma = \hat{\gamma} \cup \tilde{\gamma}$.

Lemma 4.1. *Suppose that $F - (\gamma \cup a)$ is incompressible in N. Suppose also that if $\partial_1 N[a] \neq T(\gamma)$, then there is no essential annulus in N with boundary on $\hat{\gamma} \cup a$. Then $\tilde{\gamma}$ can be chosen so that $(N, \gamma \cup a)$ is $\varnothing$-taut and so that $(N[a], \gamma)$ is $\overline{\alpha}$-taut. Furthermore, if $c \subset \partial_1 N[a]$ is a collection of disjoint, nonparallel curves such that*

- $|c| \leq 2$;
- *all components of c are on the same component of $\partial_1 N[a]$;*
- *no curve of c cobounds an essential annulus in N with a curve of $\hat{\gamma} \cup a$;*
- *if $|c| = 2$, then there is no essential annulus in N with boundary c; and*
- *if $|c| = 2$ and a is separating, there is no essential thrice punctured sphere in N with boundary $c \cup a$,*

then $\tilde{\gamma}$ can be chosen to be disjoint from c.

The main ideas of the proof are contained in [Scharlemann 1990, Section 5] and [Lackenby 1997, Theorem 2.1]. Scharlemann considers "special" collections of curves on a nontorus component of ∂N. These curves cut the component into thrice punctured spheres. Exactly two of the curves in the collection bound once punctured tori. In those tori are two curves of the collection that are called "redundant". The redundant curves are removed, and the remaining curves form the desired sutures. Scharlemann shows how to construct such a special collection that is disjoint from a set of given curves and that gives rise to a taut-sutured manifold structure on the manifold under consideration. Lackenby uses essentially the same construction (but with fewer initial hypotheses) to construct a collection of curves cutting the nontorus components of ∂N into thrice punctured spheres, but where all the curves are nonseparating. We need to allow the sutures to contain separating curves as c may contain separating curves. By slightly adapting Scharlemann's work, in the spirit of Lackenby, we can make do with the hypotheses of the lemma, which are slightly weaker than what a direct application of Scharlemann's work would allow.

Proof. Let τ be the number of once punctured tori in ∂N with boundary some component of $c \cup a$. Since all components of c are on the same component of ∂N, $\tau \leq 4$ with $\tau \geq 3$ only if a is separating.

Say that a collection of curves on ∂N is *pantsless* if, whenever a thrice punctured sphere has its boundary a subset of the collection, all components of the boundary are on the same component of ∂N. If a is nonseparating, then $\tau \leq 2$. Hence, either $\tau \leq 2$ or $c \cup a \cup \hat{\gamma}$ is pantsless.

Scharlemann shows how to extend the set c to a collection Γ such that there is no essential annulus in N with boundary on $\Gamma \cup a \cup \hat{\gamma}$ and the curves Γ cut ∂N into tori, once punctured tori, and thrice punctured spheres. Furthermore, if $c \cup a \cup \hat{\gamma}$ is pantsless, then so is $\Gamma \cup a \cup \hat{\gamma}$. An examination of Scharlemann's construction shows that all curves of $\Gamma - c$ may be taken to be nonseparating. Thus, the number of once punctured tori in ∂N with boundary on some component of $\Gamma \cup a$ is still τ. If Γ cannot be taken to be a collection of sutures on ∂N, then, by construction, $|c| = 2$, and one curve of c bounds a once punctured torus in ∂N containing the other curve of c. The component of c in the once punctured torus is "redundant" in Scharlemann's terminology. If no curve of c is redundant, let $\tilde{\gamma} = \Gamma$; otherwise, form $\tilde{\gamma}$ by removing the redundant curve from Γ. Let $\gamma' = \tilde{\gamma} \cup a \cup \hat{\gamma}$. We now have a sutured manifold (N, γ'). Notice that the number of once punctured torus components of $\partial N - \gamma'$ is equal to τ.

We now desire to show that (N, γ') is $\varnothing$-taut. If it is not taut, then $R_\pm(\gamma)$ is not norm-minimizing in $H_2(N, \eta(\partial R_\pm))$. Let J be an essential surface in N with $\partial J = \partial R_\pm = \gamma'$. Notice that $\chi_\varnothing(R_\pm) = -\chi(\partial N)/2$ and that $|\gamma'| = -3\chi(\partial N)/2 - \tau$.

Recall that either $\tau \leq 2$ or γ' is pantsless. Suppose first that $\tau \leq 2$. Since no component of J can be an essential annulus, by the arguments of Scharlemann and Lackenby, $\chi_\varnothing(J) \geq |\partial J|/3 = |\gamma'|/3$. Hence, $\chi_\varnothing(J) \geq -\chi(\partial N)/2 - \tau/3$. Since $\tau \leq 2$ and since $\chi_\varnothing(J)$ and $-\chi(\partial N)/2$ are integers, $\chi_\varnothing(J) \geq |\partial N|/2 = \chi_\varnothing(R_\pm)$. Thus, (N, γ') is a $\varnothing$-taut sutured manifold when $\tau \leq 2$.

Suppose therefore that γ' is pantsless. Recall that $\tau \leq 4$. We first examine the case when each component of J has its boundary contained on a single component of ∂M. Let J_0 be all the components of J with boundary on a single component T of ∂N. Let τ_0 be the number of once punctured torus components of $T - \gamma'$. Notice that $\tau_0 \leq 2$. The proof for the case when $\tau \leq 2$ shows that $\chi_\varnothing(J_0) \geq \chi_\varnothing(R_\pm \cap T)$. Summing over all component of ∂N shows that $\chi_\varnothing(J) \geq \chi_\varnothing(R_\pm)$, as desired.

We may therefore assume that some component J_0 of J has boundary on at least two components of ∂N. Since γ' is pantsless, $\chi_\varnothing(J_0) \geq (|\partial J_0| + 2)/3$. For the other components of J we have $\chi_\varnothing(J - J_0) \geq |\partial(J - J_0)|/3$. Thus

$$\chi_\varnothing(J) \geq \frac{|\gamma'|+2}{3} \geq -\frac{\chi(\partial N)}{2} + \frac{2-\tau}{3}.$$

Since $\tau \leq 4$ and since $\chi_\varnothing(J)$ and $-\chi(\partial N)/2$ are integers, we have as desired that $\chi_\varnothing(J) \geq -\chi(\partial N)/2 = \chi_\varnothing(R_\pm)$. Hence $(N, \gamma') = (N, \gamma \cup a)$ is $\varnothing$-taut. $\square$

Remark. The assumption that all components of c are contained on the same component of $\partial_1 N[a]$ can be weakened to a hypothesis on the number τ. For what follows, however, our assumption suffices.

We will be interested in when a component of $\partial N - F$ becomes compressible upon attaching a 2-handle to $a \subset F$ and also becomes compressible upon attaching a 2-handle to $b \subset F$. If such occurs, the curves c of the previous lemma will be the boundaries of the compressing discs for that component of ∂N. Obviously, in order to apply the lemma we will need to make assumptions on how that component compresses.

5. Constructing Q

The typical way in which we will apply the main theorem is as follows. Suppose that a and b are simple closed curves on a genus 2 component $F \subset \partial N$ and that there is an "interesting" surface $\overline{R} \subset N[b]$. We will want to use this surface to show that either $-2\chi(\overline{R}) \geq K(\overline{R})$ or $N[a]$ is taut. *A priori*, though, the surface $R = \overline{R} \cap N$ may have a-boundary compressing discs or a-torsion $2g$-gons. The purpose of this section is to show how, given the surface $\overline{R}$, we can construct another surface $\overline{Q}$ which will hopefully have similar properties to $\overline{R}$ but be such that $Q = \overline{Q} \cap N$ does not have a-boundary compressing discs or a-torsion $2g$-gons. This goal will not be entirely achievable, but Theorem 5.1 shows how close we can come. Throughout we assume that N is a compact, orientable, irreducible 3-manifold with $F \subset \partial N$ a component having genus equal to 2. Let a and b be two essential simple closed curves on F so that a and b intersect minimally and nontrivially. As before, let $\partial_1 N = \partial_1 N[b] = \partial N - F$ and let $\partial_0 N[b] = \partial N[b] - \partial_1 N[b]$. Let T_0 and T_1 be the components of $\partial_0 N[b]$. If b is nonseparating, then $T_0 = T_1$.

Before stating the theorem, we make some important observations about $N[b]$ and surfaces in $N[b]$. If b is nonseparating, there are multiple ways to obtain a manifold homeomorphic to $N[b]$. Certainly attaching a 2-handle to b is one such way. If b^* is any curve in F that cobounds in F with $\partial\eta(b)$ a thrice punctured sphere, then attaching 2-handles to both b^* and b creates a manifold with a spherical boundary component. Filling in that sphere with a 3-ball creates a manifold homeomorphic to $N[b]$. We will often think of $N[b]$ as obtained in this fashion. Say that a surface $\overline{Q} \subset N[b]$ is *suitably embedded* if each component of $\partial Q - \partial\overline{Q}$ is a curve parallel to b or to some b^*. We denote the number of components of $\partial Q - \partial\overline{Q}$ parallel to b by $q = q(\overline{Q})$ and the number parallel to b^* by $q^* = q^*(\overline{Q})$. Let $\tilde{q} = q + q^*$. If b is separating, define $b^* = \varnothing$. Define $\Delta = |b \cap a|$, $\Delta^* = |b^* \cap a|$, $\nu = |b \cap \gamma|$, and $\nu^* = |b^* \cap \gamma|$. We then have

$$K(\overline{Q}) = (\Delta - \nu - 2)q + (\Delta^* - \nu^* - 2)q^* + \Delta_\partial - \nu_\partial.$$

Define a *slope* on a component of $\partial N[b]$ to be an isotopy class of pairwise disjoint, pairwise nonparallel curves on that component. The set of curves is allowed to be the empty set. Place a partial order on the set of slopes on a component of $\partial N[b]$ by declaring $r \leq s$ if there is some set of curves representing r that is contained in a set of curves representing s. Notice that $\varnothing \leq r$ for every slope r. Say that a surface $\overline{R} \subset N[b]$ has boundary slope $\varnothing$ on a component of ∂N if $\partial\overline{Q}$ is disjoint from that component. Say that a surface $\overline{R} \subset N[b]$ has boundary slope $r \neq \varnothing$ on a component of ∂N if each curve of $\partial\overline{R}$ on that component is contained in some representative of r and every curve of a representative of r is isotopic to some component of $\partial\overline{R}$.

Define a surface to be *essential* if it is incompressible, boundary incompressible and has no component that is boundary-parallel or that is a 2-sphere bounding a 3-ball. The next theorem takes as input an essential surface $\overline{R} \subset N[b]$ and gives as output a surface $\overline{Q}$ such that $Q = \overline{Q} \cap N$ can (in many circumstances) be effectively used as a parameterizing surface. The remainder of the section will be spent proving it.

Theorem 5.1. *Suppose that $\overline{R} \subset N[b]$ is a suitably embedded essential surface and suppose either*

(I) *$\overline{R}$ is a collection of essential spheres and discs, or*

(II) *$N[b]$ contains no essential sphere or disc.*

Then there is a suitably embedded incompressible and boundary-incompressible surface $\overline{Q} \subset N[b]$ with the following properties. (The properties have been organized for convenience. The properties marked with a "" are optional and need not be invoked.)*

- *$\overline{Q}$ is no more complicated than $\overline{R}$:*

(C1) $(-\chi(\overline{Q}), \tilde{q}(\overline{Q})) \leq (-\chi(\overline{R}), \tilde{q}(\overline{R}))$ *in lexicographic order.*

(C2) *The sum of the genera components of $\overline{Q}$ is no bigger than the sum of the genera of components of $\overline{R}$.*

(C3) *$\overline{Q}$ and $\overline{R}$ represent the same class in $H_2(N[b], \partial N[b])$.*

- *The options for compressions, a-boundary compressions, and a-torsion $2g$-gons are limited:*

(D0) *Q is incompressible.*

(D1) *Either there is no a-boundary compressing disc for Q or $\tilde{q} = 0$.*

(*D2) *If no component of $\overline{R}$ is separating and if $\tilde{q} \neq 0$, then there is no a-torsion $2g$-gon for Q.*

(D3) *If $\overline{Q}$ is a disc or 2-sphere, then either $N[b]$ has a lens space connected summand or there is no a-torsion $2g$-gon for Q with $g \geq 2$.*

(D4) *If $\overline{Q}$ is a planar surface, then either there is no a-torsion $2g$-gon for Q with $g \geq 2$ or attaching 2-handles to $\partial N[b]$ along $\partial\overline{Q}$ produces a 3-manifold with a lens space connected summand.*

- *The boundaries are not unrelated:*

(*B1) *Suppose that (II) holds, that we are refilling meridians, that no component of $\overline{R}$ separates, and that $\partial\overline{R}$ has exactly one nonmeridional component on each component of $\partial_0 N[b]$. Then $\overline{Q}$ has exactly one boundary component on each component of $\partial_0 N[b]$ and the slopes are the same as those of $\partial\overline{R} \cap \partial_0 N[b]$.*

(B2) *If $\partial\overline{R} \cap \partial_1 N$ is contained on torus components of $\partial_1 N$ or if neither* (*D2) *or* (*B1) *are invoked, then the boundary slope of $\overline{Q}$ on a component of $\partial_1 N[b]$ is less than or equal to the boundary slope of $\overline{R}$ on that component.*

(B3) *If* (*D2) *is not invoked and if the boundary slope of $\overline{R}$ on a component of $\partial_0 N[b]$ is nonempty, then the boundary slope of $\overline{Q}$ on that component is less than or equal to the boundary slope of $\overline{R}$.*

Property (*B1), which is the most unpleasant to achieve, is present to guarantee that if $\overline{R}$ is a Seifert surface for L_β, then $\overline{Q}$ (possibly after discarding components) is a Seifert surface for L_β. This is not used subsequently in this paper, but future work is planned which will make use of it. However, achieving property (*D2), which is used here, requires similar considerations. Here, we will often want to achieve (*D2), though it is incompatible with (B3). However, [Taylor 2008] does not need (*D2), and so we state the theorem in a fairly general form.

The only difficulty in proving the theorem is keeping track of the listed properties of $\overline{Q}$ and $\overline{R}$. Eliminating a-boundary compressions is psychologically easier than eliminating a-torsion $2g$-gons, so we first go through the argument that a surface $\overline{Q}$ exists that has all but properties (*D2)–(D4). The argument may be easier to follow if, on a first reading, $\overline{R}$ is considered to be a sphere or essential disc. The proof is based on similar work in [Scharlemann 2008], which restricts $\overline{R}$ to being a sphere or disc.

The main purpose of assumptions (I) and (II) is to easily guarantee that the process for creating $\overline{Q}$ described below terminates. We will show that if $\tilde{q}(\overline{R}) \neq 0$ and there is an a-boundary compressing disc or a-torsion $2g$-gon for $R = \overline{R} \cap N$, then there is a sequence of operations on $\overline{R}$ each of which reduces a certain complexity but preserves the properties listed above (including essentiality of $\overline{R}$). If (I) holds, the complexity is $(\tilde{q}(\overline{R}), -\chi(R))$, with lexicographic ordering. If (II) holds, the complexity is $(-\chi(\overline{R}), \tilde{q}(\overline{R}))$, also with lexicographic ordering. If (II) holds, it is clear that $-\chi(\overline{R})$ is always nonnegative. It will be evident that each measure of complexity has a minimum. The process stops either when $\tilde{q} = 0$ or when the minimum complexity is reached.

5.1. ***Eliminating compressions.*** Suppose that R is compressible, and let D be a compressing disc. Since $\overline{R}$ is incompressible, ∂D is inessential on $\overline{R}$. Compress $\overline{R}$ using D. Let $\overline{Q}$ be the new surface. $\overline{Q}$ consists of a surface of the same topological type as $\overline{R}$ and an additional sphere. We have $\tilde{q}(\overline{Q}) = \tilde{q}(\overline{R})$. If we are assuming (II), the sphere component must be inessential in $N[b]$ and so may be discarded. Notice that in both cases (I) and (II) the complexity has decreased. Since R can be compressed only finitely many times, the complexity cannot be decreased arbitrarily far by compressions.

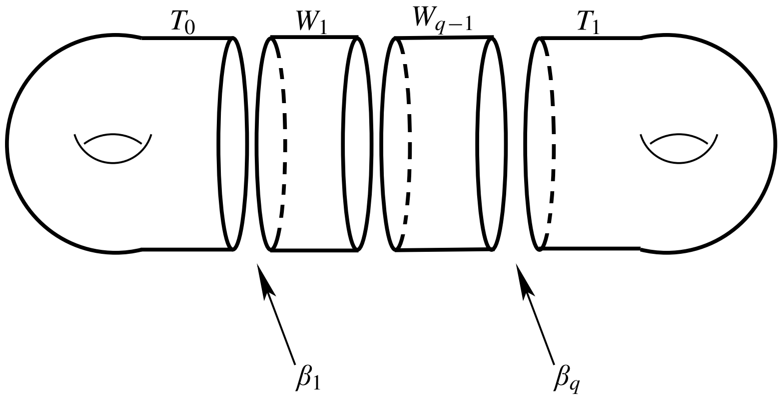

Figure 6. The tori and 1-handles W_j.

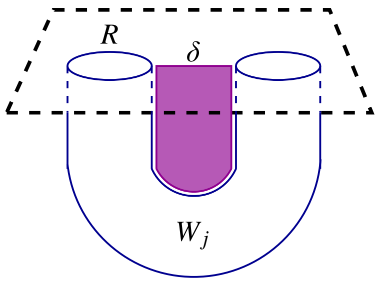

Figure 7. The disc D describes an isotopy of $\overline{R}$.

5.2. ***Eliminating a-boundary compressions.*** Assume that $\tilde{q} \neq 0$ and that there is an a-boundary compressing disc D for R with $\partial D = \delta \cup \epsilon$, where ϵ is a subarc of some essential circle in $\eta(a)$. There is no harm in considering $\epsilon \subset a - \partial R$.

Case 1: b separates W. In this case, $\eta(\overline{\beta}) - \operatorname{int} \overline{R}$ consists of $q-1$ copies of $D^2 \times I$, labeled $W_1, \dots, W_{q-1}$, and $\partial_0 N[b] = \partial N[b] - \partial N$ has two components T_0 and T_1, both tori. The frontiers of the W_j in $\eta(\overline{\beta})$ are discs $\beta_1, \dots, \beta_q$, each parallel to β, the core of the 2-handle attached to b. Each 1-handle W_j lies between β_j and β_{j+1}. The torus T_0 is incident to β_1 and the torus T_1 is incident to β_q. See Figure 6.

The interior of the arc $\epsilon \subset F$ is disjoint from ∂R. Consider the options for how ϵ could be positioned on W:

Case 1.1: ϵ lies in $\partial W_j \cap F$ for some $1 \leq j \leq q-1$. In this case, ϵ must span the annulus $\partial W_j \cap F$. The 1-handle W_j can be viewed as a regular neighborhood of the arc ϵ. The disc D can then be used to isotope W_j through $\partial D \cap R$, reducing $|\overline{R} \cap \overline{\beta}|$ by 2. See Figure 7. This maneuver decreases $\tilde{q}(\overline{R})$. Alternatively, the disc E describes an isotopy of $\overline{R}$ to a surface $\overline{Q}$ in $N[b]$ reducing $\tilde{q}$. Clearly, $\overline{Q}$ satisfies the (C) and (B) properties.

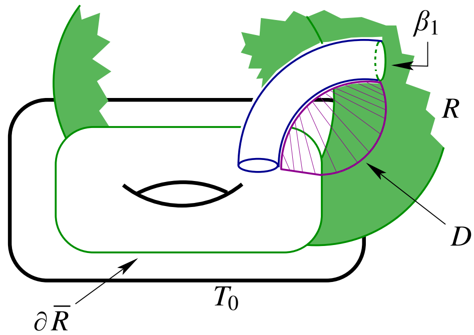

Figure 8. The disc D describes an isotopy of $\overline{R}$.

Suppose, then, that ϵ is an arc on T_0 or T_1. Without loss of generality, we may assume it is on T_0.

Case 1.2: ϵ lies in T_0 and has both endpoints on $\partial\overline{R}$. This is impossible since $\overline{R}$ was assumed to be essential in $N[b]$ and $\tilde{q} > 0$.

Case 1.3: ϵ lies in T_0 and has one endpoint on $\partial\beta_1$ and the other on $\partial\overline{R}$. The disc D guides a proper isotopy of $\overline{R}$ to a surface $\overline{Q}$ in $N[b]$ that reduces $\tilde{q}$. See Figure 8. Clearly, the (C) and (D) properties are satisfied.

Case 1.4: ϵ lies in T_0 and has endpoints on $\partial\beta_1$. Boundary-compressing $\overline{R} - \mathring{\beta}_1$ produces a surface $\overline{J}$ with two new boundary components on T_0, both of which are essential curves. They are oppositely oriented and bound an annulus containing β_1. If $\partial\overline{R} \cap T_0 \neq \varnothing$, then these two new components have the same slope on T_0 as $\partial\overline{R}$, showing that property (B2) is satisfied. It is easy to check that $\chi(\overline{J}) = \chi(\overline{R})$ and that $\tilde{q}(\overline{J}) = \tilde{q}(\overline{R}) - 1$, so that (C1) is satisfied. Clearly, (C2), (C3), and (B3) are also satisfied.

If $\overline{J}$ were compressible, there would be a compressing disc for $\overline{R}$ by an outermost arc/innermost disc argument. Thus, $\overline{J}$ is incompressible. Suppose that E is a boundary-compressing disc for $\overline{J}$ in $N[b]$ with $\partial E = \kappa \cup \lambda$, where κ is an arc in $\partial N[b]$ and λ is an arc in $\overline{J}$. Since $\overline{R}$ is boundary-incompressible, the arc κ must lie on T_0 (and not on T_1). Since T_0 is a torus, either some component of $\overline{J}$ is a boundary-parallel annulus or $\overline{J}$ (and therefore $\overline{R}$) is compressible. We may assume the former. If $\overline{J}$ has other components apart from the boundary-parallel annulus, discarding the boundary-parallel annulus leaves a surface $\overline{Q}$ satisfying the (C) and (B) properties. We may therefore assume that $\overline{J}$ in its entirety is a boundary-parallel annulus.

Since $\chi(\overline{R}) = \chi(\overline{J})$, since $\overline{J}$ is a boundary-parallel annulus, and since $\partial\overline{J}$ has two more components then $\partial\overline{R}$, $\overline{R}$ is an essential torus. However, using D to

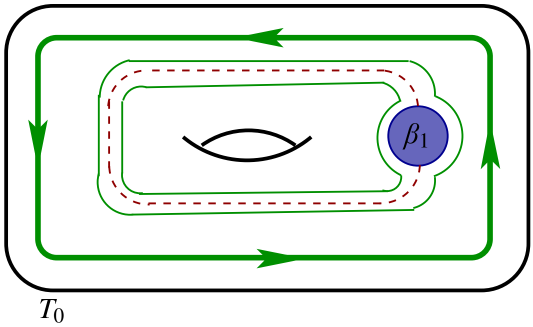

Figure 9. The annulus A lies between $\partial\overline{R}$ and one of the new boundary components of $\overline{L}$.

isotope $\eta(\delta) \subset \overline{R}$ into T_0 and then isotoping $\overline{J}$ into T_0 gives a homotopy of $\overline{R}$ into T_0, showing that it is not essential, a contradiction.

Thus, after possibly discarding a boundary-parallel annulus from $\overline{J}$ to obtain $\overline{L}$, we obtain a nonempty essential surface in $N[b]$ satisfying the first five required properties. If we do not desire property (*B1) to be satisfied, take $\overline{Q} = \overline{L}$. Notice that this step may, for example, convert an essential sphere into two discs or an essential disc with boundary on $\partial_1 N[b]$ into an annulus and a disc with boundary on $\partial_0 N[b]$. This fact accounts for the delicate phrasing of the (B) properties.

Suppose therefore that we wish to satisfy (*B1). Among other properties, we assume that $\overline{R}$ has a single boundary component on T_0.

There is an annulus $A \subset T_0$ that is disjoint from $\beta_1 \subset T_0$, that has interior disjoint from $\partial\overline{L}$, and that has its boundary two of the two or three components of $\partial\overline{L}$. See Figure 9, in which the dashed line represents the arc ϵ. The two circles formed by joining ϵ to $\partial\beta_1$ are the two new boundary components of $\overline{L}$. Since they came from a boundary-compression, they are oppositely oriented. If $\partial\overline{R}$ has a single component on T_0 (indicated by the curve with arrows in the figure), it must be oriented in the opposite direction from one of the new boundary components of $\partial\overline{L}$. Attaching A to $\overline{L}$ creates an orientable surface and does not increase negative Euler characteristic or $\tilde{q}$.

Thus, $\overline{L} \cup A$ is well defined if $|\partial\overline{R} \cap T_0| \leq 1$. It may however be compressible or boundary-compressible. Since it represents the homology class $[\overline{R}]$ in $H_2(N[b], \partial N[b])$, as long as that class is nonzero we may thoroughly compress and boundary-compress it, obtaining a surface $\overline{J}$. Discard all null-homologous components of $\overline{J}$ to obtain a surface $\overline{Q}$. By assumption (II), we never discard an essential sphere or disc. Note that since $\partial\overline{R}$ has a single boundary component on T_1, the surface $\overline{Q}$ will also have a single boundary component on T_1. That is, discarding separating components of $\overline{J}$ does not discard the component with boundary on T_1.

Boundary-compressing $\overline{J}$ may change the slope of $\partial\overline{J}$ on nontorus components of $\partial_1 N[b]$. Discarding separating components may convert a slope on a torus component to the empty slope. Nevertheless, properties (B2) and (B3) still hold.

If a component of $\overline{J}$ is an inessential sphere, then either $\overline{L}_A$ contained an inessential sphere or the sphere arose from compressions of $\overline{L}_A$. Suppose that the latter happened. Then after some compressions $\overline{L}_A$ contains a solid torus and compressing that torus creates a sphere component. Discarding the torus instead of the sphere shows that this process does not increase negative Euler characteristic. If $\overline{L}_A$ contains an inessential sphere, this component is either a component of $\overline{L}$ and therefore of $\overline{R}$ or it arose by attaching A to two disc components, D_1 and D_2, of $\overline{L}$. The first is forbidden by the assumption that $\overline{R}$ is essential and the second by (II). Consequently, negative Euler characteristic is not increased.

Notice that, in general, compressing $\overline{L}_A$ may increase $\tilde{q}$, but because $-\chi(\overline{Q})$ is decreased, property (C1) is still preserved and complexity is decreased. Since we assume (II) for the maneuver, if (I) holds at the end of this case, we can still conclude that $\tilde{q}$ was decreased. (This is an observation needed to show that the construction of $\overline{Q}$ for the conclusion of the theorem terminates.)

Case 2: b is nonseparating and $q^* \neq 0$. This is very similar to Case 1. In what follows, only the major differences are highlighted.

Since $q^* \neq 0$, the cocore $\overline{\beta}^*$ of the 2-handle attached to b^* and the cocore $\overline{\beta}$ form an arc with a loop at one end. Let $U = \eta(\overline{\beta}^* \cup \overline{\beta})$. Then $U - \overline{R}$ consists of a solid torus, $q^* - 1$ copies of $D^2 \times I$ labeled $W_1^*, \ldots, W_{q^*-1}^*$, a 3-ball $\mathcal{P}$, and $q - 1$ copies of $D^2 \times I$ labeled $W_1, \ldots, W_{q-1}$. The cylinders $W_1^*, \ldots, W_{q^*-1}^*$ have frontiers in U consisting of discs $\beta_1^*, \ldots, \beta_{q^*}^*$ all parallel to β^* (the core of the 2-handle attached to b^*). The ball $\mathcal{P}$ has frontier in U consisting of 3 discs β_{q*}^*, β_1, and β_q. The cylinders $W_1, \ldots, W_{q-1}$ have with frontiers $\beta_1, \ldots, \beta_q$ consisting of discs $\beta_1, \ldots, \beta_{q^*}$ all parallel to β. See Figure 10. $\partial_0 N[b]$ consists of a single torus T_0.

Case 2.1: ϵ is not located in $\mathcal{P}$. This is nearly identical to Case 1. To achieve (*B1), an annulus attachment trick like that in Case 1.4 is necessary.

Case 2.2: ϵ is located in $\mathcal{P}$. Since $\partial\overline{R}$ is essential in $N[b]$ and since $\overline{R}$ is embedded, $\partial\overline{R}$ is disjoint from $\mathcal{P}$. The arc ϵ has its endpoints on exactly two of $\{\partial\beta_{q^*}^*, \partial\beta_1, \partial\beta_q\}$. Denote by x and y the two discs containing $\partial\epsilon$, and denote the third by z. That is, $\{\partial x, \partial y, \partial z\} = \{\partial\beta_{q^*}^*, \partial\beta_1, \partial\beta_q\}$. Boundary-compressing $\mathrm{cl}(\overline{Q} - (x \cup y))$ along D removes ∂x and ∂y as boundary-components of R and adds another boundary-component parallel to ∂z. Attach a disc in F parallel to z to this new component, forming $\overline{J}$. Such $\overline{J}$ is isotopic in $N[b]$ to $\overline{R}$ (see Figure 11) and is therefore essential and satisfies the (C) and (B) properties.

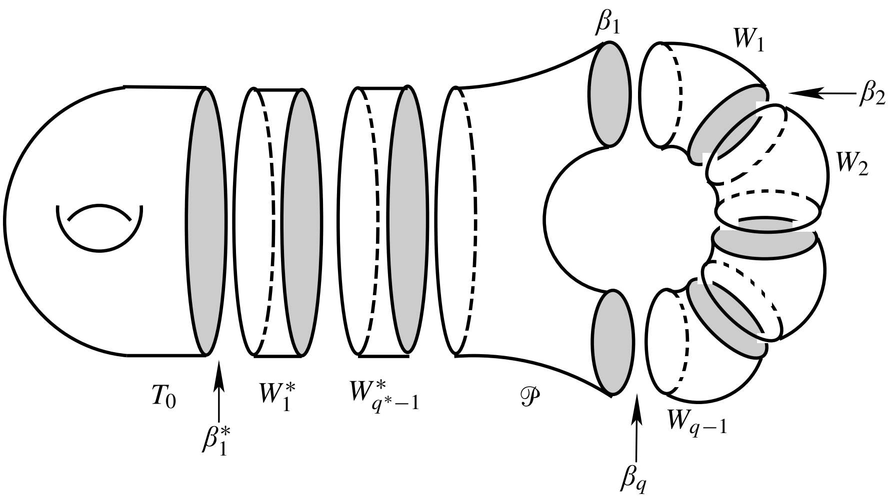

Figure 10. The torus, pair of pants, and 1-handles.

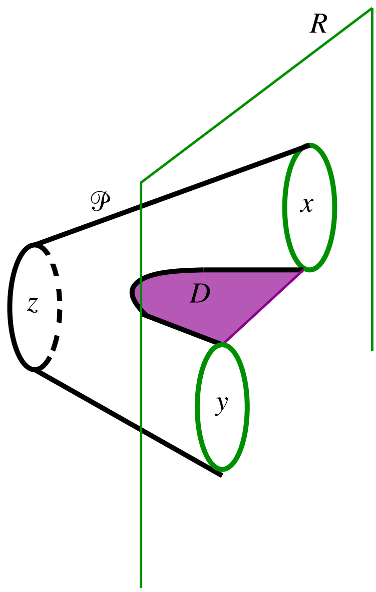

Figure 11. The disc D in Case 2.2.

Case 3: b is nonseparating and $q^* = 0$. Since b is nonseparating, $\eta(\overline{\beta}) - \overline{Q}$ consists of copies of $D^2 \times I$ labeled $W_1, \ldots, W_{q-1}$ that are separated by discs $\beta_1, \ldots, \beta_q$ each parallel to β so that each W_i is adjacent to β_i and β_{i+1}, where the indices run mod q. $\partial_0 N[b]$ is a single torus T_0. See Figure 12.

We need only consider two more cases, as the others are similar to prior cases.

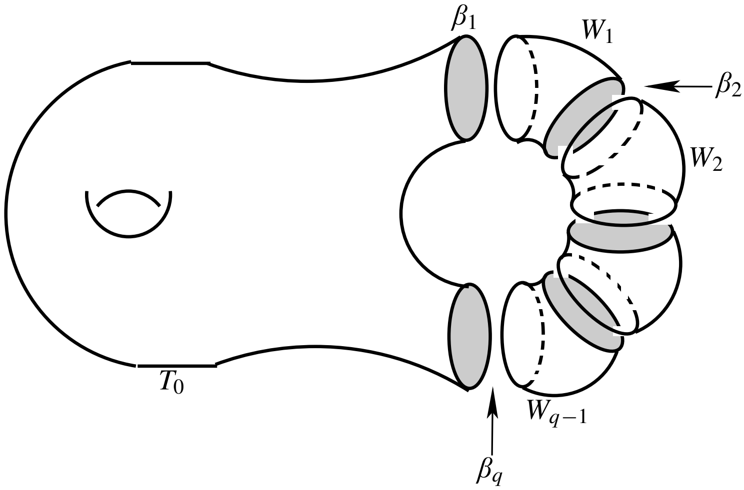

Figure 12. The solid torus and 1-handles W_j.

Case 3.4: ϵ is located on T_0 and either both endpoints are on $\partial\beta_1$ or both are on $\partial\beta_q$. The arc ϵ is a meridional arc. Suppose without loss of generality that $\partial\epsilon \subset \partial\beta_1$. Boundary-compress $\overline{R} - \mathring{\beta}_1$ along D. This creates a surface $\overline{J}$ with boundary on T_0. After possibly discarding a boundary-parallel annulus, $\overline{J}$ is essential and the (C) properties hold as well as (B2) and (B3). We need to show that (*B1) can be achieved, if desired.

Suppose that we are in the situation of refilling meridians, so that $N \subset M$ and F bounds a genus 2 handlebody W in $M - N$ with a and b bounding discs in W. Then since the endpoints of ϵ are on the same component of $\partial\eta(a) \subset F$, ϵ is a meridional arc of $b - a$. If $\partial\overline{R}$ is not meridional on T_0, this case therefore cannot occur. Thus, the (C) and (B) properties hold.

Case 3.5: ϵ is located on T_0 and has one endpoint on β_1 and the other on β_q. The disc D guides an isotopy of $\overline{R}$ to a surface $\overline{Q}$ that is suitably embedded in $N[b]$ and has $q^*(\overline{Q}) = 1$. We have $\tilde{q}(\overline{Q}) = \tilde{q}(\overline{R}) - 1$. The surface $\overline{Q}$ can also be created by boundary-compressing $\overline{R} - (\beta_1 \cup \beta_q)$ with D and then adding a disc β^* to the new boundary component. See Figure 11. Clearly, the (C) and (B) properties hold.

The previous cases have each described an operation on $\overline{R}$ that produces an essential surface $\overline{Q}$ having the (C) and (B) properties. Furthermore, the maneuver described in each case strictly decreases complexity. Thus, after repeating the operation enough times, either the surface $\overline{Q}$ will have $\tilde{q}(\overline{Q}) = 0$ or there will be no a-boundary compressions for Q. That is, the (C) and (B) properties hold, and in addition, (D0) and (D1) hold.

5.3. ***Eliminating a-torsion 2g-gons.*** We may now assume there is an a-torsion $2g$-gon D for Q with $g \geq 2$ (since an a-torsion 2-gon is an a-boundary compressing disc). For ease of notation, relabel and let $\overline{R} = \overline{Q}$ and $R = Q$. By the definition of a-torsion $2g$-gon, there is a rectangle E containing the parallel arcs $\partial D \cap F$ which, when attached to R, creates an orientable surface. Two opposite edges of ∂E lie on ∂R and the other two are parallel (as unoriented arcs) to the arcs of $\partial D \cap F$. Denote the components of ∂R containing the two edges of ∂E by ∂_x and ∂_y. It is entirely possible that $\partial_x = \partial_y$. If ∂_x is a component of $\partial R - \partial \overline{R}$, let β_x denote the disc in $\overline{R} - R$ that it bounds. Similarly define β_y.

Suppose that $\overline{R}$ is a planar surface or 2-sphere. Let $\widehat{N}$ be the 3-manifold obtained from $N[b]$ by attaching 2-handles to $\partial N[b]$ so that each component of $\partial \overline{J}$ but one bounds a disc in $\widehat{N}$. Attach these discs to $\overline{R}$, forming a surface $\widehat{R}$. Since $\overline{R}$ was a planar surface or 2-sphere, $\widehat{R}$ is a disc or 2-sphere. A regular neighborhood of $\widehat{R} \cup E$ is a solid torus, and the disc D is in the exterior of that solid torus and winds longitudinally around it $n \geq 2$ times. Thus $\eta(\widehat{R} \cup E \cup D)$ is a lens space connected summand of $\widehat{N}$. Hence, redefining $\overline{Q} = \overline{J}$ we satisfy the (C), (B), and (D) properties.

We may therefore assume that $\overline{R}$ is not a planar surface or 2-sphere. We need to show that we can achieve (*D2) in addition to the (C), (B), (D0), and (D1) properties. The surface $\overline{R}' = (\overline{R} - (\beta_x \cup \beta_y)) \cup E$ is compressible by the disc D. Compress it to obtain an orientable surface $\overline{J}$. Notice that

$$(-\chi(\overline{J}), \tilde{q}(\overline{J})) < (-\chi(\overline{R}), \tilde{q}(\overline{R})).$$

Analyzing the position of E as we did the position of ϵ in the previous section and possibly performing the annulus attachment trick, we can guarantee that the (C) and (B) properties are satisfied. If the ends of E are both on $\partial \overline{R}$, the boundary of $\overline{J}$ may have different slope from the boundary of $\overline{R}$. Whether or not we perform the annulus attachment trick, the surface $\overline{J}$ may be inessential. Compressing, boundary compressing, and discarding null-homologous components produces a nonempty essential surface $\overline{Q}$ satisfying properties (B) and (C). Considerations similar to those necessary for achieving (*B1) in Case 1.4 explain why (B2) is phrased as it is. (B3) is incompatible with (*D2) since discarding components may discard $\partial \overline{R} \cap \partial_0 N[b]$, converting a nonempty slope to an empty slope. A future attempt to eliminate an a-boundary compressing disc or a-torsion $2g$-gon may then introduce new boundary components on $\partial_0 N[b]$ of different slope.

As before, complexity has been strictly decreased under both assumptions (I) and (II). Of course, we may now have additional compressing discs, a-boundary compressing discs, or a-torsion $2g$-gons to eliminate as in the previous sections. Since all these operations lower complexity, the process terminates with the required surface $\overline{Q}$. □

The surface $\overline{Q}$ produced by the previous theorem may be disconnected. (For example, if b is separating it is possible we could start with $\overline{R}$ being a disc with boundary on T_0 and end up with $\overline{Q}$ being the union of an annulus with boundary on $T_0 \cup T_1$ and a disc with boundary on T_1.) The next corollary puts our minds at rest by elucidating when we can discard components to arrive at a connected surface $\overline{Q}$.

Corollary 5.2.
- *If $\overline{R}$ is a collection of spheres or discs then after discarding components of the surface $\overline{Q}$ created by Theorem 5.1, we may assume that $\overline{Q}$ is an essential sphere or disc such that $\tilde{q}(\overline{Q}) \leq \tilde{q}(\overline{R})$ and conclusions* (B2), (B3), (D0), (D1), (D3), *and* (D4) *hold.*
- *If $N[b]$ does not contain an essential disc or sphere, then we may assume the $\overline{Q}$ produced by Theorem 5.1 to be connected and conclusions* (C1), (C2), (B2), *and* (D0)–(D4) *hold. Furthermore, if $\overline{R}$ is nonseparating, so is $\overline{Q}$.*

Proof. Suppose that $\overline{R}$ is a collection of spheres or a discs, and let $\widetilde{Q}$ be the surface produced by Theorem 5.1. Notice that each component of $\widetilde{Q} \cap N$ is incompressible. Since $-\chi(\overline{R}) < 0$, by conclusions (C1) and (C2) of that theorem, $-\chi(\widetilde{Q}) < 0$ and each component of $\widetilde{Q}$ is a planar surface or $\widetilde{Q}$ is a sphere. Indeed, at least one component $\overline{Q}$ of $\widetilde{Q}$ is a sphere or disc. By conclusion (D1), either $\widetilde{Q}$ is disjoint from $\overline{\beta}$ or there is no a-boundary compressing disc for $\widetilde{Q} \cap N$. If there is an a-boundary compressing disc for $\overline{Q} \cap N$, then an outermost arc argument shows that there would be one for $\widetilde{Q} \cap N$. Thus, either $\overline{Q}$ is disjoint from $\overline{\beta}$ or there is no a-boundary compressing disc for $\overline{Q}$. As argued in the proof of Theorem 5.1, if there is an a-torsion $2g$-gon for Q, then $N[b]$ contains a lens space connected summand. It is clear, therefore, that the required conclusions hold.

Suppose that $N[b]$ contains no essential disc or sphere. Let $\widetilde{Q}$ be the surface produced by Theorem 5.1. Notice that $\widetilde{Q}$ contains no disc or sphere components, and also that each component of $\widetilde{Q} \cap N$ is incompressible. Choose a component $\widetilde{Q}_0$ of $\widetilde{Q}$ and discard the other components. Neither negative Euler characteristic nor $\tilde{q}$ are raised. If $\overline{R}$ was nonseparating, choose $\widetilde{Q}_0$ to be nonseparating. Either $\widetilde{Q}_0$ satisfies the conclusion of the corollary or $\tilde{q}(\widetilde{Q}_0) > 0$, and there is an a-boundary compressing disc or a-torsion $2g$-gon for $\widetilde{Q}_0 \cap N$. Apply the theorem with $\overline{R} = \widetilde{Q}_0$, and notice that the surface $\widetilde{Q}_1$ produced has strictly smaller complexity. Thus, repeating this process, each time discarding all but one component, we eventually obtain the connected surface $\overline{Q}$ promised by corollary. □

6. Refilling meridians

We now turn to applying the main theorem to "refilling meridians". For the remainder, suppose that M is a 3-manifold containing an embedded genus 2 handlebody W. Let $N = M - \mathring{W}$. Let α and β be two essential discs in W isotoped to intersect minimally and nontrivially. Let $a = \partial\alpha$, $b = \partial\beta$, $b^* = \partial\beta^*$, $M[\alpha] = N[a]$, and $M[\beta] = N[b]$. Recall that L_α and L_β are the cores of the solid tori produced by cutting W along α and β respectively. If we need to place sutures $\hat{\gamma}$ on $F = \partial W$ we will do so as described in Section 4. We begin by briefly observing that for any suitably embedded surface $\overline{Q} \subset M[\beta]$, with boundary disjoint from $\gamma \cap \partial M$, $K(\overline{Q}) \geq 0$.

If α is separating,

$$K(\overline{Q}) = q(\Delta - 2) + q^*(\Delta^* - 2) + \Delta_\partial.$$

Since b, b^*, and a all bound discs in W, Δ is at least two. If $q^* \neq 0$, then Δ^* is also at least two. Thus, $K(\overline{Q}) \geq 0$.

Recall from Section 4 that if α is nonseparating, any arc of $b - \mathring{\eta}(a)$ with endpoints on the same component of $\partial\eta(a)$ is a meridional arc of $b - a$. The number of these meridional arcs is denoted $\mathcal{M}_a(b)$, and it is always even and always at least two since there are the same number of meridional arcs based at each component of $\partial\eta(a) \subset F$. The sutures $\hat{\gamma}$ are disjoint from these meridional arcs. Since any arc of $b - a$ that is not a meridional arc intersects exactly one suture exactly once, we have

$$\Delta - \nu = \mathcal{M}_a(b) \geq 2 \quad \text{and} \quad \Delta^* - \nu^* \geq \mathcal{M}_a(b^*) \geq 2.$$

Since $\partial\overline{Q}$ is disjoint from $b \cup b^*$, it is also disjoint from the meridional arcs of $b - a$. Consequently, each arc of $\partial\overline{Q} - a$ intersects $\hat{\gamma}$ at most once. Hence, $\Delta_\partial - \nu_\partial \geq 0$. When α is nonseparating, we therefore have

$$K(\overline{Q}) \geq q(\mathcal{M}_a(b) - 2) + q^*(\mathcal{M}_a(b^*) - 2) + \Delta_\partial - \nu_\partial \geq 0.$$

6.1. ***Scharlemann's conjecture.*** Studying the operation of refilling meridians in [2008], Scharlemann was led to the following definitions and conjecture.

Define (M, W) to be *admissible* if

(A0) every sphere in M separates,

(A1) M contains no lens space connected summands,

(A2) any two curves in ∂M which compress in M are isotopic in ∂M,

(A3) $M - W$ is irreducible, and

(A4) ∂M is incompressible in N.

Conjecture (Scharlemann). *If (M, W) is admissible, then one of the following occurs*:

- *$M = S^3$ and W is unknotted (that is, N is a handlebody).*
- *At least one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible.*
- *α and β are "aligned" in W.*

The definition of "aligned" is rather complicated and is not needed for what follows, so I will not define it here. I will only remark that it is a notion that is independent of the embedding of W in M.

Scharlemann proved the following for admissible pairs (M, W):

Theorem.

- *If ∂W compresses in N, then the conjecture is true.*
- *If $\Delta \leq 4$, then the conjecture is true.*
- *If α is separating and M contains no summand that is a nontrivial rational homology sphere, then one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible.*
- *If both α and β are separating, then the conjecture is true. If in addition $\Delta \geq 6$, then one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible.*

With a slight variation on the notion of admissible, Scharlemann's conjecture can now be completed for a large class of manifolds.

Define the pair (M, W) to be *licit* if the following hold:

(L0) $H_2(M) = 0$.

(L1) $H_1(M)$ is torsion-free.

(L2) No curve on a nontorus component of ∂M that compresses in M bounds an essential annulus in N with a meridional curve of ∂W (that is, a curve on ∂W that bounds a disc in W).

(L3) N is irreducible.

(L4) ∂M is incompressible in N.

The major improvement provided by the next theorem is that the case of non-separating meridians can be effectively dealt with. The theorem nearly completes Scharlemann's conjecture for pairs (M, W) that are both licit and admissible. The one major aspect of Scharlemann's conjecture that is not covered by this theorem is the question of whether or not both of $M[\alpha]$ and $M[\beta]$ can be solid tori. In [Taylor 2008], this case is resolved.

Theorem 6.1 (Modified Scharlemann conjecture). *Suppose that (M, W) is licit and that α and β are two essential discs in W. Suppose ∂W is incompressible in N. Then either α and β can be isotoped to be disjoint or all of the following hold*:

- *One of $M[\alpha]$ or $M[\beta]$ is irreducible.*
- *If one of $M[\alpha]$ or $M[\beta]$ is reducible, then no curve on ∂M compresses in the other.*
- *No curve on ∂M compresses in both $M[\alpha]$ and $M[\beta]$.*

- *If one of $M[\alpha]$ or $M[\beta]$ is a solid torus, then the other is not reducible.*

Conditions (L0) and (L1) are stronger than conditions (A0) and (A1) but are used to guarantee that $H_1(M[\alpha])$ and $H_1(M[\beta])$ are torsion-free; this is required for the application of the main theorem. Condition (L2) is neither stronger nor weaker than condition (A2) since we allow multiple curves on ∂M to compress in M but forbid the existence of certain annuli. To show that some condition like (A2) was required, Scharlemann points out an example:

Example. Let M be a genus 2-handlebody, and let $W \subset M$ so that $M - \mathring{W}$ is a collar on ∂W. (That is, M is a regular neighborhood of W.) Then conditions (A0), (A1), (A3), (A4), (L0), (L1), (L3), and (L4) are all satisfied. But given any essential disc $\alpha \subset W$, $M[\alpha]$ is obviously boundary-reducible. Both (A2) and (L2) rule out this example.

The modified Scharlemann conjecture is simply a "symmetrized" version of the following theorem, in which the incompressibility assumption has been weakened for later applications.

Theorem 6.2. *Suppose that (M, W) is licit and that α and β are two essential discs, isotoped to intersect minimally, with $\Delta > 0$. Suppose that $M[\beta]$ is reducible or boundary-reducible. If α is separating, assume that $\partial W - a$ is incompressible in N. If β is nonseparating, assume that there is no essential disc in $M[\beta]$ that is disjoint from both $\overline{\beta}$ and a. Then*

- *$M[\alpha]$ is irreducible;*
- *if $M[\beta]$ is reducible, no essential curve in ∂M compresses in $M[\alpha]$; and*
- *if $M[\beta]$ is boundary-reducible, no essential curve of ∂M compresses in both $M[\beta]$ and $M[\alpha]$.*

Proof. We begin by showing that $H_1(M[\alpha])$ is torsion-free. Consider M as the union of $V = W - \mathring{\eta}(\alpha)$ and $M[\alpha]$. Using assumption (L0) that $H_2(M) = 0$, we see that the Mayer–Vietoris sequence gives the exact sequence

$$0 \to H_1(\partial V) \xrightarrow{\phi} H_1(M[\alpha]) \oplus H_1(V) \xrightarrow{\psi} H_1(M) \to 0.$$

Suppose that x is an element of $H_1(M[\alpha])$ and that $n \in \mathbb{N}$ is such that $nx = 0$. Then $n\psi(x, 0) = \psi(nx, 0) = 0$. Since $H_1(M)$ is torsion-free, $\psi(x, 0) = 0$. Thus, by exactness, $(x, 0)$ is in the image of ϕ. Let $y \in H_1(\partial V)$ be in the preimage of $(x, 0)$. Also, $\phi(ny) = n\phi(y) = (nx, 0) = (0, 0)$. From exactness, we know that ϕ is injective. Hence, $ny = 0 \in H_1(\partial V)$. The boundary of V is a collection of tori, and therefore $H_1(\partial V)$ is torsion-free. Consequently, $y = 0$. Therefore, $x = 0$ and $H_1(M[\alpha])$ is torsion-free.

We now proceed with the theorem by choosing appropriate sutures on ∂M. If ∂M is compressible in $M[\beta]$, let c_β be a curve on ∂M that compresses in $M[\beta]$. If $c_\beta = \varnothing$, let c be any curve on ∂M that compresses in M; otherwise let $c = c_\beta$.

By Lemma 4.1, we may choose sutures γ on $\partial M[\alpha]$ so that $\hat{\gamma} = \gamma \cap \partial_0 M[\alpha]$ is chosen as usual and so that $\gamma \cap c = \varnothing$ and $(M[\alpha], \gamma)$ is an $\overline{\alpha}$-taut sutured manifold. Let $\overline{R}$ be either an essential sphere, an essential disc with boundary $c_\beta = c$, or an essential disc with boundary on $\partial_0 M[\beta]$. Let $\overline{Q}$ be the result of applying Corollary 5.2 to $\overline{R}$. $\overline{Q}$ is an essential sphere, an essential disc with boundary c_β, or an essential disc with boundary on $\partial_0 M[\beta]$.

If $\overline{Q}$ is a sphere or disc with boundary c_β, then, since N is irreducible and ∂M is incompressible in N, $\tilde{q} > 0$. By Corollary 5.2, there is no compressing disc, a-boundary compressing disc, or a-torsion $2g$-gon for $Q = \overline{Q} \cap N$. Suppose, for the moment, that $\overline{Q}$ is a disc with boundary on ∂W. If $\tilde{q} > 0$, then Q is not disjoint from a. By Corollary 5.2 there is no compressing disc, a-boundary compressing disc or a-torsion $2g$-gon for Q. If $\tilde{q} = 0$, then by hypothesis $Q = \overline{Q}$ is not disjoint from a. Since $Q = \overline{Q}$ is a disc, there are no essential arcs in Q and so there is no compressing disc, a-boundary compressing disc, or a-torsion $2g$-gon in this case either.

Since in all cases $\partial \overline{Q}$ is disjoint from the sutures on ∂M, $K(\overline{Q}) \geq 0$ as noted in the introduction to this section. Since $\overline{Q}$ is a sphere or disc, we also have $-2\chi(\overline{Q}) < 0$. Hence, by the main theorem, $(M[\alpha], \gamma)$ is $\varnothing$-taut. This implies that $M[\alpha]$ is irreducible and that $R_\pm(\gamma)$ does not compress in $M[\alpha]$. Consequently, c does not compress in $M[\alpha]$. □

Remark. At the cost of adding hypotheses on the embedding of W in M, the conditions for being licit can be significantly weakened. For example, the hypotheses on the curves c, a, and b of Lemma 4.1 can be substituted for (L2). An examination of the homology argument at the beginning of the proof shows that (L0) can be replaced with the assumption that L_α and L_β are null-homologous in M.

7. Rational tangle replacement

In this section, we show how the main theorem combined with Theorem 5.1 can be used to give new proofs of several theorems concerning rational tangle replacement. Following [Eudave-Muñoz 1988], we define a few relevant terms.

A *tangle* (B, τ) is a properly embedded pair of arcs τ in a 3-ball B. Two tangles (B, τ) and (B, τ') are *equivalent* if they are homeomorphic as pairs. They are *equal* if there is a homeomorphism of pairs that is the identity on ∂B. The *trivial tangle* is the pair $(D^2 \times I, \{.25, .75\} \times I)$. A *rational tangle* is a tangle equivalent to the trivial tangle. Each rational tangle (B, r) has a disc $D_r \subset B$ separating the strands of r (each of which is isotopic into ∂B). The disc D_r is called a *trivializing* disc for (B, r). The *distance* $d(r, s)$ between two rational tangles (B, r) and (B, s) is simply the minimal intersection number $|D_r \cap D_s|$. We will often write $d(D_r, D_s)$ instead of $d(r, s)$. A *prime* tangle (B, τ) is one without local knots (that is, every meridional annulus is boundary-parallel) and where no disc in B separates the strands of τ.

Given a knot $L_\beta \subset M$ and a 3-ball B' intersecting L_β in two arcs such that $(B', B' \cap L_\beta) = (B', r_\beta)$ is a rational tangle, to replace (B', r_β) with a rational tangle (B', r_α) is to do a *rational tangle replacement* on L_β. Note that $\eta(L_\beta) \cup B'$ is a genus 2 handlebody W. The knots or links L_β and L_α can be obtained by refilling the meridians β and α respectively. If $M = S^3$, then $(B, \tau) = (S^3 - \mathring{B}', L_\beta - \mathring{B}')$ is a tangle. We assume that no component of L_β is disjoint from B.

Before stating the applications, we state and prove some lemmas that allow the terminology of tangle sums and rational tangle replacement to be converted into the terminology of boring.

7.1. ***Boring and rational tangle replacement.***

Lemma 7.1. *Let (B, τ) be a tangle and $N = B - \mathring{\eta}(\tau)$. Suppose that c is an essential curve on $\partial B - \tau$ that separates ∂N. If $\partial N - c$ is compressible in N then c compresses in N.*

Proof. Let d be an essential curve in $\partial N - c$ that bounds a disc $D \subset N$. Since c is separating and ∂N has genus 2, d is a curve in a once punctured torus. Thus, it is either nonseparating or parallel to c. In the latter case, we are done, so suppose that d is nonseparating. Let D_+ and D_- be parallel copies of D so that d is contained in an annulus between ∂D_+ and ∂D_-. Use a loop that intersects d exactly once to band together D_+ and D_-, forming a disc D'. The boundary of D' is an essential separating curve in the once punctured torus. $\partial D'$ is therefore parallel to c. Hence, c compresses in N. □

Lemma 7.2. *Suppose that (B, τ) and (B', r_α) are tangles embedded in S^3 with (B', r_α) a rational tangle such that $\partial B = \partial B'$ and $\partial \tau = \partial r_\alpha$. Suppose that (B', r_β) is rational tangle of distance at least one from (B', r_α). Define the sutures $\gamma \cup a$ on ∂N as before. If*

- *α is nonseparating in the handlebody $W = B' \cup \eta(\tau)$, or*
- *if (B, τ) is a prime tangle, or*
- *if (B, τ) is a rational tangle and $\partial \alpha$ does not bound a trivializing disc for (B, τ), or*
- *if $\partial \alpha$ does not compress in (B, τ),*

then $\partial W - (\gamma \cup a)$ is incompressible in N. Consequently, $(N, \gamma \cup a)$ is $\varnothing$-taut and $(N[a], \gamma)$ is $\overline{\alpha}$-taut.

Proof. If α is nonseparating, then any compressing disc for $\partial W - (\gamma \cup \partial\alpha)$ would have meridional boundary, implying that S^3 had a nonseparating 2-sphere. Thus, we may suppose that α is separating. If (B, τ) is prime, there is no disc separating the strands of τ. Similarly, if (B, τ) is a rational tangle but a does not bound a trivializing disc, then a does not compress in (B, τ). Thus, for the remaining three hypotheses, we may assume that a does not compress in (B, τ). By Lemma 7.1, $\partial N - a$ is incompressible in N, as desired. By Lemma 4.1, $(N, \gamma \cup a)$ is taut and $(N[a], \gamma)$ is $\overline{\alpha}$-taut. □

One pleasant aspect of working with rational tangle replacements is that we can make explicit calculations of $K(\overline{Q})$. Here are two lemmas which we jointly call the *Tangle Calculations*.

Tangle Calculations I (β separating). *Suppose that L_β is a link obtained from L_α by a rational tangle replacement of distance d using W. Let $\overline{Q}$ be a suitably embedded surface in the exterior $S^3[\beta]$ of L_β. Let $\partial_1\overline{Q}$ be the components of $\partial\overline{Q}$ on one component of $\partial S^3[\beta]$, and let $\partial_2\overline{Q}$ be the components on the other. Let n_i be the minimum number of times a component of $\partial_i\overline{Q}$ intersects a meridian of $\partial S^3[\beta]$.*

- *If L_α is a link, then*

$$K(\overline{Q}) \geq 2q(d-1) + d(|\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2).$$

- *If L_α is a knot, then*

$$K(\overline{Q}) \geq 2q(d-1) + (d-1)(|\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2).$$

Proof. Since L_β is a link, β is separating. Thus, $q^* = 0$. Since a and b are contained in $\partial B' = \partial B$, every arc of $b - a$ is an meridional arc. Hence, $\nu = 0$. By definition $2d = \Delta$.

Let T be a component of $\partial S^3[\beta]$. Without loss of generality, suppose that the components of $\partial\overline{Q}$ on T are $\partial_1\overline{Q}$. Since every arc of $a - b$ is meridional, there exist d meridional arcs on each component of $\partial S^3[\beta]$. Thus, each component of $\partial_1\overline{Q}$ intersects a at least dn_1 times. Each component of $\partial_2\overline{Q}$ intersects a at least dn_2 times. Consequently, $|\partial_1\overline{Q} \cap a| \geq |\partial_1\overline{Q}|n_1 d$. Similarly, $|\partial_2\overline{Q} \cap a| \geq |\partial_2\overline{Q}|n_2 d$. Hence

$$\Delta_\partial \geq d(|\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2).$$

If α is nonseparating, the curves γ are also meridian curves of L_β. Thus, γ is intersected n_i times by each component of $\partial_i\overline{Q}$. Hence, if L_α is a knot,

$$\nu_\partial = |\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2.$$

The result follows. □

Tangle Calculations II (β nonseparating). Suppose that L_β is a knot obtained from L_α by a rational tangle replacement of distance d using W. Let $\overline{Q}$ be a suitably embedded surface in the exterior $S^3[\beta]$ of L_β. Suppose that each component of $\partial\overline{Q}$ intersects n times a meridian of $\partial S^3[\beta]$.

- If L_α is a link, then

$$K(\overline{Q}) \geq 2q(d-1) + 2q^*(2d-1) + 2d|\partial\overline{Q}|n.$$

- If L_β is a knot, then

$$K(\overline{Q}) \geq 2(d-1)(q+2q^*) + 2(d-1)|\partial\overline{Q}|n.$$

Proof. These calculations are similar to the calculations of the previous lemma, so we make only a few remarks. First, since b^* and $\partial\eta(b)$ cobound a thrice punctured sphere, every meridional arc of $a-b$ intersects b^* at least twice. Since every arc of $a-b$ is meridional, there are Δ such arcs. Hence $\Delta^* \geq 4d$. Second, if L_α is a knot, then b^* intersects γ twice and b intersects γ not at all. Thus

$$q(\Delta-\nu-2) + q^*(\Delta^*-\nu^*-2) \geq q(2d-2) + q^*(4d-4).$$

The given inequality follows. □

7.2. *Discs, spheres, and meridional planar surfaces.* In [1988], Eudave-Muñoz states six related theorems. In this section, we give new proofs for three of them. Gordon and Luecke [1994] have also given different proofs for some of them. The new proofs will follow from the following generalization. Using completely different sutured manifold theory techniques, [Taylor 2008] further extends this theorem.

Theorem 7.3. *Suppose that L_β is a knot or link obtained by a rational tangle replacement of distance $d \geq 1$ on the split link L_α. Suppose that $\partial W - \partial\alpha$ does not compress in N. Then L_β is not a split link or unknot. Furthermore, if L_β has an essential properly embedded meridional planar surface with m boundary components, it contains such a surface $\overline{Q}$ with $|\partial\overline{Q}| \leq m$ such that either $\overline{Q}$ is disjoint from $\overline{\beta}$ or*

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial\overline{Q}| - 2.$$

Proof. By Lemma 7.2, $(N, \gamma \cup a)$ is a taut sutured manifold. Notice that the pair (S^3, W) is licit and that since L_α and L_β are related by rational tangle replacement, no essential disc in $S^3[\beta]$ is disjoint from a. Thus by Theorem 6.2, L_β is neither a split link nor an unknot.

Suppose therefore that $S^3[\beta]$ contains an essential meridional surface $\overline{R}$ with m boundary components. Use Corollary 5.2 to obtain the connected planar surface $\overline{Q} \subset S^3[\beta]$, and assume that $\overline{Q}$ is not disjoint from $\overline{\beta}$. That is, assume that $\tilde{q} > 0$.

Since $\overline{Q}$ is connected and has Euler characteristic not lower than our original planar surface, $|\partial\overline{Q}| \leq m$. The boundary of $\overline{Q}$ is meridional, by construction, since each arc of $a - b$ is meridional. Corollary 5.2 allows us to conclude that there is no compressing disc, a-boundary compressing disc, or a-torsion $2g$-gon for Q. Also, $S^3[\alpha]$ is reducible and $H_1(S^3[\alpha])$ is torsion-free.

The main theorem implies, therefore, that $K(\overline{Q}) \leq -2\chi(\overline{Q})$. Since $\partial\overline{Q}$ is disjoint from $a \cup \gamma$ and since L_α is a link, the tangle calculations tell us that

$$2q(d-1) + 2q^*(2d-1) \leq -2\chi(\overline{Q}).$$

Since $4q^*(d-1) \leq 2q^*(2d-1)$, we conclude that $2(q+2q^*)(d-1) \leq -2\chi(\overline{Q})$. Because $\overline{Q}$ is a planar surface with $|\partial\overline{Q}|$ boundary components, we conclude that $-2\chi(\overline{Q}) = 2|\partial\overline{Q}| - 4$. Plugging into our inequality and dividing by two, we obtain

$$(q+2q^*)(d-1) \leq |\partial\overline{Q}| - 2.$$

A slight isotopy pushing the discs in $\overline{Q}$ with boundary parallel to b^* converts each such disc into two discs each with boundary parallel to b. Hence, after the isotopy $|\overline{Q} \cap \overline{\beta}| = q + 2q^*$. Consequently,

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial\overline{Q}| - 2. \qquad \square$$

As corollaries, we have two classical results.

Theorem [Eudave-Muñoz 1988]. *If (B, τ) is prime, if L_α is a split link, and if L_β is composite, then $d(\alpha, \beta) \leq 1$.*

Proof. Suppose that $d \geq 1$. Since (B, τ) is prime and α is separating, Lemma 7.2 shows that $\partial W - a$ is incompressible in N. Since L_β contains an essential meridional annulus, we may apply Theorem 7.3 with $m = 2$. Since there are no meridional discs, $\overline{Q}$ is also a meridional annulus. Since (B, τ) is prime, $\overline{Q}$ is not disjoint from $\overline{\beta}$. The inequality from the theorem shows that $d = 1$. $\square$

Theorem [Eudave-Muñoz 1988]. *If (B, τ) is any tangle and if L_α and L_β are split links, then $r_\alpha = r_\beta$.*

Proof. It suffices to show that α and β are disjoint. Suppose not, so that $d \geq 1$. If $\partial W - a$ is incompressible in N, then by Theorem 7.3 L_β is not a split link. Thus $\partial W - a$ compresses in N. By reversing the roles of α and β, we can also conclude that $\partial W - b$ compresses in N. Since both α and β are separating, Lemma 7.1 shows that both a and b compress in N.

There is therefore a disc D_a in B with boundary a separating the strings of τ. Similarly, there is a disc D_b in B with boundary $b = \partial\beta$ separating the strings of τ. An easy innermost disc/outermost arc argument shows that D_a and D_b are isotopic. In particular, a and b are isotopic in $\partial B - \tau$, which implies that $r_\alpha = r_\beta$.

Thus we may assume without loss of generality that $\partial W - \partial\alpha$ is not compressible in N. Let $\overline{R}$ be an essential sphere in $S^3[\beta]$, and apply Corollary 5.2 to obtain an essential sphere or disc $\overline{Q}$. Since $a - b$ consists of meridional arcs, $\overline{Q}$ is not disjoint from $\eta(a)$. If $\overline{Q}$ were a disc disjoint from $\overline{\beta}$, there would be no a-boundary compressing disc for $\overline{Q}$. If $\overline{Q}$ is a sphere, $\tilde{q} > 0$. Thus, we may apply the main theorem to conclude that $S^3[\alpha]$ is irreducible or that α and β are disjoint. If the latter is true, $r_\alpha = r_\beta$. □

Theorem [Scharlemann 1985]. *If (B, τ) is any tangle and L_β is a trivial knot and L_α a split link, then (B, τ) is a rational tangle and $d \leq 1$.*

Proof. Suppose $d \geq 1$. If $\partial W - a$ were incompressible in N, then L_β would not be the unknot by Theorem 7.3. Hence $\partial W - a$ is compressible in N. Since α is separating, Lemma 7.1 shows that a compresses in N. Since L_β is the unknot, τ has no local knots. Thus, (B, τ) is a rational tangle with trivializing disc having boundary a.

It remains to prove that $d = 1$. Since L_β is the unknot, a double-branched cover of S^3 with branch set L_β is S^3. The preimage $\widetilde{B}$ of B is an unknotted solid torus. There is a correspondence between rational tangle replacement and Dehn-surgery in the double-branched cover. Replacing (B', r_β) with (B', r_α) converts the double-branched cover to a lens space, S^3 or $S^1 \times S^2$. In the double branched cover, the Dehn surgery is achieved by making a curve in $\partial\widetilde{B}$ that intersects a meridian of $\widetilde{B}$ d times bound a disc in the complementary solid torus. Since L_α is a split link, the double branched cover of S^3 over L_α is reducible. Thus, it must be $S^1 \times S^2$ and d must be one, as desired. □

Remark. In the proof of the previous theorem, note that even without proving $d \leq 1$, we have provided a new proof of Scharlemann's band sum theorem [1985]: If $K = K_1 \#_b K_2$ is the unknot, then the band sum is the connected sum of unknots. To see this note that W is $\eta(K_1 \cup K_2 \cup b)$, where b is the band. The tangle (B, τ) is $(S^3 - \mathring{\eta}(b), (K_1 \cup K_2) - \mathring{\eta}(b))$. Since $\partial\beta$ is a loop that encircles the band, $\partial\beta$ only bounds a disc in (B, τ) when the band sum is a connected sum and K_1 and K_2 are unknots.

[Taylor 2008] gives other significant applications of sutured manifold theory to problems involving rational tangle replacement.

8. Intersections of ∅-taut surfaces

The main theorem is useful for studying a homology class in $H_2(N[a], \partial N[a])$ that is not represented by a surface disjoint from $\overline{\beta}$. The propositions of this section consist of observations that can dramatically simplify the combinatorics of such a situation. Let N be a compact, orientable 3-manifold with $F \subset \partial M$ a genus 2 boundary component. Let $a, b \subset F$ be essential curves that cannot be isotoped to be disjoint, and suppose that $(N[a], \gamma)$ is $\overline{\alpha}$-taut, as in Section 4.

8.1. ***Intersection graphs.***

Proposition 8.1. *Let $(N[a], \gamma)$ and b be as above, and let $z \in H_2(N[a], \partial N[a])$ be a nontrivial homology class. Suppose that $N[a]$ does not contain an essential disc disjoint from $\overline{\alpha}$. Then z is represented by an embedded conditioned $\overline{\alpha}$-taut surface $\overline{P}$. Furthermore, for any such $\overline{P}$, either $\overline{P}$ is disjoint from $\overline{\alpha}$ or $P = \overline{P} \cap N$ has no compressing discs, b-boundary compressing discs or b-torsion $2g$-gons.*

Proof. Let $\overline{P}$ be a conditioned $\overline{\alpha}$-taut surface. (Such a surface is guaranteed to exist by [Scharlemann 1989, Theorem 2.6].) Suppose that $\overline{P}$ is not disjoint from $\overline{\alpha}$. Recall from the definition of "$\overline{\alpha}$-taut" that $\overline{\alpha}$ intersects $\overline{P}$ always with the same sign. Because $\overline{P}$ is $\overline{\alpha}$-taut, P is incompressible. Suppose that D is a b-torsion $2g$-gon for P. If $g = 1$, D is a b-boundary compressing disc for P. Let ϵ_i be the arcs $\partial D \cap F$. Let R be the rectangle containing the ϵ_i from the definition of b-torsion $2g$-gon. Suppose that the ends of R are on components of $\partial P - \partial \overline{P}$. The endpoints of the ϵ_i have signs arising from the intersection of ∂D with ∂P. Since $\overline{\alpha}$ always intersects $\overline{P}$ with the same sign, an arc ϵ_i has the same sign of intersection at both its head and tail. Since the arcs are all parallel, all heads and tails of all the ϵ_i have the same sign of intersection. However, an arc of $\partial D \cap P$ must have opposite signs of intersection, arising as it does from the intersection of two surfaces. This implies that the head of some ϵ_i has a sign different from the tail of some ϵ_i, a contradiction. Hence, at least one end of R must lie on a component of $\partial \overline{P}$.

If one end of R is on $\partial P - \partial \overline{P}$, denote that component by a_1 and denote by α_1 the disc that it bounds in $\overline{P}$. If both ends of R are on $\partial \overline{P}$, let $\alpha_1 = \varnothing$. Attach R to $\overline{P} - \alpha_1$, creating a surface $\widetilde{P}$. The disc D is contained in N and, therefore, had interior disjoint from $\overline{\alpha}$. Compress $\widetilde{P}$ using D and continue to call the result $\widetilde{P}$.

An easy calculation shows that

$$\text{if } \alpha_1 \neq \varnothing, \quad \text{then } \chi(\widetilde{P}) = \chi(\overline{P}) \text{ but } |\overline{\alpha} \cap \widetilde{P}| = |\overline{\alpha} \cap \overline{P}| - 1;$$
$$\text{if } \alpha_1 = \varnothing, \quad \text{then } -\chi(\widetilde{P}) = -\chi(\overline{P}) - 1 \text{ and } |\overline{\alpha} \cap \widetilde{P}| = |\overline{\alpha} \cap \overline{P}|.$$

If $\chi_{\overline{\alpha}}(\overline{P}) \neq |\overline{\alpha} \cap \overline{P}| - \chi(\overline{P})$, then a component of $\overline{P}$ is a disc disjoint from $\overline{\alpha}$ or a sphere intersected by $\overline{\alpha}$ once. Either of these contradict our hypotheses on $N[a]$. Suppose therefore that $\chi_{\overline{\alpha}}(\overline{P}) = |\overline{\alpha} \cap \overline{P}| - \chi(\overline{P})$.

Similarly, $\chi_{\overline{\alpha}}(\widetilde{P}) = |\overline{\alpha} \cap \widetilde{P}| - \chi(\widetilde{P})$. Hence $\chi_{\overline{\alpha}}(\widetilde{P}) = \chi_{\overline{\alpha}}(\overline{P}) - 1$. Since $\overline{\alpha}$ always intersects $\widetilde{P}$ with the same sign, $\overline{P}$ is not $\overline{\alpha}$-taut, a contradiction. Hence, there are no b-torsion $2g$-gons for P. $\square$

Remark. As Scharlemann notes [2008], when a and b are nonseparating it can be difficult to use combinatorial methods to analyze the intersection of surfaces in $N[a]$ and $N[b]$. The primary reason for this is the need to work with a^* and b^* boundary components on the surfaces. The previous proposition shows that when the surfaces in question are $\overline{\alpha}$-taut and $\overline{\beta}$-taut and not disjoint from $\overline{\alpha}$ and $\overline{\beta}$, respectively, there is no need to consider a^* and b^* curves.

The remainder of this section develops notation for studying the intersection graphs of such surfaces. Let $\overline{P} \subset N[a]$ be an $\overline{\alpha}$-taut surface, and let $\overline{Q} \subset N[b]$ be a $\overline{\beta}$-taut surface. Suppose that $\overline{P}$ and $\overline{Q}$ are not disjoint from $\overline{\alpha}$ and $\overline{\beta}$, respectively. Suppose also that there is no b-torsion $2g$-gon for $P = \overline{P} \cap N$ and no a-torsion $2g$-gon for $Q = \overline{Q} \cap N$. It is clear that P and Q are incompressible.

In Section 3, we defined intersection graphs between $\overline{Q}$ and a disc D. We now define, in a similar fashion, intersection graphs between $\overline{P}$ and $\overline{Q}$. Orient P (respectively, Q) so that all boundary components of $\partial P - \partial\overline{P}$ (respectively, $\partial Q - \partial\overline{Q}$) are parallel on $\eta(\overline{\alpha})$ (respectively, $\eta(\overline{\beta})$. The intersection of P and Q forms graphs Λ_α and Λ_β on $\overline{P}$ and $\overline{Q}$. A component of $\partial P - \partial\overline{P}$ or $\partial Q - \partial\overline{Q}$ is called an *interior boundary component*. The vertex of Λ_α or Λ_β to which it corresponds is called an *interior vertex*.

Label the components of $\partial Q \cap \eta(a)$ as $1, \ldots, \mu_Q$ and those of $\partial P \cap \eta(b)$ as $1, \ldots, \mu_P$. The labels should be in order around $\eta(a)$ and $\eta(b)$. An endpoint of an edge on an interior vertex of Λ_α corresponds to an arc of $\partial Q \cap \partial\eta(\alpha)$. Give the endpoint of the edge the label associated to that arc. Similarly, label all endpoints of edges on interior vertices of Λ_β. A *Scharlemann cycle* is a type of cycle that bounds a disc in $\overline{P}$ ($\overline{Q}$, respectively). The interior of the disc must be disjoint from Λ_α (Λ_β) and all of the vertices of the cycle must be interior vertices. Furthermore, the cycle can be oriented so that the tail end of each edge has the same label. This is the same notion of Scharlemann cycle as in Section 3, but adapted to the possibly nonplanar surfaces $\overline{P}$ and $\overline{Q}$.

Lemma 8.2. *There is no Scharlemann cycle in Λ_α or Λ_β.*

Proof. Were there a trivial loop at an interior vertex or a Scharlemann cycle in Λ_α or Λ_β, the interior would be an a or b-torsion $2g$-gon, which contradicts Proposition 8.1. □

Although we will not use it here, the next lemma may be a useful observation in the future.

Lemma 8.3. *If $\overline{P}$ is a disc, then every loop in Λ_α is based at $\partial\overline{P}$.*

Proof. Suppose that $\overline{P}$ is a disc and that there is a loop based at an interior vertex of Λ_α. A component X of the complement of the loop in $\overline{P}$ does not contain $\partial\overline{P}$. The loop is an x-cycle and Lemma 3.3 then guarantees the existence of a Scharlemann cycle in X, contrary to Lemma 8.2. □

8.2. ***When the exterior of W is anannular.*** We conclude this section with an application to refilling meridians of a genus 2 handlebody whose exterior is irreducible, boundary-irreducible, and anannular. It is based on the ideas in [Scharlemann and Wu 1993]. Suppose that M is the exterior of a link in S^3. Suppose that $W \subset M$ is a genus 2 handlebody embedded in M. Let $N = M - \mathring{W}$.

Theorem 8.4. *Suppose that N is irreducible, boundary-irreducible and anannular. Suppose that α and β are nonseparating meridians of W such that $\Delta > 0$. Suppose that neither $M[\alpha]$ nor $M[\beta]$ contain an essential disc or sphere. Suppose also that in $H_2(M[\alpha], \partial M)$ there is a homology class z_a that cannot be represented by a surface disjoint from $\overline{\alpha}$ and that in $H_2(M[\beta], \partial M)$ there is a homology class z_b that cannot be represented by a surface disjoint from $\overline{\beta}$. Then there is a $\varnothing$-taut surface $\overline{P} \subset M[\alpha]$ representing z_a intersecting $\overline{\alpha}$ p times and an $\varnothing$-taut surface $\overline{Q} \subset M[\beta]$ representing z_b intersecting $\overline{\beta}$ q times such that one of the following occurs*:

(1) $-2\chi(\overline{P}) \geq p(\mathcal{M}_b(a) - 2)$.

(2) $-2\chi(\overline{Q}) \geq q(\mathcal{M}_a(b) - 2)$.

(3) *All of the following occur*:
- $\overline{Q}$ *is* $\overline{\beta}$*-taut.*
- $\overline{P}$ *is* $\overline{\alpha}$*-taut.*
- $pq\Delta \leq 18(p - \chi(\overline{P}))(q - \chi(\overline{Q}))$.
- $\Delta < \frac{9}{2}\mathcal{M}_a(b)\mathcal{M}_b(a)$.

Proof. Notice that the right hand side of the inequalities in (1) and (2) are $K(\overline{P})$ and $K(\overline{Q})$, respectively. Choose a taut representative in $M[\beta]$ for z_b and apply Theorem 5.1, obtaining $\overline{Q}$. Since negative Euler characteristic is not increased and $M[\beta]$ does not contain an essential disc or sphere, $\overline{Q}$ is also taut. If (1) holds, we are done, so assume that $-2\chi(\overline{Q}) < K(\overline{Q})$. Recall that $\overline{Q}$ is not disjoint from $\overline{\beta}$. Apply the main theorem to obtain a surface $\overline{P} \subset M[\alpha]$ representing z_a. (The surface $\overline{P}$ is the surface S in the statement of that theorem.) $\overline{P}$ is both $\overline{\alpha}$-taut and $\varnothing$-taut. If (2) holds, we are done, so assume $-2\chi(\overline{P}) < K(\overline{P})$. Applying the main theorem again, with α and β reversed, we find a $\overline{\beta}$-taut and $\varnothing$-taut surface in $M[\beta]$ representing z_b. We may call this surface $\overline{Q}$, forgetting the previous one. Consider the graphs formed by the intersection of P and Q; let Λ_α be the graph on $\overline{P}$ and Λ_β the graph on $\overline{Q}$. Lemma 8.2 assures us that there is no trivial loop based at an interior vertex of either graph.

Lemma 8.5. $pq\Delta \leq 18(p - \chi(\overline{P}))(q - \chi(\overline{Q}))$.

Proof of Lemma 8.5. By [Scharlemann and Wu 1993, Lemma 2.1], if two edges of $P \cap Q$ are parallel in both Λ_α and Λ_β, there is an essential annulus in N, contrary to our assumption that N is anannular. The proof proceeds as in [Scharlemann and Wu 1993].

Each interior boundary component of P intersects ∂Q at $q\Delta$ places. Therefore $|\partial Q \cap \partial P| \geq pq\Delta$. Thus, Λ_α and Λ_β each have at least $pq\Delta/2$ edges.

Claim. *Λ_α has at least $pq\Delta/(6(p-\chi(\overline{P})))$ mutually parallel edges.*

This claim is similar to work in [Gordon and Litherland 1984]. Let Λ' be the graph obtained by combining each set of parallel edges of Λ_α into a single edge. Since Λ' has no loops at interior vertices and no parallel edges, by applying the formula for the Euler characteristic of a closed surface we obtain

$$\begin{aligned}\chi(\overline{P}) + |\partial\overline{P}| &= V - E + F \\ &\leq p + |\partial\overline{P}| - E + (2/3)E \\ &= p + |\partial\overline{P}| - (1/3)E,\end{aligned}$$

where V, E, and F represent the number of vertices, edges, and faces of Λ'. Thus, $E \leq 3(p-\chi(\overline{P}))$. Let n be the largest number of mutually parallel edges in Λ_α. Then, since there are at least $pq\Delta/2$ edges in Λ_α, we have

$$pq\Delta/(2n) \leq E \leq 3(p-\chi(\overline{P})).$$

The claim follows.

A similar argument shows that if a graph in $\overline{Q}$ has more than $3(q-\chi(\overline{Q}))$ edges, then two of them are parallel. Hence, since there are no mutually parallel edges in Λ_α and Λ_β we must have

$$\frac{pq\Delta}{6(p-\chi(\overline{P}))} \leq 3(q-\chi(\overline{Q})),$$

whence the lemma and the first inequality of conclusion (3) follow. □

We now proceed with the proof of the theorem. Since we are assuming that neither (1) nor (2) hold, we have

$$\begin{aligned}-\chi(\overline{P}) &< K(\overline{P})/2 = p(\mathcal{M}_b(a)-2)/2, \\ -\chi(\overline{Q}) &< K(\overline{Q})/2 = q(\mathcal{M}_a(b)-2)/2\end{aligned}$$

Plugging into the inequality from the lemma, we obtain

$$pq\Delta < 18pq\Big(1+\frac{\mathcal{M}_b(a)-2}{2}\Big)\Big(1+\frac{\mathcal{M}_a(b)-2}{2}\Big).$$

Since neither p nor q is zero, we divide and simplify to obtain

$$\Delta < 9\mathcal{M}_b(a)\mathcal{M}_a(b)/2.$$ □

Remark. The point of the previous theorem is that, under the specified conditions, either we obtain a bound on the Euler characteristic of surfaces representing the homology classes z_a or z_b or we obtain a restriction on the number of nonmeridional arcs of $a-b$ and $b-a$. For example, suppose that discs α and β are chosen so that z_a is represented by a once punctured torus, and so that $\mathcal{M}_b(a)=\mathcal{M}_a(b)=6$. Then $-2\chi(\overline{P})=2<4p=K(\overline{P})$. Then if z_b is also represented by a once punctured torus, we have $\Delta < 162$. Since Δ is even, this implies $\Delta \leq 160$.

Acknowledgments

This paper is part of my PhD dissertation at the University of California, Santa Barbara. I am grateful to my advisor, Martin Scharlemann, for suggesting I learn sutured manifold theory and for many helpful conversations. This paper benefited greatly from his comments on early drafts. I am also grateful to Ryan Blair and Robin Wilson for our conversations and to Stephanie Taylor for suggesting the "boring" terminology. Portions of this paper were written at Westmont College and the research was partially supported by a grant from the National Science Foundation.

References

[Culler et al. 1987] M. Culler, C. M. Gordon, J. Luecke, and P. B. Shalen, "Dehn surgery on knots", *Ann. of Math.* (2) **125**:2 (1987), 237–300. MR 88a:57026 Zbl 0633.57006

[Eudave-Muñoz 1988] M. Eudave-Muñoz, "Primeness and sums of tangles", *Trans. Amer. Math. Soc.* **306**:2 (1988), 773–790. MR 89g:57005 Zbl 0657.57003

[Fenn and Rourke 1979] R. Fenn and C. Rourke, "On Kirby's calculus of links", *Topology* **18**:1 (1979), 1–15. MR 80c:57005 Zbl 0413.57006

[Gabai 1987] D. Gabai, "Genus is superadditive under band connected sum", *Topology* **26**:2 (1987), 209–210. MR 88h:57005 Zbl 0621.57004

[Gabai 1989] D. Gabai, "Surgery on knots in solid tori", *Topology* **28**:1 (1989), 1–6. MR 90h:57005 Zbl 0678.57004

[Gordon and Litherland 1984] C. M. Gordon and R. A. Litherland, "Incompressible planar surfaces in 3-manifolds", *Topology Appl.* **18**:2-3 (1984), 121–144. MR 86e:57013 Zbl 0554.57010

[Gordon and Luecke 1994] C. M. Gordon and J. Luecke, "Links with unlinking number one are prime", *Proc. Amer. Math. Soc.* **120**:4 (1994), 1271–1274. MR 94k:57010 Zbl 0818.57008

[Kirby 1978] R. Kirby, "A calculus for framed links in S^3", *Invent. Math.* **45**:1 (1978), 35–56. MR 57 #7605 Zbl 0377.55001

[Lackenby 1997] M. Lackenby, "Dehn surgery on knots in 3-manifolds", *J. Amer. Math. Soc.* **10**:4 (1997), 835–864. MR 98h:57034 Zbl 0889.57014

[Scharlemann 1985] M. Scharlemann, "Smooth spheres in $\mathbf{R}^4$ with four critical points are standard", *Invent. Math.* **79**:1 (1985), 125–141. MR 86e:57010 Zbl 0559.57019

[Scharlemann 1989] M. Scharlemann, "Sutured manifolds and generalized Thurston norms", *J. Differential Geom.* **29**:3 (1989), 557–614. MR 90e:57021 Zbl 0673.57015

[Scharlemann 1990] M. Scharlemann, “Producing reducible 3-manifolds by surgery on a knot”, *Topology* **29**:4 (1990), 481–500. MR 91i:57003 Zbl 0727.57015

[Scharlemann 2008] M. Scharlemann, “Refilling meridians of genus 2 handlebodies”, *Geometry and Topology Monographs* **14** (2008), 451–475.

[Scharlemann and Wu 1993] M. Scharlemann and Y. Q. Wu, “Hyperbolic manifolds and degenerating handle additions”, *J. Austral. Math. Soc. Ser. A* **55**:1 (1993), 72–89. MR 94e:57019 Zbl 0802.57005

[Taylor 2008] S. A. Taylor, “Adding 2-handles to sutured manifolds”, preprint, 2008. arXiv 0806.1572v2

Received June 30, 2008.

SCOTT A. TAYLOR
DEPARTMENT OF MATHEMATICS
COLBY COLLEGE
5832 MAYFLOWER HILL
WATERVILLE, ME 04901
UNITED STATES

sataylor@colby.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 241, No. 1, 2009

METAPLECTIC TORI OVER LOCAL FIELDS

MARTIN H. WEISSMAN

Smooth irreducible representations of tori over local fields have been parameterized by Langlands, using class field theory and Galois cohomology. This paper extends this parameterization to some central extensions of such tori, which arise naturally in the setting of nonlinear covers of reductive groups.

1. Introduction

Motivation. Let $\boldsymbol{T}$ be an algebraic torus over a local field F; let $T = \boldsymbol{T}(F)$. Let L/F be a finite Galois extension over which $\boldsymbol{T}$ splits, with $\Gamma = \mathrm{Gal}(L/F)$. Let $\mathscr{X}(T)$ denote the group of continuous characters of T with values in $\mathbb{C}^\times$. In a preprint from 1968, now appearing as [Langlands 1997] (see [Labesse 1985]), Langlands proves the following:

Theorem 1.1. *There is a natural isomorphism*

$$\mathscr{X}(T) \cong H_c^1(\mathscr{W}_{L/F}, \hat{\mathscr{T}}),$$

where $\mathscr{W}_{L/F}$ denotes the Weil group of L/F, $\hat{\mathscr{T}}$ denotes the complex dual torus of $\boldsymbol{T}$, and H_c^1 denotes the continuous group cohomology.

MSC2000: 11F70, 22E50.
Keywords: metaplectic, torus, Langlands, local field.

We may consider $\boldsymbol{T}$ as a sheaf of groups, on the big Zariski site over F. In addition, we may consider $\boldsymbol{K}_2$ as such a sheaf, using Quillen's algebraic K-theory. Let $\boldsymbol{T}'$ be a central extension of $\boldsymbol{T}$ by $\boldsymbol{K}_2$ in the category of sheaves of groups on the big Zariski site over F. Such objects are introduced and studied extensively by Brylinski and Deligne [2001].

Let $T' = \boldsymbol{T}'(F)$ be the resulting extension of T by $K_2 = \boldsymbol{K}_2(F)$. If $F \not\cong \mathbb{C}$ and F has sufficiently many n-th roots of unity, one may push forward the central extension T' via the Hilbert symbol to obtain a central extension $\tilde{T}$ as

$$1 \to \mu_n \to \tilde{T} \to T \to 1.$$

We are interested in the set $\mathcal{I}_\epsilon(\tilde{T})$ of irreducible genuine representations of $\tilde{T}$, as defined in Section 3. Such representations arise frequently in the literature on "metaplectic groups", especially when considering principal series representations of nonlinear covers of reductive groups (see among others [Savin 2004; Kazhdan and Patterson 1984; Adams et al. 2007]). This paper's goal is to parameterize the set $\mathcal{I}_\epsilon(\tilde{T})$ in a way that naturally generalizes Theorem 1.1.

Main results. Associated to the central extension $\boldsymbol{T}'$, Deligne and Brylinski associate two functorial invariants: an integer-valued quadratic form Q on the cocharacter lattice Y of $\boldsymbol{T}$, and a Γ-equivariant central extension $\tilde{Y}$ of Y by $L^\times$. Associated to Q is a symmetric bilinear form $B_Q \colon Y \otimes_{\mathbb{Z}} Y \to \mathbb{Z}$.

Define

$$Y^\# = \{y \in Y \text{ such that } B_Q(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y\}.$$

Similarly, define

$$Y^{\Gamma\#} = \{y \in Y \text{ such that } B_Q(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y^\Gamma\}.$$

Associated to the inclusion $\iota \colon Y^\# \hookrightarrow Y$ is an isogeny $\hat{\iota} \colon \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#$ of complex tori. This isogeny is also a morphism of $\mathcal{W}_{L/F}$-modules. Associated to the sequence of inclusions $Y^\# \subset Y^{\Gamma\#} \subset Y$ are F-isogenies $\boldsymbol{T}^\# \to \boldsymbol{T}^{\Gamma\#} \to \boldsymbol{T}$ of F-tori. The main results of this paper are Theorems 4.8, 5.17, and 7.7. Putting these theorems together yields the following:

Theorem 1.2. *Suppose that one of these conditions is satisfied*:

(1) $\boldsymbol{T}$ *is a split torus.*

(2) F *is nonarchimedean with residue field* $\mathbf{f}$, *the torus* $\boldsymbol{T}$ *splits over an unramified extension of* F, *and* n *is relatively prime to the characteristic of* $\mathbf{f}$.

(3) $F \cong \mathbb{R}$.

Then, there exists a finite-to-one map

$$\Phi \colon \mathcal{I}_\epsilon(\tilde{T}) \to H^1_c(\mathcal{W}_{L/F}, \hat{\mathcal{T}}) \big/ H^1_c(\mathcal{W}_{L/F}, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#)$$

that intertwines canonical actions of $H^1_c(\mathcal{W}_{L/F}, \hat{\mathcal{T}})$. *The finite fibres of this map are torsors for a finite group* $\mathcal{X}(P^\dagger) = \operatorname{Hom}(P^\dagger, \mathbb{C}^\times)$. *In the three cases above, the "packet group"* $P^\dagger$ *can be respectively described by the three conditions that*

(1) $P^\dagger$ *is trivial*;

(2) $P^\dagger = \operatorname{Im}(\overline{\boldsymbol{T}}^{\Gamma\#}(\mathbf{f}) \to \overline{\boldsymbol{T}}(\mathbf{f})) \big/ \operatorname{Im}(\overline{\boldsymbol{T}}^{\#}(\mathbf{f}) \to \overline{\boldsymbol{T}}(\mathbf{f}))$;

(3) $P^\dagger = \operatorname{Im}(\pi_0 \boldsymbol{T}^{\Gamma\#}(\mathbb{R}) \to \pi_0 \boldsymbol{T}(\mathbb{R})) \big/ \operatorname{Im}(\pi_0 \boldsymbol{T}^{\#}(\mathbb{R}) \to \pi_0 \boldsymbol{T}(\mathbb{R}))$.

In this theorem, H^1_c denotes the continuous group cohomology or hypercohomology, as discussed by Kottwitz and Shelstad [1999]. The parameterization Φ of irreducible genuine representations is not unique; rather, it depends upon the choice of a base point. The choice of this base point is a significant problem. We identify a natural class of "pseudospherical" representations, following previous authors such as [Savin 2004] and [Adams et al. 2007]. We also parameterize pseudospherical irreducible representations as a torsor for a complex algebraic torus in Section 6; perhaps more naturally, the category of pseudospherical representations can be identified with the category of modules over a "quantum dual torus".

2. Background

Fields and sheaves. F will always denote a local field. F_{Zar} will denote the big Zariski site over F. By this, we mean that F_{Zar} is the full subcategory of the category of schemes over F, whose objects are schemes of finite type over F, endowed with the Zariski topology. $\mathfrak{Set}_F$ will denote the topos of sheaves of sets over F_{Zar}, and $\mathfrak{Gp}_F$ will denote the topos of sheaves of groups over F_{Zar}.

Any scheme or algebraic group over F will be identified with its functor of points, that is, the associated object of $\mathfrak{Set}_F$ or $\mathfrak{Gp}_F$, respectively. Quillen's K-theory [1973] yields sheaves $\boldsymbol{K}_n$ of abelian groups on F_{Zar}. We only work with $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, viewed as objects of $\mathfrak{Gp}_F$.

For any field L, the group $\boldsymbol{K}_2(L)$ is identified as a quotient

$$\boldsymbol{K}_2(L) = \frac{L^\times \otimes_{\mathbb{Z}} L^\times}{\langle x \otimes (1-x) \rangle_{1 \neq x \in L^\times}}.$$

If $l_1, l_2 \in L$, and $l_1, l_2 \notin \{0, 1\}$, then we write $\{l_1, l_2\}$ for the image of $l_1 \otimes l_2$ in $\boldsymbol{K}_2(L)$. The bilinear form $\{\cdot, \cdot\}$ is called the *universal symbol*. The relation $\{x, 1-x\} = 1$ implies that $\{x, -x\} = 1$ for all $x \in L^\times$. This implies that the universal symbol is skew-symmetric. It is usually not alternating, but $\{x, x\} = \{x, -1\}$ for all $x \in L^\times$. Proofs of these facts can be found in [Milnor 1971, Chapter 11].

Local nonarchimedean fields. Suppose that F is a nonarchimedean local field. Then $\mathcal{O}_F$ will denote the valuation ring of F, and $\mathbf{f}$ the residue field of $\mathcal{O}_F$. We let p denote the characteristic of $\mathbf{f}$ and assume that the value group of F is $\mathbb{Z}$. We let q denote the cardinality of $\mathbf{f}$.

There is a canonical short exact sequence

$$1 \to \mathcal{O}_F^\times \to F^\times \to \mathbb{Z} \to 1$$

of abelian groups, given by inclusion and valuation. It is sometimes convenient to split this sequence of abelian groups by choosing a uniformizing element $\varpi \in F^\times$. However, our main results do not depend on which uniformizing element is chosen.

Reduction yields another canonical short exact sequence

$$1 \to \mathcal{O}_F^{\times 1} \to \mathcal{O}_F^\times \to \mathbf{f}^\times \to 1.$$

This sequence is split by the Teichmüller lifting $\Theta \colon \mathbf{f}^\times \to \mathcal{O}_F^\times$.

The Weil group. We let $\mathcal{W}_F$ denote a Weil group of F as in [Tate 1979]. In particular, we follow Tate's choices and normalize the reciprocity isomorphism $rec \colon F^\times \to \mathcal{W}_F^{ab}$ of nonarchimedean local class field theory so that uniformizing elements of $F^\times$ act as the *geometric* Frobenius via *rec*.

When L is a finite Galois extension of F, we continue to follow [Tate 1979] and define $\mathcal{W}_{L/F} = \mathcal{W}_F / \overline{[\mathcal{W}_L, \mathcal{W}_L]}$. There is then a short exact sequence

$$1 \to L^\times \to \mathcal{W}_{L/F} \to \mathrm{Gal}(L/F) \to 1.$$

The Hilbert symbol. We say that F has *enough* n-th roots of unity if $\boldsymbol{\mu}_n(F)$ has n elements. When F has enough n-th roots of unity and $F \not\cong \mathbb{C}$, the Hilbert symbol provides a nondegenerate skew-symmetric bilinear map

$$(\,\cdot\,,\,\cdot\,)_{F,n} \colon \frac{F^\times}{F^{\times n}} \otimes_{\mathbb{Z}} \frac{F^\times}{F^{\times n}} \to \boldsymbol{\mu}_n(F).$$

In general, the Hilbert symbol is not alternating. The Hilbert symbol factors through $\boldsymbol{K}_2(F)$ via the universal symbol.

The definition of the Hilbert symbol relies on a choice of reciprocity isomorphism in local class field theory — this choice has been made earlier in sending a uniformizing element of $F^\times$ to a geometric Frobenius.

If F is nonarchimedean and $(p, n) = 1$, then we say that the Hilbert symbol $(\,\cdot\,,\,\cdot\,)_{F,n}$ is tame. If p is odd, then in the tame case, $(\varpi, \varpi)_{F,n} = (-1)^{(q-1)/n}$ for every uniformizing element $\varpi \in F^\times$. When $p = 2$, in the tame case, $(\varpi, \varpi)_{F,n} = 1$. When $F \cong \mathbb{R}$, we have $(-1, -1)_{F,2} = -1$.

Tori. Henceforth $\boldsymbol{T}$ will always denote an algebraic torus over F. Let L be a finite Galois extension of F, over which $\boldsymbol{T}$ splits, and define $\Gamma = \mathrm{Gal}(L/F)$. We write $X = \mathrm{Hom}(\boldsymbol{T}, \boldsymbol{G}_m)$ for the character group and Y for the cocharacter group $\mathrm{Hom}(\boldsymbol{G}_m, \boldsymbol{T})$. We view X and Y as finite rank free $\mathbb{Z}$-modules endowed with actions of Γ. The groups X and Y are in canonical Γ-invariant duality.

The dual torus $\hat{\boldsymbol{T}}$ is the split torus $\mathrm{Spec}(\mathbb{Z}[Y])$ over $\mathbb{Z}$, with the resulting action of Γ. We write $\hat{\mathcal{T}} = \hat{\boldsymbol{T}}(\mathbb{C}) \equiv X \otimes_{\mathbb{Z}} \mathbb{C}^\times$ for the resulting $\mathbb{C}$-torus, also endowed with the action of Γ.

Central extensions of tori by $\boldsymbol{K}_2$. Let $\mathfrak{CExt}(\boldsymbol{T}, \boldsymbol{K}_2)$ be the category of central extensions of $\boldsymbol{T}$ by $\boldsymbol{K}_2$ in $\mathfrak{Sp}_F$. Let $\mathfrak{CExt}_\Gamma(Y, L^\times)$ be the category of Γ-equivariant extensions of Y by $L^\times$.

In [2001], Deligne and Brylinski study a category we will call $\mathfrak{DB}_{\boldsymbol{T}}$. Its objects are pairs $(Q, \tilde{Y})$, where

- $Q\colon Y \to \mathbb{Z}$ is a Γ-invariant quadratic form;
- $\tilde{Y}$ is a Γ-equivariant central extension of Y by $L^\times$; and
- the resulting commutator map $C\colon \bigwedge^2 Y \to L^\times$ satisfies
$$C(y_1, y_2) = (-1)^{B_Q(y_1,y_2)} \quad \text{for all } y_1, y_2 \in Y,$$
where B_Q is the symmetric bilinear form associated to Q.

If $(Q_1, \tilde{Y}_1)$, and $(Q_2, \tilde{Y}_2)$ are two objects of $\mathfrak{DB}_{\boldsymbol{T}}$, then a morphism from $(Q_1, \tilde{Y}_1)$ to $(Q_2, \tilde{Y}_2)$ exists only if $Q_1 = Q_2$, in which case the morphisms of $\mathfrak{DB}_{\boldsymbol{T}}$ are the just those from $\tilde{Y}_1$ to $\tilde{Y}_2$ in $\mathfrak{CExt}_\Gamma(Y, L^\times)$.

In [2001, Section 3.10], Deligne and Brylinski go on to construct an equivalence of categories from $\mathfrak{CExt}(\boldsymbol{T}, \boldsymbol{K}_2)$ to the category $\mathfrak{DB}_{\boldsymbol{T}}$. In particular, given a central extension $\boldsymbol{T}'$ of $\boldsymbol{T}$ by $\boldsymbol{K}_2$, their work (in part following [Esnault et al. 1998]) yields a quadratic form $Q\colon Y \to \mathbb{Z}$, and a central extension $\tilde{Y}$ of Y by $L^\times$. Considering the central extension $\boldsymbol{T}'(L)$ in
$$1 \to \boldsymbol{K}_2(L) \to \boldsymbol{T}'(L) \to \boldsymbol{T}(L) \to 1,$$
they show that the resulting commutator $C_L\colon \bigwedge^2 \boldsymbol{T}(L) \to \boldsymbol{K}_2(L)$ satisfies
$$C_L(y_1(l_1), y_2(l_2)) = \{l_1, l_2\}^{B_Q(y_1,y_2)} \quad \text{for all } y_1, y_2 \in Y \text{ and } l_1, l_2 \in L^\times.$$

Locally compact abelian groups. An *LCA group* is a locally compact Hausdorff separable abelian topological group. We work here in the category $\mathfrak{LCAb}$ whose objects are LCA groups and whose morphisms are continuous homomorphisms. Suppose that we are have a short exact sequence of LCA groups and continuous homomorphisms given by
$$0 \to A \to B \to C \to 0.$$

Given a fourth LCA group D, the functor $\mathrm{Hom}(\bullet, D)$ is left-exact, yielding an exact sequence

$$0 \to \mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D).$$

2.0.1. *Continuous characters.* When A is an LCA group, we write $\mathscr{X}(A)$ for the group of continuous homomorphisms from A to the LCA group $\mathbb{C}^\times$, under pointwise multiplication. We call elements of $\mathscr{X}(A)$ characters (or continuous characters) of A. If $\chi \in \mathscr{X}(A)$ and $|\chi(a)| = 1$ for all $a \in A$, then we say that χ is a unitary character. We write $\hat{A}$ for the Pontryagin dual of A, that is, the set of unitary characters of A, with its natural topology as an LCA group.

We say that A is an elementary LCA group if $A \cong \mathbb{R}^a \times \mathbb{Z}^b \times (\mathbb{R}/\mathbb{Z})^c \times F$ for some finite group F and some nonnegative integers a, b, c. When A is elementary, $\mathscr{X}(A)$ has a natural structure as a complex algebraic group. In the case above, $\mathscr{X}(A) \cong \mathbb{C}^a \times (\mathbb{C}^\times)^b \times \mathbb{Z}^c \times \hat{F}$.

If A is generated by a compact neighborhood of the identity, then A is canonically isomorphic to the inverse limit of its elementary quotients by compact subgroups. In this case, $\mathscr{X}(A)$ is endowed with the (inductive limit) structure of a complex algebraic group. In this paper, all LCA groups will be generated by a compact neighborhood of the identity, and thus $\mathscr{X}(A)$ will be viewed as a complex algebraic group.

2.0.2. *Exactness criteria.* Given a short exact sequence

$$0 \to A \to B \to C \to 0,$$

there are two important cases in which the induced map $\mathscr{X}(B) \to \mathscr{X}(A)$ is surjective, leading to an exact sequence

$$0 \to \mathscr{X}(C) \to \mathscr{X}(B) \to \mathscr{X}(A) \to 0.$$

Proposition 2.1. *Suppose that A is compact. Then $\mathscr{X}(B) \to \mathscr{X}(A)$ is surjective.*

Proof. If A is compact, every continuous character of A is unitary. The exactness of Pontryagin duality implies that every unitary character of A extends to a unitary character of B. Hence $\mathscr{X}(B)$ surjects onto $\mathscr{X}(A)$. □

Proposition 2.2. *Suppose that the map from A to B is an open embedding. Then $\mathscr{X}(B) \to \mathscr{X}(A)$ is surjective.*

Proof. The proof, which is not difficult, follows directly from [Hoffmann and Spitzweck 2007, Proposition 3.3], for example. □

Complex varieties and groups. We use a script letter, such as $\mathcal{M}$, to denote the (complex) points of a complex algebraic variety. It is unnecessary for us to distinguish between complex varieties and their complex points. If R is a commutative reduced finitely-generated $\mathbb{C}$-algebra, then we write $\mathcal{M} = \mathcal{MS}(R)$ for the maximal ideal spectrum of A, viewed as a complex variety. We view $\mathbb{C}^\times$ as a complex algebraic variety, identifying $\mathbb{C}^\times \equiv \mathcal{MS}(\mathbb{C}[\mathbb{Z}])$, where $\mathbb{C}[\mathbb{Z}]$ denotes the group ring. We view $\mathbb{C}$ itself as an algebraic variety (the affine line over the field $\mathbb{C}$).

Let $\mathcal{G}$ be a complex algebraic group, or in other words, a group in the category of complex algebraic varieties. A $\mathcal{G}$-variety is a complex algebraic variety $\mathcal{M}$ endowed with an action $\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ that is complex-algebraic. A $\mathcal{G}$-torsor is a $\mathcal{G}$-variety $\mathcal{M}$ such that the induced map $\mathcal{G} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ sending (g, m) to $(g \cdot m, m)$ is an isomorphism of complex algebraic varieties.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{G}$-varieties, then a morphism of $\mathcal{G}$-varieties is a complex algebraic map from $\mathcal{M}_1$ to $\mathcal{M}_2$ that intertwines the action of $\mathcal{G}$. Morphisms of torsors are defined in the same way.

3. Genuine representations of metaplectic tori

In this section, we fix notation as follows:

- F will be a local field, with $F \not\cong \mathbb{C}$, and n will be a positive integer such that F has enough n-th roots of unity.
- $\boldsymbol{T}$ will be a torus over a local field F which splits over a finite Galois extension L/F, with $\Gamma = \mathrm{Gal}(L/F)$. X and Y will be the resulting character and cocharacter groups.
- $\boldsymbol{T}'$ will be an extension of $\boldsymbol{T}$ by $\boldsymbol{K}_2$ in $\mathfrak{Sp}_F$.
- $(Q, \tilde{Y})$ will be the Deligne–Brylinski invariants of $\boldsymbol{T}'$. B will be the symmetric bilinear form associated to Q.
- $\epsilon\colon \boldsymbol{\mu}_n(F) \to \mathbb{C}^\times$ will be a fixed injective character.

Heisenberg groups. Suppose that S is an LCA group and A is a finite cyclic abelian group endowed with a faithful unitary character $\epsilon\colon A \to \mathbb{C}^\times$. Suppose that $\tilde{S}$ is a locally compact group that is a central extension of S by A (in the category of locally compact groups and continuous homomorphisms:

$$1 \to A \to \tilde{S} \to S \to 1.$$

In this situation, the commutator on $\tilde{S}$ descends to a unique alternating form

$$C\colon \textstyle\bigwedge^2 S \to A.$$

Let $Z(\tilde{S})$ be the center of $\tilde{S}$. Then $Z(\tilde{S})$ is the preimage of a subgroup $Z^\dagger(S) \subset S$, where $Z^\dagger(S) = \{s \in S \text{ such that } C(s, s') = 1 \text{ for all } s' \in S\}$. Throughout this paper, the following condition will be satisfied, and hence we assume that

$Z^\dagger(S)$ is an open subgroup of finite index in S.

We define two sets:

- The set $\mathscr{X}_\epsilon(\tilde{S})$ of continuous genuine characters of $\tilde{S}$. These are elements of $\mathscr{X}(\tilde{S})$ whose restriction to A equals ϵ.
- The set $\mathscr{I}_\epsilon(\tilde{S})$ of irreducible genuine representations of $\tilde{S}$. These are irreducible (algebraic) representations (π, V) of $\tilde{S}$ on a complex vector space, on which $Z(\tilde{S})$ acts via a continuous genuine character. In particular, since $Z(\tilde{S})$ will always have finite index in $\tilde{S}$, these are finite-dimensional representations.

We often use the following analogue of the Stone–von Neumann theorem:

Theorem 3.1. *Suppose that $\chi \in \mathscr{X}_\epsilon(Z(\tilde{S}))$ is a genuine continuous character. Let $\tilde{M}$ denote a maximal commutative subgroup of $\tilde{S}$. Then there exists an extension $\tilde{\chi} \in \mathscr{X}(\tilde{M})$ of χ to $\tilde{M}$. Define a representation of $\tilde{S}$ by*

$$(\pi_\chi, V_\chi) = \mathrm{Ind}_{\tilde{M}}^{\tilde{S}} \tilde{\chi}.$$

Algebraic induction suffices here, since we always assume that $Z(\tilde{S})$ has finite index in $\tilde{S}$. Then

(1) *the representation (π_χ, V_χ) is irreducible;*

(2) *the representation (π_χ, V_χ) has central character χ;*

(3) *the isomorphism class of (π_χ, V_χ) depends only upon χ and not upon the choices of subgroup $\tilde{M}$ and extension $\tilde{\chi}$;*

(4) *every irreducible representation of $\tilde{S}$ on which $Z(\tilde{S})$ acts via χ is isomorphic to (π_χ, V_χ).*

Proof. Extension of χ to $\tilde{M}$ follows from Proposition 2.2. All but the last claim are proved in [Kazhdan and Patterson 1984, Section 0.3] and follow directly from Mackey theory. The last claim follows from the previous claims and Frobenius reciprocity. □

Metaplectic tori over local fields. The central extension of $\boldsymbol{T}$ by $\boldsymbol{K}_2$ yields a central extension of groups given by

$$1 \to \boldsymbol{K}_2(F) \to \boldsymbol{T}'(F) \to \boldsymbol{T}(F) \to 1.$$

Since F is assumed to have enough n-th roots of unity, the Hilbert symbol allows us to push forward this extension to get

$$1 \to \mu_n \to \tilde{T} \to T \to 1,$$

where $\mu_n = \boldsymbol{\mu}_n(F)$ and $T = \boldsymbol{T}(F)$. By results of [Brylinski and Deligne 2001, Sections 10.2 and 10.3], which followed [Moore 1964], this is a topological central extension of the LCA group T by the LCA group μ_n.

In this case, the center $Z(\tilde{T})$ has finite index in $\tilde{T}$. Furthermore, Theorem 3.1 implies this:

Proposition 3.2. *There is a natural bijection between the set $\mathcal{I}_\epsilon(\tilde{T})$ of irreducible genuine representations of $\tilde{T}$ and the set $\mathcal{X}_\epsilon(Z(\tilde{T}))$ of genuine characters of $Z(\tilde{T})$.*

There is a short exact sequence $1 \to \mu_n \to Z(\tilde{T}) \to Z^\dagger(T) \to 1$ of LCA groups. Proposition 2.1 then implies:

Proposition 3.3. *The space $\mathcal{X}_\epsilon(Z(\tilde{T}))$ of genuine continuous characters of $Z(\tilde{T})$ is a $\mathcal{X}(Z^\dagger(T))$-torsor.*

Corollary 3.4. *The set $\mathcal{I}_\epsilon(\tilde{T})$ is a $\mathcal{X}(Z^\dagger(T))$-torsor.*

In particular, we give $\mathcal{I}_\epsilon(\tilde{T})$ the structure of a complex algebraic variety so that it is a complex algebraic $\mathcal{X}(Z^\dagger(T))$-torsor.

Since $Z^\dagger(T)$ is a finite index subgroup of T, restriction of continuous characters yields a surjective homomorphism $res\colon \mathcal{X}(T) \to \mathcal{X}(Z^\dagger(T))$ of complex algebraic groups. As a result, the set $\mathcal{I}_\epsilon(\tilde{T})$ is a homogeneous space for $\mathcal{X}(T)$, or equivalently (by Langlands's theorem [1997]), a homogeneous space for $H^1_c(\mathcal{W}_{L/F}, \hat{\mathcal{T}})$.

4. Split tori

In this section, we keep the assumptions of the previous section. In addition, we assume that $\boldsymbol{T}$ is a split torus of rank r over F. Thus, there is a canonical identification $\boldsymbol{T}(F) \equiv Y \otimes_{\mathbb{Z}} F^\times$. We are interested in parameterizing $\mathcal{I}_\epsilon(\tilde{T})$. By the results of the previous section, we may describe this set, up to a choice of base point, by describing the set $Z^\dagger(T)$.

An isogeny. Recall that $B\colon Y \otimes_{\mathbb{Z}} Y \to \mathbb{Z}$ is the symmetric bilinear form associated to Q. It allows us to construct a subgroup of finite index $Y^\# \subset Y$ by setting

$$Y^\# = \{y \in Y \text{ such that } B(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y\}.$$

Note that we suppress mention of Q, B, and n in our notation $Y^\#$.

The subgroup $Y^\#$ can be related to the "Smith normal form" of the bilinear form B. Namely, there exists a pair of group isomorphisms α and β with diagram

$$\mathbb{Z}^r \xleftarrow{\beta} Y \xrightarrow{\alpha} \mathbb{Z}^r$$

such that one has $B_Q(y_1, y_2) = D(\alpha(y_1), \beta(y_2))$ for all $y_1, y_2 \in Y$, and D is a symmetric bilinear form on $\mathbb{Z}^r$ represented by a diagonal matrix with entries

$(d_1, \dots, d_r)$ (the elementary divisors). Let e_j denote the smallest positive integer such that $d_j e_j \in n\mathbb{Z}$ for every $1 \le j \le r$. Then we find that

$$Y^\# = \alpha^{-1}(e_1\mathbb{Z} \oplus e_2\mathbb{Z} \oplus \cdots \oplus e_r\mathbb{Z}).$$

Let $\iota\colon Y^\# \to Y$ denote the inclusion of $\mathbb{Z}$-modules. Since $Y^\#$ has finite index in Y, this corresponds to an F-isogeny $\boldsymbol{\iota}\colon \boldsymbol{T}^\# \to \boldsymbol{T}$ of split tori, where $\boldsymbol{T}^\#$ is the split algebraic torus with cocharacter lattice $Y^\#$. From the previous observations, we find that

$$\iota(T^\#) = \iota(\boldsymbol{T}^\#(F)) = \alpha^{-1}(F^{\times e_1} \times \cdots \times F^{\times e_r}).$$

Describing the center. Recall that extension $\tilde{T}$ of T by μ_n yields a commutator $C\colon \bigwedge^2 T \to \mu_n$. This commutator can be directly related to the bilinear form B; see [Brylinski and Deligne 2001]. If $u_1, u_2 \in F^\times$ and $y_1, y_2 \in Y$, then one may directly compute

$$C(y_1(u_1), y_2(u_2)) = (u_1, u_2)_n^{B(y_1, y_2)}.$$

The diagonalization of B via group isomorphisms α, β yields two isomorphisms of F-tori, given by

$$\boldsymbol{G}_m^r \xleftarrow{\ \boldsymbol{\beta}\ } \boldsymbol{T} \xrightarrow{\ \boldsymbol{\alpha}\ } \boldsymbol{G}_m^r \ .$$

One arrives at a bilinear form on $(F^\times)^r$, given by

$$\Delta(\vec{z}_1, \vec{z}_2) = \prod_{j=1}^r (z_1^{(j)}, z_2^{(j)})_n^{d_j}.$$

This is related to the commutator C by $C(t_1, t_2) = \Delta(\alpha(t_1), \beta(t_2))$.

We can now characterize $Z^\dagger(T)$:

Proposition 4.1. *The subgroup $Z^\dagger(T)$ of T is equal to the image of the isogeny ι on the F-rational points, that is, $Z^\dagger(T) = \iota(T^\#)$.*

Proof. We find that

$$\begin{aligned} t_1 \in Z^\dagger(T) \quad &\text{if and only if} & C(t_1, t_2) &= 1 \text{ for all } t_2 \in T \\ &\text{if and only if} & \Delta(\alpha(t_1), \beta(t_2)) &= 1 \text{ for all } t_2 \in T \\ &\text{if and only if} & \Delta(\alpha(t_1), \vec{z}_2) &= 1 \text{ for all } \vec{z}_2 \in (F^\times)^r \\ &\text{if and only if} & \alpha(t_1) &\in (F^{\times e_1} \times \cdots \times F^{\times e_r}) \\ &\text{if and only if} & t_1 &\in \iota(T^\#). \end{aligned}$$

The penultimate step follows from the nondegeneracy of the Hilbert symbol. □

We have now proved this:

Theorem 4.2. *If F is a local field, then $\mathcal{I}_\epsilon(\tilde{T})$ is a torsor for $\mathcal{X}(\iota(T^\#))$.*

Character groups. The previous theorem motivates the further analysis of the group $\mathscr{X}(\imath(T^\#))$. We write $\imath^*$ for the pullback homomorphism $\imath^*\colon \mathscr{X}(T) \to \mathscr{X}(T^\#)$.

Proposition 4.3. *There is a natural identification* $\mathscr{X}(\imath(T^\#)) \equiv \mathrm{Im}(\imath^*)$.

Proof. There are short exact sequences

$$1 \to \ker(\imath) \to T^\# \to \imath(T^\#) \to 1 \quad \text{and} \quad 1 \to \imath(T^\#) \to T \to \mathrm{cok}(\imath) \to 1$$

of LCA groups. Using Propositions 2.1 and 2.2, we arrive at short exact sequences

$$\begin{gathered} 1 \to \mathscr{X}(\imath(T^\#)) \to \mathscr{X}(T^\#) \to \mathscr{X}(\ker(\imath)) \to 1, \\ 1 \to \mathscr{X}(\mathrm{cok}(\imath)) \to \mathscr{X}(T) \to \mathscr{X}(\imath(T^\#)) \to 1 \end{gathered}$$

of character groups. Since $\mathscr{X}(T)$ surjects onto $\mathscr{X}(\imath(T^\#))$, we find that the image of $\imath^*\colon \mathscr{X}(T) \to \mathscr{X}(T^\#)$ equals the image of the injective map $\mathscr{X}(\imath(T^\#)) \to \mathscr{X}(T^\#)$. $\square$

The dual complex. The isogeny of split F-tori $\boldsymbol{\imath}\colon \boldsymbol{T}^\# \to \boldsymbol{T}$ yields an isogeny $\hat{\imath}\colon \hat{\mathscr{T}} \to \hat{\mathscr{T}}^\#$ of the complex dual tori. One may pull back continuous characters via $\imath^*\colon \mathscr{X}(T) \to \mathscr{X}(T^\#)$.

The following result follows from local class field theory, and demonstrates the naturality of Langlands's classification [1997].

Proposition 4.4. *There is a commutative diagram*

$$\begin{array}{ccc} \mathscr{X}(T) & \longrightarrow & H^1_c(\mathscr{W}_F, \hat{\mathscr{T}}) \\ \Big\downarrow{\scriptstyle \imath^*} & & \Big\downarrow{\scriptstyle \hat{\imath}} \\ \mathscr{X}(T^\#) & \longrightarrow & H^1_c(\mathscr{W}_F, \hat{\mathscr{T}}^\#) \end{array}$$

of complex algebraic groups, whose rows are the reciprocity isomorphisms of local class field theory.

Note that since $\boldsymbol{T}$ and $\boldsymbol{T}^\#$ are split tori, the continuous cohomology groups are simply given by $H^1_c(\mathscr{W}_F, \hat{\mathscr{T}}) = \mathrm{Hom}_c(\mathscr{W}_F, \hat{\mathscr{T}})$.

Corollary 4.5. *There is a natural identification*

$$\mathscr{X}(Z^\dagger(T)) \equiv \mathrm{Im}(\hat{\imath}\colon H^1_c(\mathscr{W}_F, \hat{\mathscr{T}}) \to H^1_c(\mathscr{W}_F, \hat{\mathscr{T}}^\#)).$$

Parameterization by hypercohomology. We may now parameterize representations using the hypercohomology of the complex of tori $\hat{\mathscr{T}} \xrightarrow{\hat{\imath}} \hat{\mathscr{T}}^\#$ concentrated in degrees zero and one. We follow the treatment in the appendices of [Kottwitz and Shelstad 1999] when discussing continuous hypercohomology of Weil groups with coefficients in complexes of tori. In particular, we concentrate the complexes in degrees 0 and 1 following [Kottwitz and Shelstad 1999], and not in degrees -1 and 0 as in [Borovoi 1998].

There is a long exact sequence in cohomology that includes

$$H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#) \xrightarrow{\eta} H_c^1(\mathcal{W}_F, \hat{\mathcal{T}}) \longrightarrow H_c^1(\mathcal{W}_F, \hat{\mathcal{T}}^\#).$$

Lemma 4.6. *The homomorphism η is injective.*

Proof. Extending the long exact sequence above, it suffices to prove the surjectivity of the preceding homomorphism $H_c^0(\mathcal{W}_F, \hat{\mathcal{T}}) \to H_c^0(\mathcal{W}_F, \hat{\mathcal{T}}^\#)$. But $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}^\#$ are complex tori, trivial as $\mathcal{W}_F$-modules, and $\hat{\imath}$ is an isogeny. Therefore, the map above is surjective. □

From this lemma, we identify the hypercohomology group $H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#)$ with a subgroup of $H_c^1(\mathcal{W}_F, \hat{\mathcal{T}})$.

Lemma 4.7. *The group $H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#)$ is finite.*

Proof. Since $\hat{\imath}$ is an isogeny, it has finite kernel and cokernel. The lemma follows because there is a long exact sequence that includes the terms

$$H_c^1(\mathcal{W}_F, \ker(\hat{\imath})) \to H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#) \to H_c^0(\mathcal{W}_F, \operatorname{cok}(\hat{\imath})). \qquad □$$

This leads to the first main theorem:

Theorem 4.8. *There exists an isomorphism*

$$\mathcal{I}_\epsilon(\tilde{T}) \cong H_c^1(\mathcal{W}_F, \hat{\mathcal{T}}) \big/ H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#)$$

in the category of varieties over $\mathbb{C}$ endowed with an action of $H_c^1(\mathcal{W}_F, \hat{\mathcal{T}})$.

Remark 4.9. The global analogue of this result also seems to hold. Let $\mathcal{I}_\epsilon^{\mathrm{aut}}(\tilde{T}_{\mathbb{A}})$ denote the appropriate set of genuine automorphic representations of $\tilde{T}_{\mathbb{A}}$; it seems likely that

$$\mathcal{I}_\epsilon^{\mathrm{aut}}(\tilde{T}_{\mathbb{A}}) \cong H_c^1(\mathcal{W}_F, \hat{\mathcal{T}}) \big/ H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#)$$

when $\boldsymbol{T}$ is a split torus over a global field F. The proof follows the same techniques (together with the Hasse principle for isogenies of split tori), but requires some analytic care with extension of continuous characters and the appropriate Stone–von Neumann theorem. We hope to treat this global theorem in a future paper.

5. Unramified tori

For split tori, the cohomology groups that arise in Theorem 4.8 are quite simple, since $\mathcal{W}_F$ acts trivially on $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}^\#$. In fact, the statement of this theorem makes sense even when $\boldsymbol{T}$ is a nonsplit torus. However, for general nonsplit tori, it seems that our explicit methods are insufficient to prove such a result. For "tame covers" of unramified tori over local nonarchimedean fields, such a paramaterization is possible.

In this section, we fix these notations:

- $\boldsymbol{T}$ will be a local nonarchimedean field F, which splits over a finite *unramified* Galois extension L/F, with $\Gamma = \mathrm{Gal}(L/F)$. X and Y will be the resulting character and cocharacter groups.
- We define $d = [L : F]$ and write $\mathbf{l}$ for the residue field of $\mathbb{O}_L$ (note that $\mathbf{l}$ has cardinality q^d).
- We fix a uniformizer ϖ of $F^\times$ (and hence of $L^\times$ as well).
- We let γ be the generator of Γ that acts upon $\mathbf{l}$ via $\gamma(x) = x^q$. We let $r = (q^d - 1)/(q - 1) = \#(\mathbf{l}^\times / \mathbf{f}^\times)$.
- $\boldsymbol{T}'$ will be an extension of $\boldsymbol{T}$ by $\boldsymbol{K}_2$ in $\mathfrak{Sp}_F$.
- $(Q, \tilde{Y})$ will be the Deligne–Brylinski invariants of $\boldsymbol{T}'$. B will be the symmetric bilinear form associated to Q.
- n will be a positive integer such that F has enough n-th roots of unity. We also assume that $(p, n) = 1$.
- $\epsilon \colon \boldsymbol{\mu}_n(F) \to \mathbb{C}^\times$ will be a fixed injective character.
- If W is a subgroup of Y, then we write W^Γ for the subgroup of Γ-fixed elements of W. We also define

$$W^\# = \{y \in Y \text{ such that } B(y, w) \in n\mathbb{Z} \text{ for all } w \in W\}.$$

$\mathbb{Z}[\Gamma]$-modules. Γ is a cyclic group generated by γ and of order d. Let $\mathbb{Z}[\Gamma]$ denote the integral group ring of Γ. We define the following elements of $\mathbb{Z}[\Gamma]$:

- Let $\mathrm{Tr} = \sum_{i=0}^{d-1} \gamma^i$, and let $\mathrm{Tr}_q = \sum_{i=0}^{d-1} q^i \gamma^i$.
- Let $\delta = \gamma - 1$, and let $\delta_q = q\gamma - 1$.

Note that $\mathrm{Tr} \circ \delta = 0$ and $\mathrm{Tr}_q \circ \delta_q = q^d - 1$. When M is an $\mathbb{Z}[\Gamma]$-module, we let $\overline{M} = M/(q^d - 1)M$. We write M^Γ for the Γ-invariant $\mathbb{Z}$-submodule of M. Therefore,

$$M^\Gamma = \{m \in M \text{ such that } \delta m = 0\}.$$

We define

$$\overline{M}^{\Gamma,q} = \{\overline{m} \in \overline{M} \text{ such that } \delta_q \overline{m} = 0\}.$$

Proposition 5.1. *Suppose that M is an $\mathbb{Z}[\Gamma]$-module. Then*

$$\mathrm{Tr}(M) \subset M^\Gamma \quad \textit{and} \quad \mathrm{Tr}_q(\overline{M}) \subset \overline{M}^{\Gamma,q}.$$

Proof. The first inclusion is obvious. For the second inclusion, suppose that $\overline{m} \in \overline{M}$. Then $\delta_q \mathrm{Tr}_q \overline{m} = \mathrm{Tr}_q \delta_q \overline{m} = (q^d - 1)\overline{m} = 0$. □

Proposition 5.2. *Suppose that M is a $\mathbb{Z}[\Gamma]$-module that is free as an $\mathbb{Z}$-module. Then δ_q and Tr_q act as injective endomorphisms of M, and*

$$\mathrm{Im}(\delta_q) = \{m \in M \textit{ such that } \mathrm{Tr}_q m \in (q^d - 1)M\}.$$

Proof. Since M is free as an $\mathbb{Z}$-module, $\delta_q \circ \mathrm{Tr}_q = \mathrm{Tr}_q \circ \delta_q = q^d - 1$ acts as an injective endomorphism of M. Hence δ_q and Tr_q must also act as injective endomorphisms of M, proving the first assertion.

Since $\mathrm{Tr}_q \circ \delta_q = q^d - 1$, it follows that

$$\mathrm{Im}(\delta_q) \subset \{m \in M \text{ such that } \mathrm{Tr}_q m \in (q^d - 1)M\}.$$

In the other direction, if $\mathrm{Tr}_q m \in (q^d - 1)M$, then $\mathrm{Tr}_q m = \mathrm{Tr}_q \delta_q m'$, for some $m' \in M$. Since Tr_q acts via an injective endomorphism, it follows that $m = \delta_q m'$. □

Unramified tori. Much of our treatment of *unramified tori* is inspired by Ono [1961, Section 2]. Recall that X and Y are naturally $\mathbb{Z}[\Gamma]$-modules, and the pairing is Γ-invariant.

We fix a smooth model $\underline{\boldsymbol{T}}$ of $\boldsymbol{T}$ over $\mathcal{O}_F$. We make the identifications

$$T_L = \boldsymbol{T}(L) \equiv Y \otimes_{\mathbb{Z}} L^{\times} \quad \text{and} \quad T_F = \boldsymbol{T}(F) \equiv (Y \otimes_{\mathbb{Z}} L^{\times})^{\Gamma}.$$

Similarly, for the integral points, we identify

$$T_L^{\circ} = \underline{\boldsymbol{T}}(\mathcal{O}_L) \equiv Y \otimes_{\mathbb{Z}} \mathcal{O}_L^{\times} \quad \text{and} \quad T_F^{\circ} = \underline{\boldsymbol{T}}(\mathcal{O}_F) \equiv (Y \otimes_{\mathbb{Z}} \mathcal{O}_L^{\times})^{\Gamma}.$$

We write $\overline{\boldsymbol{T}}$ for the special fibre of $\underline{\boldsymbol{T}}$. Then, we also identify

$$\overline{T}_{\mathbf{l}} = \overline{\boldsymbol{T}}(\mathbf{l}) \equiv Y \otimes_{\mathbb{Z}} \mathbf{l}^{\times} \quad \text{and} \quad \overline{T}_{\mathbf{f}} = \overline{\boldsymbol{T}}(\mathbf{f}) \equiv (Y \otimes_{\mathbb{Z}} \mathbf{l}^{\times})^{\Gamma}.$$

There are natural reduction homomorphisms $T_L^{\circ} \to \overline{T}_{\mathbf{l}}$ and $T_F^{\circ} \to \overline{T}_{\mathbf{f}}$. Let T_L^1 and T_F^1 denote the kernels of these reduction maps. The reduction morphisms are split by the Teichmüller lift, and we arrive at a decomposition $T_L^{\circ} \equiv T_L^1 \times \overline{T}_{\mathbf{l}}$ of $\mathbb{Z}[\Gamma]$-modules. Together with the valuation map, we arrive at a short exact sequence $1 \to T_L^1 \times \overline{T}_{\mathbf{l}} \to T_L \to Y \to 1$ of $\mathbb{Z}[\Gamma]$-modules. The choice of (Γ-invariant) uniformizing element ϖ splits this exact sequence, leading to a decomposition of $\mathbb{Z}[\Gamma]$-modules given by $T_L \equiv Y \times \overline{T}_{\mathbf{l}} \times T_L^1$.

We use this decomposition to "get our hands on" elements of T_L. First, every element of T_L can be expressed as $y(\varpi)t^{\circ}$ for uniquely determined $y \in Y$ and $t^{\circ} \in T_L^{\circ}$. Let $\theta_{\mathbf{l}}$ denote a generator of the cyclic group $\mathbf{l}^{\times}$, and let $\theta_{\mathbf{f}} = \theta_{\mathbf{l}}^r$. Thus $\theta_{\mathbf{f}}$ is a generator of the cyclic group $\mathbf{f}^{\times}$. Let $\vartheta_L \in \mathcal{O}_L^{\times}$ and $\vartheta_F \in \mathcal{O}_F^{\times}$ denote the Teichmüller lifts of $\theta_{\mathbf{l}}$ and $\theta_{\mathbf{f}}$, respectively.

Let $\zeta_L = (\varpi, \vartheta_L)_{L,q^d-1}$. Let $\zeta_F = \zeta_L^r$. Note that ζ_L is a primitive $(q^d - 1)$-st root of unity, and ζ_F is a primitive $(q-1)$-st root of unity.

Recall that $\overline{Y} = Y/(q^d - 1)Y$; thus, for $\bar{y} \in \overline{Y}$, it makes sense to write $\bar{y}(\vartheta_L)$ as an element of T_L°. According to the decomposition $T_L \equiv Y \times \overline{T}_{\mathbf{l}} \times T_L^1$, every element $t \in T_L$ has a unique expression as $t = y_1(\varpi)\bar{y}_2(\vartheta_L)t^1$, where $y_1 \in Y$, $\bar{y}_2 \in \overline{Y}$, and $t^1 \in T_L^1$. To determine when such an expression lies in T_F, we have the following characterization:

Proposition 5.3. *An element $y_1(\varpi)\bar{y}_2(\vartheta_L)t^1$ of T_L, with y_1, $\bar{y}_2$, and t^1 as above, lies in T_F if and only if*

- $y_1 \in Y^\Gamma$, *that is*, $\delta(y_1) = 0$;
- $\bar{y}_2 \in \bar{Y}^{\Gamma,q}$, *that is*, $\delta_q(\bar{y}_2) = 0$; *and*
- $t^1 \in T_F^1$.

Proof. By the Γ-invariance of the decomposition $T_L \equiv Y \times \bar{T}_{\mathbf{l}} \times T_L^1$, we find that $y_1(\varpi)\bar{y}_2(\vartheta_L)t^1 \in T_F$ if and only if the three factors are fixed by Γ. The proposition follows from three observations:

- Since $\varpi \in F$, we have $y_1(\varpi) \in T_L^\Gamma$ if and only if $y_1 \in Y^\Gamma$.
- Since $\gamma(\vartheta_L) = \vartheta_L^q$, we find that $\bar{y}_2(\vartheta_L) \in T_L^\Gamma$ if and only if $\bar{y}_2 = q\gamma(\bar{y}_2)$ in $\bar{Y}$.
- Since the reduction map intertwines the action of Γ, we have $t^1 \in (T_L^1)^\Gamma$ if and only if $t^1 \in T_F^1$. □

Tame metaplectic unramified tori. The structure of $\boldsymbol{T}'(L)$ and $\boldsymbol{T}'(F)$ is based on [Brylinski and Deligne 2001, Sections 12.8–12.12]. In particular, if we let $T_L' = \boldsymbol{T}'(L)$ and $T_F' = \boldsymbol{T}'(F)$, there is a natural commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \boldsymbol{K}_2(F) \arrow[r] \arrow[d] & T_F' \arrow[r] \arrow[d] & T_F \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \boldsymbol{K}_2(L) \arrow[r] & T_L' \arrow[r] & T_L \arrow[r] & 1. \end{tikzcd}$$

There is a natural action of Γ on the bottom row such that $\boldsymbol{K}_2(F)$ maps to $\boldsymbol{K}_2(L)^\Gamma$, $T_F = T_L^\Gamma$, and T_F' maps to $(T_L')^\Gamma$. The tame symbols yield a commutative diagram

$$\begin{tikzcd} \boldsymbol{K}_2(F) \arrow[r, "t"] \arrow[d] & \mathbf{f}^\times \arrow[d] \\ \boldsymbol{K}_2(L) \arrow[r, "t"] & \mathbf{l}^\times, \end{tikzcd}$$

where the downward arrows arise from the functoriality of $\boldsymbol{K}_2$ and $\boldsymbol{K}_1$. The bottom row is a morphism of $\mathbb{Z}[\Gamma]$-modules. Pushing forward T_F' and T_L' via the tame symbols yields a commutative diagram of locally compact groups, with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{f}^\times \arrow[r] \arrow[d] & \tilde{T}_F^t \arrow[r] \arrow[d] & T_F \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbf{l}^\times \arrow[r] & \tilde{T}_L^t \arrow[r] & T_L \arrow[r] & 1. \end{tikzcd} \tag{5-1}$$

The downward arrows arise from the inclusion of F in L, and of $\mathbf{f}$ in $\mathbf{l}$. Deligne and Brylinski, in [2001, Section 12.8], note the following:

Proposition 5.4. *In the commutative diagram* (5-1), *the groups in the top row are precisely the* Γ*-invariant subgroups of the bottom row. In other words,* $\mathbf{f}^\times = (\mathbf{l}^\times)^\Gamma$, $T_F = T_L^\Gamma$, *and* $\tilde{T}_F^t = (\tilde{T}_L^t)^\Gamma$.

We may push forward the covers further to obtain all *tame covers*. Recall that $(p, n) = 1$ and F has enough n-th roots of unity. Then, we find n divides $(q-1)$, and one obtains a natural surjective map

$$\psi_F : \mathbf{f}^\times \to \boldsymbol{\mu}_n(F),$$

by first applying the Teichmüller map (from $\mathbf{f}^\times$ to $\boldsymbol{\mu}_{q-1}(F)$), and then raising to the power $m = (q-1)/n$. Recall that $r = (q^d - 1)/(q-1)$. One gets a similar map

$$\psi_L : \mathbf{l}^\times \to \boldsymbol{\mu}_{nr}(L)$$

by applying the Teichmüller map (from $\mathbf{l}^\times$ to $\boldsymbol{\mu}_{q^d-1}(L)$) and then raising to the power $m = (q-1)/n$. The compatibility of these maps yields a new commutative diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & \boldsymbol{\mu}_n(F) \arrow[r] \arrow[d] & \tilde{T}_F \arrow[r] \arrow[d] & T_F \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \boldsymbol{\mu}_{nr}(L) \arrow[r] & \tilde{T}_L \arrow[r] & T_L \arrow[r] & 1. \end{tikzcd}$$

With this construction, we say that $\tilde{T}_F$ is a *tame* metaplectic cover of T_F, and $\tilde{T}_L$ is a tame metaplectic cover of T_L as well. $\tilde{T}_F$ is identified as a subgroup of $\tilde{T}_L$.

Note that the commutator map for $\tilde{T}_L$ satisfies

$$C_L(y_1(u), y_2(v)) = (u, v)_{L,nr}^{B(y_1,y_2)} = (u, v)_{L,q^d-1}^{mB(y_1,y_2)},$$

where $(\cdot,\cdot)_{L,nr}$ and $(\cdot,\cdot)_{L,q^d-1}$ denote the appropriate Hilbert symbols (in this case, norm residue symbols) on $L^\times$. The commutator on T_F is simply the restriction of C_L. As a result, $Z^\dagger(T_F) \supset Z^\dagger(T_L) \cap T_F$, where the preimage of $Z^\dagger(T_F)$ is the center of $\tilde{T}_F$ and the preimage of $Z^\dagger(T_L)$ is the center of $\tilde{T}_L$.

Computation of the center. We now recall that the set $\mathcal{I}_\epsilon(\tilde{T}_F)$ is a torsor for $\mathcal{X}(Z^\dagger(T_F))$. Therefore we wish to study the group $Z^\dagger(T_F)$ in more detail. To this end, we first observe this:

Proposition 5.5. *The group* T_L^1 *is contained in* $Z^\dagger(T_L)$. *Similarly,* T_F^1 *is contained in* $Z^\dagger(T_F)$.

Proof. Since $(q^d - 1, p) = 1$, the Hilbert symbol (in this case, a norm-residue symbol) is trivial when one of its "inputs" is contained in $\mathcal{O}_L^1$. Hence the commutator $C_L(\cdot,\cdot)$ is trivial when one of its inputs is contained in T_L^1. Hence $T_L^1 \subset Z^\dagger(T_L)$. Since $Z^\dagger(T_F) \supset Z^\dagger(T_L) \cap T_F$, we find that $T_F^1 \subset Z^\dagger(T_F)$ as well. $\square$

Since T_F^1 is contained in $Z^\dagger(T_F)$, $Z^\dagger(T_F)$ corresponds to a subgroup of T_F/T_F^1. Our choice of uniformizing element, together with the previously mentioned splittings, yields a decomposition $T_L/T_L^1 \equiv Y \times \overline{T}_{\mathbf{1}}$ of $\mathbb{Z}[\Gamma]$-modules. Namely, every element t of T_L/T_L^1 can be represented by $y_1(\varpi)\bar{y}_2(\vartheta_L)$, for uniquely determined $y_1 \in Y$ and $\bar{y}_2 \in \overline{Y}$.

In order to describe $Z^\dagger(T_F)$, we work with a number of subgroups of Y. Recall that $Y^{\Gamma\#} = \{y \in Y \text{ such that } B(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y^\Gamma\}$. Note that $Y^{\Gamma\#} \supset Y^\#$. Also, it is important to distinguish between $Y^{\Gamma\#}$ and $Y^{\#\Gamma} = (Y^\#)^\Gamma$.

Lemma 5.6. *There are inclusions of* $\mathbb{Z}[\Gamma]$*-modules, of finite index in* Y*, given by*

$$Y \supset Y^{\Gamma\#} \supset Y^\# \supset (q^d - 1)Y.$$

Furthermore, $\delta_q(Y) \subset Y^{\Gamma\#}$*, and* $\mathrm{Tr}_q(Y^{\Gamma\#}) \subset Y^\#$.

Proof. The inclusions are clear, since n divides $q^d - 1$. If $y \in Y$ and $y' \in Y^\Gamma$, then we find

$$\begin{aligned} B(\delta_q y, y') &= B(q\gamma y - y, y') \\ &= qB(y, \gamma^{-1} y') - B(y, y') \\ &= (q-1)B(y, y') \in n\mathbb{Z} \quad (\text{since } q - 1 = mn). \end{aligned}$$

Hence $\delta_q(Y) \subset Y^{\Gamma\#}$.

Now, suppose that $w \in Y^{\Gamma\#}$ and $y' \in Y$. Then, we find

$$\begin{aligned} B(\mathrm{Tr}_q(w), y') &= \sum_{i=0}^{d-1} B(q^i \gamma^i w, y') \\ &\equiv \sum_{i=0}^{d-1} B(\gamma^i w, y') \pmod{n} \quad (\text{since } q - 1 = mn) \\ &\equiv B(w, \mathrm{Tr}(y')) \in n\mathbb{Z} \qquad (\text{since } \mathrm{Tr}(y') \in Y^\Gamma). \end{aligned}$$

Hence $\mathrm{Tr}_q(Y^{\Gamma\#}) \in Y^\#$. □

Now, we fully describe $Z^\dagger(T_F)$ with two results:

Theorem 5.7. *Let* $y_1 \in Y$ *and* $\bar{y}_2 \in \overline{Y}$*. Then if the element* $t = y_1(\varpi)\bar{y}_2(\vartheta_L)$ *is contained in* $Z^\dagger(T_F)$*, then for every lift* $y_2 \in Y$ *of* $\bar{y}_2$*,*

$$y_1, y_2 \in Y^\# \quad \textit{and} \quad \delta_q y_2 \in (q^d - 1)Y^{\Gamma\#}.$$

Proof. For reference during this proof, we recall that

$$nm = q - 1, \qquad r = 1 + q + \cdots + q^{d-1}, \qquad nmr = q^d - 1.$$

Suppose furthermore that $y_1', y_2' \in Y$, and let $\bar{y}_2' \in \overline{Y}$ be the reduction of y_2'. Then we find that $[\mathrm{Tr}(y_1')](\varpi)$ and $[\mathrm{Tr}_q(\bar{y}_2')](\vartheta_L)$ are elements of T_F. It follows that

$$C_L(y_1(\varpi)\bar{y}_2(\vartheta_L), [\mathrm{Tr}(y_1')](\varpi)) = 1 \quad \text{and} \quad C_L(y_1(\varpi)\bar{y}_2(\vartheta_L), [\mathrm{Tr}_q(\bar{y}_2')](\vartheta_L)) = 1.$$

The explicit formula for the commutator C_L yields

$$\begin{aligned}
1 &= C_L\left(y_1(\varpi)y_2(\vartheta_L), [\mathrm{Tr}_q(\bar{y}_2')](\vartheta_L)\right) \\
&= \prod_{i=0}^{d-1} (\varpi, \vartheta_L)_{L,q^d-1}^{mq^i B(y_1, \gamma^i y_2')} \\
&= \zeta_L^{\sum_{i=0}^{d-1} mq^i B(y_1, \gamma^i y_2')} && \text{(since } (\varpi, \vartheta_L)_{L,q^d-1} = \zeta_L) \\
&= \zeta_L^{\sum_{i=0}^{d-1} mq^i B(\gamma^{d-i} y_1, y_2')} && \text{(by the } \Gamma\text{-invariance of } B) \\
&= \zeta_L^{\sum_{i=0}^{d-1} mq^i B(y_1, y_2')} && \text{(by the } \Gamma\text{-invariance of } y_1) \\
&= \zeta_L^{mrB(y_1, y_2')} && \text{(by summing a partial geometric series)} \\
&= \zeta_F^{mB(y_1, y_2')} && \text{(since } \zeta_F = \zeta_L^r).
\end{aligned}$$

Since $1 = \zeta_F^{mB(y_1, y_2')}$ for all $y_2' \in Y$, we find that $y_1 \in Y^\#$.

Carrying out a similar analysis, an explicit computation yields

$$1 = C_L\left(y_1(\varpi)y_2(\vartheta_L), [\mathrm{Tr}(y_1')](\varpi)\right) = \prod_{i=0}^{d-1} (\varpi, \varpi)_{L,q^d-1}^{mB(y_1, \gamma^i y_1')} (\varpi, \vartheta_L)_{L,q^d-1}^{mB(y_2, \gamma^i y_1')}.$$

Now if p is odd, we find that $q-1$ is even. Since $(\varpi, \varpi)_{L,q^d-1} = \pm 1$, and $mB(y_1, \gamma^i y_1') \in mn\mathbb{Z} = (q-1)\mathbb{Z} \subset 2\mathbb{Z}$ (since $y_1 \in Y^\#$), we find that

$$(\varpi, \varpi)_{L,q^d-1}^{mB(y_1, \gamma^i y_1')} = 1.$$

On the other hand, if $p = 2$, then $(\varpi, \varpi)_{L,q^d-1} = 1$, and once again the equality above holds. Continuing our computations yields

$$\begin{aligned}
1 &= \prod_{i=0}^{d-1} (\varpi, \varpi)_{L,q^d-1}^{mB(y_1, \gamma^i y_1')} (\varpi, \vartheta_L)_{L,q^d-1}^{mB(y_2, \gamma^i y_1')} \\
&= \prod_{i=0}^{d-1} \zeta_L^{mB(y_2, \gamma^i y_1')} && \text{(since } (\varpi, \varpi)_{L,q^d-1}^{mB(y_2, \gamma^i y_1')} = 1 \text{ and } (\varpi, \vartheta_L)_{L,q^d-1} = \zeta_L\text{)} \\
&= \zeta_L^{\sum_{i=0}^{d-1} mq^i B(y_2, y_1')} && \text{(since } q\gamma(\bar{y}_2) = \bar{y}_2) \\
&= \zeta_L^{mrB(y_2, y_1')} && \text{(by summing a partial geometric series)} \\
&= \zeta_F^{mB(y_2, y_1')} && \text{(since } \zeta_F = \zeta_L^r).
\end{aligned}$$

Hence, we find that $y_2 \in Y^{\#}$.

Finally, we prove that $\delta_q y_2 \in (q^d-1)Y^{\Gamma\#}$. Note that $\delta_q y_2 \in (q^d-1)Y$ because $\bar{y}_2 \in \overline{Y}^{\Gamma,q}$. Thus, $\delta_q y_2 = (q^d-1)y_3$ for some $y_3 \in Y$. It suffices to prove that $y_3 \in Y^{\Gamma\#}$.

Now, to prove that $y_3 \in Y^{\Gamma\#}$, suppose that $y' \in Y^{\Gamma}$. It follows that

$$\begin{aligned} 1 &= C_L\left(y_1(\varpi)\bar{y}_2(\vartheta_L), y'(\varpi)\right) \\ &= (\varpi,\varpi)_{q^d-1}^{mB(y_1,y')}(\vartheta_L,\varpi)_{q^d-1}^{mB(y_2,y')} = \zeta_L^{mB(y_2,y')}. \end{aligned}$$

Hence $B(y_2,y') \in nr\mathbb{Z}$. It follows that

$$\begin{aligned} B(y_3,y') &= (q^d-1)^{-1}B(\delta_q y_2, y') \\ &= (q^d-1)^{-1}(B(q\gamma y_2,y') - B(y_2,y')) = r^{-1}B(y_2,y') \in n\mathbb{Z}. \end{aligned}$$

Thus $y_3 \in Y^{\Gamma\#}$. □

Theorem 5.8. *Suppose that* $y_1, y_2 \in Y^{\#}$. *Also suppose that* $y_1 \in Y^{\Gamma}$ *and* $\bar{y}_2 \in \overline{Y}^{\Gamma,q}$. *Finally, suppose that* $\delta_q y_2 \in (q^d-1)Y^{\Gamma\#}$. *Then* $y_1(\varpi)\bar{y}_2(\vartheta_L) \in Z^{\dagger}(T_F)$.

Proof. Since $y_1 \in Y^{\Gamma}$ and $\bar{y}_2 \in \overline{Y}^{\Gamma,q}$, it follows that $y_1(\varpi)\bar{y}_2(\vartheta_L) \in T_F$. Now we may compute some commutators.

Suppose that $y_1' \in Y^{\Gamma}$, $y_2' \in Y$, and $\bar{y}_2' \in \overline{Y}^{\Gamma,q}$. Thus $y_1'(\varpi)$ and $\bar{y}_2'(\vartheta_L)$ are elements of T_F. We begin by computing

$$C_L(y_1(\varpi), y_1'(\varpi)) = (\varpi,\varpi)_{L,q^d-1}^{mB(y_1,y_1')}.$$

If p is odd, then $mn = q-1$ is even, and thus $mB(y_1,y_1')$ is even. Hence the commutator is trivial. If p is even, then q^d-1 is odd, and hence $(\varpi,\varpi)_{L,q^d-1} = 1$. In either case, the commutator is trivial.

Now, consider the commutator $C_L(y_1(\varpi), \bar{y}_2'(\vartheta_L)) = \zeta_L^{mB(y_1,y_2')}$. We claim that $mB(y_1,y_2') \in (q^d-1)\mathbb{Z}$. Indeed, we have

$$B(y_1,y_2') = B(\gamma y_1, y_2') = B(y_1, \gamma^{-1}y_2') = B(y_1, qy_2' + (q^d-1)y_3'),$$

for some $y_3' \in Y$. Since $y_1 \in Y^{\#}$, we have $B(y_1,(q^d-1)y_3') \in n(q^d-1)\mathbb{Z}$. It follows that $(q-1)B(y_1,y_2') \in n(q^d-1)\mathbb{Z}$. From this, we find $B(y_1,y_2') \in nr\mathbb{Z}$. Hence $mB(y_1,y_2') \in mnr\mathbb{Z} = (q^d-1)\mathbb{Z}$. This proves our claim, and we have proved that $C_L(y_1(\varpi), \bar{y}_2'(\vartheta_L) = 1$.

Next, consider the commutator $C_L(\bar{y}_2(\vartheta_L), y_1'(\varpi)) = \zeta_L^{-mB(y_2,y_1')}$. We claim now that $mB(y_2,y_1') \in (q^d-1)\mathbb{Z}$. Indeed, we have

$$B(y_2,y_1') = B(q\gamma y_2 + (q^d-1)y_3, y_1') = qB(y_2,y_1') + (q^d-1)B(y_3,y_1')$$

for some $y_3 \in Y^{\Gamma\#}$. In particular, $B(y_3, y_1') \in n\mathbb{Z}$ since $y_1' \in Y^\Gamma$. It follows that $(q-1)B(y_2, y_1') \in n(q^d-1)\mathbb{Z}$. From this we find that $B(y_2, y_1') \in nr\mathbb{Z}$, from which the claim follows. We have proved that $C_L(\bar{y}_2(\vartheta_L), y_1'(\varpi)) = 1$.

Finally, note that $(\vartheta_L, \vartheta_L)_{L,q^d-1} = 1$. Hence $C_L(\bar{y}_2(\vartheta_L), \bar{y}_2'(\vartheta_L)) = 1$. We have proved that $y_1(\varpi)$ and $\bar{y}_2(\vartheta_L)$ commute with a set of generators for T_F/T_F^1. Since $T_F^1 \in Z^\dagger(T_F)$, this suffices to prove that $y_1(\varpi)\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$. □

The previous two theorems fully characterize the subgroup $Z^\dagger(T_F)$.

Corollary 5.9. *Suppose that $y_1 \in Y$, $\bar{y}_2 \in \bar{Y}$, and $t^1 \in T_F^1$. Then $t = y_1(\varpi)\bar{y}_2(\vartheta_L)t^1$ belongs to $Z^\dagger(T_F)$ if and only if*

- *$y_1 \in Y^{\#\Gamma}$,*
- *$y_2 \in Y^\#$ for any choice of representative y_2 of $\bar{y}_2$, and*
- *$\delta_q y_2 \in (q^d-1)Y^{\Gamma\#}$ for any choice of representative y_2 of $\bar{y}_2$.*

Proof. This corollary follows directly from the previous two theorems. One important observation is that the latter two conditions do not depend upon the choice of representative $y_2 \in Y$ for a given $\bar{y}_2 \in \bar{Y}$.

Indeed, suppose that $y_2' = y_2 + (q^d-1)z$ for some $z \in Y$, so that y_2 and y_2' are representatives for $\bar{y}_2$. Since $Y^\# \subset nY$ and n divides (q^d-1), we find that $y_2 \in Y^\#$ if and only if $y_2' \in Y^\#$.

Similarly, we find that $\delta_q y_2' = \delta_q y_2 + (q^d-1)\delta_q z$. By Lemma 5.6, $\delta_q z \in Y^{\Gamma\#}$. It follows that $\delta_q y_2 \in (q^d-1)Y^{\Gamma\#}$ if and only if $\delta_q y_2' \in (q^d-1)Y^{\Gamma\#}$. □

The above corollary implies that $y_1(\varpi) \in Z^\dagger(T_F)$ for a given $y_1 \in Y$ if and only if $y_1 \in Y^{\#\Gamma}$. It also implies the following:

Corollary 5.10. *Suppose that $\bar{y}_2 \in \bar{Y}$. Then $\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$ if and only if*

$$\bar{y}_2 \in \operatorname{Im}(\operatorname{Tr}_q(\overline{Y^{\Gamma\#}}) \to \bar{Y}). \tag{5-2}$$

Proof. The previous corollary implies that $\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$ if and only if

(1) $y_2 \in Y^\#$ for some (equivalently, every) representative y_2 of $\bar{y}_2$, and

(2) $\delta_q y_2 \in (q^d-1)Y^{\Gamma\#}$ for some (equivalently, every) representative y_2 of $\bar{y}_2$.

Given these conditions and a representative y_2 of $\bar{y}_2$, there exists $w \in Y^{\Gamma\#}$ such that $\delta_q(y_2) = (q^d-1)w$. Hence $\delta_q(y_2) = \delta_q \operatorname{Tr}_q(w)$. The injectivity of δ_q implies that $y_2 = \operatorname{Tr}_q(w)$. It follows that $\bar{y}_2$ is the image of $\operatorname{Tr}_q(\bar{w})$ in $\bar{Y}$. Hence, conditions (1) and (2) imply the one condition (5-2) of this corollary.

Conversely, suppose that Equation (5-2) is satisfied. Then we may choose $w \in Y^{\Gamma\#}$ such that $\bar{y}_2$ equals the image of $\operatorname{Tr}_q(\bar{w})$ in $\bar{Y}$. Thus $y_2 = \operatorname{Tr}_q(w)$ is a representative for $\bar{y}_2$ in $\bar{Y}$. Since $\operatorname{Tr}_q(Y^{\Gamma\#}) \subset Y^\#$ by Lemma 5.6, condition (1) is satisfied. Since $\delta_q y_2 = \operatorname{Tr}_q \delta_q w = (q^d-1)w$, condition (2) is satisfied as well. Therefore, $\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$. □

The image of an isogeny. For split metaplectic tori, we found a useful characterization of $Z^\dagger(T_F)$ as the image of an isogeny on F-rational points. The same isogeny makes sense for nonsplit tori; however there is a small but important difference between the image of the isogeny and $Z^\dagger(T_F)$. We view this difference as accounting for "packets" of representations of metaplectic tori, with the same parameter.

Consider the inclusion of $\mathbb{Z}[\Gamma]$-modules $\imath\colon Y^\# \hookrightarrow Y$. Note that we use the fact that Q is a Γ-invariant quadratic form, so that $Y^\#$ is a $\mathbb{Z}[\Gamma]$-submodule. This inclusion corresponds to an isogeny $\boldsymbol{\imath}\colon \boldsymbol{T}^\# \to \boldsymbol{T}$ of algebraic tori over F. Our description of the F-rational and L-rational points for $\boldsymbol{T}$ is also valid, mutatis mutandis, for $\boldsymbol{T}^\#$. When $y \in Y^\#$ and $u \in L^\times$, we simply write $(y \otimes u)$ for the corresponding element of $\boldsymbol{T}^\#(L) \equiv Y^\# \otimes L^\times$. We choose this notation rather than $y(u)$ since we do not wish to confuse cocharacters of $\boldsymbol{T}$ with cocharacters of $\boldsymbol{T}^\#$. Since $Y^\#$ is a $\mathbb{Z}[\Gamma]$-module, we find this:

Proposition 5.11. *The torus $\boldsymbol{T}^\#$ splits over an unramified extension of F. Suppose that $y_1, y_2 \in Y^\#$. Then $(y_1 \otimes \varpi)(y_2 \otimes \vartheta_L) \in T^\# = \boldsymbol{T}^\#(F)$ if and only if*

$$y_1 \in Y^{\#\Gamma} \quad \textit{and} \quad \bar{y}_2 \in (\overline{Y^\#})^{\Gamma,q}.$$

The isogeny $\boldsymbol{\imath}$ acts on L-rational points by

$$\boldsymbol{\imath}(y \otimes u) = y(u) \quad \text{for all } y \in Y^\#,\ u \in L^\times \text{ and } (y \otimes u) \in \boldsymbol{T}^\#(L).$$

Proposition 5.12. *Suppose that $y_1 \in Y$, $\bar{y}_2 \in \bar{Y}$, and $t^1 \in T_L^1$. Then $y_1(\varpi)\bar{y}_2(\vartheta)t^1$ is an element of the image of $\imath\colon \boldsymbol{T}^\#(F) \to \boldsymbol{T}(F)$ if and only if*

- $y_1 \in Y^{\#\Gamma}$,
- *there exists a $y_2 \in Y^\#$ representing $\bar{y}_2$ such that $\delta_q y_2 \in (q^d - 1)Y^\#$, and*
- $t^1 \in T_F^1$.

Proof. Since $(n, p) = 1$, the image of $\imath$ contains T_F^1. It suffices only to consider the images $\imath((y_1 \otimes \varpi)(y_2 \otimes \vartheta_L))$ for all $y_1 \in Y^{\#\Gamma}$ and $y_2 \in Y^\#$ such that $\bar{y}_2 \in (\overline{Y^\#})^{\Gamma,q}$. □

Corollary 5.13. *Suppose that $\bar{y}_2 \in \bar{Y}$. Then $\bar{y}_2(\vartheta_L) \in \imath(\boldsymbol{T}^\#(F))$ if and only if*

$$\bar{y}_2 \in \operatorname{Im}(\operatorname{Tr}_q(\overline{Y^\#}) \to \bar{Y}).$$

Proof. If $\bar{y}_2 \in \operatorname{Im}(\operatorname{Tr}_q(\overline{Y^\#}) \to \bar{Y})$, there exists an element $y_3 \in Y^\#$ such that $\bar{y}_2$ equals the image of $\operatorname{Tr}_q(y_3)$ in $\bar{Y}$. If $y_2 = \operatorname{Tr}_q(y_3)$, then y_2 is a representative for $\bar{y}_2$ in Y. Note that $y_2 \in Y^\#$ since $y_3 \in Y^\#$. Also $\delta_q y_2 = \delta_q \operatorname{Tr}_q(y_3) = (q^d - 1)y_3 \in (q^d - 1)Y^\#$. Hence $\bar{y}_2(\vartheta_L) \in \imath(\boldsymbol{T}^\#(F))$ by the previous proposition.

Conversely, suppose that $\bar{y}_2(\vartheta_L) \in \imath(\boldsymbol{T}^\#(F))$. By the previous proposition, there exists a representative y_2 of $\bar{y}_2$ in Y such that $y_2 \in Y^\#$ and $\delta_q(y_2) \in (q^d - 1)Y^\#$. It follows that $\delta_q(y_2) = \delta_q \mathrm{Tr}_q(y_3)$ for some $y_3 \in Y^\#$. Hence $y_2 = \mathrm{Tr}_q(y_3)$. Hence $\bar{y}_2$ is contained in the image of $\mathrm{Tr}_q(\overline{Y^\#})$ in $\overline{Y}$. $\square$

The packet group. From the previous two sections, we have described the groups $Z^\dagger(T_F)$ and $\iota(\boldsymbol{T}^\#(F))$. They are quite similar, with one exception. For given $\bar{y}_2 \in \overline{Y}$, we have

- $\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$ if and only if $\bar{y}_2 \in \mathrm{Im}(\mathrm{Tr}_q(\overline{Y^{\Gamma\#}}) \to \overline{Y})$, and
- $\bar{y}_2(\vartheta_L) \in \iota(\boldsymbol{T}^\#(F))$ if and only if $\bar{y}_2 \in \mathrm{Im}(\mathrm{Tr}_q(\overline{Y^\#}) \to \overline{Y})$.

Define a finite group $P^\dagger_{\theta_{\mathbf{l}}}$ by

$$P^\dagger_{\theta_{\mathbf{l}}} = \mathrm{Im}(\mathrm{Tr}_q(\overline{Y^{\Gamma\#}}) \to \overline{Y}) \,\big/\, \mathrm{Im}(\mathrm{Tr}_q(\overline{Y^\#}) \to \overline{Y}).$$

It follows from Proposition 5.12 and Corollary 5.9 that there is a natural short exact sequence

$$1 \to \iota(\boldsymbol{T}^\#(F)) \to Z^\dagger(T_F) \to P^\dagger_{\theta_{\mathbf{l}}} \to 1. \tag{5-3}$$

However, this sequence depends upon the choice of generator $\theta_{\mathbf{l}}$ of $\mathbf{l}^\times$. We identify $P^\dagger_{\theta_{\mathbf{l}}}$ here so that this sequence is independent of the choice of generator.

The $\mathbb{Z}[\Gamma]$-modules $Y^{\Gamma\#}$ and $Y^\#$ correspond to a pair $\overline{\boldsymbol{T}}^{\Gamma\#}$ and $\overline{\boldsymbol{T}}^\#$ of $\mathbf{f}$-tori that split over $\mathbf{l}$. Moreover, the inclusions $Y^\# \subset Y^{\Gamma\#} \subset Y$ correspond to $\mathbf{f}$-isogenies of $\mathbf{f}$-tori via $\overline{\boldsymbol{T}}^\# \to \overline{\boldsymbol{T}}^{\Gamma\#} \to \overline{\boldsymbol{T}}$. The choice of generator $\theta_{\mathbf{l}}$ of $\mathbf{l}^\times$ corresponds to the identifications

$$\overline{\boldsymbol{T}}^{\Gamma\#}(\mathbf{l}) \equiv \overline{Y^{\Gamma\#}} \quad \text{and} \quad \overline{\boldsymbol{T}}^\#(\mathbf{l}) \equiv \overline{Y^\#}.$$

Furthermore, the trace map Tr_q corresponds to the norm maps. For example, there is a commutative diagram

$$\begin{tikzcd} \overline{\boldsymbol{T}}^{\Gamma\#}(\mathbf{l}) \arrow[r, "\equiv"] \arrow[d, "N_{\mathbf{l}/\mathbf{k}}"] & \overline{Y^{\Gamma\#}} \arrow[d, "\mathrm{Tr}_q"] \\ \overline{\boldsymbol{T}}^{\Gamma\#}(\mathbf{f}) \arrow[r, "\equiv"] & \overline{Y^{\Gamma\#}}. \end{tikzcd}$$

Now, Lang's theorem [1956] implies that the norm map is surjective. In other words, the commutative diagram above yields the identifications

$$\overline{\boldsymbol{T}}^{\Gamma\#}(\mathbf{f}) \equiv \mathrm{Tr}_q \overline{Y^{\Gamma\#}} \quad \text{and} \quad \overline{\boldsymbol{T}}^\#(\mathbf{f}) \equiv \mathrm{Tr}_q \overline{Y^\#}.$$

A proposition follows:

Proposition 5.14. *Define a finite group $P^\dagger$ by*

$$P^\dagger = \mathrm{Im}(\overline{\boldsymbol{T}}^{\Gamma\#}(\mathbf{f}) \to \overline{\boldsymbol{T}}(\mathbf{f})) \,\big/\, \mathrm{Im}(\overline{\boldsymbol{T}}^\#(\mathbf{f}) \to \overline{\boldsymbol{T}}(\mathbf{f})).$$

Then $P^\dagger_{\theta_{\mathbf{l}}} \cong P^\dagger$.

The construction of the group $P^\dagger$ does not depend upon the choice of generator $\theta_{\mathbf{l}}$. This yields a canonical short exact sequence

$$1 \to \iota(\boldsymbol{T}^\#(F)) \to Z^\dagger(T_F) \to P^\dagger \to 1,$$

which depends neither on the choice of uniformizing element ϖ nor on the choice of generator $\theta_{\mathbf{l}}$.

The main theorem of Langlands [1997], which parameterizes smooth characters of tori over local fields, determines isomorphisms

$$\mathscr{X}(T_F) \cong H^1_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}}) \quad \text{and} \quad \mathscr{X}(T_F^\#) \cong H^1_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}}^\#).$$

As before, the characters of the image of an isogeny can be parameterized cohomologically:

Proposition 5.15. *The Langlands parameterization yields a finite-to-one parameterization of the smooth characters of $Z^\dagger(T_F)$:*

$$1 \to \mathscr{X}(P^\dagger) \to \mathscr{X}(Z^\dagger(T_F)) \to H^1_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}}) \big/ H^1_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}} \to \hat{\mathscr{T}}^\#) \to 1.$$

Remark 5.16. To view $H^1_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}} \to \hat{\mathscr{T}}^\#)$ as a subgroup of $H^1_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}})$ as above, we must know that the map $H^0_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}}) \to H^0_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}}^\#)$ is surjective. This follows from the identifications

$$H^0_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}}) \equiv \mathrm{Hom}_{\mathbb{Z}}(Y^\Gamma, \mathbb{C}^\times) \quad \text{and} \quad H^0_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}}^\#) \equiv \mathrm{Hom}_{\mathbb{Z}}(Y^{\#\Gamma}, \mathbb{C}^\times)$$

and the fact that $Y^{\#\Gamma}$ has finite index in Y^Γ.

This leads directly, via a Stone–von Neumann theorem, to a main theorem for tame covers of unramified tori:

Theorem 5.17. *Suppose that we have tame metaplectic cover of an unramified torus given by*

$$1 \to \mu_n \to \tilde{T} \to T \to 1.$$

Then, with the sublattices $Y^\# \subset Y^{\Gamma\#} \subset Y$ defined as before and the resulting isogenies $\boldsymbol{T}^\# \to \boldsymbol{T}^{\Gamma\#} \to \boldsymbol{T}$ of unramified tori, we find that

- *there is a finite-to-one surjective map*

$$\Phi\colon \mathscr{I}_\epsilon(\tilde{T}) \to H^1_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}}) \big/ H^1_c(\mathscr{W}_{L/F}, \hat{\mathscr{T}} \to \hat{\mathscr{T}}^\#)$$

  intertwining the natural action of $H^1(\mathscr{W}_{L/F}, \hat{\mathscr{T}})$, and

- *the fibres of this map are torsors for the finite group $\mathscr{X}(P^\dagger)$, where*

$$P^\dagger = \mathrm{Im}(\overline{\boldsymbol{T}}^{\Gamma\#}(\mathbf{f}) \to \overline{\boldsymbol{T}}(\mathbf{f})) \big/ \mathrm{Im}(\overline{\boldsymbol{T}}^\#(\mathbf{f}) \to \overline{\boldsymbol{T}}(\mathbf{f})).$$

Remark 5.18. We do not know if a parameterization such as that above holds for general metaplectic tori over local fields. Namely, we have not been able to describe the center of such metaplectic tori for the case in which $\boldsymbol{T}$ is ramified or an unramified torus but the cover is not tame. We hope that such a parameterization is possible, though the packets might be substantially different.

Remark 5.19. In proving the previous theorem, we chose a uniformizing element $\varpi \in F^\times$ and a root of unity θ_L. However, this choice does not have any effect on the parameterization given above. The sublattices $Y^\#$ and $Y^{\Gamma\#}$ clearly do not depend upon such a choice. Moreover, the action of $\mathscr{X}(P^\dagger)$ on the fibres of Φ does not depend on such a choice.

6. Pseudospherical and pseudotrivial representations

We maintain all of the conventions of the previous section. In particular, we have a tame metaplectic cover of an unramified torus given by

$$1 \to \mu_n \to \tilde{T}_F \to T_F \to 1.$$

We have shown that the irreducible genuine representations of $\tilde{T}_F$ can be parameterized by the points of a homogeneous space on which $H^1(\mathscr{W}_{L/F}, \hat{\mathscr{T}})$ acts transitively. However, such a parameterization is not unique; one must choose a base point in the space of irreducible genuine representations of $\tilde{T}_F$ in order to choose a specific morphism

$$\Phi\colon \mathscr{I}_\epsilon(\tilde{T}_F) \to H^1(\mathscr{W}_{L/F}, \hat{\mathscr{T}}) \,\big/\, H^1(\mathscr{W}_{L/F}, \hat{\mathscr{T}} \to \hat{\mathscr{T}}^\#)$$

of homogeneous spaces.

In this section, we discuss the data that determines such base points. Such choices arise frequently in treatments of metaplectic groups, often as choices of square roots of -1 in $\mathbb{C}$.

The residual extension. Recall that the unramified torus $\boldsymbol{T}$ has a smooth model $\underline{\boldsymbol{T}}$ over $\mathcal{O}_F$, and $T_F^\circ = \underline{\boldsymbol{T}}(\mathcal{O}_F)$. In this case, T_F° is the maximal compact subgroup of T, and we let $\tilde{T}_F^\circ$ be its preimage in $\tilde{T}_F$. Also, $\overline{\boldsymbol{T}}$ denotes the special fibre of $\underline{\boldsymbol{T}}$ that is a torus over $\mathbf{f}$. Recall that $\boldsymbol{T}'$ is a central extension of $\boldsymbol{T}$ by $\boldsymbol{K}_2$. Pushing forward via the tame symbol led to the tame central extension

$$1 \to \mathbf{f}^\times \to T_F^t \to T_F \to 1.$$

We write $T_F^{t\circ}$ for the preimage of T_F° in T_F^t.

Deligne and Brylinski, in [2001, Section 12.11], construct an extension $\overline{\boldsymbol{T}}'$ of $\overline{\boldsymbol{T}}$ by $\boldsymbol{G}_m$ (in the category of groups over $\mathbf{f}$). We call $\overline{\boldsymbol{T}}'$ the *residual extension* associated to $\boldsymbol{T}'$. The residual extension fits into the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{f}^\times \arrow[r] \arrow[d] & T_F^{t\circ} \arrow[r] \arrow[d] & T_F^\circ \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbf{f}^\times \arrow[r] & \overline{T}'_{\mathbf{f}} \arrow[r] & \overline{T}_{\mathbf{f}} \arrow[r] & 1. \end{tikzcd}$$

Here, the map from $\mathbf{f}^\times$ to itself is the identity, the map from T_F° to $\overline{T}_{\mathbf{f}}$ is the reduction map, and the diagram identifies the top row with the pullback of the bottom row via reduction.

As an extension of $\overline{\boldsymbol{T}}$ by $\boldsymbol{G}_m$ over $\mathbf{f}$, the group $\overline{\boldsymbol{T}}'$ is an algebraic torus over $\mathbf{f}$. Note that the category of extensions of $\overline{\boldsymbol{T}}$ by $\boldsymbol{G}_m$, in the category of groups over $\mathbf{f}$, is equivalent to the category of extensions of Y by $\mathbb{Z}$, in the category of $\mathbb{Z}[\Gamma]$-modules (where $\mathbb{Z}$ is given the trivial module structure). In this way, the construction of [Brylinski and Deligne 2001, Section 12.11] associates an extension of Y by $\mathbb{Z}$ to any extension of an unramified torus $\boldsymbol{T}$ by $\boldsymbol{K}_2$.

Remark 6.1. Recall that $\tilde{Y}$ is a Γ-equivariant extension of Y by $L^\times$, constructed as a functorial invariant of the extension $\boldsymbol{T}'$ of $\boldsymbol{T}$ by $\boldsymbol{K}_2$. Let Y' be the extension of Y by $\mathbb{Z}$, obtained by pushing forward $\tilde{Y}$ via the valuation map $L^\times \to \mathbb{Z}$, that is,

$$0 \to \mathbb{Z} \to Y' \to Y \to 0.$$

We do not know whether this extension is naturally isomorphic to the exact sequence of cocharacter groups of the residual extension of tori described above

Definition 6.2. Let $\mathrm{Spl}(\overline{\boldsymbol{T}}')$ denote the set of splittings, in the category of algebraic groups over $\mathbf{f}$, of the short exact sequence

$$1 \to \boldsymbol{G}_m \to \overline{\boldsymbol{T}}' \to \overline{\boldsymbol{T}} \to 1.$$

We say the extension $\boldsymbol{T}'$ of the unramified torus $\boldsymbol{T}$ is a *residually split extension* if $\mathrm{Spl}(\overline{\boldsymbol{T}}')$ is nonempty.

In particular, if $\boldsymbol{T}$ is a split torus, then $\boldsymbol{T}'$ is residually split.

Proposition 6.3. *If* $\mathrm{Spl}(\overline{\boldsymbol{T}}')$ *is nonempty, then* $\mathrm{Spl}(\overline{\boldsymbol{T}}')$ *is a torsor for the abelian group* X^Γ.

Proof. Any two algebraic splittings are related by an element of $\mathrm{Hom}_{\mathbf{f}}(\overline{\boldsymbol{T}}, \boldsymbol{G}_m)$. This group may be identified with the Γ-fixed characters of $\boldsymbol{T}$. □

Pseudospherical representations. Suppose now that $\boldsymbol{T}'$ is a residually split extension of $\boldsymbol{T}$ by $\boldsymbol{K}_2$. Fix a splitting $s \in \mathrm{Spl}(\overline{\boldsymbol{T}}')$. The splitting lifts to a splitting $\sigma : T_F^\circ \to T_F^{t\circ}$. Pushing forward via the m-th power map, we may also view σ as a splitting $T_F^\circ \to \tilde{T}_F^\circ$. From such a splitting s, we let $\theta_s^\circ \colon \tilde{T}_F^\circ \to \mathbb{C}^\times$ denote the character obtained by projecting onto μ_n (via the splitting σ) and then applying the injective homomorphism $\epsilon\colon \mu_n \to \mathbb{C}^\times$.

Let $Z_{\tilde{T}_F}(\tilde{T}_F^\circ)$ be the centralizer of $\tilde{T}_F^\circ$ in $\tilde{T}_F$.

Proposition 6.4. *The group $Z_{\tilde{T}_F}(\tilde{T}_F^\circ)$ is the preimage of a subgroup $Z_{T_F}^\dagger(T_F^\circ) \subset T_F$. Consider the valuation map*

$$\mathrm{val}\colon T_F \to Y^\Gamma$$

whose kernel is T_F°. Then $Z_{T_F}^\dagger(T_F^\circ)$ is equal to the preimage of $Y^{\#\Gamma}$.

Proof. Since $Z_{T_F}^\dagger(T_F^\circ) \supset T_F^\circ$, it suffices to identify the set of $y \in Y^\Gamma$ such that

$$C_L(y(\varpi), \bar{y}'(\vartheta_L)) = 1 \quad \text{for all } \bar{y}' \in \bar{Y}^{\Gamma,q}.$$

In fact, the set of such y has been identified in the proofs of Theorems 5.7 and 5.8. The above condition is satisfied if and only if $y \in Y^{\#\Gamma}$. □

Corollary 6.5. *The group $Z_{\tilde{T}_F}(\tilde{T}_F^\circ)$ is abelian.*

Proof. As $Z_{\tilde{T}_F}(\tilde{T}_F^\circ)$ is the centralizer of the abelian group $\tilde{T}_F^\circ$, it suffices to prove that $C(y(\varpi), y'(\varpi)) = 1$ for all $y, y' \in Y^{\#\Gamma}$. This is proved in the beginning of the proof of Theorem 5.8. □

Directly following [Savin 2004, Section 4], we find this:

Proposition 6.6. *There is a natural bijection between the following two sets*:

- *The set $\mathcal{I}_{s,\epsilon}^{\mathrm{sph}}(\tilde{T}_F)$ of pseudospherical irreducible representations of $\tilde{T}_F$ (for the splitting s). These are the genuine irreducible representations of $\tilde{T}_F$ whose restriction to $\tilde{T}_F^\circ$ via the splitting s contains a nontrivial θ_s°-isotypic component.*
- *The set of extensions of θ_s° to the group $Z_{\tilde{T}_F}(\tilde{T}_F^\circ)$.*

Namely, if (π, V) is a pseudospherical irreducible representation, its θ_s°-isotypic subrepresentation is an extension of θ_s° to the group $Z_{\tilde{T}_F}(\tilde{T}_F^\circ)$. Conversely, given such an extension θ_s^1 of θ_s° to a character of $Z_{\tilde{T}_F}(\tilde{T}_F^\circ)$, the induced representation $\mathrm{Ind}_{Z_{\tilde{T}_F}(\tilde{T}_F^\circ)}^{\tilde{T}_F} \theta_s^1$ is a pseudospherical irreducible representation.

One may rephrase the bijection above slightly: the splitting s yields an injective homomorphism from T_F° onto a normal subgroup of $Z_{\tilde{T}_F}(\tilde{T}_F^\circ)$. This fits into a commutative diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & 1 \arrow[r] \arrow[d] & T_F^\circ \arrow[r] \arrow[d, "s"] & T_F^\circ \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[d] & Z_{\tilde{T}_F}(\tilde{T}_F^\circ) \arrow[r] \arrow[d] & Z^\dagger_{T_F}(T_F^\circ) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[d] & \tilde{Y}^{\#\Gamma} \arrow[r] \arrow[d] & Y^{\#\Gamma} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

with exact rows and columns. Hence, the splitting s determines an extension $\tilde{Y}^{\#\Gamma}$ of $Y^{\#\Gamma}$ by μ_n. A standard diagram chase now yields this:

Proposition 6.7. *There is a natural bijection*

$$\mathscr{I}^{\mathrm{sph}}_{s,\epsilon}(\tilde{T}_F) \leftrightarrow \mathscr{X}_\epsilon(\tilde{Y}^{\#\Gamma}).$$

Corollary 6.8. *The space $\mathscr{I}^{\mathrm{sph}}_{s,\epsilon}(\tilde{T}_F)$ is naturally a torsor for the complex algebraic torus $\mathscr{X}(Y^{\#\Gamma})$.*

Remark 6.9. One may view $\mathscr{X}_\epsilon(\tilde{Y}^{\#\Gamma})$ as the set of irreducible representations of a "quantum torus". Indeed, the ring

$$\mathbb{C}_\epsilon[\tilde{Y}^{\#\Gamma}] = \mathbb{C}[\tilde{Y}^{\#\Gamma}] \big/ \langle \zeta - \epsilon(\zeta) \rangle_{\zeta \in \mu_n},$$

can be viewed as (the coordinate ring of) a quantum complex torus. This torus, which we call $\hat{\mathscr{T}}^{\#\Gamma}_\epsilon$, is the quantization of a complex torus, at a root of unity. Quasicoherent sheaves on this quantum torus (that is, modules over its coordinate ring) correspond naturally to pseudospherical representations of $\tilde{T}_F$.

Pseudotrivial representations. In many practical situations, the extension $\tilde{Y}^{\#\Gamma}$ of $Y^{\#\Gamma}$ by μ_n splits over a quite large submodule of $Y^{\#\Gamma}$. For example, in many cases, the extension splits over $Y^{\#\Gamma} \cap 2Y$.

Suppose that $V \subset Y^{\#\Gamma}$ is a finite index subgroup, endowed with a splitting υ of the resulting exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \mu_n \arrow[r] & \tilde{V} \arrow[r] & V \arrow[l, bend right, "\upsilon"'] \arrow[r] & 1.
\end{tikzcd}$$

Let $U = Y^{\#\Gamma}/V$ denote the quotient. The splitting υ yields an extension of finite abelian groups given by $1 \to \mu_n \to \tilde{U} \to U \to 1$.

Pulling back yields natural inclusions $\mathscr{X}_\epsilon(\tilde{U}) \hookrightarrow \mathscr{X}_\epsilon(\tilde{Y}^{\#\Gamma}) \equiv \mathscr{I}^{\text{sph}}_{s,\epsilon}(\tilde{T}_F)$.

Therefore, within the set of pseudospherical representations of $\tilde{T}_F$, we find a *finite* set of "pseudotrivial" representations (relative to the choice of splitting subgroup (V, υ) of $Y^{\#\Gamma}$:

Definition 6.10. The genuine pseudotrivial representations of $\tilde{T}_F$ are those irreducible pseudospherical genuine representations that are in the image of $\mathscr{X}_\epsilon(\tilde{U})$. This definition depends upon the choice of

- the splitting s (to determine the pseudospherical representations), and
- the splitting subgroup (V, υ) (to determine the pseudotrivial representations).

Remark 6.11. Most often, one chooses a pseudotrivial "base point" in the space $\mathscr{I}_\epsilon(\tilde{T}_F)$. Very often (see the examples of [Savin 2004]), $\tilde{U}$ is a finite abelian group of exponent 4. It follows that pseudotrivial representations may often be given by specifying certain characters of an abelian group of exponent 4. This explains why choosing fourth roots of unity is often necessary in work on metaplectic groups.

7. Tori over $\mathbb{R}$

In this section, we fix these notations:

- $\boldsymbol{T}$ will be a torus over $\mathbb{R}$, and $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$. X and Y will be the resulting character and cocharacter groups, viewed as $\mathbb{Z}[\Gamma]$-modules.
- $\boldsymbol{T}'$ will be an extension of $\boldsymbol{T}$ by $\boldsymbol{K}_2$ in $\mathfrak{Sp}_{\mathbb{R}}$.
- $(Q, \tilde{Y})$ will be the Deligne–Brylinski invariants of $\boldsymbol{T}'$, and B will be the symmetric bilinear form associated to Q.
- We fix $n = 2$, so that $\mathbb{R}$ has enough n-th roots of unity.
- If W is a subgroup of Y, then we write W^Γ for the subgroup of Γ-fixed elements of W. We also define
$$W^\# = \{y \in Y \text{ such that } B(y, w) \in 2\mathbb{Z} \text{ for all } w \in W\}.$$
- $\epsilon \colon \boldsymbol{\mu}_2(\mathbb{R}) \to \mathbb{C}^\times$ will be the unique injective character.
- We view $T = \boldsymbol{T}(\mathbb{R})$ as a real Lie group. We write T° for the connected component of the identity element, and $\pi_0 T$ for the component group of T.
- The extension $\boldsymbol{T}'$, and the quadratic Hilbert symbol, yields an extension of Lie groups given by
$$1 \to \mu_2 \to \tilde{T} \to T \to 1.$$

We are interested in parameterizing the irreducible genuine representations of $\tilde{T}$, which, as before, we call $\mathcal{I}_\epsilon(\tilde{T})$.

Structure of metaplectic tori over $\mathbb{R}$. There is a short exact sequence

$$1 \to T^\circ \to T \to \pi_0 T \to 1$$

of Lie groups. Let $\boldsymbol{T}^\Gamma$ be the split real torus with cocharacter group Y^Γ. Let T^Γ denote the real points of $\boldsymbol{T}^\Gamma$. Then, we find that $\pi_0 T^\Gamma$ is canonically isomorphic to $\overline{Y^\Gamma} = Y^\Gamma \otimes_{\mathbb{Z}} \mu_2 \equiv Y^\Gamma / 2Y^\Gamma$. Moreover, the inclusion of $\mathbb{R}$-tori from $\boldsymbol{T}^\Gamma$ into $\boldsymbol{T}$ induces a surjection $\pi_0 T^\Gamma \twoheadrightarrow \pi_0 T$ of component groups. Therefore, every element t of T has a (often nonunique) decomposition $t = t^\circ \bar{y}(-1)$ for some $t^\circ \in T^\circ$ and $\bar{y} \in \overline{Y^\Gamma}$. In other words, there is a natural surjective homomorphism $\overline{Y^\Gamma} \to \pi_0 T$.

Now, we consider the metaplectic cover $1 \to \mu_2 \to \tilde{T} \to T \to 1$ of T. The commutator $C\colon T \times T \to \mu_2$ is bimultiplicative and continuous. It follows that the commutator is trivial when either of its inputs is in T°. Hence:

Proposition 7.1. *T° is a subgroup of $Z^\dagger(T)$.*

Description of the center. It follows from the previous proposition that to describe $Z^\dagger(T)$, it suffices to describe its image in T/T°. Hence, it suffices to determine the $\bar{y} \in \overline{Y^\Gamma}$ for which $\bar{y}(-1) \in Z^\dagger(T)$. We must be able to compute the commutator $C(\bar{y}(-1), \bar{y}'(-1))$ for arbitrary $\bar{y}, \bar{y}' \in \overline{Y^\Gamma}$.

Here, we note that such elements $\bar{y}(-1)$ and $\bar{y}'(-1)$ are contained in the real points of the maximal $\mathbb{R}$-split subtorus $\boldsymbol{T}^\Gamma \hookrightarrow \boldsymbol{T}$. Restricting the central extension of $\boldsymbol{T}$ by $\boldsymbol{K}_2$ to the split subtorus $\boldsymbol{T}^\Gamma$, the formula of [Brylinski and Deligne 2001, Corollary 3.14] is valid for computing commutators.

Proposition 7.2. *If $y, y' \in Y^\Gamma$, then $C(\bar{y}(-1), \bar{y}'(-1)) = (-1)^{B(y,y')}$.*

Proof. This follows directly from [Brylinski and Deligne 2001, Corollary 3.14] and the Hilbert symbol over $\mathbb{R}$, which satisfies $(-1,-1)_{\mathbb{R},2} = -1$. □

Proposition 7.3. *Given $\bar{y} \in \overline{Y^\Gamma}$, we have $\bar{y}(-1) \in Z^\dagger(T)$ if and only if every representative y of $\bar{y}$ in Y satisfies $y \in Y^{\Gamma\#\Gamma}$.*

Proof. Suppose $\bar{y}, \bar{y}' \in \overline{Y^\Gamma}$. Let y be a representative of $\bar{y}$ in Y. The commutator has been computed as $C(\bar{y}(-1), \bar{y}'(-1)) = (-1)^{B(\bar{y},\bar{y}')}$. Thus, we find that $C(\bar{y}(-1), \bar{y}'(-1)) = 1$ for all $\bar{y}' \in \overline{Y^\Gamma}$ if and only if $B(y, y') \in 2\mathbb{Z}$ for all representatives y of all $\bar{y}' \in \overline{Y^\Gamma}$. This occurs if and only if $B(y, y') \in 2\mathbb{Z}$ for all $y' \in Y^\Gamma$, that is, $y \in Y^{\Gamma\#}$.

Thus, $\bar{y}(-1) \in Z^\dagger(T)$ for $\bar{y} \in \overline{Y^\Gamma}$ if and only if $y \in Y^{\Gamma\#} \cap Y^\Gamma = Y^{\Gamma\#\Gamma}$. □

Corollary 7.4. *Let $\boldsymbol{T}^{\Gamma\#}$ be the real torus with cocharacter group $Y^{\Gamma\#}$. Suppose $T^{\Gamma\#} = \boldsymbol{T}^{\Gamma\#}(\mathbb{R})$. Then the quotient $Z^\dagger(T)/T^\circ$ is isomorphic to* $\mathrm{Im}(\pi_0 T^{\Gamma\#} \to \pi_0 T)$.

Proof. The diagram

$$\begin{tikzcd} Y^{\Gamma\#\Gamma} \otimes_{\mathbb{Z}} \mu_2 \arrow[r, "\eta"] \arrow[d, "p^{\Gamma\#}"] & Y^{\Gamma} \otimes_{\mathbb{Z}} \mu_2 \arrow[d, "p"] \\ \pi_0 T^{\Gamma\#} \arrow[r, "\rho"] & \pi_0 T \end{tikzcd}$$

of finite abelian groups commutes. The previous proposition demonstrates that $Z^{\dagger}(T)/T^{\circ}$ can be identified with the image of $p \circ \eta$. The commutativity of the above diagram, together with the surjectivity of $p^{\Gamma\#}$, implies that this image is the same as the image of ρ. $\square$

The image of an isogeny. The inclusion $Y^{\#} \hookrightarrow Y$ of $\mathbb{Z}[\Gamma]$-modules corresponds, as in the nonarchimedean case, to an isogeny $\boldsymbol{\iota}\colon \boldsymbol{T}^{\#} \to \boldsymbol{T}$ of tori over $\mathbb{R}$. We are interested in the resulting continuous homomorphism $\iota\colon T^{\#} \to T$ of real Lie groups. Since $\boldsymbol{\iota}$ is an isogeny, we find that $\iota(T^{\#}) \supset T^{\circ}$. Thus, in order to fully describe $\iota(T^{\#})$, it suffices to determine the $\bar{y} \in \overline{Y^{\Gamma}}$ for which $\bar{y}(-1) \in \iota(T^{\#})$.

Proposition 7.5. *Let $\bar{y} \in \overline{Y^{\Gamma}}$. Then $\bar{y}(-1) \in \iota(T^{\#})$ if and only if $\bar{y} \in \mathrm{Im}(\overline{Y^{\#\Gamma}} \to \overline{Y^{\Gamma}})$.*

Proof. We find that $\bar{y}(-1) \in \iota(T^{\#})$ if and only if there exists a $y \in Y^{\#\Gamma}$ that represents $\bar{y}$. The proposition follows. $\square$

Corollary 7.6. *We can identify the quotient $\iota(T^{\#})/T^{\circ}$ with* $\mathrm{Im}(\pi_0 T^{\#} \to \pi_0 T)$.

Comparing the image of the isogeny ι to the group $Z^{\dagger}(T)$ yields a short exact sequence $1 \to \iota(T^{\#}) \to Z^{\dagger}(T) \to P^{\dagger} \to 1$, where we may identify the finite group

$$P^{\dagger} \equiv \mathrm{Im}(\pi_0 T^{\Gamma\#} \to \pi_0 T) \big/ \mathrm{Im}(\pi_0 T^{\#} \to \pi_0 T).$$

Parameterization. As for the case of nonarchimedean fields, we choose to parameterize the genuine irreducible representations of $\tilde{T}$ through a finite-to-one map and a description of the fibres. Over $\mathbb{R}$, the previous two sections imply that the space $\mathcal{I}_{\epsilon}(\tilde{T})$ can be identified (via Theorem 3.1) with the complex variety of genuine characters $\mathcal{X}_{\epsilon}(Z(\tilde{T}))$. This is a torsor for the complex algebraic group of characters $\mathcal{X}(Z^{\dagger}(T))$. There is a short exact sequence

$$1 \to \mathcal{X}(P^{\dagger}) \to \mathcal{X}(Z^{\dagger}(T)) \to H^1_c(\mathcal{W}_{\mathbb{R}}, \hat{\mathcal{T}}) \big/ H^1_c(\mathcal{W}_{\mathbb{R}}, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^{\#}) \to 1.$$

Hence:

Theorem 7.7. *Suppose that we are given a metaplectic cover*

$$1 \to \mu_n \to \tilde{T} \to T \to 1$$

of a real torus. Then, with the sublattices $Y^{\#} \subset Y^{\Gamma\#} \subset Y$ defined as before and the resulting isogenies $\boldsymbol{T}^{\#} \to \boldsymbol{T}^{\Gamma\#} \to \boldsymbol{T}$, we find that

- *there is a finite-to-one surjection*

$$\Phi\colon \mathcal{I}_\epsilon(\tilde{T}) \to H^1_c(\mathcal{W}_{\mathbb{R}}, \hat{\mathcal{T}}) \big/ H^1_c(\mathcal{W}_{\mathbb{R}}, \hat{\mathcal{T}} \to \hat{\mathcal{T}}^\#)$$

that intertwines the natural action of $H^1(\mathcal{W}_{\mathbb{R}}, \hat{\mathcal{T}})$, *and*

- *the fibres of this map are torsors for the finite group* $\mathcal{X}(P^\dagger)$, *where*

$$P^\dagger = \mathrm{Im}(\pi_0 T^{\Gamma\#} \to \pi_0 T) \big/ \mathrm{Im}(\pi_0 T^\# \to \pi_0 T).$$

Note that this theorem is quite similar to the parameterization of $\mathcal{I}_\epsilon(\tilde{T})$ for tame covers of unramified tori over nonarchimedean local fields. The only difference is that points of residual tori are replaced by component groups.

Acknowledgments

I would like to thank Jeffrey Adams, Gordan Savin, and the referee for providing some advice and insight related to this paper. In addition, I am thankful for the hospitality and excellent working environment provided by the Hausdorff Institute for Mathematics in Bonn during the preparation of this paper. I thank the University of Michigan for its hospitality while this paper was being finished, during which Brian Conrad and Stephen DeBacker provided helpful conversations.

I am heavily indebted to Pierre Deligne, who read an early draft of this paper and gave extensive helpful comments. His generosity has led to significant improvements in presentation and content.

References

[Adams et al. 2007] J. Adams, D. Barbasch, A. Paul, P. Trapa, and D. A. Vogan, Jr., "Unitary Shimura correspondences for split real groups", *J. Amer. Math. Soc.* **20**:3 (2007), 701–751. MR 2008i:22008 Zbl 1114.22009

[Borovoi 1998] M. Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132:**626, Amer. Math. Soc., Providence, R.I., 1998. MR 98j:20061 Zbl 0918.20037

[Brylinski and Deligne 2001] J.-L. Brylinski and P. Deligne, "Central extensions of reductive groups by $\mathbf{K}_2$", *Publ. Math. Inst. Hautes Études Sci.* 94 (2001), 5–85. MR 2004a:20049 Zbl 1093.20027

[Esnault et al. 1998] H. Esnault, B. Kahn, M. Levine, and E. Viehweg, "The Arason invariant and mod 2 algebraic cycles", *J. Amer. Math. Soc.* **11**:1 (1998), 73–118. MR 98d:14010 Zbl 1025.11009

[Hoffmann and Spitzweck 2007] N. Hoffmann and M. Spitzweck, "Homological algebra with locally compact abelian groups", *Adv. Math.* **212**:2 (2007), 504–524. MR 2329311 Zbl 1123.22002

[Kazhdan and Patterson 1984] D. A. Kazhdan and S. J. Patterson, "Metaplectic forms", *Inst. Hautes Études Sci. Publ. Math.* 59 (1984), 35–142. MR 85g:22033 Zbl 0559.10026

[Kottwitz and Shelstad 1999] R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque **255**, Société Mathématique de France, Paris, 1999. MR 2000k:22024 Zbl 0958.22013

[Labesse 1985] J.-P. Labesse, "Cohomologie, L-groupes et fonctorialité", *Compositio Math.* **55**:2 (1985), 163–184. MR 86j:11117 Zbl 0595.12004

[Lang 1956] S. Lang, "Algebraic groups over finite fields", *Amer. J. Math.* **78** (1956), 555–563. MR 19,174a Zbl 0073.37901

[Langlands 1997] R. P. Langlands, "Representations of abelian algebraic groups", *Pacific J. Math.* Special Issue (1997), 231–250. MR 99b:11125

[Milnor 1971] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies **72**, Princeton University Press, 1971. MR 50 #2304 Zbl 0237.18005

[Moore 1964] C. C. Moore, "Extensions and low dimensional cohomology theory of locally compact groups. I, II", *Trans. Amer. Math. Soc.* **113** (1964), 40–63. MR 30 #2106 Zbl 0131.26902

[Ono 1961] T. Ono, "Arithmetic of algebraic tori", *Ann. of Math.* (2) **74** (1961), 101–139. MR 23 #A1640 Zbl 0119.27801

[Quillen 1973] D. Quillen, "Higher algebraic K-theory, I", pp. 85–147 in *Algebraic K-theory, I: Higher K-theories* (Seattle, WA, 1972), edited by H. Bass, Lecture Notes in Math. **341**, Springer, Berlin, 1973. MR 49 #2895 Zbl 0292.18004

[Savin 2004] G. Savin, "On unramified representations of covering groups", *J. Reine Angew. Math.* **566** (2004), 111–134. MR 2005a:22014 Zbl 1032.22006

[Tate 1979] J. Tate, "Number theoretic background", pp. 3–26 in *Automorphic forms, representations and L-functions, II* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, R.I., 1979. MR 80m:12009 Zbl 0422.12007

Received November 7, 2008.

MARTIN H. WEISSMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA CRUZ, CA 95064
UNITED STATES

weissman@ucsc.edu
http://people.ucsc.edu/~weissman/

Guidelines for Authors

Authors may submit manuscripts at pjm.math.berkeley.edu/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 241 No. 1 May 2009

0030-8730(200905)241:1;1-C